The Structure of Orthomorphism Graph of $\mathbb{Z}_2 \times \mathbb{Z}_4$

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Abstract: In this paper, we gave a theoretical proof of the fact that Orthomorphism graph of group $\mathbb{Z}_2 \times \mathbb{Z}_4$ has maximal clique 2, by determining the structure of the graph.

Keywords: Orthomorphism Graph
2020 Mathematical Subject Classification: 05B15

1 Introduction

Let G be a group. A bijection θ: $G \to G$ for which map ϕ: $x \mapsto x^{-1}\theta(x)$ is also a bijection of G is called *Orthomorphism of G*. Two orthomorphisms θ_1 and θ_2 of G are called *orthogonal*, written $\theta_1 \perp \theta_2$, if $x \mapsto \theta_1(x)^{-1}\theta_2(x)$ is a bijection of G. An orthomorphism of a group which fixes identity element of the group is called *normalised* orthomorphism. Now onwards by an orthomorphism we mean normalised orthomorphism. We denote the set of orthomorphism of a group G by *Orth(G)*. A graph in which vertices are orthomorphisms of G and adjacency being synonymous with orthogonality is called *orthomorphism graph* of G, which is also denoted by $\mathrm{Orth}(G)$. The order of the largest complete subgraph of a graph is called *clique number* of the graph. Clique number of orthomorphism graph of a group G is denoted by $\omega(G)$. In this paper, we prove that $\omega(\mathbb{Z}_2 \times \mathbb{Z}_4) = 2$.

In 1961, Johnson, Dulmage, and Mendelsohn showed that $|Orth(\mathbb{Z}_2 \times \mathbb{Z}_4)| = 48$ and $\omega(\mathbb{Z}_2 \times \mathbb{Z}_4) = 2$, via a computer search. In 1964, through exhaustive hand computation, Chang, Hsiang, and Tai [1] also found that $\omega(\mathbb{Z}_2 \times \mathbb{Z}_4) = 2$ and in 1986, via a computer search, Jungnickel and Grams [2] also confirm above fact. In 1992, Evans and Perkel found that $Orth(\mathbb{Z}_2 \times \mathbb{Z}_4)$ consists of 12 disjoint 4-cycles using Cayley (a forerunner of the computer algebra system Magma)[3] and asked for theoretical proof of this fact [3, Problem 19]. In 2021, Evans gave a theoretical proof of this in [5]. In this paper we gave another proof of this fact.

An *automorphisms* A of $\mathrm{Orth}(G)$ is a bijection on $\mathrm{Orth}(G)$ such that $A(\theta_1) \perp A(\theta_2)$ if and only if $\theta_1 \perp \theta_2$ where $\theta_1, \theta_2 \in \mathrm{Orth}(G)$.

For $f \in \mathrm{Aut}(G)$, the group of automorphism of G, the map H_f: $\mathrm{Orth}(G) \to \mathrm{Orth}(G)$ defined as $H_f(\theta) = f\theta f^{-1}$ is known as *homology* of $\mathrm{Orth}(G)$. Homology is an example of automorphism of $\mathrm{Orth}(G)$ [4, Theorem 8.6, p.206]. Any unexplained notation used in the paper is from [4].

2 Basic Results

Suppose $G = \{g_1 = e, g_2, \ldots, g_n\}$ is a finite group. e is identity element of G. Suppose τ denotes the regular left permutation representation of G. Let us identify G with $\tau(G)$. So $G \leqslant Sym(G)$. Further, let us identify $Sym(G)$ with $Sym\{1, 2, \ldots, n\}$ (denoted as S_n) by identifying g_i with i. Clearly, if $\theta \in \mathrm{Orth}(G)$, then $\theta \in$ Sym{2,3,. . .,n}(denoted as S_{n-1}). For $\theta \in \mathrm{Orth}(G)$, the map ϕ_θ: $G \to G$ defined as $\phi_\theta(x) = x^{-1}\theta(x)$ is called the *complete mapping* associated with θ. Clearly $\phi_\theta \in S_n$. Also note that $\theta(x) = x$ if and only if x is identity element of G.

Remark. For $\theta \in Orth(\mathbb{Z}_2 \times \mathbb{Z}_4)$, take $a, y \in \mathbb{Z}_2 \times \mathbb{Z}_4$ such that $o(a) = 4$ and $o(y) = o(\theta(y)) = 2$. Then $\{x \in \mathbb{Z}_2 \times \mathbb{Z}_4 \mid o(x) = 4\} = \{a, ay, a\theta(y), ay\theta(y)\}$ and $\{x \in \mathbb{Z}_2 \times \mathbb{Z}_4 \mid o(x) = 2\} = \{y, \theta(y), y\theta(y)\}$.

Lemma 2.1. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ be a group and θ is a map from G to G. Define A_{ij} $i, j \in \{2, 4\}$ as follows*

$$\begin{aligned}
A_{44} =&\{x \in G \mid o(x) = 4, o(\theta(x)) = 4\},\\
A_{42} =&\{x \in G \mid o(x) = 4, o(\theta(x)) = 2\},\\
A_{24} =&\{x \in G \mid o(x) = 2, o(\theta(x)) = 4\},\\
A_{22} =&\{x \in G \mid o(x) = 2, o(\theta(x)) = 2\}.
\end{aligned}$$

(1) θ is a bijection fixing identity element of G if and only if

(a) $\{\theta(x) \mid x \in A_{44}\} \sqcup \{\theta(x) \mid x \in A_{24}\} = \{x \in G \mid o(x) = 4\}$, *and*

(b) $\{\theta(x) \mid x \in A_{42}\} \sqcup \{\theta(x) \mid x \in A_{22}\} = \{x \in G \mid o(x) = 2\}$.

(2) θ is an orthomorphism if and only if θ is bijection fixing identity and

(a) $\{x^{-1}\theta(x) \mid x \in A_{42}\} \sqcup \{x^{-1}\theta(x) \mid x \in A_{24}\} = \{x \in G \mid o(x) = 4\}$, *and*

(b) $\{x^{-1}\theta(x) \mid x \in A_{44}\} \sqcup \{x^{-1}\theta(x) \mid x \in A_{22}\} = \{x \in G \mid o(x) = 2\}$.

Proof. Follows from the definition of bijection and orthomorphism. □

Corollary 2.2. *If $\theta \in Orth(\mathbb{Z}_2 \times \mathbb{Z}_4)$, then* $|A_{44}| = |A_{42}| = |A_{24}| = 2$ *and* $|A_{22}| = 1$.

Corollary 2.3. *For $\theta \in Orth(\mathbb{Z}_2 \times \mathbb{Z}_4)$*

(i) $|A_{44} \cap \theta(A_{44})| = |A_{44} \cap \theta(A_{24})| = |A_{42} \cap \theta(A_{44})| = |A_{42} \cap \theta(A_{24})| = 1$.

(ii) If $A_{22} = \{x\}$, *then* $\theta(A_{42}) = \{x, x\theta(x)\}$ *and* $A_{24} = \{\theta(x), x\theta(x)\}$.

(iii) $\{x^{-1}\theta(x) \mid x \in A_{44}\} = \{x, \theta(x)\}$.

Proof. (i) From Corollary 2.2, exactly two element of order 4 will map to order 4 elements. Since $a^{-1}(ay) = (ay)^{-1}a = y$, where y is an element of order 2 and $a \in A_{44}$, so $A_{44} \neq \theta(A_{44})$. Suppose $A_{44} \cap \theta(A_{44}) = \phi$. Then if $\{a, ay\} \in A_{44}$ then $\{az, azy\} \in \theta(A_{44})$, where $z \neq y$ and z, y are elements of order 2. Then $a^{-1}(az) = (ay)^{-1}(azy) = z$ or $a^{-1}(azy) = (ay)^{-1}(az) = zy$ which contradicts the bijectivity of ϕ_θ. Hence $|A_{44} \cap \theta(A_{44})| = 1$. As $\theta(A_{44})$ is in partition with $\theta(A_{24})$ so $|A_{44} \cap \theta(A_{24})| = 1$, also A_{44} and A_{42} are in partition, therefore $|A_{42} \cap \theta(A_{44})| = |A_{42} \cap \theta(A_{24})| = 1$.

(ii) Suppose $A_{22} = \{x\}$. Then $\theta(x) \neq x$. So $\{x, \theta(x), x\theta(x)\}$ is the set of all elements of order 2 in $\mathbb{Z}_2 \times \mathbb{Z}_4$. $\theta(A_{42}) \sqcup \theta(A_{22}) = \{x, \theta(x), x\theta(x)\}$. Clearly, $\theta(A_{42}) = \{x, x\theta(x)\}$. As $x \notin A_{24}$, so $A_{24} = \{\theta(x), x\theta(x)\}$.
(iii) If θ is an orthomorphism then, $\{x^{-1}\theta(x) \mid x \in A_{44}\} \sqcup \{x^{-1}\theta(x) \mid x \in A_{22}\} = \{x, \theta(x), x\theta(x)\}$. Therefore $\{x^{-1}\theta(x) \mid x \in A_{44}\} = \{x, \theta(x)\}$. □

Proposition 2.4. *If* $\theta \in Orth(\mathbb{Z}_2 \times \mathbb{Z}_4)$ *and* $x \in A_{22}$, $a \in A_{44} \setminus \theta(A_{44})$, *then* $\theta(a) = ax$. *Moreover* $A_{44} = \{a, ax\}, \theta(A_{44}) = \{ax, ax\theta(x)\}$ *and* $A_{42} = \{ax\theta(x), a\theta(x)\}$.

Proof. Suppose $x \in A_{22}$ and $a \in A_{44} \setminus \theta(A_{44})$. By Corollary 2.3 (iii), $a^{-1}\theta(a) \in \{x, \theta(x)\}$. Assume $\theta(a) = a\theta(x)$. By Corollary 2.3 (i), $A_{44} = \{a, a\theta(x)\}$. Then by Corollary 2.3 (iii), $\theta(a\theta(x)) = ax\theta(x)$ and $A_{42} = \{ax, ax\theta(x)\}$. By Corollary 2.3 (ii), $\theta(A_{42}) = \{x, x\theta(x)\}$.
Case(a): If $\theta(ax) = x$ and $\theta(ax\theta(x)) = x\theta(x)$, then $\phi_\theta(ax) = a^{-1}$ and $\phi_\theta(ax\theta(x)) = a^{-1}$, which contradicts the bijectivity of ϕ_θ.
Case(b): If $\theta(ax) = x\theta(x)$ and $\theta(ax\theta(x)) = x$, then $\phi_\theta(ax) = a^{-1}\theta(x)$ and $\phi_\theta(ax\theta(x)) = a^{-1}\theta(x)$, which again contradicts the bijectivity of ϕ_θ.
Thus, $\theta(a) = ax$ and $\theta(ax) = ax\theta(x)$. Hence $A_{44} = \{a, ax\}$, $\theta(A_{44}) = \{ax, ax\theta(x)\}$ and $A_{42} = \{ax\theta(x), a\theta(x)\}$. □

This can be shown by the Figure 1

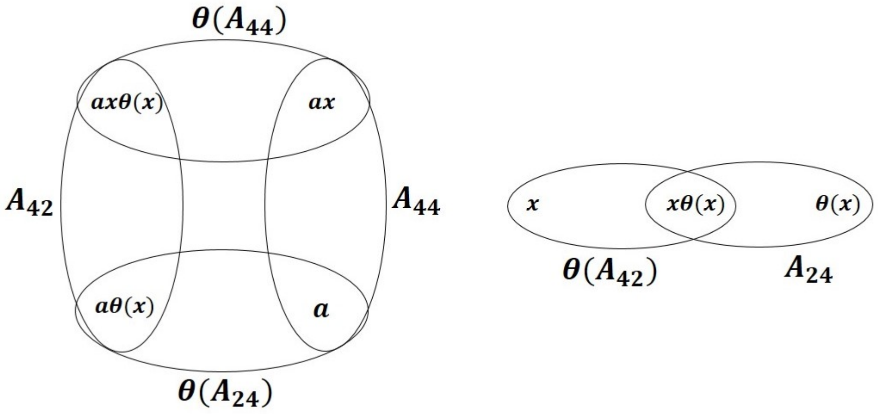

Figure 1

3 The Structure of Orth($\mathbb{Z}_2 \times \mathbb{Z}_4$)

Theorem 3.1. *Suppose $\theta \in Orth(\mathbb{Z}_2 \times \mathbb{Z}_4)$, $A_{22} = \{x\}$, $a \in A_{44} \setminus \theta(A_{44})$. Then θ has one of the following form*

(i) $(a, ax, ax\theta(x), x\theta(x), a\theta(x), x, \theta(x))$, *where* $x\theta(x) = a^2$.

(ii) $(a, ax, ax\theta(x), x\theta(x))(\theta(x), a\theta(x), x)$, *where* $x\theta(x) \neq a^2$.

(iii) $(a, ax, ax\theta(x), x, \theta(x))(a\theta(x), x\theta(x))$, *where* $x\theta(x) \neq a^2$.

(iv) $(a, ax, ax\theta(x), x, \theta(x), a\theta(x), x\theta(x))$, *where* $x\theta(x) = a^2$.

and $|Orth(G)| = 48$.

Proof. Suppose $A_{22} = \{x\}$ and $a \in A_{44} \setminus \theta(A_{44})$. Then by Proposition 2.4, $A_{44} = \{a, ax\}$, $\theta(A_{44}) = \{ax, ax\theta(x)\}$ and $A_{42} = \{ax\theta(x), a\theta(x)\}$.

Case(i): Assume $\theta(ax\theta(x)) = x\theta(x)$. Clearly, $\theta(a\theta(x)) = x$. Then $\phi_\theta(ax\theta(x)) = a^{-1}$ and $\phi_\theta(a\theta(x)) = a^{-1}x\theta(x)$.

Subcase(a): Assume $\theta(\theta(x)) = a$. Then $\theta(x\theta(x)) = a\theta(x)$, $\phi_\theta(\theta(x)) = a\theta(x)$ and $\phi_\theta(x\theta(x)) = ax$. Bijectivity of ϕ_θ implies, $a^{-1}x\theta(x) = a$ or $x\theta(x) = a^2$. Thus, if $x\theta(x) = a^2$, then θ is an orthomorphism, given by (i). We have 4 choices for a and 2 choices for x as $x \neq a^2$. Hence, we have 8 orthomorphisms of this form in $Orth(G)$.

Subcase(b): Assume $\theta(\theta(x)) = a\theta(x)$. Then $\theta(x\theta(x)) = a$, $\phi_\theta(\theta(x)) = a$ and $\phi_\theta(x\theta(x)) = ax\theta(x)$. Bijectivity of ϕ_θ implies $x\theta(x) \neq a^2$. Thus, if $x\theta(x) \neq a^2$, then θ is an orthomorphism given by (ii). Clearly, we have 16 orthomorphisms of this form in $Orth(G)$.

Case(ii): Assume $\theta(ax\theta(x)) = x$. Clearly, $\theta(a\theta(x)) = x\theta(x)$. Then $\phi_\theta(ax\theta(x)) = a^{-1}\theta(x)$ and $\phi_\theta\ (a\theta(x)) = a^{-1}x$.

Subcase(a): Assume $\theta(\theta(x)) = a$. Then $\theta(x\theta(x)) = a\theta(x)$, $\phi_\theta(\theta(x)) = a\theta(x)$ and $\phi_\theta(x\theta(x)) = ax$. Bijectivity of ϕ_θ implies, $x\theta(x) \neq a^2$. Thus, if $x\theta(x) \neq a^2$, then θ is an orthomorphism given by (iii). Clearly, we have 16 orthomorphisms of this form in $Orth(G)$.

Subcase(b): Assume $\theta(\theta(x)) = a\theta(x)$. Then $\theta(x\theta(x)) = a$, $\phi_\theta(\theta(x)) = a$ and $\phi_\theta(x\theta(x)) = ax\theta(x)$. Bijectivity of ϕ_θ implies $x\theta(x) = a^2$. Thus, if $x\theta(x) = a^2$, then θ is an orthomorphism given by (iv). Clearly, we have 8 orthomorphisms of this form in $Orth(G)$. Hence, $|Orth(G)|= 48$. $\square$

4 Clique in $Orth(\mathbb{Z}_2 \times \mathbb{Z}_4)$

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ and A_{ij}, A'_{ij} denotes the partition of $\mathbb{Z}_2 \times \mathbb{Z}_4$ with respect to θ_1, θ_2 respectively as defined in Lemma 2.1.

Lemma 4.1. *If θ_1 and $\theta_2 \in Orth(G)$, then $\theta_1 \perp \theta_2$ if and only if*

(a) $$\{\theta_1(x)^{-1}\theta_2(x) \mid x \in A_{44} \cap A'_{44}\} \sqcup \{\theta_1(x)^{-1}\theta_2(x) \mid x \in A_{24} \cap A'_{24}\} \sqcup \{\theta_1(x)^{-1}\theta_2(x) \mid x \in A_{42} \cap A'_{42}\} \sqcup \{\theta_1(x)^{-1}\theta_2(x) \mid x \in A_{22} \cap A'_{22}\} = \{x \in G \mid o(x) = 2\}.$$

(b) $$\{\theta_1(x)^{-1}\theta_2(x) \mid x \in A_{44} \cap A'_{42}\} \sqcup \{\theta_1(x)^{-1}\theta_2(x) \mid x \in A_{42} \cap A'_{44}\} \sqcup \{\theta_1(x)^{-1}\theta_2(x) \mid x \in A_{24} \cap A'_{22}\} \sqcup \{\theta_1(x)^{-1}\theta_2(x) \mid x \in A_{22} \cap A'_{24}\} = \{x \in G \mid o(x) = 4\}.$$

Proof. Follows from the bijectivity of map $x \mapsto \theta_1(x)^{-1}\theta_2(x)$. $\square$

Proposition 4.2. *Let* θ_1,$\theta_2 \in Orth(G)$ *such that* $|A_{44} \cap A^{'}_{44}|= 2$. *Then* $\theta_1 \not\perp \theta_2$.

Proof. Since $|A_{44} \cap A^{'}_{44}|= 2$, $|A_{42} \cap A^{'}_{42}|= 2$.
So, $|\{\theta_1(x)^{-1}\theta_2(x) \mid x \in A_{44} \cap A^{'}_{44}\}|+|\{\theta_1(x)^{-1}\theta_2(x) \mid x \in A_{24} \cap A^{'}_{24}\}|= 4 > |\{x \in G \mid o(x) = 2\}|$. Therefore, by Lemma 4.1, $\theta_1 \not\perp \theta_2$. $\square$

Proposition 4.3. *Let* $\theta_1, \theta_2 \in Orth(G)$ *such that* $|A_{44} \cap A^{'}_{44}|= 1$. *Then* $\theta_1 \not\perp \theta_2$.

Proof. Suppose $A_{44} \cap A^{'}_{44} = \{a\}$. Then by Proposition 2.4, $A_{44} = \{a, ax\}$ and $A^{'}_{44} = \{a, ax'\}$, where $A_{22} = \{x\}$ and $A^{'}_{22} = \{x'\}$. Clearly, $x \neq x'$.
Case(1): Assume $a \in A_{44} \cap \theta_1(A_{24})$ and $a \in A^{'}_{44} \cap \theta_2(A^{'}_{24})$. Then $\theta_1(a)^{-1}\theta_2(a) = (ax)^{-1}ax' = xx'$.
Subcase(a): Assume $\theta_1(x) = \theta_2(x')$. Then $A_{22} \cup A_{24} = \{x, x', \theta_1(x)\}$. Also, $\theta_1(a)^{-1}\theta_2(a) = xx' = \theta_1(x)$ and $a\theta_1(x) \in A_{42} \cap A^{'}_{42}$. Clearly, $\theta_1(A_{42}) = \{x, x'\} = \theta_2(A^{'}_{42})$. So, $\theta_1(a\theta_1(x))^{-1}\theta_2(a\theta_2(x'))$ $= e$ or $\theta_1(x)$. This is a contradiction to Lemma 4.1. Hence $\theta_1 \not\perp \theta_2$.
Subcase(b): Assume $\theta_1(x) \neq \theta_2(x')$. Since $\theta_2(x') \in \{x, x', \theta_1(x)\}$ and $\theta_2(x') \neq x'$, $\theta_2(x') = x$. So, $a\theta_1(x) = ax'\theta_2(x') \in A_{42} \cap A^{'}_{42}$ and $\theta_1(x) = x'\theta_2(x') \in A_{24} \cap A^{'}_{24}$.

(i) If $\theta_1(a\theta_1(x)) = x$ and $\theta_2(ax'\theta_2(x')) = x'$, then $\theta_1(a\theta_1(x))^{-1}$ $\theta_2(ax'\theta_2(x')) = xx' = \theta_1(x)$. By Lemma 4.1, $\theta_1 \not\perp \theta_2$.

(ii) If $\theta_1(a\theta_1(x)) = x\theta_1(x)$ and $\theta_2(ax'\theta_2(x')) = x'\theta_2(x')$, then $\theta_1(a\theta_1(x))^{-1}\theta_2(ax'\theta_2(x')) = x$. Also $\theta_1(x) = x'\theta_2(x') \in A_{24} \cap A^{'}_{24}$.
If $x\theta_1(x) = a^2$, then $x'\theta_2(x') \neq a^2$. By Theorem 3.1 (iv) and (ii), $\theta_1(\theta_1(x))^{-1}\theta_2(x'\theta_2(x')) = (a\theta_1(x))^{-1}a = \theta_1(x)$. This is a contradiction to Lemma 4.1.
If $x\theta_1(x) \neq a^2$ and $x'\theta_2(x') = a^2$, then by Theorem 3.1 (iii) and (i), $\theta_1(\theta_1(x))^{-1}\theta_2(x'\theta_2(x')) = a^{-1}a\theta_2(x') = \theta_2(x') = x$.

This is a contradiction to Lemma 4.1.
If $x\theta_1(x) \neq a^2$ and $x'\theta_2(x') \neq a^2$, then by Theorem 3.1 (iii) and (ii), $\theta_1(\theta_1(x))^{-1}\theta_2(x'\theta_2(x')) = a^{-1}a = e$. This is a contradiction to Lemma 4.1.

(iii) If $\theta_1(a\theta_1(x)) = x$ and $\theta_2(ax'\theta_2(x')) = x'\theta_2(x')$, then $\theta_1(a\theta_1(x))^{-1}$ $\theta_2(ax'\theta_2(x')) = x'$.
If $x\theta_1(x) = a^2$, then $x'\theta_2(x') \neq a^2$. Then by Theorem 3.1 (i) and (ii), $\theta_1(\theta_1(x))^{-1}\theta_1(x'\theta_2(x')) = a^{-1}a = e$. This is a contradiction to Lemma 4.1.
If $x\theta_1(x) \neq a^2$ and $x'\theta_2(x') = a^2$, then by Theorem 3.1 (ii) and (i), $\theta_1(\theta_1(x))^{-1}\theta_2(x'\theta_2(x')) = (a\theta_1(x))^{-1}a\theta_2(x') = x'$. This is a contradiction to Lemma 4.1.
If $x\theta_1(x) \neq a^2$ and $x'\theta_2(x') \neq a^2$, then by Theorem 3.1 (ii) and (ii), $\theta_1(\theta_1(x))^{-1}$ $\theta_2(x'\theta_2(x')) = (a\theta_1(x))^{-1}a = \theta_1(x)$. This is a contradiction to Lemma 4.1.

(iv) If $\theta_1(a\theta_1(x)) = x\theta_1(x)$ and $\theta_2(ax'\theta_2(x')) = x'$, then $\theta_1(a\theta_1(x))^{-1}$ $\theta_2(ax'\theta_2(x')) = x\theta_1(x)x' = e$ This is a contradiction to Lemma 4.1. Hence, $\theta_1 \not\angle \theta_2$ when $a \in A_{44} \cap \theta_1(A_{24})$ and $a \in A^{'}_{44} \cap \theta_2(A^{'}_{24})$.

Case(2): If $a \in A_{44} \cap \theta_1(A_{24})$ and $a \in A^{'}_{44} \cap \theta_2(A^{'}_{44})$, then $\theta_1(a) = ax, \theta_1(ax) = ax\theta_1(x)$ and $\theta_2(ax') = a, \theta_2(a) = a\theta_2(x')$. Clearly, $a \in A_{44} \cap A^{'}_{44}$ and $\theta_1(a)^{-1}\theta_2(a) = (ax)^{-1}a\theta_2(x') = x\theta_2(x')$.
Subcase(a): Assume $\theta_1(x) = \theta_2(x')$. Then $A_{22} \cup A_{24} = \{x, x', \theta_1(x)\}$ and $\theta_1(a)^{-1}\theta_2(a) = x'$. Clearly, $a\theta_1(x) = a\theta_2(x') \in A_{42} \cap A^{'}_{42}$ and $\theta_1(x) = \theta_2(x') \in A_{24} \cap A^{'}_{24}$.

(i) If $\theta_1(a\theta_1(x)) = x$ and $\theta_2(a\theta_2(x')) = x'$, then $\theta_1(a\theta_1(x))^{-1}\theta_2(a$ $\theta_2(x')) = xx' = \theta_1(x)$.
If $x\theta_1(x) = a^2$, then $x'\theta_2(x') \neq a^2$. By Theorem 3.1 (i) and (ii), $\theta_1(\theta_1(x))^{-1}\theta_2(\theta_2(x')) = a^{-1}a\theta_1(x) = \theta_1(x)$. This is a contradiction to Lemma 4.1.
If $x\theta_1(x) \neq a^2$ and $x'\theta_2(x') = a^2$, then by Theorem 3.1 (ii) and (i), $\theta_1(\theta_1(x))^{-1}\theta_2(\theta_2(x')) = (a\theta_1(x))^{-1}a = \theta_1(x)$. This is a contradiction to Lemma 4.1.

If $x\theta_1(x) \neq a^2$ and $x'\theta_2(x') \neq a^2$, then by Theorem 3.1 (ii) and (ii), $\theta_1(\theta_1(x))^{-1}\theta_2(\theta_2(x')) = (a\theta_1(x))^{-1}a\theta_2(x') = e$. This is a contradiction to Lemma 4.1.

(ii) If $\theta_1(a\theta_1(x)) = x\theta_1(x)$ and $\theta_2(a\theta_2(x')) = x'\theta_2(x')$, then $\theta_1(a\theta_1(x))^{-1}$ $\theta_2(a\theta_2(x)) = x\theta_1(x)x'\theta_2(x') = \theta_1(x)$.
If $x\theta_1(x) = a^2$, then $x'\theta_2(x') \neq a^2$. Then by Theorem 3.1 (iv) and (iii), $\theta_1(\theta_1(x))^{-1}\theta_2(\theta_2(x')) = (a\theta_1(x))^{-1}a = \theta_1(x)$. This is a contradiction to Lemma 4.1.
If $x\theta_1(x) \neq a^2$ and $x'\theta_2(x') = a^2$, then by Theorem 3.1 (iii) and (iv), $\theta_1(\theta_1(x))^{-1}\theta_2(\theta_2(x')) = a^{-1}a\theta_2(x') = \theta_1(x)$. This is a contradiction to Lemma 4.1.
If $x\theta_1(x) \neq a^2$ and $x'\theta_2(x') \neq a^2$, then by Theorem 3.1 (iii) and (iii), $\theta_1(\theta_1(x))^{-1}\theta_2(\theta_2(x')) = a^{-1}a = e$. This is a contradiction to Lemma 4.1.

(iii) If $\theta_1(a\theta_1(x)) = x$ and $\theta_2(a\theta_2(x')) = x'\theta_2(x')$, then $\theta_1(a\theta_1(x))^{-1}$ $\theta_2(a\theta_2(x')) = xx'\theta_2(x') = e$. This is a contradiction to Lemma 4.1.

(iv) If $\theta_1(a\theta_1(x)) = x\theta_1(x)$ and $\theta_2(a\theta_2(x')) = x'$, then $\theta_1(a\theta_1(x))^{-1}$ $\theta_2(a\theta_2(x')) = x\theta_1(x)x' = e$. This is a contradiction to Lemma 4.1. Hence, $\theta_1 \not\perp \theta_2$.

Subcase(b): Assume $\theta_1(x) \neq \theta_2(x')$. Since $\theta_2(x^{'}) \in \{x, x^{'}, \theta_1(x)\}$ and $\theta_2(x') \neq x'$, $\theta_2(x') = x$. As $a \in A_{44} \cap A^{'}_{44}$, so $\theta_1(a)^{-1}\theta_2(a) = x\theta_2(x') = e$. This is a contradiction to Lemma 4.1. Thus, $\theta_1 \not\perp \theta_2$ when $a \in A_{44} \cap \theta_1(A_{24})$ and $a \in A^{'}_{44} \cap \theta_2(A^{'}_{44})$. □

Proposition 4.4. *Let $\theta_1, \theta_2 \in Orth(G)$ and $|A_{44} \cap A^{'}_{44}| = 0$. Then $\theta_1 \perp \theta_2$.*

Proof. If $|A_{44} \cap A^{'}_{44}| = 0$ then $|A_{22} \cap A^{'}_{22}| = 1$ and $|A_{24} \cap A^{'}_{24}| = 2$. Also $|A_{44} \cap A^{'}_{42}| = 2$ and $|A_{42} \cap A^{'}_{44}| = 2$. Consider the orthomorphism of the form $\theta_1 = (a, ax, ax\theta_1(x), x\theta_1(x), a\theta_1(x), x, \theta_1(x))$ where $x\theta_1(x) = a^2$. Here, $A_{22} = \{x\}$, $A_{44} = \{a, ax\}$, $A_{42} = \{ax\theta_1(x), a\theta_1(x)\}$ and $A_{24} = \{\theta_1(x), x\theta_1(x)\}$. If θ_2 is orthogonal to θ_1 then, $A^{'}_{22} = \{x\}$, $A^{'}_{44} = \{ax\theta_1(x),\, a\theta_1(x)\}$, $A^{'}_{42} = \{a, ax\}$

and $A'_{24} = \{\theta_1(x), x\theta_1(x)\}$.
Case(1): Assume $a\theta_1(x) \in A'_{44} \cap \theta_2(A'_{24})$. Then by Proposition 2.4, $\theta_2(a\theta_1(x)) = ax\theta_1(x)$ and $\theta_2(ax\theta_1(x)) = a$ as $\theta_2(x) = x\theta_1(x)$. Since $\theta_1(x\theta_1(x)) = a\theta_1(x)$, $\theta_2(x\theta_1(x)) = ax$ and $\theta_2(\theta_1(x)) = a\theta_1(x)$. Then $\phi_{\theta_2}(a\theta_1(x)) = x, \phi_{\theta_2}(ax\theta_1(x)) = x\theta_1(x), \phi_{\theta_2}(x) = \theta_1(x), \phi_{\theta_2}(x\theta_1(x)) = a\theta_1(x), \phi_{\theta_2}(\theta_1(x)) = a$.
Subcase(a): Assume $\theta_2(a) = x$. Then $\theta_2(ax) = \theta_1(x)$, $\phi_{\theta_2}(a) = a^{-1}x$ and $\phi_{\theta_2}(ax) = (ax)^{-1}\theta_1(x) = a$, which is not an orthomorphism.
Subcase(b): Assume $\theta_2(a) = \theta_1(x)$. Then $\theta_2(ax) = x$, $\phi_{\theta_2}(a) = a^{-1}\theta_1(x) = ax$ and $\phi_{\theta_2}(ax) = (ax)^{-1}x = a^{-1}$. In this case θ_2 becomes an orthomorphism given by $\theta_2 = (a\theta_1(x), ax\theta_1(x), a, \theta_1(x))(x, x\theta_1(x), ax)$.
Now,

$$\theta_1(y)^{-1}\theta_2(y) = \begin{cases} \theta_1(x)x\theta_1(x) = x & y = x \in A_{22} \cap A'_{22} \\ a^{-1}a\theta_1(x) = \theta_1(x) & y = \theta_1(x) \in A_{24} \cap A'_{24} \\ a^{-1}\theta_1(x)ax = x\theta_1(x) & y = x\theta_1(x) \in A_{24} \cap A'_{24} \\ a^{-1}x\theta_1(x) = a & y = a \in A_{44} \cap A'_{42} \\ a^{-1}x\theta_1(x)x = ax & y = ax \in A_{44} \cap A'_{42} \\ xax\theta_1(x) = a\theta_1(x) & y = a\theta_1(x) \in A_{42} \cap A'_{44} \\ x\theta_1(x)a = ax\theta_1(x) & y = ax\theta_1(x) \in A_{42} \cap A'_{44} \end{cases}$$

Clearly, $y \mapsto \theta_1(y)^{-1}\theta_2(y)$ is a bijective map. Thus, $\theta_1 \perp \theta_2$.
Case(2): Assume $a\theta_1(x) \in A'_{44} \cap \theta_2(A'_{44})$. Then by Proposition 2.4, $\theta_2(a\theta_1(x)) = ax, \theta_2(ax\theta_1(x)) = a\theta_1(x)$ as $\theta_2(x) = x\theta_1(x)$. Since $\theta_1(\theta_1(x)) = a$, $\theta_2(\theta_1(x)) = ax\theta_1(x)$ and $\theta_2(x\theta_1(x)) = a$. Then $\phi_{\theta_2}(a\theta_1(x)) = x\theta_1(x), \phi_{\theta_2}(ax\theta_1(x)) = x, \phi_{\theta_2}(x) = \theta_1(x)$, $\phi_{\theta_2}(x\theta_1(x)) = ax\theta_1(x) = a^{-1}, \phi_{\theta_2}(\theta_1(x)) = ax$.
Subcase(a): Assume $\theta_2(a) = \theta_1(x)$. Then $\theta_2(ax) = x$, $\phi_{\theta_2}(a) = a^{-1}\theta_1(x)$ and $\phi_{\theta_2}(ax) = (ax)^{-1}x = a^{-1}$ which is not an orthomorphism.
Subcase(b): Assume $\theta_2(a) = x$. Then $\theta_2(ax) = \theta_1(x)$, $\phi_{\theta_2}(a) = a^{-1}x$ and $\phi_{\theta_2}(ax) = (ax)^{-1}\theta_1(x)$. In this case θ_2 becomes an or-

thomorphism given by $\theta_2 = (ax\theta_1(x), a\theta_1(x), ax, \theta_1(x))\ (x, x\theta_1(x), a)$. Now,

$$\theta_1(y)^{-1}\theta_2(y) = \begin{cases} \theta_1(x)x\theta_1(x) = x & y = x \in A_{22} \cap A_{22}^{'} \\ a^{-1}ax\theta_1(x) = x\theta_1(x) & y = \theta_1(x) \in A_{24} \cap A_{24}^{'} \\ a^{-1}\theta_1(x)a = \theta_1(x) & y = x\theta_1(x) \in A_{24} \cap A_{24}^{'} \\ a^{-1}xx = ax\theta_1(x) & y = a \in A_{44} \cap A_{42}^{'} \\ a^{-1}x\theta_1(x)\theta_1(x) = a\theta_1(x) & y = ax \in A_{44} \cap A_{42}^{'} \\ xax = a & y = a\theta_1(x) \in A_{42} \cap A_{44}^{'} \\ x\theta_1(x)a\theta_1(x) = ax & y = ax\theta_1(x) \in A_{42} \cap A_{44}^{'} \end{cases}$$

Clearly, $y \mapsto \theta_1(y)^{-1}\theta_2(y)$ is a bijective map. Thus, $\theta_1 \perp \theta_2$. Hence, If $\theta_1 = (a, ax, ax\theta_1(x), x\theta_1(x), a\theta_1(x), x, \theta_1(x))$ where $x\theta_1(x) = a^2$ then θ_1 is orthogonal to $\theta_2 = (a\theta_1(x), ax\theta_1(x), a, \theta_1(x))(x, x\theta_1(x), ax)$ and $\theta_3 = (ax\theta_1(x), a\theta_1(x), ax, \theta_1(x))(x, x\theta_1(x), a)$. $\square$

Similarly, calculating the other cases, the following Table 1 has been constructed:

θ_1	θ_2, θ_3
$(a, ax, ax\theta_1(x), x\theta_1(x), a\theta_1(x), x, \theta_1(x))$ where $x\theta_1(x) = a^2$	$(a\theta_1(x), ax\theta_1(x), a, \theta_1(x))(x, x\theta_1(x), ax)$, $(ax\theta_1(x), a\theta_1(x), ax, \theta_1(x))(x, x\theta_1(x), a)$.
$(a, ax, ax\theta_1(x), x, \theta_1(x), a\theta_1(x), x\theta_1(x))$ where $x\theta_1(x) = a^2$	$(a\theta_1(x), ax\theta_1(x), a, x, x\theta_1(x))(ax, \theta_1(x))$, $(ax\theta_1(x), a\theta_1(x), ax, x, x\theta_1(x))(a, \theta_1(x))$.
$(a, ax, ax\theta_1(x), x\theta_1(x))(\theta_1(x), a\theta_1(x), x)$ where $x\theta_1(x) \neq a^2$ and $x = a^2$	$(a\theta_1(x), ax\theta_1(x), a, x, x\theta_1(x))(ax, \theta_1(x))$, $(ax\theta_1(x), a\theta_1(x), ax, x, x\theta_1(x))(a, \theta_1(x))$.
$(a, ax, ax\theta_1(x), x\theta_1(x))(\theta_1(x), a\theta_1(x), x)$ where $x\theta_1(x) \neq a^2$ and $\theta_1(x) = a^2$	$(a\theta_1(x), ax\theta_1(x), a, \theta_1(x), ax, x, x\theta_1(x))$, $(ax\theta_1(x), a\theta_1(x), ax, \theta_1(x), a, x, x\theta_1(x))$.
$(a, ax, ax\theta_1(x), x, \theta_1(x))(a\theta_1(x), x\theta_1(x))$ where $x\theta_1(x) \neq a^2$ and $x = a^2$	$(a\theta_1(x), ax\theta_1(x), a, \theta_1(x))(ax, x, x\theta_1(x))$, $(ax\theta_1(x), a\theta_1(x), ax, \theta_1(x))(a, x, x\theta_1(x))$.
$(a, ax, ax\theta_1(x), x, \theta_1(x))(a\theta_1(x), x\theta_1(x))$ where $x\theta_1(x) \neq a^2$ and $\theta_1(x) = a^2$	$(a\theta_1(x), ax\theta_1(x), a, x, x\theta_1(x), ax, \theta_1(x))$, $(ax\theta_1(x), a\theta_1(x), ax, x, x\theta_1(x), a, \theta_1(x))$.

Table 1: $\theta_1 \perp \theta_2$ and $\theta_1 \perp \theta_3$

Corollary 4.5. $\omega(\mathbb{Z}_2 \times \mathbb{Z}_4) = 2$.

Proof. Clearly, there are two orthomorphism orthogonal to a given orthomorphism and they cannot be orthogonal to each other as their A_{44} are same. Hence, $\omega(\mathbb{Z}_2 \times \mathbb{Z}_4) = 2$. □

Corollary 4.6. *(i) Two orthomorphism ψ_1 and ψ_2 which are orthogonal to θ are conjugate to each other by an element $\alpha = (a, ax)(ax\theta(x), a\theta(x))$ in $Aut(Z_2 \times Z_4)$ where $A_{44} = \{a, ax\}$ and $A_{22} = \{x\}$ of θ. Also ψ_1 and ψ_2 are also orthogonal to $\alpha\theta\alpha^{-1} = \theta^\alpha$.*

(ii) $Orth(Z_2 \times Z_4)$ consists of 12 disjoint 4-cycles. Each 4-cycle is given by Figure 2.

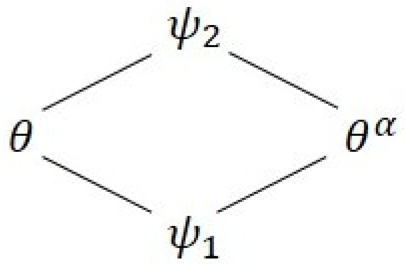

Figure 2

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