A Unified Approach to Unimodality of Gaussian Polynomials

Christoph Koutschan
RICAM
Austrian Academy of Sciences
4040 Linz, Austria
christoph.koutschan@oeaw.ac.at

Ali K. Uncu
RICAM, Austrian Academy of Sciences & University of Bath
Austria & United Kingdom
akuncu@ricam.oeaw.ac.at

Elaine Wong
Oak Ridge National Laboratory
Oak Ridge, TN 37380
United States of America
wongey@ornl.gov

ABSTRACT

In 2013, Pak and Panova proved the strict unimodality property of q-binomial coefficients $\left[\begin{smallmatrix}\ell+m\\ m\end{smallmatrix}\right]_q$ (as polynomials in q) based on the combinatorics of Young tableaux and the semigroup property of Kronecker coefficients. They showed it to be true for all $\ell, m \geq 8$ and a few other cases. We propose a different approach to this problem based on computer algebra, where we establish a closed form for the coefficients of these polynomials and then use cylindrical algebraic decomposition to identify exactly the range of coefficients where strict unimodality holds. This strategy allows us to tackle generalizations of the problem, e.g., to show unimodality with larger gaps or unimodality of related sequences. In particular, we present proofs of two additional cases of a conjecture by Stanley and Zanello.

CCS CONCEPTS

• **Computing methodologies** → **Symbolic and algebraic manipulation**; • **Mathematics of computing** → **Combinatorics**.

KEYWORDS

Gaussian polynomial, q-binomial coefficient, cylindrical algebraic decomposition, unimodality

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1 INTRODUCTION

In recent years, we have witnessed the increased development of computer algebra tools that can handle questions which are combinatorial in nature, enabling the resolution of open problems and the establishment of new conjectures (see for example [2, 11, 15, 23, 35]). In this paper, we showcase how some of these tools, notably cylindrical algebraic decomposition [10], can be put into action. We present a method that can be applied to answer unimodality questions related to q-binomial coefficients. Such questions have been around for decades, and we detail some of the rich history before presenting our approach.

Definition 1. *A finite sequence of real numbers $a_1, \ldots, a_n$ is called d-strictly increasing (resp. decreasing) if $a_{k+1} - a_k \geq d$ (resp. $a_k - a_{k+1} \geq d$) holds for all $1 \leq k < n$. A sequence is called* unimodal *if for some $m \in \mathbb{N}$ we have non-decreasing (i.e., 0-strictly increasing) behavior up to m and subsequently non-increasing behavior:*

$$a_1 \leq a_2 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_n. \tag{1}$$

The sequence is called strictly unimodal *if all inequalities in (1) are strict. More generally, we call a sequence d-strictly unimodal if for some $m \in \{1, \ldots, n\}$ the subsequence $a_1, \ldots, a_m$ is d-strictly increasing and $a_m, \ldots, a_n$ is d-strictly decreasing.*

Definition 2. *For $\ell, m \in \mathbb{Z}_{\geq 0}$ the* q-binomial coefficient*, also called* Gaussian polynomial*, is a polynomial in q defined by*

$$\begin{bmatrix}\ell+m\\ m\end{bmatrix}_q := \frac{(q^{\ell+1};q)_m}{(q;q)_m} = \prod_{i=1}^{m}\frac{1-q^{\ell+i}}{1-q^i} = \sum_{k=0}^{\ell m} p_k(\ell,m)\cdot q^k,$$

and 0 for other combinations of ℓ and m. Here, $(a;q)_m$ denotes the q-Pochhammer symbol (see [1]).

The (d-strict) unimodality of q-binomial coefficients refers to the fact that the sequence of coefficients of the corresponding Gaussian polynomial is a (d-strictly) unimodal sequence. It should however be noted that when m and ℓ are both odd integers, we have two equal elements at the peak, which does not quite fit Definition 1 for strict unimodality.

An *integer partition* $\pi = (\pi_1, \pi_2, \ldots)$ of k is a finite list of non-increasing positive integers that add up to k, denoted by $\pi \vdash k$ [1]. The elements π_i of a partition are called *parts* and the number of all parts in π is denoted by $\#(\pi)$. Classically, one denotes the number of partitions of an integer k by $p(k)$. By convention, the empty sequence is the only partition of 0, hence $p(0) = 1$. The coefficients $p_k(\ell, m)$ can be interpreted as the number of partitions of k with at most m parts, each of size at most ℓ (equivalently, the number of partitions of k whose Young diagram fits inside an $\ell \times m$ box).

The Gaussian polynomials are palindromic, i.e.,

$$p_{\lfloor \ell m/2\rfloor - k}(\ell,m) = p_{\lceil \ell m/2\rceil + k}(\ell,m) \tag{2}$$

is satisfied for every $k = 0, \ldots, \lfloor \ell m/2\rfloor$. This is immediately clear if we view partitions as Young diagrams in an $\ell \times m$ box: for each partition there exists the complementary partition that is obtained by interpreting the complement of the Young diagram in the box as the Young diagram of a new partition (rotated by 180 degrees).

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However, the observation that

$$p_k(\ell, m) \leq p_{k+1}(\ell, m) \qquad (3)$$

for all $k = 0, \ldots, \lfloor \ell m/2 \rfloor - 1$ is known to be a hard question. First conjectured by Cayley [8], the properties (2) and (3) together imply that the coefficients of the Gaussian polynomials are in fact unimodal. Cayley's conjecture was first proven by Sylvester [34] using invariant theory of binary forms, where he shows that the difference $p_{k+1}(\ell, m) - p_k(\ell, m)$ represents the number of degree-ℓ and weight-m semi-invariants, implying its nonnegativity. Since then, several different proofs of unimodality were found, based on invariant theory [16], Lie algebras [31], linear algebra [29], algebraic geometry [32], and Pólya theory [36]. In 1988, O'Hara [25] gave the first constructive proof of the unimodality of Gaussian polynomials. For more context, the interested reader is referred to the expository article by Zeilberger [38], where the combinatorial meaning, the elements, and the importance of O'Hara's groundbreaking proof are detailed. Zeilberger [39] also formulated O'Hara's argument in algebraic terms and devised the following formula, widely referred to as (KOH) formula in the literature:

$$\begin{bmatrix} \ell + m \\ m \end{bmatrix}_q = \sum_{\pi \vdash m} q^{2\sum_{i\geq 1}\binom{\pi_i}{2}} \prod_{j=1}^{\#(\pi)} \begin{bmatrix} j(\ell+2) - Y_{j-1} - Y_{j+1} \\ \pi_j - \pi_{j+1} \end{bmatrix}_q, \qquad \text{(KOH)}$$

where $Y_j := \sum_{i=1}^{j} \pi_i$ with the end values $Y_0 = 0$ and $Y_{\#(\pi)+1} = m$ since $\pi_{\#(\pi)+1} = 0$ by convention. The (KOH) formula is constructed in such a way that each summand on the right-hand side is a polynomial with a unimodal coefficient sequence such that the sum of the lowest and highest exponent of q with nonzero coefficients is equal to ℓm. Therefore, this (finite) sum adds up a sequence of unimodal polynomials with the same midpoint at $\ell m/2$. This is enough to prove the unimodality of Gaussian polynomials, as was illustrated by Bressoud in 1992 [5].

We demonstrate the (KOH) formula with $\ell = 8$ and $m = 5$ in Figure 1, where we plot the coefficients of the partial sums from the right-hand side of (KOH). For each of these polynomials, the term $a_k q^k$ is plotted at (k, a_k). In this example, the bottom-most layer corresponds to the summand in (KOH) corresponding to the partition $\pi = (5) \vdash 5$, the next layer above that is the total of the (KOH) summands corresponding to the partitions (5) and $(4,1) \vdash 5$, and so on. The top-most layer is the sum of all the summands on the right-hand side of (KOH), and is therefore the graphical representation of the coefficients of $\begin{bmatrix} 13 \\ 5 \end{bmatrix}_q$.

Recently, the question about *strict* unimodality of the coefficients of Gaussian polynomials attracted quite some interest. This is a natural extension of Cayley's conjecture, where one looks for (3) with strict inequalities. However, this requires us to start from $k = 1$ in (3) since $p_0(\ell, m) = p_1(\ell, m) = 1$ for all $\ell, m \in \mathbb{N}$. Moreover, one has to take into account that there is an exception with two equal maximal coefficients when ℓ and m are both odd.

Pak and Panova [26] (correction of [27], which does not identify all of the exceptional cases) prove that the sequence $p_k(\ell, m)$ is strictly unimodal for $\ell = m = 2$ or $\ell, m \geq 5$ with the following finite list of exceptional (ℓ, m) pairs: $(5, 6)$, $(5, 10)$, $(5, 14)$, $(6, 6)$, $(6, 7)$, $(6, 9)$, $(6, 11)$, $(6, 13)$, $(7, 10)$. Without loss of generality, only those pairs with $\ell \leq m$ are displayed, the rest follows by symmetry.

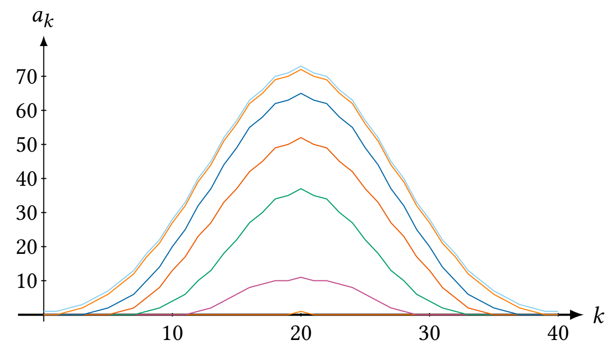

Figure 1: Graphical representation of the (KOH) summation with $\ell = 8$ and $m = 5$.

Although the problem is highly combinatorial, their proof uses technical algebraic tools to show that $p_{k+1}(\ell, m) - p_k(\ell, m) > 0$ for all $1 \leq k \leq \lfloor \ell m/2 \rfloor - 1$. Then, in the same spirit as (KOH), they proceed by putting together strictly unimodal sequences that are aligned at their midpoints as the induction step. The induction argument works smoothly for the cases $\ell, m \geq 8$, but for $\ell \leq 7$ some case distinctions are necessary due to the mentioned exceptions.

At the end of their paper [26], they raise some important points. They suggest that (KOH) can be a way to prove the strict unimodality of q-binomial coefficients. This was achieved by Zanello [37] in 2015. Zanello identifies explicit summands in (KOH) that are strictly unimodal, which is sufficient because the right-hand side of (KOH) is a sum of unimodal polynomials with nonnegative coefficients. There are alternative proofs of strict unimodality in the literature. For example, Pak and Panova prove strict unimodality for $\ell, m \geq 8$ using bounds on Kronecker coefficients [28].

They also muse about when d-strict unimodality might hold. Similar to the 1-strict case, we need to modify the definition of d-strict unimodality slightly. For a fixed d, let $L(d)$ be the smallest natural number that satisfies $p(L(d) + 1) - p(L(d)) \geq d$. We call a Gaussian polynomial d-strictly unimodal if

$$p_{k+1}(\ell, m) - p_k(\ell, m) \geq d \qquad (4)$$

holds for all $k = L(d), \ldots, \lfloor \ell m/2 \rfloor - 1$. The belief is that except for a list of identifiable exceptional cases (ℓ, m), the Gaussian polynomials are d-strictly unimodal. In other words, for every $d \geq 2$ there is some $n_d \in \mathbb{N}$, such that all Gaussian polynomials are d-strictly unimodal for $\ell, m \geq n_d$.

It is clear that as d gets larger, $L(d)$ should also get larger [18]. We display the values of $L(d)$ for small consecutive d, where the missing $L(d)$ for $d < 22$ are obtained by $L(d) = L(d-1)$ (e.g., $L(7) = L(6) = L(5) = 7$ or $L(15) = \cdots = L(21) = 11$):

d	0	1	2	3	5	8	9	13	15	22
$L(d)$	0	1	3	5	7	8	9	10	11	12

The algebraic techniques used in [26] do not easily apply to d-strict questions. Furthermore, the lower bounds in [28] do not tell us exactly when the property of d-strict monotonicity actually begins. However, [28, Theorem 1.2] guarantees that Gaussian polynomials become d-strict eventually. Zanello [37, Proposition 4] also showed that the peaks of Gaussian polynomials will eventually satisfy (4) using (KOH). It is worth noting that around the same time, Dhand [13] gave a combinatorial proof of the strict unimodality of Gaussian polynomials.

The second-named author met Panova at the Algebraic and Enumerative Combinatorics thematic event, held in 2017 at the Erwin Schrödinger Institute [17]. Following a talk on an elementary analysis of the maximum absolute coefficients of q-Pochhammer symbols [3, 4], she asked whether it would be possible to prove strict unimodality of Gaussian polynomials for $m \le 7$, using some similar analysis. In the present paper, we approach the problem by developing a unified approach that is directly applicable to all d-strict considerations for the coefficients of Gaussian polynomials and their generalizations. We propose to study the coefficients $p_k(\ell, m)$ from the viewpoint of Taylor expansions. This allows us to obtain closed-form formulas for $p_k(\ell, m)$ for fixed choices of m and for symbolic ℓ, containing complex numbers. We then establish the validity of the condition $p_{k+1}(\ell, m) - p_k(\ell, m) \ge d$ in the range $k = L(d), \dots, \lfloor \ell m/2 \rfloor - 1$ for the given d of interest. This can be done by cylindrical algebraic decomposition (CAD) [10], after the complex numbers have been eliminated by performing case distinctions. It is known that the worst-case complexity of CAD is doubly exponential [6, 12]. However, in many applications, including this one, we experience fast returns. A broad exposition on the versatility and applicability of CAD is given in [19].

Using this approach, we give a new proof of strict unimodality for small m and confirm the exceptional cases of Pak and Panova [26]. We describe our approach in Section 2 and provide an illustrative sampling of computational results in Section 3.1 for small cases of d and m. Section 3.2 includes notes on what would be needed for a full induction proof, in order to extend them to arbitrary ℓ, m. These results show that the proposed approach can answer specific questions about d-strict unimodality, thanks to our closed-form representation of the coefficients. It turns out that it is also applicable to unimodality questions for combinations of q-binomial coefficients, and we showcase such examples in Section 4.

2 THE SYMBOLIC APPROACH

In this section, we describe our approach in a general setting, of which the q-binomial coefficient is a special case. Let $D \in \mathbb{Z}[q]$ be a univariate polynomial, all of whose zeros are roots of unity, i.e., $D(q) = \prod_{i=1}^{r}(1 - q^{e_i})$ with $e_1, \dots, e_r \in \mathbb{N}$ (not necessarily distinct), and let $N \in \mathbb{Q}[q, X, q^{-1}, X^{-1}]$ be a multivariate Laurent polynomial with $X = X_1, \dots, X_n$. For $\ell_1, \dots, \ell_n \in \mathbb{Z}$, we define $c_k(\ell_1, \dots, \ell_n)$ to be the coefficient of q^k in the series expansion of the following rational function:

$$c_k := c_k(\ell_1, \dots, \ell_n) := \langle q^k \rangle \frac{N(q, q^{\ell_1}, \dots, q^{\ell_n})}{D(q)}$$

(and use the short-hand notation c_k whenever there is no ambiguity). For example, for any concrete integer $m \in \mathbb{N}$ one can define

$$\begin{aligned} N(q, q^\ell) &= (1 - q^{\ell+1})(1 - q^{\ell+2}) \cdots (1 - q^{\ell+m}) \\ D(q) &= (1 - q)(1 - q^2) \cdots (1 - q^m) \end{aligned}$$

and obtain for c_k the partition numbers introduced in Section 1:

$$c_k = \langle q^k \rangle \frac{N(q, q^\ell)}{D(q)} = \langle q^k \rangle \begin{bmatrix} \ell + m \\ m \end{bmatrix}_q = p_k(\ell, m).$$

For a prescribed set $\Omega \subseteq \mathbb{Z}^n$ (typically $|\Omega| = \infty$) defined by polynomial inequalities, and for given $d \in \mathbb{Z}$, the goal is to prove that for all $(\ell_1, \dots, \ell_n) \in \Omega$ the sequence (c_k) is d-strictly increasing in a certain range $a \le k \le b$, where the bounds a and b may depend on $\ell_1, \dots, \ell_n$. Our strategy is the following:

(1) Derive a closed form for c_k as an exponential polynomial in k and $\ell_1, \dots, \ell_n$, with bases being the roots of $D(q)$.
(2) Build the difference $c_{k+1} - c_k$ and perform an appropriate case distinction such that all complex roots of unity are eliminated, and thus each instance is reduced to a polynomial in k and $\ell_1, \dots, \ell_n$.
(3) Apply CAD to each case to show that $c_{k+1} - c_k \ge d$ for all k in the corresponding range of interest.

2.1 Expanding the denominator

In order to derive a closed form for the coefficients c_k, we first study the coefficients d_k in the Taylor expansion of the rational function

$$\frac{1}{D(q)} = \sum_{k=0}^{\infty} d_k q^k.$$

By partial fraction decomposition, the k-th coefficient in the Taylor expansion of a univariate rational function can be expressed as an exponential polynomial in k, where the bases of the exponentials are the reciprocals of the denominator roots. Since by assumption, all roots of $D(q)$ are roots of unity, it does not matter whether we consider the roots themselves or their reciprocals. Denoting the distinct roots of $D(q)$ by $\omega_1, \dots, \omega_s$, we have

$$d_k = \sum_{i=1}^{s} p_i(k) \cdot \omega_i^k, \tag{5}$$

for all $k \ge 0$, where each p_i is a polynomial in $\mathbb{Q}(\omega_1, \dots, \omega_s)[k]$ of degree less than the multiplicity of the root ω_i. The smallest field that contains $\mathbb{Q}$ and all of these roots is the cyclotomic field $\mathbb{Q}(\omega)$ where ω is chosen to be the primitive root of unity $\exp(2\pi i/L)$ with $L \in \mathbb{N}$ being the smallest integer such that $\omega_1^L = \dots = \omega_s^L = 1$.

The closed form for d_k can be derived by writing the polynomials p_i with undetermined coefficients, and by equating d_k with the ansatz (5) for $k = 0, \dots, \deg(D) - 1$. The required first values for d_k can easily be obtained from the Taylor expansion of $1/D(q)$. The unknown coefficients in the ansatz can now be determined by solving a linear system of equations over $\mathbb{Q}(\omega)$.

Remark 3. *Alternatively, one can set up the linear system by instantiating the ansatz (5) with $k = -\deg(D) + 1, \dots, 0$ and forcing $d_k = 0$ for $k < 0$. To see that this is equivalent to the previous linear system and therefore yields the same solution, extend the range of the sum in $D(q) \cdot \sum_{k \ge 0} d_k q^k = 1$ to start at $k = 1 - \deg(D)$. As a consequence, the closed form for d_k produces correct values not only for $k \ge 0$, but also for $k_0 \le k < 0$ with $k_0 = 1 - \deg(D)$. Note however, that in general it produces nonzero values for $k < k_0$.*

Example 4. *We consider the q-binomial coefficient $\left[\begin{smallmatrix} \ell+3 \\ 3 \end{smallmatrix}\right]_q$, hence*

$$D(q) = (1 - q)(1 - q^2)(1 - q^3).$$

All roots of $D(q)$ can be expressed as powers of $\omega = \exp(2\pi i/L)$ with $L = 6$: they are $\omega^0 = 1$ (with multiplicity 3), $\omega^3 = -1$, $\omega^2 = (-1 + i\sqrt{3})/2$, and $\omega^4 = (-1 - i\sqrt{3})/2$ (each with multiplicity 1). According to (5), we make an ansatz by introducing undetermined coefficients $u_1, \ldots, u_6$, and by equating it to the Taylor expansion:

$$\frac{1}{D(q)} = \sum_{k=0}^{\infty} \left(u_1 + u_2 k + u_3 k^2 + u_4 \omega^{3k} + u_5 \omega^{2k} + u_6 \omega^{4k}\right) q^k$$
$$= 1 + q + 2q^2 + 3q^3 + 4q^4 + 5q^5 + 7q^6 + \ldots,$$

and coefficient comparison with respect to $q^0, \ldots, q^5$ yields a 6×6 linear system over $\mathbb{C}$ whose solution gives the following closed form:

$$d_k = \frac{47}{72} + \frac{k}{2} + \frac{k^2}{12} + \frac{\omega^{3k}}{8} + \frac{\omega^{2k}}{9} + \frac{\omega^{4k}}{9}.$$

Remark 5. *We found it expedient to keep ω as a symbol and exploit the well-known fact that the cyclotomic field we are working in is isomorphic to the field $\mathbb{Q}(\omega)/(\Phi_L(\omega))$ where Φ_L is the L-th cyclotomic polynomial. Each element of this field can be represented canonically as a polynomial in ω of degree less than $\phi(L)$, where ϕ is Euler's totient function. That is, we perform the reductions modulo $\Phi_L(\omega)$ ourselves, as well as extended polynomial gcd's for taking inverses. This produces a significant speed-up compared to using Mathematica's built-in data type* `AlgebraicNumber`*, and is of course much more efficient than computing with explicit complex numbers, independent of which format they are written in (radicals, trigonometric functions, complex exponential function, etc.).*

2.2 Including the numerator

We write the numerator $N(q, q^{\ell_1}, \ldots, q^{\ell_n})$ in expanded form,

$$\frac{N(q, q^{\ell_1}, \ldots, q^{\ell_n})}{D(q)} = \sum_{i=1}^{r} \gamma_i q^{a_{i,1}\ell_1 + \cdots + a_{i,n}\ell_n + b_i} \cdot \frac{1}{D(q)},$$

with $a_{i,j}, b_i \in \mathbb{Z}$. For a closed-form representation of c_k, each summand of the form $q^{a_{i,1}\ell_1+\cdots+a_{i,n}\ell_n+b_i}/D(q)$ contributes a term $d_{k-a_{i,1}\ell_1-\cdots-a_{i,n}\ell_n-b_i}$, so that c_k can be written as a $\mathbb{Q}$-linear combination of shifts of d_k:

$$c_k = \sum_{i=1}^{r} \gamma_i d_{k-a_{i,1}\ell_1-\cdots-a_{i,n}\ell_n-b_i}.$$

However, there is a caveat here: although $d_k = 0$ for all $k < 0$ by definition, this is not the case for the closed form of d_k that was derived in Section 2.1. To compensate for this, the domain

$$\Omega' = \left\{(\ell_1, \ldots, \ell_n, k) \,\middle|\, (\ell_1, \ldots, \ell_n) \in \Omega, a \le k \le b\right\}$$

is divided into finitely many regions such that in each region the expressions $k - a_{i,1}\ell_1 - \cdots - a_{i,n}\ell_n - b_i$, $1 \le i \le r$, are sign-invariant (< 0 or ≥ 0). Consequently, in each of these regions, $c_k(\ell_1, \ldots, \ell_n)$ is defined only by those terms for which the exponent is nonnegative:

$$c_k(\ell_1, \ldots, \ell_n) = \sum_{\substack{i=1 \\ k-a_{i,1}\ell_1-\cdots-a_{i,n}\ell_n-b_i \ge 0}}^{r} \gamma_i d_{k-a_{i,1}\ell_1-\cdots-a_{i,n}\ell_n-b_i}.$$

As a result, we obtain a closed-form expression for c_k, which is given as a piecewise expression, the number of cases corresponding to the number of regions of Ω'.

Remark 6. *In practice, we can take advantage of the fact that the closed form for d_k from Section 2.1 is valid for all $k \ge k_0$, and not only for $k \ge 0$. On the one hand, this gives us some freedom as to where to put the boundaries between two neighboring regions, which can lead to the complete elimination of some regions, resulting in a piecewise expression with fewer case distinctions. On the other hand, the definitions may partly overlap, in the sense that two expressions of neighboring pieces produce the same values in a certain range, whose size depends on k_0. This will be exploited when considering the difference $c_{k+1} - c_k$, by not having to introduce extra case distinctions.*

Example 7 (continuation of Example 4). *First we note that the closed form for d_k derived in Example 4 evaluates to 0 precisely for $-5 \le k \le -1$, hence $k_0 = -5$. The expanded form of the numerator is*

$$N(q, q^{\ell}) = 1 - q^{\ell+1} - q^{\ell+2} - q^{\ell+3} + q^{2\ell+3} + q^{2\ell+4} + q^{2\ell+5} - q^{3\ell+6}.$$

By the symmetry of the Gaussian polynomial, we focus on $k \le \frac{3}{2}\ell$ only, i.e., the first half of the coefficients $c_k = p_k(\ell, 3)$, and ignore all q-powers of the form $q^{2\ell+a}$ and $q^{3\ell+a}$ to obtain

$$p_k(\ell, 3) = d_k - d_{k-\ell-1} - d_{k-\ell-2} - d_{k-\ell-3} \quad \left(0 \le k \le \tfrac{3}{2}\ell\right).$$

Using the closed form for d_k from Example 4, we get the following piecewise expression:

$$p_k(\ell, 3) = \begin{cases} \frac{47}{72} + \frac{1}{2}k + \frac{1}{12}k^2 + \frac{1}{8}\omega^{3k} + \frac{1}{9}\omega^{2k} + \frac{1}{9}\omega^{4k}, & 0 \le k < \ell, \\ \frac{19}{36} + \frac{1}{2}\ell - \frac{1}{6}k^2 + \frac{1}{2}k\ell - \frac{1}{4}\ell^2 & \\ \quad + \frac{1}{8}\omega^{3k} + \frac{1}{8}\omega^{3k+3\ell} + \frac{1}{9}\omega^{2k} + \frac{1}{9}\omega^{4k}, & \ell \le k < 2\ell. \end{cases}$$

Note that $k_0 = -5$ allows us to reduce the four cases that result from the conditions $0 \le k < \ell + 1$, $\ell + 1 \le k < \ell + 2$, $\ell + 2 \le k < \ell + 3$, and $\ell + 3 \le k \le \frac{3}{2}\ell$, to only two case distinctions. Moreover, one finds that the first expression is also valid for $k = \ell$ (because $q^{\ell+1}$ is the smallest q-power of the form $q^{\ell+a}$), while the second line actually produces correct values for $\ell - 2 \le k \le 2\ell + 2$ (because $q^{\ell+3}$ is the largest q-power of the form $q^{\ell+a}$ and $k_0 + 3 = -2$, and because $q^{2\ell+3}$ is the smallest q-power of the form $q^{2\ell+a}$).

2.3 Proving d-strict monotonicity

Recall that our final goal is to prove that the coefficient sequence $\left(c_k(\ell_1, \ldots, \ell_n)\right)_{a \le k \le b}$ is d-strictly increasing for given fixed d, and for symbolic $\ell_1, \ldots, \ell_n$ subject to certain conditions on the ℓ_i. This amounts to showing that $c_k + d \le c_{k+1}$ for all $a \le k \le b - 1$. With the results of the two previous subsections, we now have a closed-form expression of the difference $\Delta := c_{k+1} - c_k$ at our disposal, and we wish to show that $\Delta \ge d$. The closed form for Δ is again a piecewise expression, for different ranges of k, and $\ell_1, \ldots, \ell_n$.

Since this closed form not only involves complex numbers, but also powers of $\omega^k, \omega^{\ell_1}, \ldots, \omega^{\ell_n}$, we cannot directly apply known tools for inequality proving. However, recalling that $\omega^L = 1$, these powers can easily be eliminated by substituting $k \to Lk' + \kappa$ and $\ell_i \to L\ell'_i + \lambda_i$, where $k', \ell'_1, \ldots, \ell'_n$ are new variables taking integral values, and $\kappa, \lambda_1, \ldots, \lambda_n \in \{0, \ldots, L-1\}$ are concrete integers. The possible choices for κ and for the λ_i amount to L^{n+1} case distinctions, thereby converting the exponential polynomial into a quasi-polynomial. Each of these L^{n+1} cases then reduces to several polynomial expressions in $\mathbb{Q}[k', \ell'_1, \ldots, \ell'_n]$, which correspond to the different cases of the piecewise expression. By construction, the coefficients of these polynomials do not involve ω any more. We then apply CAD to each of these $(n+1)$-variate polynomials, in order to show that it is $\geq d$ under the assumption on the conditions on $k, \ell_1, \ldots, \ell_n$ in the current piece.

Example 8 (continuation of Example 7). *For computing the difference $\Delta := p_{k+1}(\ell,3) - p_k(\ell,3)$ using the piecewise closed form from Example 7, one can benefit from the fact that the first line is also valid for $k=\ell$, since one does not need to introduce another case distinction for $k = \ell - 1$:*

$$\Delta = \begin{cases} \frac{7}{12} + \frac{k}{6} - \frac{1}{4}\omega^{3k} + \frac{1}{9}(\omega-2)\omega^{2k} - \frac{1}{9}(\omega+1)\omega^{4k}, & 0 \leq k < \ell, \\ -\frac{1}{6} - \frac{1}{3}k + \frac{1}{2}l - \frac{1}{4}\omega^{3k} - \frac{1}{4}\omega^{3k+3l} & \\ \quad + \frac{1}{9}(\omega-2)\omega^{2k} - \frac{1}{9}(\omega+1)\omega^{4k}, & \ell \leq k < 2\ell. \end{cases}$$

Next, the case distinction for k and ℓ modulo 6 yields 36 cases. For the sake of demonstration, we focus on one of them, say $\kappa = 4$ and $\lambda = 2$. After the substitution $k \longrightarrow 6k' + 4$ and $\ell \longrightarrow 6\ell' + 2$, the expression Δ simplifies as follows:

$$\Delta_{4,2} = \begin{cases} k' + 1, & 0 \leq 6k' + 4 \leq 6\ell' + 1, \\ 3\ell' - 2k' - 1, & 6\ell' + 2 \leq 6k' + 4 \leq 12\ell' + 3. \end{cases}$$

Assume we want to prove strict unimodality, i.e., that $p_k(\ell,3)$ is strictly increasing for $0 \leq k \leq \frac{3}{2}\ell$. Since $k'+1$ is obviously positive, we focus on the second line. Applying CAD to the input formula

$$k' \geq 0 \wedge \ell' \geq 0 \wedge 6\ell' \leq 6k' + 2 \leq 9\ell' \implies 3\ell' - 2k' - 1 \geq 1$$

yields the output

$$\begin{aligned} \ell' < \tfrac{2}{9} &\vee \left(\tfrac{2}{9} \leq \ell' \leq \tfrac{1}{3} \wedge \left(k' < 0 \vee k' > \tfrac{1}{6}(9\ell' - 2)\right)\right) \\ &\vee \left(\tfrac{1}{3} < \ell' < \tfrac{4}{3} \wedge \left(k' < \tfrac{1}{3}(3\ell' - 1) \vee k' > \tfrac{1}{6}(9\ell' - 2)\right)\right) \\ &\vee \left(\ell' \geq \tfrac{4}{3} \wedge \left(k' \leq \tfrac{1}{2}(3\ell' - 2) \vee k' > \tfrac{1}{6}(9\ell' - 2)\right)\right). \end{aligned}$$

Since ℓ' is assumed to take on integer values, the first and third clauses deal with the special cases $\ell' = 0$ and $\ell' = 1$, respectively, while the second clause does not yield any solutions in the integers (recall that CAD works over the reals). Hence, the most interesting one is the last line, which says the formula is false if $\frac{3}{2}\ell' - 1 < k' \leq \frac{3}{2}\ell' - \frac{1}{3}$. There is no such k' if ℓ' is even, but there are solutions for odd ℓ'. Hence let $\ell' = 2j+1$. Determining all integer solutions for k' (there is just one) and backsubstituting yields the infinite family $(k,\ell) = (18j+10, 12j+8)$, $j \in \mathbb{Z}_{\geq 0}$, of pairs where $p_k(\ell,3)$ is not strictly increasing. For example, for $\ell = 8$, we see this violation at $k = 10$, since q^{10} and q^{11} have the same coefficient:

$$\begin{bmatrix} 11 \\ 3 \end{bmatrix}_q = 1 + q + 2q^2 + 3q^3 + 4q^4 + 5q^5 + 7q^6 + 8q^7 + 10q^8 \\ + 11q^9 + 12q^{10} + 12q^{11} + 13q^{12} + 12q^{13} + 12q^{14} + \ldots$$

As the m in $\left[\begin{smallmatrix} \ell+m \\ m \end{smallmatrix}\right]_q$ increases, the polynomial inequalities to be proven turn out to have higher degrees and are therefore less trivial. The same analysis could be done using quasi-polynomials and implementing the case distinctions from the start (see Castillo et al.[7]), but we found it more convenient to deal with expressions involving complex numbers.

Remark 9. *Note that the CAD algorithm works intrinsically over the reals, but we are interested in integer solutions. Nevertheless, it turned out to be most efficient to first compute the cylindrical decomposition and then identify the exceptional values over the integers.*

Table 1: Ranges and exceptions for d-strict unimodality of q-binomial coefficients (see Theorem 10).

d	m	$L(m,d)$	$U(m,d)$	Exceptions (ℓ)
1	3	1	3	None
	4	1	2	4
	5	1	0	$1,\ldots,4,6,10,14$
	6	1	0	$1,\ldots,7,9,11,13$
	7	1	0	$1,\ldots,4,6,10$
2	3	7	6	None
	4	5	2	$5,\ldots,8,10$
	5	3	0	$1,\ldots,10,14$
	6	3	0	$1,\ldots,9,11,13,15,17$
	7	3	0	$1,\ldots,5,6,10$
3	3	13	9	None
	4	7	2	$5,\ldots,14,16$
	5	5	0	$1,\ldots,12,14,18,22,26$
	6	5	0	$1,\ldots,11,13,15,17,19$
	7	5	0	$1,\ldots,4,6,10$
4	3	19	12	None
	4	9	2	$6,\ldots,20,22$
	5	7	0	$1,\ldots,15,18,22,26,30$
	6	7	0	$1,\ldots,11,13,15,17,19,21$
	7	7	0	$1,\ldots,8,10$
5	3	25	15	None
	4	11	2	$7,\ldots,26,28$
	5	7	0	$1,\ldots,18,22,26,30,34$
	6	7	0	$1,\ldots,13,15,17,19,21,23$
	7	7	0	$1,\ldots,10,14$

3 STRICT UNIMODALITY RESULTS FOR GAUSSIAN POLYNOMIALS

We present the results from our approach for small values of d, ℓ, m, and this will serve as base cases for an induction argument presented in the section afterwards.

3.1 Computational results for small m

We apply the approach described in Section 2 to establish d-strict monotonicity of q-binomial coefficients for small values of d and m.

Theorem 10. *Let $d, \ell, m \in \mathbb{N}$ such that $1 \leq d \leq 5$ and $3 \leq m \leq 7$, and let $p_k(\ell,m)$ be as in Definition 2. Then there exist positive integers $L(m,d)$ and $U(m,d)$ such that (4) holds for all*

$$L(m,d) \leq k \leq \lfloor \ell m/2 \rfloor - 1 - U(m,d)$$

and almost all $\ell \geq 1$, with a finite number of exceptions that are summarized in Table 1.

Proof. For each m in the specified range, we derive a closed form for $p_k(\ell,m)$ in terms of $\omega = \exp(2\pi i/L)$ with $L = \mathrm{lcm}(1,\ldots,m)$, as described in Sections 2.1 and 2.2. This closed form is a piecewise expression, defined differently for $0 \leq k < \ell$, $\ell \leq k < 2\ell$, etc. We compute a similar expression for the forward difference, eliminate all occurrences of ω by case distinctions $k, \ell \pmod L$, and

Table 2: Computations for proving Theorem 10, where t_0 is the time for eliminating ω, and t_d is the time for the CAD computations, for $d = 1, 2, 5$ (timings are given in seconds and were measured on Intel Core i7-8550U CPU @ 1.80GHz).

m	L	cases	t_0	t_1	t_2	t_5
3	6	72	0.01	0.47	0.31	0.31
4	12	288	0.12	7.58	32.09	166.05
5	60	10800	3.05	44.22	46.06	44.37
6	60	10800	4.43	75.16	73.27	76.28
7	420	705600	1950.08	7694.77	7232.02	7656.09

apply CAD to the obtained bivariate polynomials, according to Section 2.3. Some measurements are given in Table 2, but the detailed computations can be found in the accompanying notebook [22]. ◻

For the case $d = 1$, our results for $m = 5, 6, 7$ align with the previously known exceptions [26]. Our method allows us to say even more: we can identify for every listed exceptional pair (ℓ, m) the precise locations k where those exceptions occur. We choose not to list all of these locations here, but they can be found in [22].

For the cases $m = 3, 4$, we can also say more. While previous results [13, 26] only indicated a negative answer to the question of strict unimodality, we can identify the largest intervals $L(m, d) \le k \le \lfloor \ell m/2\rfloor - 1 - U(m, d)$ for which the d-strict monotonicity occurs with only a finite number of exceptions. If we choose to expand those intervals, i.e., by choosing smaller values of $L(m, d)$ or $U(m, d)$, we would be able to identify infinite families of exceptions to the d-strict monotonicity.

In principle, our approach can be applied to any $m \ge 8$ and $d \ge 6$, with the tradeoff being increased computational time (cf. Table 2). However, our choice to stop at $m = 7$ was not arbitrary given that the strict unimodality of q-binomial coefficients has already been known for all $\ell, m \ge 8$. On the other hand, our choice to stop at $d = 5$ did not come with a specific reason.

3.2 Induction argument for large m

For any given $d \ge 2$, we can experimentally identify a lower bound $n_d \ge L(d)$ such that for all $\ell, m \ge n_d$ we have that $\begin{bmatrix}\ell+m\\ m\end{bmatrix}_q$ is d-strictly unimodal. We can also identify and prove where the d-strict unimodality holds for all pairs (ℓ, m) with $m \le n_d$ using the method outlined in Section 2.

Next, we recall two parity-dependent results of Reiner and Stanton. First, [30, Theorem 1] states that the difference

$$\begin{bmatrix}\ell+m\\ m\end{bmatrix}_q - \begin{bmatrix}\ell+m\\ m-1\end{bmatrix}_q \tag{6}$$

is a unimodal polynomial with nonnegative coefficients if $\ell + m \equiv 1 \bmod 2$ and $m \le \ell + 1$. Second, [30, Theorem 5] asserts that the difference

$$\begin{bmatrix}\ell+m\\ m\end{bmatrix}_q - q^{\ell}\begin{bmatrix}\ell+(m-2)\\ m-2\end{bmatrix}_q \tag{7}$$

is a unimodal polynomial with nonnegative coefficients if ℓ is even. The difference (7) is in the spirit of (KOH); that is an expression with unimodal sequences aligned at their peaks.

These properties are observably true without the parity conditions. In other words, if we were allowed to drop these parity restrictions on $\ell + m$ and ℓ in (6) and (7), respectively, we can easily give an induction proof of d-strict unimodality by first proving that d-strict unimodality holds for all ℓ such that $\ell \ge m = n_d$ and $\ell \ge m = n_d + 1$. Then (7) can be used to show d-strict unimodality close to the peak, while (6) is used for the early terms. Nevertheless, we can still prove the following theorem.

Theorem 11. *Let $d \ge 2$ and let n_d be an even positive integer greater than $L(d)$. The Gaussian polynomials $\begin{bmatrix}\ell+m\\ m\end{bmatrix}_q$ are d-strictly unimodal for $\ell, m > n_d$ with $\ell + m \equiv 1$ (mod 2), provided that the d-strict unimodality is proven for $\ell \ge m = n_d$ and $\ell \ge m = n_d - 1$.*

Proof. First, we prove the claim for $m = n_d + 1$ and $\ell = n_d + 2i$ with $i \in \mathbb{N}$. The d-strict unimodality of $\begin{bmatrix}\ell+(m-2)\\ m-2\end{bmatrix}_q$ from the assumption and the unimodality of (7) imply that $\begin{bmatrix}\ell+m\\ m\end{bmatrix}_q$ satisfies (4) for $k = L(d)+\ell, \dots, \lfloor \ell m/2\rfloor - 1$. Similarly, the d-strict unimodality of $\begin{bmatrix}\ell+m\\ m-1\end{bmatrix}_q$ from the assumption and the unimodality of (6) imply that $\begin{bmatrix}\ell+m\\ m\end{bmatrix}_q$ satisfies (4) for $k = L(d), \dots, \lfloor(\ell+1)(m-1)/2\rfloor - 1$. Then it is a simple matter of checking that $\lfloor(\ell+1)(m-1)/2\rfloor \ge L(d)+\ell$, which can be seen to hold with the assumption $n_d > L(d)$ for all $n_d \ge 3$. Note that any d-strictly unimodal sequence is also $(d-1)$-strictly unimodal, and we interpret n_d as the smallest point where d-strict unimodality starts, which implies $n_d \ge n_{d-1}$. Pak and Panova [26] proved that $n_1 = 8$. Hence, $n_d \ge 3$ is expected and satisfied.

Next, we move on to $m = n_d + 2$. From the symmetries of the arguments of Gaussian polynomials, the first instance $(\ell, m) = (n_d + 1, n_d + 2)$ is already proven to be d-strictly unimodal. This is useful and in general it allows us to restrict ourselves to cases where $\ell > m$. This is desirable since we would like to employ (6). For $m = n_d + 2$, let $\ell = n_d + 2i + 1$ for $i \in \mathbb{N}$. We use induction over i. Here if we use (7) on $\begin{bmatrix}\ell+m\\ \ell\end{bmatrix}_q$ (i.e., with ℓ and m switched places) we see that the Gaussian polynomial satisfies (4) for $k = L(d) + m, \dots, \lfloor \ell m/2\rfloor - 1$. Similarly, now (6) (used in the normal fashion as before) shows that it satisfies (4) for $k = L(d), \dots, \lfloor(\ell + 1)(m - 1)/2\rfloor - 1$. Note that while using (6) we use the d-strict unimodality cases that we prove on the $m = n_d+1$ line. Once again showing that $\lfloor(\ell+1)(m-1)/2\rfloor \ge L(d) + m$ proves the d-strict unimodality.

Now, by repeating these steps at each fixed $m > n_d$, we prove that $\begin{bmatrix}\ell+m\\ m\end{bmatrix}_q$ is d-strictly unimodal for all $\ell > n_d$ s.t. $\ell \not\equiv m$ (mod 2). ◻

4 STANLEY AND ZANELLO'S CONJECTURE

Some other problems we can tackle with this method are the Reiner–Stanton conjectures [30], Stanley and Zanello's generalization of those conjectures [33], and similar results (e.g., see Chen and Jia [9]). Reiner and Stanton predicted that certain differences

$$\begin{bmatrix}\ell+m\\ m\end{bmatrix}_q - q^{\ell-(m-2)(2r-1)}\begin{bmatrix}\ell+m+4(r-1)\\ m-2\end{bmatrix}_q \tag{8}$$

are unimodal with nonnegative coefficients assuming that $\ell + m$ is even and r, m are nonnegative integers with

$$\ell - (m-2)(2r-1) \ge 0.$$

They established some preliminary evidence for this using Lie algebras. This more-than-20-year-old conjecture is still open. Then

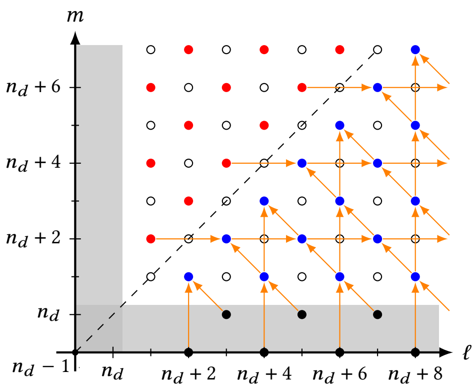

Figure 2: Induction scheme in the proof of Theorem 11. Each vertical (resp. horizontal) arrow is a direct (resp. mirrored) application of (7) for ℓ (resp. m) even. Each northwest pointing arrow is an application of (6). Both together imply the d-strict unimodality for the target pair (indicated by a solid blue dot). The pairs corresponding to red dots follow by symmetry. The solid black dots in the greyed out region represent the base cases for the induction.

in 2020, Stanley and Zanello [33] extended Reiner and Stanton's claim by conjecturing that

$$\begin{bmatrix} \ell+m \\ m \end{bmatrix}_q - q^{\frac{m(\ell-b)}{2}+b} \begin{bmatrix} b+m-2 \\ m-2 \end{bmatrix}_q \tag{9}$$

has nonnegative and unimodal coefficients for large enough ℓ and for $b \le \ell m/(m-2)$ such that $mb \equiv \ell m \pmod 2$, with the only exception $b = (\ell m - 2)/(m-2)$ whenever it is an integer. They use (KOH) to show the $m = 5$ case, and characterize the $m \le 5$ cases. By letting $b = l + 4r - 2$ in (9), we obtain (8) without the restriction of $\ell + m$ being even.

Now using our approach as described in Section 2, we construct a closed form for (9). This allows us to do a similar analysis as with a single q-binomial coefficient, but with increased computational difficulty due to the additional parameter. As a result, we can confirm the unimodality of (9) for the cases $m = 6$ and $m = 7$ (see Theorems 12 and 13 below).

Theorem 12. *The coefficient sequence of the polynomial*

$$\sum_{k=0}^{6\ell} c_k q^k := \begin{bmatrix} \ell+6 \\ 6 \end{bmatrix}_q - q^{3\ell-2b} \begin{bmatrix} b+4 \\ 4 \end{bmatrix}_q \tag{10}$$

is unimodal for all integers $\ell > 25$ and $0 \le b \le \frac{3}{2}\ell$, except when $b = \frac{1}{2}(3\ell - 1)$ for odd ℓ.

Proof. The difference of q-binomials (10) can be written as a rational function $N(q, q^\ell, q^b)/D(q)$ where $D(q) = (q;q)_6$ and N has the following support (as a Laurent polynomial in q^ℓ and q^b):

$$1,\ q^\ell,\ q^{2\ell},\ q^{3\ell}q^{-2b},\ q^{3\ell}q^{-b},\ q^{3\ell},\ q^{3\ell}q^{b},\ q^{3\ell}q^{2b},\ q^{4\ell},\ q^{5\ell},\ q^{6\ell}.$$

For the purpose of deriving a closed form for c_k for $0 \le k \le 3\ell$, one can omit all terms from $q^{3\ell}$ on. We apply the framework of Section 2 with $n = 2$, $\Omega = \left\{(\ell, b) \,\middle|\, \ell \ge 0,\ 0 \le b \le \frac{3}{2}\ell\right\}$, and

$$\Omega' = \left\{(\ell, b, k) \,\middle|\, \ell \ge 0,\ 0 \le b \le \tfrac{3}{2}\ell,\ 0 \le k \le 3\ell\right\} \subseteq \mathbb{Z}^3.$$

Using the fact $k_0 = -20$, the set Ω' is divided into only eight regions (see Figure 3 for a two-dimensional slice for arbitrary but fixed ℓ). More concretely, the regions are defined by the following inequalities, which ensure, after close inspection of the q-powers occurring in N, that also the difference $c_{k+1} - c_k$ is correctly evaluated:

1. $0 \le k \le \ell - 1 \wedge 0 \le 2b \le 3\ell - k - 2$,
2. $0 \le k \le \ell - 1 \wedge 3\ell - k - 1 \le 2b \le 3\ell$,
3. $\ell \le k \le 2\ell - 1 \wedge 0 \le 2b \le 3\ell - k - 2$,
4. $\ell \le k \le 2\ell - 1 \wedge 3\ell - k - 1 \le 2b \le 6\ell - 2k - 1 \wedge 2b \le 3\ell$,
5. $\ell \le k \le 2\ell - 1 \wedge 6\ell - 2k \le 2b \le 3\ell$,
6. $2\ell \le k \le 3\ell - 1 \wedge 0 \le 2b \le 3\ell - k - 2$,
7. $2\ell \le k \le 3\ell - 1 \wedge 3\ell - k - 1 \le 2b \le 6\ell - 2k - 1$,
8. $2\ell \le k \le 3\ell - 1 \wedge 6\ell - 2k \le 2b \le 3\ell$.

The eight exponential polynomials in $\ell, b, k, \omega, \omega^\ell, \omega^b, \omega^k$ that define c_k (resp. $c_{k+1} - c_k$) in each of the eight regions are too large to be displayed here (their number of monomials ranges from 41 to 113), but can be found in the accompanying notebook [22]. We notice that all powers of ω^b are divisible by 10, thus the substitutions $\ell = L\ell_1 + \lambda$, $b = 6b_1 + \beta$, $k = Lk_1 + \kappa$ (for $L = 60$) eliminate all occurrences of ω, forcing us to check $6L^2 = 21600$ cases. To ease these general computations, we slightly restrict the range of k by excluding the cases $k_1 = 0$ and $k = 3\ell - 1$, with the effect that all CAD proofs go through smoothly. Solving these three-variable CAD problems took about 3.5 h. The excluded special cases are then treated separately (note that they are lower-dimensional and therefore run faster). For $k_1 = \kappa = 0$ it is found that unimodality is violated for $b = \frac{1}{2}(3\ell - 1)$ at $k = 0$ for all odd ℓ. For $k_1 = 0$ and $0 < \kappa < L$ the following exceptional triples (ℓ, b, k) are identified:

$$\begin{gathered}(2,3,2),\ (3,4,4),\ (3,1,6),\ (3,2,6),\ (3,3,6),\ (3,4,6),\\ (5,7,6),\ (5,6,10),\ (5,7,10),\ (5,6,12),\ (5,7,12),\ (5,4,12),\\ (5,5,12),\ (7,7,18),\ (7,8,18),\ (7,9,18),\ (7,10,18),\ (9,13,24).\end{gathered}$$

Finally a set of exceptions of the form $(\ell, b, 3\ell - 1)$ is found for the following values of ℓ and b:

ℓ	b	ℓ	b
1	0	15	6, 8, . . . , 22
3	0, 2, 3, 4	17	6, 8, . . . , 25
5	0, 2, . . . , 7	19	12, 14, . . . , 28
7	0, 2, . . . , 10	21	18, 20, . . . , 31
9	0, 2, . . . , 13	23	24, 26, . . . , 34
11	0, 2, . . . , 16	25	36
13	0, 2, . . . , 19	27	–

Since there are no more exceptions where unimodality is violated than listed above, the proof is complete, which resolves Stanley and Zanello's conjecture for $m = 6$. □

The proof of Theorem 12 follows the framework of Section 2 pretty well and, barring some of the difficulties identifying exceptional cases, we are able to arrive at our conclusion in a reasonable amount of time. Alternatively, we can take advantage of the (KOH)

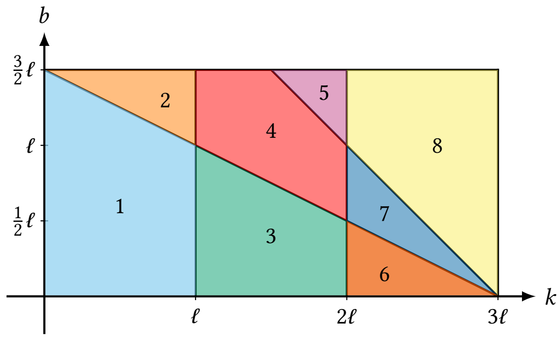

Figure 3: Subdivision of Ω' in the proof of Theorem 12 (two-dimensional slice for fixed ℓ).

formula to manually divide the problem into cases for faster (parallel) processing. In this next part, we present this strategy to prove the case $m = 7$.

Theorem 13. *The coefficient sequence of the polynomial*

$$\sum_{k=0}^{7\ell} c_k q^k := \begin{bmatrix} \ell+7 \\ 7 \end{bmatrix}_q - q^{(7\ell-5b)/2} \begin{bmatrix} b+5 \\ 5 \end{bmatrix}_q \tag{11}$$

is unimodal for all integers $\ell > 10$ and $b = \ell + 2\left\lfloor \frac{1}{5}\ell \right\rfloor - b_1$ with $b_1 \in \{0, 2, 4, 6\}$, except when $b = \frac{1}{5}(7\ell - 2)$ for $\ell \equiv 1$ (mod 5).

Proof. Note that $b = \ell + 2\left\lfloor \frac{1}{5}\ell \right\rfloor$ gives the largest integer that has the same parity as ℓ and is at most $\frac{7}{5}\ell$. To express it without the floor function, we make a case distinction for ℓ mod 5 by setting $\ell = 5\ell_1 + \lambda_1$ with $0 \le \lambda_1 \le 4$. Together with $b_1 \in \{0, 2, 4, 6\}$, there are 20 cases to check in total. In all these cases, we have $D(q) = (q;q)_7$ and therefore we have $L = \operatorname{lcm}(1, \ldots, 7) = 420$ and $k_0 = -27$. In contrast to a single q-binomial coefficient (see Section 3), it is more delicate here to determine the ranges for the piecewise definition of c_k.

We illustrate in detail the computations for the case $b_1 = \lambda_1 = 4$, the other 19 cases being analogous. The numerator $N(q, q^{\ell_1})$ has the following form:

$$\begin{aligned}
&1 - q^{14} + q^{20} + q^{21} - q^{27} \;\; - \left(q^5 + \cdots + q^{11}\right) \cdot q^{5\ell_1} \\
&+ \left(q^{15} + \cdots + q^{32}\right) \cdot q^{7\ell_1} \;\; + \left(q^{11} + \cdots + q^{21}\right) \cdot q^{10\ell_1} \\
&- \left(q^{17} + \cdots + q^{36}\right) \cdot q^{14\ell_1} - \left(q^{18} + \cdots + q^{30}\right) \cdot q^{15\ell_1} \\
&+ \left(q^{26} + \cdots + q^{38}\right) \cdot q^{20\ell_1} + \left(q^{20} + \cdots + q^{39}\right) \cdot q^{21\ell_1} \\
&- \left(q^{35} + \cdots + q^{45}\right) \cdot q^{25\ell_1} - \left(q^{24} + \cdots + q^{41}\right) \cdot q^{28\ell_1} \\
&+ \left(q^{45} + \cdots + q^{51}\right) \cdot q^{30\ell_1} + \left(q^{29} - \cdots - q^{56}\right) \cdot q^{35\ell_1}.
\end{aligned}$$

Since we focus our attention on the first half of the sequence c_k, all terms from $q^{20\ell_1}$ on are irrelevant. We cannot just divide the range for k at multiples of ℓ_1 (as we did in Section 3), because some q-exponents exceed $-k_0 = 27$, such as q^{32} in front of $q^{7\ell_1}$. However, note that the difference between the maximal and the minimal q-exponent in each prefactor does not exceed $-k_0$. Therefore the problem can be cured by defining split points $j\ell_1 + \sigma_j$ with $j \in \{0, 5, 7, 10, 14, 15\}$ such that $\sigma_j \ge d_j + k_0$, where d_j denotes the q-degree of the coefficient of $q^{j\ell_1}$. Moreover, to ensure that the split points form an increasing sequence for any nonnegative ℓ_1, we impose $\sigma_0 \le \sigma_5 \le \cdots \le \sigma_{15}$. Here, the following split points can be chosen:

$$0,\; 5\ell_1,\; 7\ell_1 + 5,\; 10\ell_1 + 5,\; 14\ell_1 + 9,\; 15\ell_1 + 9.$$

Luckily in all 20 cases a suitable choice for the σ_j exists, so that we can always split the range of k into at most seven intervals (if $\sigma_0 > 0$ then we introduce one more case for $0 \le k < \sigma_0$). A priori one would expect that in order to eliminate $\omega = \exp(2\pi i/L)$, the mod-84-behavior of ℓ_1 has to be studied (since $84 = L/5$). By inspection, we realize that all powers of ω^{ℓ_1} in the closed form of c_k are divisible by 35, and therefore it suffices to consider the mod-12-behavior of ℓ_1, as well as the mod-L-behavior of k. For the CAD computations, we exclude the case $(b_1, \lambda_1, k) = (0, 1, 0)$, since it corresponds to the exceptional case $b = \frac{1}{5}(7\ell - 2)$, mentioned in the theorem. The computations take about 10 minutes for each of the 20 choices for (b_1, λ_1), and in each of them it is confirmed that $c_{k+1} \ge c_k$ for all $0 \le k \le \frac{7}{2}\ell - 1$, except for the following pairs (ℓ, k):

b_1	exceptional pairs (ℓ, k)
0	(6, 12), (6, 16), (6, 18), (6, 20), (8, 26)
2	(2, 6), (4, 12), (10, 34)
4	(6, 20)
6	(10, 34)

The program code for this proof is contained in the electronic material [22]. ◻

Corollary 14. *Expression (11) is actually unimodal for all $0 \le b \le \frac{7}{5}\ell$ and $\ell > 10$, except for $b = \frac{1}{5}(7\ell - 2)$.*

Proof. The statement follows from Theorem 2.3 in [33], which uses the (KOH) formula to descend from the four topmost values of b (for which unimodality was proven in Theorem 13), in order to establish unimodality for all b. This resolves Stanley and Zanello's conjecture for $m = 7$. ◻

5 OUTLOOK

In a more general framework, one can also study the unimodality of the specialized Schur function [24] $s_\lambda(1, q, \ldots, q^m)$ for any fixed partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ as a polynomial in q. These polynomials are directly related to the generalized q-binomial coefficients as

$$s_\lambda(1, q, \ldots, q^m) = q^{n(\lambda)} \begin{bmatrix} m \\ \lambda' \end{bmatrix}_q = \prod_{x \in \lambda} \frac{1 - q^{m + c(x)}}{1 - q^{h(x)}},$$

where λ' is the conjugate partition of λ, $n(\lambda) = \sum_{i=1}^{\#(\lambda)} (i-1)\lambda_i$, and where $c(x)$ (resp. $h(x)$) denote the content (resp. the hook-length) of the box x in the Young diagram of λ [24, p.11]. The generalized Gaussian polynomial becomes the ordinary q-binomial coefficient when λ is a partition with a single part. It is conjectured [26] that for partitions with large enough Durfee squares (see [1]) the generalized Gaussian polynomials will also be strictly unimodal. Zanello's proof [37] of strict unimodality for Gaussian polynomials can be adapted to the generalized Gaussian polynomials, using Kirillov's generalization [21] of (KOH) and again imposing the need to prove a few initial cases as an induction base, which could be done by the approach proposed in this paper. Lastly, other conjectures proposed by Dousse, Kim, and Keith [14, 20] may also be amenable to our method.

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