Geometrical optics of first-passage functionals of random acceleration

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Random acceleration is a fundamental stochastic process encountered in many applications. In the one-dimensional version of the process a particle is randomly accelerated according to the Langevin equation $\ddot{x}(t) = \sqrt{2D}\xi(t)$, where $x(t)$ is the particle's coordinate, $\xi(t)$ is Gaussian white noise with zero mean, and D is the particle velocity diffusion constant. Here we evaluate the $A \to 0$ tail of the distribution $P_n(A|L)$ of the functional $I[x(t)] = \int_0^T x^n(t)dt = A$, where T is the first-passage time of the particle from a specified point $x = L$ to the origin, and $n \geq 0$. We employ the optimal fluctuation method akin to geometrical optics. Its crucial element is determination of the optimal path – the most probable realization of the random acceleration process $x(t)$, conditioned on specified $A \to 0$, n and L. This realization dominates the probability distribution $P_n(A|L)$. We show that the $A \to 0$ tail of this distribution has a universal essential singularity, $P_n(A \to 0|L) \sim \exp\left(-\frac{\alpha_n L^{3n+2}}{DA^3}\right)$, where α_n is an n-dependent number which we calculate analytically for $n = 0, 1$ and 2 and numerically for other n. For $n = 0$ our result agrees with the asymptotic of the previously found first-passage time distribution.

I. INTRODUCTION

The random acceleration process is governed by the Langevin equation

$$\ddot{x}(t) = \sqrt{2D}\xi(t). \tag{1}$$

This equation describes the position of a particle moving along the x-axis and subject to a random force which is modeled as a Gaussian white noise with zero mean, $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$. Alternatively, $x(t)$ can be considered as the integral of a Brownian motion over time. The random acceleration is a fundamental stochastic process in its own right. On the one hand, it serves as a simple example of a non-Markovian process (which becomes Markovian when considered in two dimensions x and $\dot{x}$, see *e.g.* Ref. [1]). On the other hand, its mathematical equivalents have found a variety of applications in physics: from a simplified description of free semiflexible polymer chains in narrow channels [2–7] to interface growth in 1+1 dimensions [8–10] and to decaying turbulence in the Burgers equation [11, 12]. In all these systems it is a spatial coordinate which plays the role of time t in Eq. (1), while the polymer shape, or the interface shape, *etc.* plays the role of x.

Here we are interested in the statistics of first-passage functionals $I[x(t)] = \int_0^T x^n(t)\,dt$ of the random acceleration $x(t)$, defined up to the time of first passage of the process T, starting say at $x = L > 0$, to a specified point in space, for example to the origin. The case $n = 0$ corresponds to the statistics of the first-passage time itself. The case $n = 1$ corresponds to the first-passage area under the graph of $x(t)$. In the context of interface growth, governed by the noisy Mullins-Herring equation [8, 9], it describes the area under the stochastic interface until it

crosses a zero level in space for the first time. The case $n = 2$ corresponds to the statistics of the moment of inertia of a semiflexible polymer chain of a given length in narrow channels. It is natural then to attempt to calculate the distribution of the values of the first-passage functional $I[x(t)] = \int_0^T x^n(t)\,dt$ for arbitrary n.

For comparison, the statistics of first-passage Brownian functionals [13, 14] – where $x(t)$ is a Brownian motion – is well studied, see Ref. [15] and references therein. For the random acceleration process, however, the problem has been solved only for $n = 0$, that is only for the statistics of the first-passage time itself [6, 16, 17].

In this work we focus on *large-deviation statistics* of the first-passage functionals of random acceleration for any $n \geq 0$. Specifically, we evaluate the $A \to 0$ tail of the probability distribution $P_n(A \to 0|L)$ of the values $I[x(t)] = A$ and show that this tail exhibit an essential singularity, see Eq. (18) below. To achieve these goals, we employ the optimal fluctuation method akin to geometrical optics. The method relies on the determination of the optimal path, that is the most likely realization of the process $x(t)$, conditioned on the specified value of $A \to 0$ at given n and L. It is this optimal path that dominates the $A \to 0$ tail of $P_n(A \to 0|L)$. Previously, the geometrical optics was applied to a plethora of problems related to statistics of Brownian motion [15, 18–26]. An extension of the method to the random acceleration is a natural next step.

Here is a plan of the remainder of the paper. We complete the formulation of the problem, establish the scaling properties of $P_n(A \to 0|L)$ and derive the governing equation of the optimal fluctuation method in Sec. II. Some analytical and numerical solutions for different n are presented in Sec. III. Section IV includes a brief summary and an extension of our results. A technical derivation is delegated to the Appendix.

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II. FORMULATION OF THE PROBLEM AND GOVERNING EQUATIONS

We start by completing the formulation of the problem. The initial and final positions of the particle are

$$x(t = 0) = L, \quad x(T) = 0, \tag{2}$$

where T is the first passage time to the origin, and L can be assumed positive without loss of generality. We assume for simplicity that the particle starts with zero velocity:

$$\dot{x}(t = 0) = 0. \tag{3}$$

We consider first-passage functionals of the form $I[x(t)] = \int_0^T x^n(t)\, dt$ and study the probability distribution $P_n(A|L)$ of their values A:

$$\int_0^T x^n(t)\, dt = A. \tag{4}$$

Equations (1)-(4) define the stochastic problem completely. Their dimensional analysis yields the following *exact* scaling behavior of $P_n(A|L)$:

$$P_n(A|L) = \frac{D^{1/3}}{L^{n+\frac{2}{3}}} F_n\left(\frac{D^{1/3}A}{L^{n+\frac{2}{3}}}\right) \tag{5}$$

with an unknown scaling function $F_n(z)$.

Rather than attempting to determine the entire scaling function $F_n(z)$, here we find its leading-order $z \to 0$ asymptotic. This asymptotic describes the large-deviation tail $A \to 0$ of the distribution $P_n(A|L)$, and it can be obtained by the optimal fluctuation method, akin to geometrical optics. We identify the action functional, corresponding to the Langevin equation (1):

$$S[x(t)] = \frac{1}{4D} \int_0^T \ddot{x}^2(t)\, dt, \tag{6}$$

and seek the optimal path $x_*(t)$ which minimizes this functional subject to the boundary conditions (2) and (3), to the positivity condition $x(t) > 0$ for $0 < t < T$, and to the integral constraint

$$I[x(t)] = \int_0^T x^n(t)\, dt = A. \tag{7}$$

The minimization must be performed not only with respect to different paths $x(t)$, but also with respect to the first-passage time T.

Let us rescale the coordinate, $\tilde{x} = x/L$. The action functional (6) takes the form

$$S[x(t)] = \frac{L^2}{2D} s(\tilde{x}), \quad \text{where} \quad s(\tilde{x}) = \frac{1}{2} \int_0^T \ddot{\tilde{x}}^2(t)\, dt. \tag{8}$$

The constraint (7) becomes

$$I[\tilde{x}(t)] = \int_0^T \tilde{x}^n(t)dt = \frac{A}{L^n}. \tag{9}$$

The minimization of the rescaled functional $s(\tilde{x})$ subject to the constraint (9) can be achieved by minimizing the modified functional

$$s_\lambda[\tilde{x}(t)] = s[\tilde{x}(t)] - \lambda I[\tilde{x}(t)]. \tag{10}$$

The Lagrange multiplier λ turns out to be negative, so we can set $\lambda = -\Lambda^4$, where $\Lambda > 0$. Now we also rescale time, $\tilde{t} = \Lambda t$. The first-passage time T also gets rescaled, $\tilde{T} = \Lambda T$. The functional (10) becomes

$$s_\lambda[\tilde{x}(\tilde{t})] = \Lambda^3 \int_0^{\tilde{T}} \left[\frac{\ddot{\tilde{x}}^2(\tilde{t})}{2} + \tilde{x}^n(\tilde{t})\right] d\tilde{t}. \tag{11}$$

and we will drop the tildes everywhere in the following. Since the rescaled functional $s_0[x(t)]$ (recall that the tildes are dropped) involves the particle acceleration $\ddot{x}(t)$, the Euler-Lagrange equation is of the fourth order (see the Appendix):

$$x^{(4)}(t) + nx^{n-1}(t) = 0, \tag{12}$$

where the superscript (4) denotes the fourth derivative with respect to time. Three boundary conditions for Eq. (12) come with the formulation of the original stochastic problem, see Eqs. (2) and (3):

$$x(0) = 1, \ \dot{x}(0) = 0, \ \text{and } x(T) = 0. \tag{13}$$

The fourth boundary condition,

$$\ddot{x}(T) = 0, \tag{14}$$

follows from minimization of the action with respect to all possible variations of the particle velocity $\dot{x}$ at $t = T$ (see the Appendix).

The general solution of the rescaled Euler-Lagrange equation (12) has four arbitrary constants. When this equation is supplemented by the four boundary conditions (13) and (14) [and the inequality $x(0 < t < T) > 0$], the problem of finding the $A \to 0$ asymptotic of $P_n(A|L)$ is determined completely only for $n = 0$ where $A = T$, and one is looking for the distribution $P_n(T|L)$ of first passage times. For all other $n > 0$ one should, in addition, minimize the action $S(A, T)$ with respect to T. The minimization yields the *optimal value* of the first-passage time $T = T_*(A)$ which dominates the probability $P_n(A|L)$ that we are after. As we show in the Appendix, this additional minimization brings about a fifth boundary condition

$$\dddot{x}(T) = 0. \tag{15}$$

Once the optimal path $x(t)$ and, for $n \neq 0$, the optimal value $T = T_*(A)$, are found, we can determine Λ from the relation

$$\Lambda = \frac{L^n}{A} \int_0^{T_*} x^n(t)\, dt, \tag{16}$$

which follows from the constraint (7) or, equivalently, (9). The original action (6) can now be written as follows:

$$S[x(t)] = \frac{L^2\Lambda^3}{2D} s_0[x(t)], \quad \text{where}$$

$$s_0[x(t)] = \frac{1}{2}\int_0^T \ddot{x}^2(t)dt. \tag{17}$$

Plugging Eq. (16) into the first line of Eq. (17) we obtain, up to a pre-exponential factor, the $A \to 0$ tail of $P_n(A|L)$. It scales as

$$-\ln P_n(A \to 0|L) \simeq S = \frac{\alpha_n L^{3n+2}}{DA^3}, \tag{18}$$

where

$$\alpha_n = \frac{1}{4}\left[\int_0^{T_*} x^n(t)dt\right]^3 \int_0^{T_*} \ddot{x}^2(t)dt. \tag{19}$$

Equation (18) describes a universal essential singularity $\sim \exp(-A^{-3})$ of the $A \to 0$ tail of the distribution. It is much steeper than the essential singularity $\sim \exp(-A^{-1})$ of the first-passage Brownian functionals [15].

In fact, the large-deviation scaling (18) (with an unknown α_n) immediately follows from the exact scaling (5) once we realize that the $A \to 0$ asymptotic of the function $F_n(\dots)$ in Eq. (5) must exhibit, up to a pre-exponent, the characteristic weak-noise scaling $F_n \sim \exp(-\Phi/D)$, where Φ depends on A and L but is independent of D. Now let us concentrate on finding the optimal path, that is on solving Eq. (12) subject to the boundary conditions (13)-(15).

III. SOLUTION

A. General

Equation (12) is easily solvable for $n = 0$, 1 and 2, when the equation is linear. We will present these solutions shortly. In the general case, there is conservation law

$$\dot{x}(t)\,\dddot{x}(t) - \frac{1}{2}\ddot{x}^2(t) + x^n(t) = C = \text{const}, \tag{20}$$

which is a higher-order analog of energy conservation in classical mechanics. The conservation law (20) reduces the order of Eq. (12) by one. Using the boundary conditions (13)-(15) at $t = T$, we find that $C = 0$ for all $n > 0$ [27].

Evaluating the left hand side of the conservation law (20) (where $C = 0$) at $t = 0$, we uncover one more universal property of the optimal path:

$$\ddot{x}(t = 0) = -\sqrt{2} \quad \text{for all} \quad n > 0. \tag{21}$$

Finally, using the conservation law (20) with $C = 0$, integration by parts and Eqs. (13) and (14), we can rewrite

the expression (19) for α_n in two equivalent alternative forms:

$$\alpha_n = \frac{1}{6}\left[\int_0^{T_*} x^n(t)dt\right]^3 = \frac{27}{32}\left[\int_0^{T_*} \ddot{x}^2(t)dt\right]^4. \tag{22}$$

B. $n = 0$: First-passage time

The first-passage time distribution $P(T|L)$ of the random acceleration process was determined quite some time ago [6, 16, 17]. Its short-time asymptotic coincides, in the leading order, with the short-time asymptotic of the propagator of the random acceleration. For the zero initial particle velocity, the exact propagator (see *e.g.* Ref. [6]) simplifies to

$$\rho(T, v) = \frac{\sqrt{3}}{2\pi DT^2} e^{-\frac{3L^2+3LTv+T^2v^2}{DT^3}}, \tag{23}$$

where $v = \dot{x}(t = T)$ is the particle velocity (in the original units) at $t = T$. We identify the action, corresponding to this distribution,

$$S_\rho(T, v) = \frac{3L^2 + 3LTv + T^2v^2}{DT^3}, \tag{24}$$

and focus on the large-deviation regime $T \to 0$, where this action is much larger than unity. Minimizing $S_\rho(T, v)$ with respect to v, we obtain the optimal value $v_* = -3L/(2T)$. The corresponding minimum of the action,

$$S_\rho(T, v_*) = \frac{3L^2}{4DT^3}, \tag{25}$$

determines the small-A asymptotic of $P(T|L)$:

$$-\ln P(T|L) \simeq \frac{3L^2}{4DT^3}, \tag{26}$$

which obeys our asymptotic scaling relation (18) with $\alpha_0 = 3/4$. Now we will rederive the asymptotic (26) by using the optimal fluctuation formalism.

For $n = 0$ the Euler-Lagrange equation (12) becomes trivial: $x^{(4)} = 0$. Its solution, satisfying the boundary conditions (13) and (14),

$$x(t) = 1 - \frac{3t^2}{2T^2} + \frac{t^3}{2T^3}, \tag{27}$$

is a cubic parabola. Equation (16) yields $\Lambda = 1$. Then, using Eq. (17), we arrive at Eqs. (25) and (26) as to be expected.

C. $n = 1$: First-passage area

For $n = 1$ the Euler-Lagrange equation (12) is still very simple: $x^{(4)} = -1$. Its solution is a quartic parabola.

Here we have to demand all five boundary conditions (13)-(15) which determine the four arbitrary constants and the optimal value of the first-passage time $T_* = 2^{3/4}$. The resulting rescaled optimal path,

$$x(t) = 1 - \frac{t^2}{\sqrt{2}} + \frac{t^3}{3\sqrt[4]{2}} - \frac{t^4}{24}, \qquad (28)$$

is depicted, alongside with the optimal acceleration $\ddot{x}(t)$, in Fig. 1. The optimal acceleration is nothing but the (rescaled) optimal realization of the white Gaussian noise $\xi(t)$, see Eq. (1). Needless to say, the optimal realization of the noise looks very differently from a *typical* realization of the noise. Now using Eqs. (18) and (19) for $n = 1$, we obtain

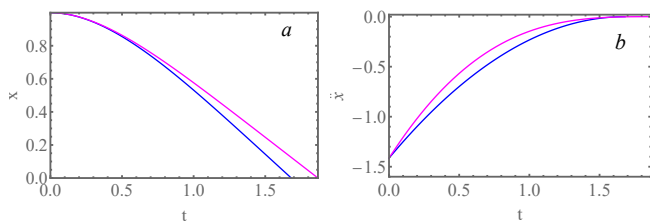$$-\ln P(A|L) \simeq \frac{108L^5}{625DA^3}, \qquad (29)$$

with $\alpha_1 = 108/625$.

FIG. 1: The rescaled optimal path $x(t)$ (a) and optimal acceleration $\ddot{x}(t)$ (b), dominating the $A \to 0$ asymptotics of $P(A|L)$ for $n = 1$ (blue) and $n = 2$ (magenta). The optimal first-passage time is $T_* = 2^{3/4}$ for $n = 1$ and $T_* = \pi/2^{3/4}$ for $n = 2$.

D. $n = 2$

Here the Euler-Lagrange equation (12) is still linear and elementary:

$$x^{(4)}(t) + 2x(t) = 0. \qquad (30)$$

The solution, obeying the boundary conditions (13)-(15), yields the rescaled optimal path:

$$x(t) = \frac{\left(1 - e^{\frac{t}{T_*}}\right)\sin\left(\frac{\pi t}{2T_*}\right) + \left(1 + e^{\frac{t}{T_*}}\right)\cos\left(\frac{\pi t}{2T_*}\right)}{(1 + e^{-\pi})\, e^{\frac{\pi t}{2T_*}}}, \qquad (31)$$

where $T_* = 2^{-3/4}\pi$ is the optimal first passage time. Figure 1 shows this optimal path alongside with the optimal acceleration $\ddot{x}(t)$.

Using Eqs. (18) and (19) for $n = 2$, we obtain

$$-\ln P(A|L) \simeq \frac{27\tanh^4\left(\frac{\pi}{2}\right)L^8}{256DA^3}, \qquad (32)$$

Here $\alpha_2 = (27/256)\tanh^4(\pi/2) = 0.074625\ldots$.

E. Numerics

For arbitrary n the optimal path can be found numerically. We used artificial relaxation in conjunction with "shooting". Artificial relaxation was implemented as follows. We introduced artificial time τ and replaced the Euler-Lagrange equation (12) by the fourth-order partial differential equation

$$\partial_\tau X(t, \tau) = -\partial_t^4 X(t, \tau) - nX^{n-1}(t, \tau), \qquad (33)$$

where the physical time t plays the role of a coordinate. The sign of the right-hand-side of Eq. (33) is chosen so as to enforce relaxation to a steady-state, $x(t) = X(t, \tau \to \infty)$ which satisfies our Eq. (12). The initial condition $X(t, \tau = 0)$ is chosen qualitatively similar to the expected steady-state solution. Since we do not know the optimal first-passage time T *a priori*, we use the "shooting" method, see e.g. Ref. [28]. We first solve Eq. (33) with boundary conditions (13) and (14) for a fixed T (the first guess of T_*, or first "shot") until the steady-state solution $x(t)$ is reached. Then we evaluate the third derivative $\partial_t^{(3)} X(t, \tau \gg 1)$ at $t = T$, and iterate T until the third derivative vanishes [as Eq. (15) demands] with desired accuracy. Alternatively, one can iterate until $\partial_t^2 X(t, \tau \gg 1)$ at $t = 0$ approaches $-\sqrt{2}$, see Eq. (21). We validated the method by comparing the numerically found $x(t)$ with the analytical solutions for $n = 1$ and 2. The accuracy was monitored by checking the conservation law (20) with $C = 0$. Once T_* and $x(t)$ are found, we can evaluate α_n from any of the equations (19) or (22). We used a standard PDE solver of "Mathematica" [29].

Figure 2 shows the numerically found optimal paths $x(t)$ and the optimal accelerations $\ddot{x}(t)$ for $n = 3$ and 4. In these cases $T_* \simeq 2.036$ and 2.185, respectively, whereas the $A \to 0$ asymptotics of $P(A|L)$ are described by Eq. (18) with $\alpha_3 \simeq 0.041$ and $\alpha_4 \simeq 0.026$. Overall, we solved the problem numerically and found the optimal first passage time T_* and the factor α_n for a range of n, see Fig. 3. As one can see, T_* increases with n, while α_n decreases.
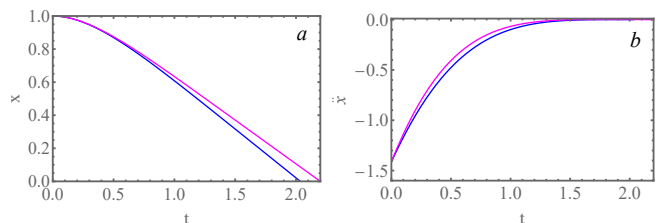

FIG. 2: Numerically found rescaled optimal path $x(t)$ (a) and optimal acceleration $\ddot{x}(t)$ (b), dominating the $A \to 0$ asymptotics of $P(A|L)$ for $n = 3$ (blue) and $n = 4$ (magenta). The optimal first-passage time is $\simeq 2.036$ for $n = 3$ and $T_* \simeq 2.20$ for $n = 4$.
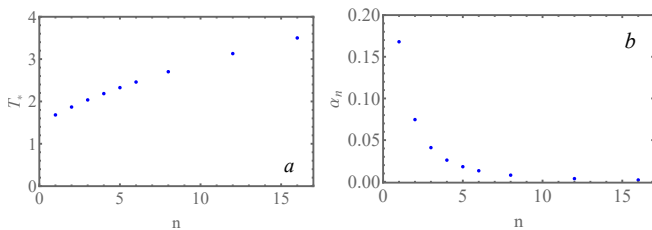
FIG. 3: The optimal first-passage time T_* (a) and the factor α_n which enters Eqs. (18) and (19) (b) are plotted as functions of n. The points $n = 1$ and $n = 2$ were obtained analytically, the rest of the points numerically.

IV. SUMMARY AND DISCUSSION

Statistics of first-passage functionals provide a useful characterization of random processes. Here we evaluated the $A \to 0$ tail of these statistics for Brownian acceleration. We also used this problem to extend the optimal fluctuation (or geometrical optics) method to a stochastic process of a higher-order. In addition to the $A \to 0$ asymptotic of the probability distribution $P_n(A|L)$, we calculated analytically and numerically the optimal paths of the conditioned processes at different n. These provide an interesting insight into the nature of large deviations in this system.

The problem of statistics of the first-passage functionals $I[x(t)] = \int_0^T x^n(t)dt$ can be extended to a whole family of processes, described by the Langevin equation $d^k x(t)/dt^k = \sqrt{2D}\xi(t)$, where k is any positive integer. The cases of $k = 1$ and $k = 2$ correspond to the Brownian motion and random acceleration, respectively. Again, let $x(0) = L$, and suppose for simplicity that all the derivatives of $x(t)$ with order less than k vanish at $t = 0$. Then the exact scaling behavior of probability distribution $P_n^{(k)}(A|L)$ of the values $I[x(t)] = A$ follows from dimensional analysis:

$$P_n^{(k)}(A|L) = \frac{D^\nu}{L^{n+2\nu}} F_n^{(k)} \left(\frac{D^\nu A}{L^{n+2\nu}} \right), \qquad (34)$$

where $F_n^{(k)}(z)$ is an unknown scaling function, and $\nu = 1/(2k-1)$. In its turn, the leading-order $A \to 0$ asymptotic of $P_k(A|L)$ must have the characteristic weak-noise form

$$- \ln P_n^{(k)}(A \to 0|L) \simeq \frac{\alpha_n^{(k)} L^{\frac{n}{\nu}+2}}{DA^{1/\nu}}, \qquad (35)$$

where $\alpha_n^{(k)}$ is a numerical factor which depends on k and n. As one can see from Eq. (35), for all these models theory predicts am essential singularity at $A \to 0$, and the singularity becomes stronger as k is increased.

Appendix A: Derivation of Eq. (12) and boundary condition (15).

Here we temporarily switch back to the original variables and consider a linear variation of the constrained action functional

$$s_\lambda[x(t), T] = \int_0^T \left(\frac{\ddot{x}^2}{2} - \lambda x^n \right) dt \qquad (A1)$$

with respect to small variations of both $x(t)$ and T: $x(t) \to x(t) + \delta x(t)$ and $T \to T + \delta T$. We need to linearize the variation

$$\delta s_\lambda = s[x(t) + \delta x(t), T + \delta T] - s[x(t), T] \qquad (A2)$$

with respect to δx and δT. The linearization yields, after simple algebra,

$$\begin{aligned} \delta s_\lambda &= \int_0^T \left(\ddot{x}\delta\ddot{x} - \lambda n x^{n-1}\delta x \right) dt \\ &+ \int_T^{T+\delta T} \left(\frac{\ddot{x}^2}{2} - \lambda x^n \right) dt. \end{aligned} \qquad (A3)$$

Performing two integrations in parts in the first integral, evaluating the second interval in the limit of $\delta T \to 0$, and taking into account the boundary conditions (13), we obtain

$$\begin{aligned} \delta s_\lambda &= \int_0^T \left(x^{(4)} - \lambda n x^{n-1} \right) \delta x \, dt + \ddot{x}(T)\delta\dot{x}(T) \\ &+ \left[\frac{\ddot{x}^2(T)}{2} - \lambda x^n(T) - \dddot{x}(T)\dot{x}(T) \right] \delta T. \end{aligned} \qquad (A4)$$

Each of the three terms in the variation must vanish independently for arbitrary δx and δT. The first term yields the Euler-Lagrange equation $x^{(4)} - \lambda n x^{n-1} = 0$ which, upon the rescaling $\Lambda t \to t$ (we recall that $\lambda \equiv -\Lambda^4$), coincides with Eq. (12). The second term yields the boundary condition (14). Using the latter condition and the condition $x(T) = 0$ in the third term, we arrive at the boundary condition (15).

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$$z''(x) = -\frac{3}{2} x^n z^{-5/3}(x),$$

which is known as the Emden-Fowler equation [30]. Unfortunately, its exactly soluble cases do not seem to extend beyond $n = 0, 1$ and 2, where it is much easier to solve Eq. (12) directly.

[28] "Shooting method", Wikipedia, https://en.wikipedia.org/wiki/Shooting_method.

[29] Wolfram Research, Inc., Mathematica, Version 13.2, (Champaign, IL, 2022).

[30] A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions of Ordinary Differential Equations* (CRC Press, Boca Raton, FL, 1995).