Cospectral graphs obtained by edge deletion

Chris Godsil*[1], Wanting Sun †[1,2], and Xiaohong Zhang[1]

[1]Department of Combinatorics & Optimization, University of Waterloo, Waterloo, Ontario, Canada
[2]Faculty of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, PR China

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Abstract

Let $M \circ N$ denote the Schur product of two matrices M and N. A graph X with adjacency matrix A is *walk regular* if $A^k \circ I$ is a constant times I for each $k \geq 0$, and X is 1-walk-regular if it is walk regular and $A^k \circ A$ is a constant times A for each $k \geq 0$. Assume X is 1-walk regular. Here we show that by deleting an edge in X, or deleting edges of a graph inside a clique of X, we obtain families of graphs that are not necessarily isomorphic, but are cospectral with respect to four types of matrices: the adjacency matrix, Laplacian matrix, unsigned Laplacian matrix, and normalized Laplacian matrix. Furthermore, we show that removing edges of Laplacian cospectral graphs in cliques of a 1-walk regular graph results in Laplacian cospectral graphs; removing edges of unsigned Laplacian cospectral graphs whose complements are also cospectral with respect to the unsigned Laplacian in cliques of a 1-walk regular graph results in unsigned Laplacian cospectral graphs.

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1 Introduction

There are a number of useful matrices we can associate with a graph X. We have, of course, the adjacency matrix $A = A(X)$. If D is the diagonal matrix with $D_{i,i}$ equal to the valency of the vertex i, we also have the *Laplacian* $L(X) = D - A$, the *unsigned Laplacian* (also called *signless Laplacian*) $S(X) = D + A$, and the *normalized Laplacian* $N(X) = D^{-1/2}LD^{-1/2}$ (assume for now that X has no isolated vertices). It has long been known that, although it is a useful invariant, the spectrum of $A(X)$ does not determine X. In this paper we provide new constructions of families of graphs X and Y that are not distinguished by the spectrum of any of the four matrices just given.

One of our main results implies the following.

1.1 Theorem. *If X is a strongly regular graph. Then for any two edges e and f of X, the graphs $X\backslash e$ and $X\backslash f$ are cospectral with respect to the adjacency matrix, the Laplacian matrix, the unsigned Laplacian and the normalized Laplacian.* □

For a concrete example, by use of a strongly regular graph with parameters $(25,12,5,6)$, we obtain a family of 150 graphs which are pairwise non-isomorphic, such that any two graphs are cospectral with cospectral complements with respect to A, L, S and N.

We discuss some related earlier results. In [6], Godsil and McKay gave several constructions for adjacency cospectral graphs. Dutta constructed unsigned Laplacian cospectral graphs in [4] and Butler [1] constructs graphs that are cospectral with respect to adjacency matrix and normalized Laplacian matrix. In [2, Corollary 1.1], Butler et al. construct graphs that are cospectral with respect to adjacency matrix, Laplacian, unsigned and normalized Laplacian matrices. In [11, Theorem 4.1], Wang et al. give a construction of pairs of graphs cospectral for all four matrices.

Let $\Delta(X)$ denote the distance matrix of that graph X, and let $\overline{X}$ denote the complement of X. In 1977, McKay [9] constructed pairs of non-isomorphic trees that are not distinguished by the spectrum of $A(T)$, $A(\overline{T})$, $L(T)$, $S(T)$ and $\Delta(T)$. (His full list of matrices is longer.) Osborne [10, Theorem 3.3.2] constructs pairs of trees that are not distinguished by the spectrum of $A(T)$, $A(\overline{T})$, $L(T)$, $S(T)$, and $N(T)$.

The graphs we construct are related by edge-deletion. For vertex deletion, we have the following. Let X be a graph with adjacency matrix A. For any subset R of $V(X)$, denote the induced subgraph of X on $V(X)\backslash R$ by $X\backslash R$, and the characteristic polynomial of X with respect to the adjacency matrix as $\phi(X,t)$. Jacobi's Theorem [5, Chapter 4] connects characteristic polynomials of graphs and determinant of a principal submatrix of the inverse of a matrix:

$$\det\left((tI-A)^{-1}_{R,R}\right)=\frac{\phi(X\backslash R,t)}{\phi(X,t)}.$$

This implies that if X is a strongly regular graph, and Y_1, Y_2 are induced subgraphs of X that are cospectral with cospectral complements, then so are $X\backslash Y_1$ and $X\backslash Y_2$.

Moreover, if we use strongly regular graphs in our construction, we obtain families of graphs such that graphs in the same family are pairwise cospectral with respect to the four matrices, and so are their complements.

We also show that with Laplacian cospectral graphs of small size, we can construct Laplacian cospectral graphs of larger size by removing the edges of the smaller cospectral pair from cliques of a 1-walk regular graph; similarly, with a pair of unsigned Laplacian cospectral graphs of small size whose complements are also unsigned Laplacian cospectral, we can obtain unsigned Laplacian cospectral graphs of larger size.

This work is motivated by quantum state transfer on graphs. Let X be a graph. Let Y_1 and Y_2 be two spanning subgraphs of X. If there is a time t such that $e^{itL(X)}L(Y_1)e^{-itL(X)}=L(Y_2)$, then we say there is subgraph transfer from Y_1 to Y_2. If subgraph transfer occurs between Y_1 and Y_2, then in addition to Y_1 and Y_2 being similar, $L(X)-L(Y_1)$ and $L(X)-L(Y_2)$ are also similar. That is, the edge-deleted subgraphs are cospectral.

2 1-walk regular graphs

A graph is *walk-regular*, if for any positive integer k, the number of closed walks of length k is the same at all vertices. If further, the number of walks from vertex u to v of length k is the same for all adjacent vertex pairs u,v, then we say X is *1-walk regular*. Let A and B be two matrices of the same size, say $m\times n$, then their *Schur product* $A\circ B$ is a matrix of the same size such that

$$(A\circ B)_{j,k}=A_{j,k}B_{j,k}.$$

In terms of this matrix product, a graph X with adjacency matrix A is 1-walk regular if and only if, for any positive integer k, there exist scalars a_k, b_k such that

$$A^k \circ I = a_k I \text{ and } A^k \circ A = b_k A.$$

Distance-regular graphs are 1-walk regular; more generally, any graph in a symmetric association scheme [5, Chapter 12] is 1-walk regular.

Let M be a Hermitian matrix. For any eigenvalue θ of M, let E_θ denote the orthogonal projection matrix onto the eigenspace associated to θ. Assume M has exactly m distinct eigenvalues $\theta_1, \dots, \theta_m$, then

$$M = \sum_{r=1}^{m} \theta_r E_r.$$

and the above equation is called the *spectral decomposition* of M. If $f(x)$ is a function which is defined on each eigenvalue of M, then $f(M) = \sum_r f(\theta_r) E_r$. In particular,

$$(tI - M)^{-1} = \sum_r \frac{1}{t - \theta_r} E_r.$$

Since each of the orthogonal projection matrices $E_1, \dots, E_m$ is a polynomial in M, the matrix algebra $\langle M \rangle$ generated by M is the same as the one generated by $\{E_1, \dots, E_m\}$. Therefore a graph X with adjacency matrix $A = \sum_r \theta_r E_r$ is 1-walk regular if and only if for each r, there exist scalars α_r and β_r such that

$$E_r \circ I = \alpha_r I, \quad E_r \circ A = \beta_r A.$$

We have seen two commonly used characterizations of 1-walk regular graphs: in terms of powers of adjacency matrix, or in terms of eigenspace projection matrices. Now we give more characterizations. In particular, 1-walk regularity is equivalent to cospectrality of certain subgraphs of X.

We denote the adjugate of a matrix M by $\operatorname{adj}(M)$. Let X be a graph, and denote its characteristic polynomial $\det(tI - A(X))$ by $\phi(X, t)$. For any $R \subseteq V(X)$, let $X \backslash R$ denote the graph obtained from X by deleting vertices in R. When $R = \{u\}$, we use $X \backslash u$ instead.

2.1 Theorem. *Let X be a graph on n vertices with adjacency matrix A. Assume $A = \sum_{r=1}^m \theta_r E_r$ is the spectral decomposition of A. The following are equivalent.*

(a) *X is 1-walk regular.*

(b) *For any positive integer k, there exist scalars a_k and b_k such that $A^k \circ I = a_k I$ and $A^k \circ A = b_k A$.*

(c) *For any integer $k = 1,\dots,n-1$, there exist scalars a_k and b_k such that $A^k \circ I = a_k I$ and $A^k \circ A = b_k A$.*

(d) *For $r = 1,\dots,m$, there exist scalars α_r and β_r such that $E_r \circ I = \alpha_r I$ and $E_r \circ A = \beta_r A$.*

(e) *There exist functions $a(t)$ and $b(t)$ such that $(tI-A)^{-1} \circ I = a(t)I$ and $(tI-A)^{-1} \circ A = b(t)A$.*

(f) *There exist polynomials $f(t)$ and $g(t)$ such that* $\operatorname{adj}(tI-A) \circ I = f(t)I$ *and* $\operatorname{adj}(tI-A) \circ A = g(t)A$.

(g) *$\phi(X\backslash u, t)$ is the same for all $u \in V(X)$, and $\phi(X\backslash\{u,v\}, t)$ is the same for adjacent vertices u, v in X.*

Proof. The equivalence between $(a), (b)$ and (d) have been shown. The equivalence of (b) and (c) follows from Cayley-Hamilton Theorem. The equivalence of (d) and (e) follows from $(tI-A)^{-1} = \sum_{r=1}^{m} \frac{1}{t-\theta_r} E_r$. The equivalence of (e) and (f) follows from $\operatorname{adj}(tI-A) = \phi(X,t)(tI-A)^{-1}$.

Finally we prove the equivalence of (f) and (g). Note that $[\operatorname{adj}(tI-A)]_{u,u} = \phi(X\backslash u, t)$, the equivalence of $\phi(X\backslash u, t)$ is the same for all $u \in V(X)$ and $\operatorname{adj}(tI-A) \circ I = f(t)I$ for some $f(t)$ follows. Denote $[\operatorname{adj}(tI-A)]_{u,v}$ as $\phi_{uv}(X,t)$, it is known that $\phi_{uv}(X,t)$ can be expressed in terms of the characteristic polynomials of subgraphs of X with at most two vertices deleted [5, Chapter 4]. In fact, for any two vertices u, v of X,

$$\phi_{u,v}(X,t) = \sqrt{\phi(X\backslash u, t)\phi(X\backslash v, t) - \phi(X,t)\phi(X\backslash\{u,v\}, t)}. \tag{2.1}$$

Therefore for a walk-regular graph X, $\phi(X\backslash\{u,v\}, t)$ is the same for all adjacent vertex pairs u and v if and only if $\phi_{u,v}(X,t)$ is, and hence (f) and (g) are equivalent. □

The above theorem implies that if X is 1-walk regular, then for a function $f(x)$ defined on the eigenvalues of the adjacency matrix A, the matrix $f(A)$ has constant diagonal, and is constant on the entries corresponding to edges of X. In fact a similar result hold for the Laplacian, unsigned Laplacian, and normalized Laplacian matrix.

2.2 Lemma. *Let X be a 1-walk regular graph. Let M denote the adjacency, Laplacian, unsigned Laplacian, or normalized Laplacian of X. If $f(x)$ is a function that is defined on all eigenvalues of M, then there exist α_f and β_f such that*

$$f(M) \circ I = \alpha_f I \text{ and } f(M) \circ A = \beta_f A.$$

Proof. The result for adjacency matrix follows from Theorem 2.1. The other cases follows from the fact a function in M is also a function in A. In fact, assume X is d-regular. then $f(L) = f(dI - A)$, $f(S) = f(dI + A)$, and $f(N) = f(I - \frac{1}{d}A)$. □

3 Constructing graphs cospectral with respect to A, L, S and N

Let X be a 1-walk regular graph. We know from Theorem 2.1 that, all the graphs in the set $\{X \backslash u \mid u \in V(X)\}$ are adjacency cospectral, and so are all graphs in $\{X \backslash \{u, v\} \mid \{u, v\} \in E(X)\}$. Now we show that deleting any edge of X results in adjacency cospectral graphs. The characteristic polynomials of these vertex-deleted or edge-deleted subgraphs of any graph are closely related. For an edge e of X, let $X \backslash e$ denote the graph obtained by deleting the edge e from X.

3.1 Lemma. [5, Chapter 4] *Let X be a graph and let $e = \{u, v\}$ be an edge of X. Then*

$$\begin{aligned}\phi(X, t) =& \phi(X \backslash e, t) - \phi(X \backslash \{u, v\}, t) \\ & - 2\sqrt{\phi(X \backslash u, t)\phi(X \backslash v, t) - \phi(X \backslash e, t)\phi(X \backslash \{u, v\}, t)}.\end{aligned} \tag{3.1}$$

Now we show delete an edge in a 1-walk regular graph results in adjacency cospectral graphs.

3.2 Theorem. *Let X be a 1-walk regular graph. Then for any two edges e and f of X, the two graphs $X \backslash e$ and $X \backslash f$ are adjacency cospectral.*

Proof. Assume X has n vertices and $e = \{u, v\}$. Solving $\phi_{u,v}(X, t)$ from (3.1), we have

$$\begin{aligned}\phi(X \backslash e, t) &= \phi(X, t) - \phi(X \backslash \{u, v\}, t) \pm 2\sqrt{\phi(X \backslash u, t)\phi(X \backslash v, t) - \phi(X, t)\phi(X \backslash \{u, v\}, t)} \\ &= \phi(X, t) - \phi(X \backslash \{u, v\}, t) \pm 2\phi_{u,v}(X, t) \quad \text{by (2.1).}\end{aligned}$$

Recall that for a graph Y on n vertices, the coefficient of x^{n-2} in $\phi(Y, t)$ is equal to $-|E(Y)|$. Now comparing the coefficient of x^{n-2} on both sides of the above equation, we know that only the plus sign is valid, that is,

$$\phi(X \backslash e, t) = \phi(X, t) - \phi(X \backslash \{u, v\}, t) + 2\phi_{u,v}(X, t),$$

Now the result follows from (f) and (g) in Theorem 2.1 (recall $\phi_{u,v}(X, t) = [\mathrm{adj}(tI - A)]_{u,v}$). □

3.3 Example. As mentioned in Section 4, there are 15 non-isomorphic strongly regular graphs with parameters SRG(25,12, 5, 6). By Theorem 3.2, for each of these graphs, by deleting an edge, we obtain adjacency cospectral graphs. In particular, two of the 15 strongly regular graphs have the property that deleting an edge from the graph results in non-isomorphic graphs. They each provides a family of 150 graphs such that any two graphs in the same family are cospectral but not isomorphic. These two graphs correspond to X_1 and X_3 in Table 1, with graph6-string being, respectively,

'X~zfCqTc{YPT'fUQidaeNRKxItIMpholosZFKjXHZGnDZDYHwuF',
'X~zfCqTc{YPR'jUQidaeNRLXIrIMphoxKsVXKixPZCnD[fBHuQl'.

In fact, as we will see in the following, a more general cospectral property holds when edges are removed from a 1-walk regular graph.

3.1 Deleting subgraphs in cliques

We have established that by deleting an edge from a 1-walk regular graph, we get a set of adjacency cospectral graphs. In fact, these graphs are also cospectral with respect to Laplacian matrix, unsigned Laplacian matrix, and normalized Laplacian matrix. We show a more general result: deleting edges of a graph from a clique of 1-walk regular graphs results in graphs cospectral with respect to A, L, S and N.

We make use of the following result about the inverse of a rank-1 update of an invertible matrix.

3.4 Theorem (Sherman-Morrison)**.** *Suppose B is an $n \times n$ invertible real matrix and $u, v \in \mathbb{R}^n$. Then $B + uv^T$ is invertible if and only if $1 + v^T B^{-1} u \neq 0$. In this case,*

$$(B + uv^T)^{-1} = B^{-1} - \frac{B^{-1} uv^T B^{-1}}{1 + v^T B^{-1} u}. \tag{3.2}$$

For two matrices C and D such that CD and DC are both defined, $\det(I - CD)$ and $\det(I - DC)$ are closely related.

3.5 Lemma. [7] *Assume C and D^T are both matrices of size $m \times n$, then*

$$\det(I_m - CD) = \det(I_n - DC).$$

In particular, if $C = u$ and $D = v^T$ for some real vectors $u, v \in \mathbb{R}^n$, we have

$$\det(I_n - uv^T) = (1 - v^T u). \tag{3.3}$$

First we prove a result concerning entries of a matrix related to edges deletion in a clique of a 1-walk regular graph.

3.6 Lemma. *Let X be a 1-walk regular graph with adjacency matrix A. Let $u_1, v_1, \dots, u_r, v_r$ be vertices in the same clique of X. Then for any function $f(x)$ defined on the eigenvalues of A, the value of*

$$e_{u_r}^T\left(tI - f(A) \pm e_{u_1}e_{v_1}^T \pm \cdots \pm e_{u_{r-1}}e_{v_{r-1}}^T\right)^{-1} e_{v_r} \tag{3.4}$$

is independent on the choice of the clique and on the ordering of vertices of the chosen clique.

Proof. The function $g(x) = (t - f(x))^{-1}$ is defined on all eigenvalues of A. By Lemma 2.2, there exist $\alpha(t), \beta(t)$ such that $\left(tI - f(A)\right)^{-1} \circ I = \alpha(t)I$ and $\left(tI - f(A)\right)^{-1} \circ A = \beta(t)A$.

We prove the result by induction. When $r = 1$, since u_1 and v_1 are in the same clique of X,

$$e_{u_1}^T\left(tI - f(A)\right)^{-1} e_{v_1} = \delta_{u_1,v_1}\alpha(t) + (1 - \delta_{u_1,v_1})\beta(t),$$

which only depends on whether u_1 and v_1 are the same or not.

Let

$$M_s = tI - f(A) \pm e_{u_1}e_{v_1}^T \pm \cdots \pm e_{u_s}e_{v_s}^T,\ s = 1, \dots, r. \tag{3.5}$$

Then (3.4) can be written as $e_{u_r}^T M_{r-1}^{-1} e_{v_r}$. Assume the result holds for $r = k$, that is, the value of $e_{u_k}^T M_{k-1}^{-1} e_{v_k}$ is independent on the choice of the clique and on the ordering of vertices of the chosen clique. Now

$$\begin{aligned}
&e_{u_{k+1}}^T M_k^{-1} e_{v_{k+1}} \\
=&e_{u_{k+1}}^T\left(M_{k-1} \pm e_{u_k}e_{v_k}^T\right)^{-1} e_{v_{k+1}} \\
=&e_{u_{k+1}}^T\left(M_{k-1}^{-1} \mp \frac{M_{k-1}^{-1}e_{u_k}e_{v_k}^T M_{k-1}^{-1}}{1 + e_{v_k}^T M_{k-1}^{-1} e_{u_k}}\right)e_{v_{k+1}} \quad \text{(by (3.2))} \\
=&e_{u_{k+1}}^T M_{k-1}^{-1} e_{v_{k+1}} \mp \frac{\left(e_{u_{k+1}} M_{k-1}^{-1} e_{u_k}\right)\left(e_{v_k}^T M_{k-1}^{-1} e_{v_{k+1}}\right)}{1 + e_{v_k}^T M_{k-1}^{-1} e_{u_k}},
\end{aligned}$$

whose value does not depend on which clique the vertices are in, and remains unchanged if we reorder the vertices insides the clique, since each term satisfies this condition by the induction hypothesis. $\square$

3.7 Theorem. *Let X be a 1-walk regular graph. Assume the clique number of X is ω. Then for any graph Y on at most ω vertices, removing the edges of Y in a clique of X results in graphs that are cospectral with respect to adjacency, Laplacian, unsigned Laplacian, and normalized Laplacian matrices.*

Proof. Let $\hat{Y}$ be the graph obtained from Y by adding $|V(X)|-|V(Y)|$ isolated vertices; order the vertices of $\hat{Y}$ so that vertices of Y correspond to a clique in X. Assume Y has m edges and the edges of $\hat{Y}$ are $e_i=\{a_i,b_i\}$, $i=1,\dots,m$. We prove the result for Laplacian matrix. That is

$$\det\left(tI-L+(e_{a_1}-e_{b_1})(e_{a_1}-e_{b_1})^T+\cdots+(e_{a_m}-e_{b_m})(e_{a_m}-e_{b_m})^T\right)$$

does not depend on which clique of X the vertex set of Y correspond to or how the vertices of Y are ordered. We prove by induction. When $m=1$,

$$\begin{aligned}
&\det\left(tI-L+(e_{a_1}-e_{b_1})(e_{a_1}-e_{b_1})^T\right)\\
&=\det(tI-L)\det\left(I+(tI-L)^{-1}(e_{a_1}-e_{b_1})(e_{a_1}-e_{b_1})^T\right)\\
&=\det(tI-L)\left(1+(e_{a_1}-e_{b_1})^T(tI-L)^{-1}(e_{a_1}-e_{b_1})\right)\quad (\text{ by (3.3)})\\
&=\det(tI-L)\Big(1+e_{a_1}^T(tI-L)^{-1}e_{a_1}-e_{a_1}^T(tI-L)^{-1}e_{b_1}-e_{b_1}^T(tI-L)^{-1}e_{a_1}\\
&\qquad+e_{b_1}^T(tI-L)^{-1}e_{b_1}\Big),
\end{aligned}$$

which is independent on the choice of the edge $\{a_1,b_1\}$, since each summand in the second factor does not by Lemma 3.6. Again define M_s as in (3.5). With $f(A)=dI-A$ and some proper choice of $\pm$ signs and choice of vertices u_i and v_i in (3.5), we have

$$M_{4m}=tI-L+(e_{a_1}-e_{b_1})(e_{a_1}-e_{b_1})^T+\cdots+(e_{a_m}-e_{b_m})(e_{a_m}-e_{b_m})^T.$$

Now the Laplacian characteristic polynomial of the graph obtained from X by deleting the edges of $\hat{Y}$ satisfy

$$\begin{aligned}
&\det\left(tI-L+(e_{a_1}-e_{b_1})(e_{a_1}-e_{b_1})^T+\cdots+(e_{a_m}-e_{b_m})(e_{a_m}-e_{b_m})^T\right)\\
&=\det\left(M_{4(m-1)}+(e_{a_m}-e_{b_m})(e_{a_m}-e_{b_m})^T\right)\\
&=\det(M_{4(m-1)})\det\left(I+\left(M_{4(m-1)}\right)^{-1}(e_{a_m}-e_{b_m})(e_{a_m}-e_{b_m})^T\right)\\
&=\det(M_{4(m-1)})\left(1+(e_{a_m}-e_{b_m})^TM_{4(m-1)}^{-1}(e_{a_m}-e_{b_m})\right)\quad (\text{ by (3.3)})\\
&=\det(M_{4(m-1)})\Big(1+e_{a_m}^TM_{4(m-1)}^{-1}e_{a_m}-e_{a_m}^TM_{4(m-1)}^{-1}e_{b_m}-e_{b_m}^TM_{4(m-1)}^{-1}e_{a_m}\\
&\qquad+e_{b_m}^TM_{4(m-1)}^{-1}e_{b_m}\Big).
\end{aligned}$$

Since by induction the first factor, and by Lemma 3.6 each summand in the second factor do not depend on the choice of the clique in X nor on the ordering of vertices of Y, the result follows.

As above, with proper choice of signs and vertices in (3.5), we can prove the case for adjacency matrix and unsigned Laplacian matrix similarly, where deleting an edge corresponds to $tI - A + e_a e_b^T + e_b e_a^T$ in the adjacency case, and corresponds to $tI - S + (e_a + e_b)(e_a + e_b)^T$ in the unsigned Laplacian case.

For normalized Laplacian case, even though the perturbation to $N = I - \frac{1}{d}A$ from an edge deletion is not just a sum of simple outer product of basis vectors e_i, but it works similarly. Let X be a connected graph with no vertices of degree 1. Since the normalized Laplacian $N(X) = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$ and $D^{-1}L$ are similar:

$$D^{-1}L(X) = D^{-\frac{1}{2}} N(X) D^{\frac{1}{2}},$$

and $D^{-1}L = D^{-1}(D - A) = I - D^{-1}A$, we can prove the claim by showing that the characteristic polynomial of $D(X\backslash e)^{-1}A(X\backslash e)$ are all the same when removing any edge $e = \{a, b\}$ from X. Note that $D(X) = dI$ here. Now

$$\begin{aligned}
&\det\big(tI - D(X\backslash e)^{-1}A(X\backslash e)\big) \\
=&\det\left[tI - D(X)^{-1}A(X) + e_a\left((\frac{1}{d} - \frac{1}{d-1})e_a^T A + \frac{1}{d-1}e_b^T\right)\right. \\
&\left.+ e_b\left((\frac{1}{d} - \frac{1}{d-1})e_b^T A + \frac{1}{d-1}e_a^T\right)\right] \\
=&\det(tI - \frac{1}{d}A)\Big((\frac{1}{d} - \frac{1}{d-1})e_a^T A(tI - \frac{1}{d}A)^{-1}e_a + \frac{1}{d-1}e_b^T(tI - \frac{1}{d}A)^{-1}e_a \\
&+ (\frac{1}{d} - \frac{1}{d-1})e_b^T A(tI - \frac{1}{d}A)^{-1}e_b + \frac{1}{d-1}e_a^T(tI - \frac{1}{d}A)^{-1}e_b\Big).
\end{aligned}$$

All the difference is the argument is that instead of having the function $f(A)$ being $f(A) = (I - \frac{1}{d}A)^{-1}$ for the base step, we also have a combination of $f(A) = (I - \frac{1}{d}A)^{-1}$ and $f(A) = A(I - \frac{1}{d}A)^{-1}$. □

3.8 Remark. Let X be a 1-walk regular graph on n vertices with adjacency matrix A. Let $f(x)$ be a function defined on the eigenvalues of A. Let B be a square matrix of size n, all of whose non-zero entries correspond to a clique of X. By use of Lemma 3.6, the argument in Theorem 3.7 can in fact be used to prove that $\det\big(tI - f(A) - B\big)$ does not depend on the choice of a clique in X to which the non-zero entries of B corresponds nor on the ordering of the vertices of X inside the clique.

As a special type of 1-walk regular graph, there is more we can say about strongly regular graphs.

3.9 Corollary. *Let X be a strongly regular graph with clique number ω and let Y be any graph on at most ω vertices. Removing edges of Y from cliques of X results in cospectral graphs with cospectral complement, with respect to adjacency, Laplacian, unsigned Laplacian, and normalized Laplacian matrix.*

Proof. For strongly regular graphs, in addition to $f(A)\circ I = \alpha_f I$, $f(A)\circ A = \beta_f A$, we also have $f(A)\circ(J-I-A) = \gamma_f(J-I-A)$ for some γ_f, since a strongly regular graph SRG$(n,k;a,c)$ with adjacency matrix A satisfies $A^2 = kI + aA + c(J-I-A)$. The same argument in the proof of Theorem 3.7 shows that adding edges of a graph Z inside a coclique of a strongly regular graph X results in graphs that are cospectral with respect to A, L, S, N. Now the result follows from the fact that deleting edges of Z in a clique of X corresponds to adding edges of Z in the corresponding cocliqe of $\bar{X}$ and the fact that $\bar{X}$ is also 1-walk regular. □

Hence the graphs we obtained in Example 3.3 are cospectral with cospectral complement with respect to A, L, S and N.

4 Some examples

There are exactly 15 non-isomorphic strongly regular graphs with parameters SRG(25,12, 5, 6). Their adjacency matrices can be found at Spence's website: `http://www.maths.gla.ac.uk/~es/srgraphs.php`. In Table 1, we denote these graphs as X_i for $i = 0, 1, \ldots, 14$, in accordance with the order of the adjacency matrices given on the website. Exactly one of these graphs, X_{14}, is edge-transitive, that is, deleting any edge from the graph, all the resulting graphs are isomorphic. This is the Latin square graph corresponding to the addition table of $\mathbb{Z}_5$.

All the 15 graphs have clique number 5: with $X_0, \ldots, X_{12}$ containing exactly 3 cliques of size 5, and X_{13} and X_{14} containing exactly 15 cliques of size 5. In Table 1, for each of the above 15 strongly regular graphs X, we show the number of pairwise non-isomorphic graphs that results when we delete the edges of a small graph from cliques of X (so the graphs are cospectral with respect to A, L, S, and N). These small graphs include K_2, K_3, K_4, P_3, and two graphs on 5 vertices. For example, removing an edge from X_0 gives a family of 81 graphs, they are pairwise non-isomorphic but cospectral with respect to A, L, S and N.

Similarly, removing edges of a triangle from X_0 gives such a family of graphs of size 132.

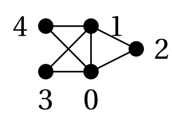
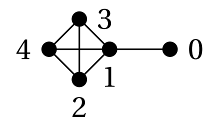
Subgraphs removed from a clique of X_i / Graphs	K_2	K_3	K_4	P_3		
X_0	81	132	48	252	17	9
X_1	150	250	89	373	30	15
X_2	50	86	32	216	10	5
X_3	150	250	89	373	30	15
X_4	81	134	50	247	17	9
X_5	79	132	46	304	16	15
X_6	29	50	17	148	6	5
X_7	31	50	19	155	7	9
X_8	50	86	32	232	10	15
X_9	33	52	20	97	8	6
X_{10}	6	10	6	24	2	3
X_{11}	79	134	48	298	19	14
X_{12}	29	49	15	101	6	4
X_{13}	5	10	5	16	5	4
X_{14}	1	2	1	5	1	1

Table 1: The number of pairwise non-isomorphic (cospectral) subgraphs of X_i obtained by deleting edges of a small graph, K_2, K_3, K_4, P_3, ect., respectively, in cliques of X_i

5 Matrices

In the next section, we provide more general ways to construct Laplacian or unsigned Laplacian cospectral graphs. In this section, we develop some tools for that.

5.1 Similar matrices

Let M_1, M_2 be two cospectral Hermitian matrices. We give a characterization of when their rank one updates $M_1 - v_1 v_1^*$ and $M_2 - v_2 v_2^*$ are also cospectral.

5.1 Theorem. *Let M_1 and M_2 be two similar Hermitian matrices of size n. Let v_1 and v_2 be two vectors in $\mathbb{C}^n$. Then the following are equivalent:*

(a) $v_1 v_1^* - M_1$ *and* $v_2 v_2^* - M_2$ *are also similar,*

(b) $v_1^*(tI - M_1)^{-1} v_1 = v_2^*(tI - M_2)^{-1} v_2,$

(c) there exists a unitary matrix U such that

$$UM_1 = M_2U, \quad \text{and} \quad Uv_1 = v_2.$$

Furthermore, if M_1, M_2, v_1 and v_2 are all real, then the unitary matrix U in condition [(c)] *can be chosen to be orthogonal.*

Proof. We first prove that conditions (a) and (b) are equivalent. Let M_1 and M_2 be two similar Hermitian matrices. For $i = 1, 2$,

$$\begin{aligned}
\det\big(tI - (v_i v_i^* - M_i)\big) &= \det(tI + M_i - v_i v_i^*) \\
&= \det\Big((tI + M_i)\big[I - (tI + M_i)^{-1} v_i v_i^*\big]\Big) \\
&= \det(tI + M_i)\big(1 - v_i^*(tI + M_1)^{-1} v_i\big). \qquad (5.1)
\end{aligned}$$

Since the matrices involved are Hermitian, we know that $v_1 v_1^* - M_1$ and $v_2 v_2^* - M_2$ are similar if and only if $\det\big(tI - (v_1 v_1^* - M_1)\big) = \det\big(tI - (v_2 v_2^* - M_2)\big)$. Therefore by equation (5.1), for two similar Hermitian matrices M_1 and M_2, $v_1 v_1^* - M_1$ and $v_2 v_2^* - M_2$ are similar if and only if $v_1^*(tI - M_1)^{-1} v_1 = v_2^*(tI - M_2)^{-1} v_2$.

We now show that (b) implies (c). Let

$$M_1 = \sum_{r=1}^{m} \theta_r E_r, \quad M_2 = \sum_{r=1}^{m} \theta_r F_r$$

be the spectral decomposition of M_1 and M_2, respectively. Then

$$v_1^*(tI - M_1)^{-1} v_1 = \sum_{r=1}^{m} \frac{v_1^* E_r v_1}{t - \theta_r}$$

and condition (b) holds if and only if

$$v_1^* E_r v_1 = v_2^* F_r v_2, \forall r \tag{5.2}$$

Now we construct a unitary matrix U such that $UM_1 = M_2U$ and $Uv_1 = v_2$. Any unitary matrix U' that maps an orthonormal basis of each eigenspace of M_1 to an orthonormal basis of the corresponding eigenspace of M_2 satisfies $U'M_1 = M_2U'$. In choosing a basis for each eigenspace, we can start with any unit vector in the eigenspace. Choose the first basis vector in the eigenspace associated to θ_r to be $\frac{1}{\sqrt{v_1^* E_r v_1}} E_r v_1$ if $E_r v_1 \neq 0$, and if $E_r v_1 = 0$ we don't put any restrictions on the orthonormal basis of the eigenspace associated to θ_r. Choose an orthonormal basis for each eigenspace of M_2 in the same way. Then the transition matrix U between the two bases is unitary and satisfies $UM_1 = M_2U$. Furthermore, and for any r such that $E_r v_1 \neq 0$, we have

$$U\left(\frac{1}{\sqrt{v_1^* E_r v_1}} E_r v_1\right) = \frac{1}{\sqrt{v_2^* F_r v_2}} F_r v_2.$$

By (5.2), we conclude that for all r, $U(E_r v_1) = F_r v_2$. Thus

$$Uv_1 = U\sum_r E_r v_1 = \sum_r F_r v_2 = v_2.$$

Now we prove (c) implies (a). Assume (c) holds, then

$$U(v_1 v_1^* - M_1) = v_2 v_1^* - UM_1 = v_2 v_2^* U - M_2 U = (v_2 v_2^* - M_2)U,$$

therefore $v_1 v_1^* - M_1$ and $v_2 v_2^* - M_2$ are similar. $\square$

When M_1 and M_2 are adjacency matrices of cospectral graphs, we have the following result of Johnson and Newman [8], and a similar result for unsigned Laplacian matrices. Denote the complement of X by $\bar{X}$, and denote the all-ones vector by $\mathbf{1}_n$.

5.2 Corollary. *(a) (Johnson and Newman)*
If X and Y are adjacency cospectral graphs with cospectral complements, then there is an orthogonal matrix Q such that

$$QA(X)Q^T = A(Y),\ QA(\bar{X})Q^T = A(\bar{Y}),\ Q\mathbf{1} = \mathbf{1}.$$

(b) *If X and Y are unsigned Laplacian cospectral graphs with cospectral complements, then there is an orthogonal matrix Q such that*

$$QS(X)Q^T = S(Y),\ QS(\bar{X})Q^T = S(\bar{Y}),\ Q\mathbf{1} = \mathbf{1}.$$

(c) *If X and Y are Laplacian cospectral graphs, then there is an orthogonal matrix Q such that*

$$QL(X)Q^T = L(Y),\ QL(\bar{X})Q^T = L(\bar{Y}),\ Q\mathbf{1} = \mathbf{1}.$$

Proof. Assume X and Y have n vertices. By assumption, $A(X)$ and $A(Y)$ are similar, and $J - A(X)$ and $J - A(Y)$ are similar. Let $M_1 = A(X)$, $M_2 = A(Y)$, and $v_1 = v_2 = \mathbf{1}_n$. By the equivalence of condition (a) and (c) in Theorem 5.1, there exists an orthogonal matrix Q such that

$$QA(X) = A(Y)Q,\ Q\mathbf{1}_n = \mathbf{1}_n.$$

Therefore

$$QA(\bar{X})Q^T = Q(J - I - A(X))Q^T = J - I - A(Y) = A(\bar{Y}).$$

The proof for (b) follows similarly with $S(\bar{X}) = J + (n-2)I - S(X)$. The proof for (c) follows from the fact if two graphs are Laplacian cospectral then so are their complement, $L(\bar{X}) = nI - J - L(X)$, and from the above similar argument. □

5.3 Remark. Let α, β be fixed real numbers. Let X be a graph and consider the weighted matrix $A_{\alpha,\beta}(X) = \alpha D(X) + \beta A(X)$. We say two graphs are $A_{\alpha,\beta}$-cospectral if their associated $A_{\alpha,\beta}$ matrices have the same characteristic polynomials. Then a similar argument as in the above corollary shows that: if X and Y are $A_{\alpha,\beta}$-cospectral with cospectral complements, then there is an orthogonal matrix Q such that

$$QA_{\alpha,\beta}(X)Q^T = A_{\alpha,\beta}(Y),\ QA_{\alpha,\beta}(\bar{X})Q^T = A_{\alpha,\beta}(\bar{Y}),\ Q\mathbf{1} = \mathbf{1}.$$

The fact the all-ones vector is in the null space of the Laplacian matrix of any graph, can be used for a different proof of (c) in Corollary 5.2.

5.4 Lemma. *Let Y_1 and Y_2 be Laplacian cospectral graphs. Then there is an orthogonal matrix Q such that $Q^T L(Y_1)Q = L(Y_2)$ and $Q\mathbf{1} = \mathbf{1}$.*

Proof. Assume Y_1 and Y_2 are on m vertices. Then $\frac{1}{\sqrt{m}}\mathbf{1}$ is a unit eigenvector associated to eigenvalue 0 of $L(Y_i)$ for $i = 1,2$. Denote the eigenvalues of Y_i as $\theta_1 = 0,\dots,\theta_m$, and let $\Lambda = \operatorname{diag}(\theta_1 = 0,\dots,\theta_m)$. Then there exist orthogonal matrices Q_i of the form $Q_i = [\frac{1}{\sqrt{m}}\mathbf{1}\ \ \hat{Q}_i]$ such that $Q_i^T L(Y_i) Q_i = \Lambda$. Let $Q = Q_1 Q_2^T$, then

$$Q^T L(Y_1) Q = Q_2 Q_1^T L(Y_1) Q_1 Q_2^T = Q_2 \Lambda Q_2^T = L(Y_2)$$

and

$$Q\mathbf{1} = [\frac{1}{\sqrt{m}}\mathbf{1}\ \ \hat{Q}_1] \begin{bmatrix} \frac{1}{\sqrt{m}}\mathbf{1}^T \\ \hat{Q}_2^T \end{bmatrix} \mathbf{1} = [\frac{1}{\sqrt{m}}\mathbf{1}\ \ \hat{Q}_1] \sqrt{m} e_1 = \mathbf{1}. \qquad \square$$

5.2 Gram matrices

Given a graph X, if we assign a direction to each edge we obtain an oriented graph $\tilde{X}$. Further, given arc (a,b), we call a its tail and b its head. The incidence matrix of an oriented graph $\tilde{X}$ is the $(0,\pm1)$-matrix with rows indexed by the vertices and columns indexed by the arcs, such that the ae-entry is equal to 1 if vertex a is the head of the arc e, -1 if a is the tail of e, and 0 otherwise. This incidence matrix is called an *oriented incidence matrix* of X.

Different orientations of X result in different oriented incidence matrices of X, but for any oriented incidence matrix B of X, we have $BB^T = L(X)$. Furthermore, different oriented incidence matrices of the same graph are related by an orthogonal matrix.

5.5 Theorem. *Let B and C be $m \times n$ matrices. Then there is a unitary matrix Q such that $QB = C$ if and only if $B^* B = C^* C$.*
If B and C are real, then there is an orthogonal matrix Q such that $QB = C$ if and only if $B^T B = C^T C$.

Proof. **1, SVD**: We prove the result for real matrix case. Since $B^T B = C^T C$ is a real symmetric matrix, it is orthogonally diagonalizable, say by $U = [u_1\ \dots\ u_n]$ to $\Lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. That is, $U^T B^T B U = \Lambda$. Assume $\operatorname{rk}(B) = r$, then $\lambda_1 \ge \cdots \ge \lambda_r > 0$, and $\lambda_{r+1} = \cdots = \lambda_n = 0$.

Let

$$v_i = \frac{1}{\sqrt{\lambda_i}} B u_i, \quad i = 1,\dots,r \tag{5.3}$$

and let

$$v_{r+1},\ \dots,\ v_m$$

be an orthonormal basis of the null space of B^T. Then

$$V = [v_1 \ \dots \ v_m]$$

is an orthogonal matrix, and

$$\begin{aligned} BU &= [Bu_1 \cdots Bu_r \ Bu_{r+1} \cdots Bu_m] \\ &= [\sqrt{\lambda_1} v_1 \cdots \sqrt{\lambda_r} v_r \ 0 \cdots 0] \quad \text{by (5.3)} \\ &= V\Sigma, \end{aligned}$$

where Σ is the $m \times n$ matrix with $\Sigma_{i,i} = \sqrt{\lambda_i}$ for $i = 1, \dots, r$ and zero elsewhere. That is,

$$B = V\Sigma U^T \tag{5.4}$$

is a singular value decomposition of B. Similarly, let $w_i = \frac{1}{\sqrt{\lambda_i}} C u_i$, $i = 1, \dots, r$, and let $w_{r+1}, \dots, w_m$ be an orthonormal basis of the null space of C^T. Then for $W = [w_1, \dots, w_m]$,

$$C = W\Sigma U^T \tag{5.5}$$

is a singular value decomposition of C. From (5.4) and (5.5) we have

$$C = W\Sigma U^T = W(V^T BU)U^T = (WV^T)B.$$

Therefore $Q = WV^T$ is an orthogonal matrix satisfies $QB = C$. □

Proof. **2, reflection induction**: We prove the result for real matrix case. Let the columns of B and C be respectively $b_1, \dots, b_n$ and $c_1, \dots, c_n$. Assume that $\mathrm{rk}(B) = r$ and that $b_1, \dots, b_r$ is a basis for the column space of B. Then $c_1, \dots, c_r$ is a basis for the column space of C.

Since b_1 and c_1 have the same length, the matrix Q_1 representing reflection in the hyperplane $(b_1 - c_1)^\perp$ is an orthogonal matrix swapping b_1 and c_1 and

$$(Q_1 B)^T Q_1 B = B^T B = C^T C;$$

we obtain a matrix $Q_1 B$ that share the same first column as C and is equivalent to C (since $(Q_1 B)^T Q_1 B = C^T C$). We denote $Q_1 B$ as B.

Now assume inductively that $b_i = c_i$ for $i = 1, \dots, k$, with $1 \le k \le r$. If y and z are two vectors such that $\langle y, y\rangle = \langle z, z\rangle$ and

$$\langle c_i, y\rangle = \langle c_i, z\rangle, \quad (i = 1, \dots, k)$$

then $y-z$ is orthogonal to $c_1,\dots,c_k$ and the reflection in $(y-z)^\perp$ fixes $c_1,\dots,c_k$ and swaps y and z. If $k<r$, take $y=Be_{r+1}$ and $z=Ce_{r+1}$, and the above implies that there is an orthogonal matrix Q_{k+1} such that the first $k+1$ columns of $Q_{k+1}B$ and C are equal.

To complete the proof, we observe that if the first r columns of B is a basis of col(B) and are equal to the first r columns of C, then $B^TB=C^TC$ implies $B=C$. The theorem follows. □

6 More (unsigned) Laplacian cospectral graphs

In Section 3, we constructed graphs cospectral with respect to A, L, S and N by removing edges of the same graph from cliques of a 1-walk regular graph. In fact, there is more we can say about the Laplacian and unsigned Laplacian case. Removing edges of Laplacian cospectral graphs from cliques of a 1-walk regular graph results in Laplacian cospectral graphs; removing edges of unsigned Laplacian cospectral graphs that have cospectral complements (with respect to unsigned Laplacian) from cliques of a 1-walk regular graph results in unsigned Laplacian cospectral graphs.

6.1 Theorem. *Let X be a 1-walk regular graph with clique number ω.*

(a) If Y_1 and Y_2 are two Laplacian cospectral graphs on at most ω vertices, then removing edges of Y_1 and Y_2, respectively, from a clique of X result in Laplacian cospectral graphs.

(b) If Y_1 and Y_2 are two unsigned Laplacian cospectral graphs with cospectral complement on at most ω vertices, then removing edges of Y_1 and of Y_2, respectively, from a clique of X results in unsigned Laplacian cospectral graphs.

Proof. (a) Assume $|V(X)|=n$ and $|V(Y_1)|=|V(Y_2)|=m$. We just need to prove the result for the case where edges of Y_1 and Y_2 are removed from the same clique of size m in X, the general case follows from this and Theorem 3.7.

For $i=1,2$, let B_i be a signed incidence matrix of Y_i, that is, $B_iB_i^T=L(Y_i)$. By Lemma 5.4, there exist an orthogonal matrix Q such that $Q^TL(Y_1)Q=L(Y_2)$ and $Q\mathbf{1}=\mathbf{1}$. Hence $Q^TB_1B_1^TQ=B_2B_2^T$. By Theorem 5.5, there exists an orthogonal matrix Q_0 such that

$$B_2=Q^TB_1Q_0. \tag{6.1}$$

Assume without loss of generality that the first m vertices of X form a clique. Let $\hat{Y}_i$ be the graph obtained from Y_i by adding $n-m$ isolated vertices so that $\hat{Y}_i$ has the same vertex set as X and the vertices of Y_i are labelled $1,\ldots,m$. For $i=1,2$, let

$$\hat{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}$$

be the matrix obtained from B_i by adding $n-m$ rows of zero. Then $\hat{B}_i\hat{B}_i^T = L(\hat{Y}_i)$. Our goal is to prove that the two graphs obtained by removing edges of $\hat{Y}_1$ or $\hat{Y}_2$, respectively, from X are Laplacian cospectral graphs. That is,

$$\det\big(tI - L(X) + L(\hat{Y}_1)\big) = \det\big(tI - L(X) + L(\hat{Y}_2)\big).$$

Since

$$\begin{aligned}\det\big(tI - L(X) + L(\hat{Y}_1)\big) &= \det\Big(\big(tI - L(X)\big)\big(I + (tI - L(X))^{-1}\,\hat{B}_1\hat{B}_1^T\big)\Big)\\ &= \phi_L(X,t)\det\big(I + \hat{B}_1^T(tI - L(X)\big)^{-1}\hat{B}_1\big) \quad (\text{ by Lemma 3.5}),\end{aligned}$$

it follows that it is equivalent to prove

$$\det\big(\hat{B}_1^T(tI - L(X)\big)^{-1}\hat{B}_1\big) = \det\big(\hat{B}_2^T(tI - L(X)\big)^{-1}\hat{B}_2\big).$$

Since X is 1-walk regular and the first m vertices form a clique of X, by Lemma 2.2, there exist scalars α,β such that $(tI - L(X))^{-1}$ is of the form

$$(tI - L(X))^{-1} = \begin{bmatrix} \alpha I - \beta(J-I) & M_1 \\ M_2 & M_3 \end{bmatrix},$$

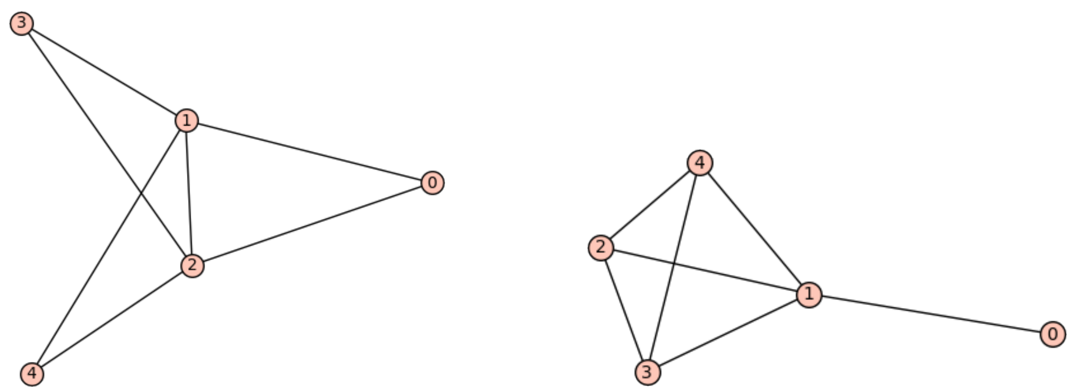
for some matrices M_1, M_2 and M_3, whose value does not matter here. Now

$$\begin{aligned}\det\big(\hat{B}_2^T(tI - L(X)\big)^{-1}\hat{B}_2\big) &= \det\left(\begin{bmatrix} B_2^T & 0\end{bmatrix}\begin{bmatrix} \alpha I - \beta(J-I) & M_1 \\ M_2 & M_3 \end{bmatrix}\begin{bmatrix} B_2 \\ 0\end{bmatrix}\right)\\ &= \det\Big(B_2^T\big(\alpha I - \beta(J-I)\big)B_2\Big)\\ &= \det\Big(Q_0^T B_1^T Q\big(\alpha I - \beta(J-I)\big)Q^T B_1 Q_0\Big) \quad \text{by (6.1)}\\ &= \det\Big(B_1^T Q\big(\alpha I - \beta(J-I)\big)Q^T B_1\Big)\\ &= \det\Big(B_1^T\big(\alpha I - \beta(J-I)\big)B_1\Big) \quad (\text{ since } Q\mathbf{1} = \mathbf{1})\\ &= \det\big(\hat{B}_1^T(tI - L(X)\big)^{-1}\hat{B}_1\big).\end{aligned}$$

(b) By use of the vertex-edge incidence matrix instead of an oriented incidence matrix of X, and Corollary 5.2 (b), the result for unsigned Laplacian case follows similarly. □

As in Remark 5.3, we mention a more general result on the weighted matrix $A_{\alpha,\beta}(X) = \alpha D(X) + \beta A(X)$ with $\alpha \geq |\beta|$. In this case, $A_{\alpha,\beta}(X)$ is positive-semidefinite, and hence there exists a matrix B such that $BB^T = A_{\alpha,\beta}$. With a similar proof as in Theorem 6.1 we have the following. Recall two graphs X and Y are $A_{\alpha,\beta}$-cospectral if $\det\big(tI - A_{\alpha,\beta}(X)\big) = \det\big(tI - A_{\alpha,\beta}(Y)\big)$.

6.2 Remark. Let α, β be real numbers with $\alpha \geq |\beta|$. Let X be a 1-walk regular graph with clique number w. Let Y_1 and Y_2 be two graphs on at most w vertices that are $A_{\alpha,\beta}$-cospectral graphs with cospectral complements. Then removing edges of Y_1 and of Y_2, respectively, from a clique of X results in $A_{\alpha,\beta}$ cospectral graphs.



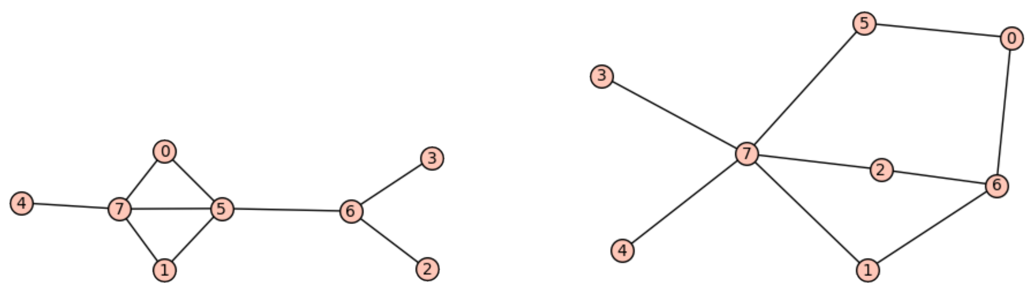
(a) Y_1 (b) Y_2

Figure 1: A pair of unsigned Laplacian cospectral graphs with cospectral complements

6.3 Example. Let Y_1 and Y_2 be the two graphs as shown in Figure 1 (also in Table 1). They are unsigned Laplacian cospectral with cospectral complements. By Theorem 6.1, removing edges of Y_1 or edges of Y_2, respectively, inside a clique of a 1-walk regular graph gives unsigned Laplacian cospectral graphs, which are not isomorphic (they have different degree sequences). Therefore, for each of the strongly regular graphs in Table 1, we can take the union of the two families of non-isomorphic graphs resulting from deleting edges of Y_1 or Y_2, respectively, and get a bigger family of pairwise non-isomorphic but unsigned Laplacian cospectral graphs. The sizes of the families of non-isomorphic graphs resulting from deleting edges of Y_1 or Y_2 from cliques of X correspond to the last two columns of Table 1. For example, deleting edges of Y_1 in a clique of

X_0 gives a family of 17 non-isomorphic but (A, L, S, N) cospectral graphs, deleting edges of Y_2 in a clique of X_0 gives such a family of size 9. They together give a family of $17 + 9 = 26$ nonisomorphic unsigned Laplacian cospectral graphs.

6.4 Example. Note that for unsigned Laplacian case, the condition that the two small graphs are unsigned Laplacian cospectral with cospectral complement is important. For example, the two graphs Y_3 and Y_4 in Figure 2 are unsigned Laplacian cospectral, but don't have cospectral complements. Removing their edges inside a clique of SRG(36,14,7,4) does not always result in unsigned Laplacian cospectral graphs.



(a) Y_3 (b) Y_4

Figure 2: A pair of unsigned Laplacian cospectral graphs with non-cospectral complements

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