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GENERALISATIONS OF THOMPSON'S GROUP V ARISING FROM PURELY INFINITE GROUPOIDS

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Abstract. We study a class of generalisations of Thompson's group V arising naturally as topological full groups of purely infinite, minimal groupoids. In the process, we show that the derived subgroup of such a group has no proper characters, and that it is 2-generated whenever it is finitely generated. We characterise the class of purely infinite, minimal groupoids through a number of group-theoretic conditions on their full groups including vigor, the existence of suitable embeddings of V, and compressibility. We moreover give a complete abstract description of those groups that arise as either topological full groups or derived subgroups of purely infinite, minimal groupoids. As an application, we obtain several new facts about the Brin-Higman-Thompson groups, including C*-simplicity and a description of all their proper characters.

1. Introduction

In his famous unpublished notes from 1967 [43], Thompson introduced three groups which are now most commonly denoted $F \subseteq T \subseteq V$. Although Thompson's motivation for considering these groups came from algebraic logic, they received a great deal of attention in the context of the von Neumann problem (which asked whether every nonamenable group must contain the free group $\mathbb{F}_2$). It is known that T and V contain a free group, so that they cannot possibly give a negative answer to the problem. On the other hand, F is known not to contain free groups, but it remains open whether it is amenable or not. Beyond the von Neumann problem, these groups are quite remarkable in that they exhibit a number of unusual properties. In the setting of group theory, the role of the group V cannot be overstated. For example, V was the first example of a simple, infinite and finitely presented group, and as such it remains a fundamental object in the study of combinatorial group theory. For a discussion and introductory notes on Thompson's groups, we refer the reader to [9, 10].

Since the introduction of V, group theorists have created vast families of groups generalising it, which for the purposes of this introduction we call *Thompson-like groups*. Examples of such groups have been studied by Higman [18], Brin [6], Cleary [12] and Stein [41]. These groups are very diverse, and one aspect that unifies them is that they have simple derived subgroups with certain finiteness properties.

In this work, we take a groupoid perspective on Thompson-like groups, and study them from the point of view of topological full groups and the associated derived

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subgroups. More explicitly, beginning with an ample, essentially principal groupoid $\mathcal{G}$, its topological full group $\mathsf{F}(\mathcal{G})$ is defined as the group of homeomorphisms of the unit space that are piecewise formed from finitely many compact open bisections. In essence, one builds global symmetries from local ones; see Definition 2.6 for the precise definition. We now know much about the subgroup structure of these groups; for example, the derived subgroup $\mathsf{D}(\mathcal{G}) = \big[\mathsf{F}(\mathcal{G}), \mathsf{F}(\mathcal{G})\big]$ is often simple, and often agrees with the so-called alternating group $\mathsf{A}(\mathcal{G})$, which is the subgroup of $\mathsf{F}(\mathcal{G})$ generated by certain elements of order three.

Since its conception, the framework of topological full groups has served as a bridge between Thompson-like groups and the groupoid models of purely infinite C*-algebras, a line of research that was pioneered by Matui [26]. In fact, promising observations by Nekrashevych [30] predate, and even motivate, the definition of topological full groups of étale groupoids by Matui. Because the underlying topological groupoids are often much more accessible to study than the Thompson-like groups themselves, this new dynamical perspective has led to progress in our understanding of the homology [22], finite presentation [21], and subgroup structure of Thompson-like groups [4].

We summarise the correspondence between Thompson-like groups and purely infinite groupoids in the table below, where $\mathcal{E}_k$ is the full shift on k generators, $\mathcal{R}_r$ is the full equivalence relation on r generators, $\mathcal{O}_k$ is the Cuntz algebra on k generators, and $\mathcal{Q}^\lambda$ is the class of Kirchberg algebras considered in [20, 11]:

C*-algebra	**Groupoid**	**Thompson-like group**
$\mathcal{O}_2$	$\mathcal{E}_2$	V
$M_r(\mathcal{O}_k)$	$\mathcal{R}_r \times \mathcal{E}_k$	$V_{k,r}$
$M_r(\bigotimes_{j=1}^n \mathcal{O}_k)$	$\mathcal{R}_r \times \mathcal{E}_k^n$	$nV_{k,r}$
$\mathcal{Q}^\lambda$	Particular partial actions	Irrational slope Thompson groups

Figure 1. Realisations of Thompson-like groups as topological full groups

The dynamical realisation of the Brin-Higman-Thompson groups is described in [28], while the realisation of irrational slope Thompson groups as topological full groups is the subject of the second author's PhD thesis [42].

In light of this correspondence between groupoids and Thompson-like groups, Matui proposed in [27] to regard the topological full groups of other generalisations of the full shift on two generators as generalisations of V, and in particular study them as Thompson-like groups. In turn, analysing the topological full groups led us to the discovery of interesting new groups, many of which exhibit properties enjoyed by Thompson's group V. This vein of research is summarised in the table below:

C*-algebra	**Groupoid $\mathcal{G}$**	Is $\mathsf{A}(\mathcal{G})$ simple and finitely presented?
Cuntz-Krieger algebras	SFT Groupoids	Yes
Tensors of Cuntz-Krieger algebras	Products of SFT groupoids	Yes
Graph algebras	Graph groupoids	Yes
Katsura-Exel-Pardo algebras	Katsura-Exel-Pardo groupoids	Yes

Figure 2. New generalisations of Thompson's group V from topological full groups

Very general structural properties have been studied for these examples in [28], and significant progress toward homological properties has been made in [22]. In particular, topological full groups inside this class give rise to many new examples of finitely presented, simple, infinite groups [22, 21, 25, 27, 32, 34, 33].

C*-algebraists have a deep source of such groupoids which have been developed primarily as a way to construct simple, purely infinite, nuclear C*-algebras – also known as Kirchberg algebras; see [40]. The relevance of these examples stems from the fact that such C*-algebras play a crucial role in the classification theory of simple, nuclear C*-algebras pioneered by Elliott; see [35]. Each of the underlying groupoids is minimal, topological principal and purely infinite in the sense of Matui [27].

In this paper, we study the topological full groups of minimal, purely infinite groupoids. We show that this class of groupoids can be characterised in a number of ways intrinsic to the topological full groups. We summarise these characterisations in the following result:

Theorem A. Let $\mathcal{G}$ be an essentially principal, ample groupoid. Then the following are equivalent:

(1) $\mathcal{G}$ is purely infinite and minimal.
(2) $\mathsf{A}(\mathcal{G})$ is a vigorous subgroup of $\mathrm{Homeo}(\mathcal{G}^{(0)})$ in the sense of Bleak-Elliott-Hyde [3].
(3) For every $x_0 \in \mathcal{G}^{(0)}$, the subgroup

$$\mathsf{A}(\mathcal{G})_{x_0} = \{g \in \mathsf{A}(\mathcal{G}) \colon \text{there exists a neighbourhood } Y \text{ of } x_0 \text{ such that } g|_Y = 1\}$$

acts compressibly on $\mathcal{G}^{(0)} \setminus \{x_0\}$ in the sense of Dudko-Medynets [13].

Moreover, if any of the above holds, then for every nontrivial compact open subset $X \subsetneq \mathcal{G}^{(0)}$, there is an embedding

$$\phi_X \colon V \to \mathsf{A}(\mathcal{G})$$

such that $X \subseteq \mathrm{supp}(\phi_X(V))$.

Theorem 3.24 completely answers Question 6.1 of [8], which asks to determine when a topological full group is vigorous. The proof of the above theorem is completed at the end of Section 3.

The last condition in the theorem above shows some form of universality enjoyed by V in the context of purely infinite, minimal groupoids and their topological full groups, which is reminiscent of similar properties enjoyed by the Cuntz algebra $\mathcal{O}_2$ among purely infinite, simple C*-algebras.

The fact that conditions (2) and (3) characterise minimality and pure infiniteness of $\mathcal{G}$ is surprising, since vigor and compressibility were introduced in a completely different context without having groupoids in mind, and with the goal of further developing the understanding of Thompson's group V and its generalisations. We discuss both vigor and compressibility separately, since our results have interesting applications in both settings.

Vigor was introduced in [3] in order to determine when a simple, finitely generated group is two-generated. More specifically, said condition was introduced in order to understand the following well-established conjecture:

Conjecture B. Every simple, finitely presented group is two-generated.

Our first corollary confirms this conjecture for a broad class of derived subgroups in topological full groups; see Corollary 3.15.

Corollary C. Let $\mathcal{G}$ be a minimal, expansive, purely infinite, essentially principal Cantor groupoid. Then $\mathsf{A}(\mathcal{G})$ is two-generated.

On the other hand, compressibility was introduced in [13] in order to understand the representation theory and dynamical properties of the Higman-Thompson groups $V_{k,r}$. This definition is an analogue of compressibility in the realm of measurable dynamics, and the introduction of this property follows the recent trend of ideas in measurable dynamics being imported directly to topological dynamics. This gives us two main facts about $\mathsf{F}(\mathcal{G})$; see Corollary 3.21 and Proposition 3.23:

Corollary D. Let $\mathcal{G}$ be a minimal, purely infinite, essentially principal Cantor groupoid. Then $\mathsf{A}(\mathcal{G})$ has no proper characters, and there are no nontrivial finite factor representations of $\mathsf{A}(\mathcal{G})$.

One way to interpret this is that $\mathsf{A}(\mathcal{G})$ is highly nonlinear, being an infinite simple group. This allows us to reduce the question of understanding the representations of $\mathsf{F}(\mathcal{G})$ to understanding the abelianisation $\mathsf{F}(\mathcal{G})_{\mathrm{ab}} = \mathsf{F}(\mathcal{G})/\mathsf{D}(\mathcal{G})$. This abelianisation can be described very concretely through Matui's AH conjecture, which has recently been verified by Xin Li [22] for the class of groupoids considered in this paper. We give a wealth of concrete computations: for example, we are able to describe concretely all characters of the Brin-Higman-Thompson groups $nV_{k,r}$; see Example 4.7. The property of compressibility also gives us dynamical information for actions of Thompson-like groups; see Proposition 4.2:

Corollary E. Let $\mathcal{G}$ be an essentially principal, purely infinite, minimal, Cantor groupoid such that $H_1(\mathsf{F}(\mathcal{G}))$ is finite (this is for example the case if $H_0(\mathcal{G})$ and $H_1(\mathcal{G})$ are finite). Then every faithful ergodic measure-preserving action of $\mathsf{F}(\mathcal{G})$ is essentially free.

The fact that V embeds into $\mathsf{F}(\mathcal{G})$ whenever $\mathcal{G}$ is minimal and purely infinite is in-and-of itself useful for understanding key properties of $\mathsf{F}(\mathcal{G})$. As a consequence, many free products are in $\mathsf{F}(\mathcal{G})$, including the product $\mathbb{F}_2 \times \mathbb{F}_2$ of the free group on two generators with itself. Also, it allows us to conclude that the generalised word problem is not solvable for $\mathsf{F}(\mathcal{G})$. Crucially, this means that many of these topological full groups, for example many groupoids in the Garside category framework introduced by Li in [21] (a broad class whose topological full groups include all Brin-Higman-Thompson groups and Stein groups), have solvable word problem but not solvable generalised word problem. (For the precise definition of the (generalised) word problem, see Definition 3.10.) In other words, these groups sit on an important boundary for computability.

The methods we develop in this work allow us to give a complete abstract characterisation of the full and derived subgroups of minimal purely infinite groupoids:

Theorem F. Let $\mathcal{C}$ denote the Cantor space.

(1) For a subgroup $F \leq \mathrm{Homeo}(\mathcal{C})$, the following are equivalent:
  - (F.1) There exists a minimal, purely infinite, essentially principal, Cantor étale groupoid $\mathcal{G}_F$ with $\mathsf{F}(\mathcal{G}_F) \cong F$, and
  - (F.2) F is vigorous and locally closed.

(2) For a subgroup $A \leq \mathrm{Homeo}(\mathcal{C})$, the following are equivalent:
  - (A.1) There exists a minimal, purely infinite, essentially principal, Cantor étale groupoid $\mathcal{G}_A$ with $\mathsf{A}(\mathcal{G}_A) \cong A$, and
  - (A.2) A is vigorous and simple.

Moreover, the groupoids $\mathcal{G}_F$ and $\mathcal{G}_A$ as in (F.1) and (A.1) above are unique up to groupoid isomorphism.

As an application, we show in Corollary 3.32 that if $A \leq F \leq \mathrm{Homeo}(\mathcal{C})$ are nested groups such that A is vigorous and simple, and F is locally closed, then any intermediate group $A \leq H \leq F$ is C*-simple. In particular, any vigorous, simple subgroup of $\mathrm{Homeo}(\mathcal{C})$ is C*-simple, and the same applies to any vigorous subgroup which is locally closed. Specifically for Thompson's group V, this had been shown by Le Boudec and Matte-Bon in [19].

Our final section concerns applications to the examples described in the tables above. To illustrate, we enumerate a number of new facts about the Brin-Higman-Thompson groups $nV_{k,r}$:

(i) $nV_{k,r}$ is C*-simple for all $n, r \geq 1$ and $k \geq 2$.
(ii) The characters and finite factor representations of $nV_{k,r}$ can be explicitly described; see Example 4.7 for details.
(iii) For any H with $\mathsf{D}(nV_{k,r}) \leq H \leq nV_{k,r}$, any faithful, ergodic, measure-preserving action of H is automatically essentially free.

The structure of this paper is as follows. First, in Section 2, we give some basic preliminaries on étale groupoids and their topological full groups. In Section 3 we prove our main results, Theorem A and F in subsection 3.4, and we derive general consequences from them. Finally, in Section 4 we give examples and applications.

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2. Preliminaries

In this section, we collect a number of basic definitions and results that will be used throughout our work. For further discussion on groupoids, we refer the reader to [39].

Definition 2.1. A *groupoid* is a small category where every morphism is an isomorphism. More explicitly, a groupoid is a set $\mathcal{G}$ with partially defined multiplication and everywhere defined involutive operation $\gamma \mapsto \gamma^{-1}$, satisfying:

(1) Associativity: If $\gamma_1\gamma_2$ and $(\gamma_1\gamma_2)\gamma_3$ are defined, then $\gamma_2\gamma_3$ is defined and $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$
(2) For every $\gamma \in \mathcal{G}$, the product $\gamma\gamma^{-1}$ is always defined.
(3) For $\gamma_1, \gamma_2 \in \mathcal{G}$, if $\gamma_1\gamma_2$ is defined, then
$$\gamma_1 = \gamma_1\gamma_2\gamma_2^{-1} \quad \text{and} \quad \gamma_2 = \gamma_1^{-1}\gamma_1\gamma_2.$$

Elements of the form $\gamma\gamma^{-1}$ are called *units*, and the space of units in $\mathcal{G}$ is denoted $\mathcal{G}^{(0)}$. The subset of $\mathcal{G} \times \mathcal{G}$ consisting of composable pairs is denoted by $\mathcal{G}^{(2)}$. We define the *source* and *range* maps $s, r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ by $s(\gamma) = \gamma^{-1}\gamma$ and $r(\gamma) = \gamma\gamma^{-1}$ for all $\gamma \in \mathcal{G}$.

A *topological groupoid* is a groupoid endowed with a topology such that the multiplication and inverse operations are continuous. We assume further that the topology on our groupoid is Hausdorff. An (open) *bisection* in $\mathcal{G}$ is an open subset $B \subseteq \mathcal{G}$ for which the restrictions of s and r are homeomorphisms and $s(B), r(B)$ are open. We will work exclusively with ample groupoids in this paper, which we define next.

Definition 2.2. Let $\mathcal{G}$ be a topological groupoid.

(1) We say that $\mathcal{G}$ is *étale* if its source and range maps are local homeomorphisms. Equivalently, the set of all open bisections in $\mathcal{G}$ forms a basis for the topology of $\mathcal{G}$.
(2) We say that it is *ample* if the set of all compact, open bisections in $\mathcal{G}$ forms a basis for the topology of $\mathcal{G}$. Equivalently, $\mathcal{G}$ is étale and $\mathcal{G}^{(0)}$ is totally disconnected

(3) We say that $\mathcal{G}$ is a *Cantor groupoid* if $\mathcal{G}^{(0)}$ is a Cantor space.

For an ample groupoid $\mathcal{G}$, we will write $\mathcal{B}(\mathcal{G})$ for the set of its compact open bisections. In this case, $\mathcal{B}(\mathcal{G})$ is an inverse monoid with respect to set multiplication and inverses.

Note that étale, Cantor groupoids are ample. We will often need to assume minimality and essential principality for our groupoids, so we define these.

Definition 2.3. Let $\mathcal{G}$ be a topological groupoid, and let $x \in \mathcal{G}^{(0)}$.

(1) We denote by $\mathcal{G}x$ the $\mathcal{G}$-*orbit* of x, which is given by
$$\mathcal{G}x = \{r(\gamma)\colon \gamma \in \mathcal{G}, s(\gamma) = x\}.$$
(2) We say that $\mathcal{G}$ is *minimal* if $\mathcal{G}x$ is dense in $\mathcal{G}^{(0)}$ for all $x \in \mathcal{G}^{(0)}$.
(3) We denote by $\mathcal{G}_x$ the *isotropy group* of x, which is given by
$$\mathcal{G}_x = \{\gamma \in \mathcal{G}\colon r(\gamma) = s(\gamma) = x\}.$$
(4) We say that $\mathcal{G}$ is *principal* if $\mathcal{G}_x = \{x\}$ for every $x \in \mathcal{G}^{(0)}$.
(5) We say that $\mathcal{G}$ is *essentially principal* if the set $\big\{x \in \mathcal{G}^{(0)}\colon \mathcal{G}_x = \{x\}\big\}$ is dense in $\mathcal{G}^{(0)}$.
(6) We say that $\mathcal{G}$ is *effective* if for all $x \in \mathcal{G}^{(0)}$, all $\gamma \in \mathcal{G}_x \setminus \{x\}$, and all open compact bisections B such that $\gamma \in B$, there exists $\gamma' \in B$ such that $s(\gamma') \neq r(\gamma')$

Some references in the literature define essential principality using the condition that the units with trivial isotropy have full measure for every invariant measure on the unit space. This notion is closely related to effectiveness, and the two agree when $\mathcal{G}$ is Hausdorff. In this text, we have a blanket assumption of $\mathcal{G}$ being effective, since this is what allows us to think of open compact bisections as local homeomorphisms without losing information.

We now give a key example for this paper: the transformation groupoid associated to a topological dynamical system.

Example 2.4. Let $\Gamma \curvearrowright X$ be a discrete group acting by homeomorphisms on a topological space X. Then we define the *transformation groupoid* $\Gamma \ltimes X$ to be $\Gamma \ltimes X$ as a set, with multiplication determined by $(g, h(x))(h, x) = (gh, x)$, and inverses given by $(g, x) = (g^{-1}, g(x))$. Here $s(g, x) = (1, x)$ and $r(g, x) = (1, g(x))$, and the unit space is canonically identified with X. Moreover, $\Gamma \ltimes X$ is étale if and only if Γ is discrete. Many properties of $\Gamma \curvearrowright X$ translate into properties of the transformation groupoid, for example:

(1) The action is *free*, namely for all $g \in \Gamma \setminus \{1\}$ and $x \in X$, one has $gx \neq x$, if and only if $\Gamma \ltimes X$ is principal.
(2) The action is *topologically free*, namely for all $g \in \Gamma \setminus \{1\}$, the set $\{x \in X\colon gx = x\}$ has empty interior, if and only if $\Gamma \ltimes X$ is essentially principal.
(3) The action is minimal, namely for each $x \in X$, the orbit Γx is dense in X, if and only if the groupoid is minimal.
(4) A basis for the (compact) open bisections consists of sets of the form (g, U), where $g \in \Gamma$ and $U \subseteq X$ is (compact and) open. In particular, $\Gamma \ltimes X$ is ample if and only if Γ is discrete and X is totally disconnected.

Crucial in our paper is the construction of a groupoid of germs.

Example 2.5 (Groupoid of Germs)**.** Let $\Gamma \curvearrowright X$ be an action as in Example 2.4. The associated *groupoid of germs* $\mathrm{Germ}(\Gamma \curvearrowright X)$ is defined as
$$\mathrm{Germ}(\Gamma \curvearrowright X) := \Gamma \ltimes X/\sim,$$

where $\sim$ is the equivalence relation given by $(g,x) \sim (h,y)$ whenever $x = y$ and there exists a neighbourhood U of x such that $g|_U = h|_U$. The groupoid structure, as well as its topology, are inherited from $\Gamma \ltimes X$. We warn the reader that $\mathrm{Germ}(\Gamma \curvearrowright X)$ may fail to be Hausdorff in general (although $\Gamma \ltimes X$ always is).

We are ready to define the topological full group of a groupoid. This concept was first introduced in the setting of Cantor minimal systems by Giordano-Putnam-Skau in [17], and later generalised by Matui to more general étale groupoids in [26]. These groups have been extensively studied in recent years, among others by Matui [28, 27] and Nekrashevych [31]. The definition below, allowing for locally compact unit spaces, is due to Nyland-Ortega [34].

Definition 2.6. Let $\mathcal{G}$ be an ample, effective groupoid. For each compact open subset $K \subseteq \mathcal{G}^{(0)}$, set

$$\mathsf{F}(\mathcal{G})_K = \big\{B \in \mathcal{B}(\mathcal{G}) \colon s(B) = r(B) = K\big\}$$

Note that $\mathsf{F}(\mathcal{G})_K$ is a group, with unit given by the (trivial) bisection K. There is a canonical action α^K of $\mathsf{F}(\mathcal{G})_K$ on K by homeomorphisms. Extending these homeomorphisms as the identity outside of K yields an action of $\mathsf{F}(\mathcal{G})_K$ on $\mathcal{G}^{(0)}$, which we also denote by α^K. If $L \subseteq \mathcal{G}^{(0)}$ is another compact open subset with $K \subseteq L$, it is then easy to see that $\alpha^K(\mathsf{F}(\mathcal{G})_K) \subseteq \alpha^L(\mathsf{F}(\mathcal{G})_L)$.

The *topological full group* of $\mathcal{G}$ is then the inductive limit

$$\mathsf{F}(\mathcal{G}) = \varinjlim_{K \subseteq \mathcal{G}^{(0)}} \alpha^K(\mathsf{F}(\mathcal{G})_K) \ \subseteq \mathrm{Homeo}(\mathcal{G}^{(0)}),$$

where K ranges over all compact open subsets of $\mathcal{G}^{(0)}$.

When $\mathcal{G}^{(0)}$ is compact, we have $\mathsf{F}(\mathcal{G}) = \mathsf{F}(\mathcal{G})_{\mathcal{G}^{(0)}}$. In particular, for étale Cantor groupoids we have:

$$\mathsf{F}(\mathcal{G}) = \big\{B \in \mathcal{B}(\mathcal{G}) \colon s(B) = r(B) = \mathcal{G}^{(0)}\big\}.$$

One can think of the topological full group as the subgroup of homeomorphisms of $\mathcal{G}^{(0)}$ that locally look like bisections. In fact, an equivalent definition is the following:

$$\mathsf{F}(\mathcal{G}) = \left\{ g \in \mathrm{Homeo}(\mathcal{G}^{(0)}) \colon \begin{array}{c} \text{for all } x \in \mathcal{G}^{(0)} \text{ there is } B \in \mathcal{B}(\mathcal{G}) \\ \text{with } x \in s(B), \text{ and } g|_{s(B)} = B \end{array} \right\}.$$

In other words, the elements of the topological full group are global symmetries (homeomorphisms) built from local symmetries (partial homeomorphisms) induced by compact open bisections.

For transformation groupoids, the following convenient description of the topological full group goes back to the seminal work of Giordano-Putnam-Skau [17]:

Example 2.7. Let Γ be a discrete group, let X be a locally compact Hausdorff space, and let $\Gamma \curvearrowright X$ be a topologically free action by homeomorphisms. Then $\mathsf{F}(\Gamma \ltimes X)$ can be naturally identified with the subgroup of homeomorphisms $g \in \mathrm{Homeo}(X)$ for which there exists a continuous map $f_g \colon X \to \Gamma$ such that $g(x) = f_g(x) \cdot x$ for all $x \in X$. Such a map f_g is called an *orbit cocycle* for g.

For the associated groupoid of germs, it follows from [32, Proposition 4.6] that $\mathsf{F}(\mathrm{Germ}(\Gamma \curvearrowright X))$ agrees with $\mathsf{F}(\Gamma \ltimes X)$, so it can be computed using orbit cocycles for $\Gamma \curvearrowright X$ as in the previous paragraph.

In [31], Nekrashevych introduced the *alternating subgroup* $\mathsf{A}(\mathcal{G})$ of $\mathsf{F}(\mathcal{G})$, in analogy with the alternating subgroup A_n of S_n, defined via so-called multisections. In this paper, since all of our orbits are infinite, we have the luxury of a much simpler definition.

Definition 2.8. Let $\mathcal{G}$ be an ample groupoid such that all orbits are infinite. Its *alternating group* $\mathsf{A}(\mathcal{G})$ is the subgroup of $\mathsf{F}(\mathcal{G})$ generated by the elements of order three coming from multisections, that is, elements $g \in \mathsf{F}(\mathcal{G})$ of order three such that there exist disjoint compact open subsets $X_0, X_1, X_2 \subseteq \mathcal{G}^{(0)}$ satisfying $g(X_j) = X_{j+1}$ for $j = 0, 1, 2$ (with indices taken modulo 3), and such that g is the identity on $\mathcal{G}^{(0)} \setminus (X_0 \sqcup X_1 \sqcup X_2)$.

When $\mathcal{G}$ is purely infinite and minimal, the definition given above agrees with Nekrashevych's definition by [31, Proposition 3.6], and thus $\mathsf{A}(\mathcal{G})$ is normal in $\mathsf{F}(\mathcal{G})$.

Another subgroup of $\mathsf{F}(\mathcal{G})$ considered by Nekrashevych is the derived subgroup:

Definition 2.9. Let $\mathcal{G}$ be an ample groupoid. Then the *commutator subgroup* (also known as the *derived subgroup*) $\mathsf{D}(\mathcal{G})$ is the subgroup of $\mathsf{F}(\mathcal{G})$ generated by the commutators $[g, f]$, for $g, f \in \mathsf{F}(\mathcal{G})$.

It is immediate from the definition that $\mathsf{D}(\mathcal{G})$ is normal in $\mathsf{F}(\mathcal{G})$. One can also show that $\mathsf{A}(\mathcal{G}) \subseteq \mathsf{D}(\mathcal{G})$; see [31, Theorem 4.1]. As it turns out, it is often the case that $\mathsf{A}(\mathcal{G}) = \mathsf{D}(\mathcal{G})$, and this is in particular always the case for minimal, purely infinite groupoids (see Proposition 3.4), which is the main focus of this work. In fact, there is presently no known example of an étale groupoid that does not have this property. In the rest of this work, we will mostly focus on alternating groups for technical reasons, but the reader should keep in mind that there is no distinction between them for purely infinite groupoids.

3. Characterisations of Pure Infiniteness and Minimality

We begin by recalling the notion of pure infiniteness for étale groupoids due to Matui [27]. All groupoids in this section are effective and ample.

Definition 3.1. A groupoid $\mathcal{G}$ is said to be *purely infinite* if for every compact and open set $X \subseteq \mathcal{G}^{(0)}$ there exist compact open bisections $B, B' \subseteq \mathcal{G}$ such that $s(B) = s(B') = X$, and $r(B), r(B')$ are disjoint and contained in X.

The terminology above is justified by the fact that the reduced groupoid C*-algebra of a minimal, essentially principal, purely infinite étale groupoid is purely infinite and simple. The following result is known to the experts [27], but we provide a proof nonetheless for the convenience of the reader. Recall that a projection in a C*-algebra is said to be *properly infinite* if it is Murray-von Neumann equivalent to two pairwise orthogonal subprojections of itself.

Proposition 3.2. Let $\mathcal{G}$ be a minimal, essentially principal, purely infinite, ample groupoid. Then $C^*_r(\mathcal{G})$ is purely infinite and simple.

Proof. Simplicity of $C^*_r(\mathcal{G})$ is well-known, so we only prove pure infiniteness. By [5, Theorem B], it suffices to show that every projection in $C_0(\mathcal{G}^{(0)})$ is properly infinite in $C^*_r(\mathcal{G})$. Let $p \in C_0(\mathcal{G}^{(0)})$ and let X denote its support, which is compact and open. Use pure infiniteness of $\mathcal{G}$ to obtain compact open bisections $B, B' \subseteq \mathcal{G}$ satisfying $s(B) = s(B') = X$ and $r(B) \sqcup r(B') \subseteq X$. Set $v = \mathbb{1}_B$ and $w = \mathbb{1}_{B'}$, which belong to $C_c(\mathcal{G})$ since B and B' are compact. The conditions on B and B' easily give

$$v^*v = w^*w = p, \quad vv^* \perp ww^*, \quad \text{and} \quad vv^*, ww^* \leq p.$$

Thus vv^* and ww^* are pairwise orthogonal subprojections of p which are Murray-von Neumann equivalent to p. We conclude that p is properly infinite, and hence that $C^*_r(\mathcal{G})$ is purely infinite, as desired. □

Note that many ample groupoids are known to be purely infinite and minimal, for example:

(1) Transformation groupoids of certain nonamenable groups acting amenably, such as those studied in [15, 14, 36].
(2) Shift of finite type groupoids [27], or more generally, large classes of graph groupoids [32].
(3) Groupoids arising from Beta expansions [25].
(4) Certain groupoids arising from left regular representations of Garside categories [21]; see also [23, Theorem A].

We will need the following result of Matui:

Lemma 3.3. ([27, Proposition 4.11]). Let $\mathcal{G}$ be an essentially principal, étale, ample groupoid. The following are equivalent:

(1) $\mathcal{G}$ is purely infinite and minimal.
(2) For all compact and open subsets $X, Y \subseteq \mathcal{G}^{(0)}$, there exists a compact open bisection $B \subseteq \mathcal{G}$ such that $s(B) = X$ and $r(B) \subseteq Y$.
(3) For all compact and open subsets $X, Y \subseteq \mathcal{G}^{(0)}$ with $X \neq \mathcal{G}^{(0)}$ and $Y \neq \emptyset$, there exists $\alpha \in \mathsf{F}(\mathcal{G})$ such that $\alpha(X) \subseteq Y$.

Next, we record the fact, essentially due to Matui, that the subgroups $\mathsf{A}(\mathcal{G})$ and $\mathsf{D}(\mathcal{G})$ agree for the class of purely infinite, minimal groupoids.

Proposition 3.4. Let $\mathcal{G}$ be an ample, effective, purely infinite and minimal groupoid. Then $\mathsf{D}(\mathcal{G})$ is simple and thus $\mathsf{A}(\mathcal{G}) = \mathsf{D}(\mathcal{G})$.

Proof. Simplicity of $\mathsf{D}(\mathcal{G})$ follows from [28, Theorem 4.16]. Thus $\mathsf{A}(\mathcal{G}) = \mathsf{D}(\mathcal{G})$ by [31, Theorem 4.1]. $\square$

3.1. **Embeddings of Thompson's group V.** In [27, Proposition 4.10], Matui proved that if $\mathcal{G}$ is purely infinite, then $\mathsf{F}(\mathcal{G})$ contains $\mathbb{Z}_2 * \mathbb{Z}_3$ as a subgroup (in fact, for this it suffices for $\mathcal{G}$ to be *properly infinite*). In particular, $\mathsf{F}(\mathcal{G})$ is nonamenable. We will strengthen this result by showing that the Thompson group V *always* embeds into $\mathsf{F}(\mathcal{G})$. In fact, we give a characterisation of pure infiniteness of $\mathcal{G}$ in terms of the existence of certain embeddings of V into $\mathsf{F}(\mathcal{G})$; see Proposition 3.7.

We need some preparation first. The following notion is due to Bleak, Elliott and Hyde [3].

Definition 3.5. ([3, Definition 1.1]). Let $\Gamma \leq \mathrm{Homeo}(\mathcal{C})$ be a subgroup of homeomorphisms of the Cantor space $\mathcal{C}$. We say that Γ is *vigorous*[1] if whenever $X, Y_1, Y_2 \subseteq \mathcal{C}$ are compact and open with $Y_1 \neq X$ and $Y_2 \neq \emptyset$, and satisfy $Y_1, Y_2 \subseteq X$, then there exists $g \in \Gamma$ such that g is the identity on $\mathcal{C} \setminus X$, and $g(Y_1) \subseteq Y_2$.

We will need the following observations.

Proposition 3.6. Let $\Gamma \leq \mathrm{Homeo}(\mathcal{C})$ be a vigorous group. Then:

(1) There is no Γ-invariant Borel probability measure on $\mathcal{C}$. In particular, Γ is not amenable.
(2) Γ has infinite conjugacy classes (ICC).

Proof. (1). Let μ be a Borel probability measure on $\mathcal{C}$. Choose nontrivial compact and open sets $Y_1, Y_2 \subseteq \mathcal{C}$ with $\mu(Y_1) < \mu(Y_2)$. By vigor (with $X = \mathcal{C}$), there is $g \in \Gamma$ with $g(Y_1) \subseteq Y_2$. Then $\mu(gY_2) \leq \mu(Y_1) < \mu(Y_2)$, so $g \cdot \mu \neq \mu$ and hence μ is not Γ-invariant.

The last claim follows since a discrete group is amenable if and only if every action on a compact Hausdorff space admits an invariant Borel probability measure.

(2). Let $g \in \Gamma \setminus \{1\}$; we will show that the conjugacy class of g is infinite. Find a compact and open subset $Y_1 \subseteq \mathcal{C}$ such that $g(Y_1) \cap Y_1 = \emptyset$, and find an infinite

[1] It would be more technically correct to say that the action of Γ on $\mathcal{C}$ is vigorous, since vigor is not a property of Γ, but rather of the way it sits in $\mathrm{Homeo}(\mathcal{C})$.

family $\{Y_2^{(n)} : n \in \mathbb{N}\}$ of pairwise disjoint compact and open subsets of Y_1. Use vigor of Γ with $X = \mathcal{C}$ to find, for every $n \in \mathbb{N}$, a group element $h_n \in \Gamma$ with support contained in $\mathcal{C} \setminus Y_1$, such that $h_n(g(Y_1)) \subseteq Y_2^{(n)}$. One readily checks that $h_n^{-1} g h_n(Y_1) \subseteq Y_2^{(n)}$ for all $n \in \mathbb{N}$; in particular, this implies that $h_n^{-1} g h_n \neq h_m^{-1} g h_m$ if $n \neq m$, as desired. □

Note that in the following theorem, we must exclude $X = \mathcal{G}^{(0)}$, since the existence of an embedding $V \hookrightarrow \mathsf{F}(\mathcal{G})$ with full support implies that $\mathcal{G}^{(0)}$ has trivial homology class.

Proposition 3.7. Let $\mathcal{G}$ be an étale Cantor groupoid which is purely infinite and minimal. Then for every nontrivial compact and open subset $X \subseteq \mathcal{G}^{(0)}$, there exists an embedding $\varphi_X : V \hookrightarrow \mathsf{A}(\mathcal{G})$ such that $\mathrm{supp}(\varphi_X(V))$ contains X.

Proof. Let $X \subseteq \mathcal{G}^{(0)}$ be nontrivial, compact and open. Use Lemma 3.3 to find a compact and open bisection B such that $s(B) = \mathcal{G}^{(0)}$ and $r(B) \subsetneq \mathcal{G}^{(0)} \setminus X$. Set $Y = \mathcal{G}^{(0)} \setminus r(B)$, which is a compact and open subset of $\mathcal{G}^{(0)}$ containing X. Moreover,

$$[Y] = [\mathcal{G}^{(0)}] - [r(B)] = [s(B)] - [r(B)] = 0.$$

Denote by $\mathcal{G}|_Y$ the *reduction* of $\mathcal{G}$ to Y, namely

$$\mathcal{G}|_Y = r^{-1}(Y) \cap s^{-1}(Y),$$

which is an étale subgroupoid of $\mathcal{G}$ with unit space equal to Y. Since Y is itself nonempty and compact and open, there is an embedding $\mathsf{F}(\mathcal{G}_Y) \hookrightarrow \mathsf{F}(\mathcal{G})$ whose support is exactly Y. Denote by $\mathcal{E}_2$ the groupoid associated to the shift of finite type corresponding to the single vertex graph with 2 loops. (This groupoid is also called the *Cuntz groupoid*, since its C*-algebra is canonically isomorphic to the Cuntz algebra $\mathcal{O}_2$.) Recall that $\mathsf{F}(\mathcal{E}_2) \cong V$ and that $H_0(\mathcal{E}_2) = \{0\}$. Since the class of the unit space of $\mathcal{G}|_Y$ is trivial in homology, by [28, Proposition 5.14] there exists a unital homomorphism $\pi : C_r^*(\mathcal{E}_2) \to C_r^*(\mathcal{G}_Y)$ satisfying

(a) $\pi(C(\mathcal{E}_2^{(0)})) \subseteq C(Y) = C(\mathcal{G}_Y^{(0)})$, and
(b) for every compact, open bisection $B \subseteq \mathcal{E}_2$, there exists a compact, open bisection $B' \subseteq \mathcal{G}_Y$ such that $\pi(\mathbb{1}_B) = \mathbb{1}_{B'}$.

Since $C_r^*(\mathcal{E}_2) \cong \mathcal{O}_2$ is simple, the map π is injective. In particular, π induces an embedding

$$\varphi_X : V \cong \mathsf{F}(\mathcal{E}_2) \hookrightarrow \mathsf{F}(\mathcal{G}_Y) \subseteq \mathsf{F}(\mathcal{G}).$$

It remains to show that the image of φ_X is actually contained in $\mathsf{A}(\mathcal{G}_Y)$. By Proposition 3.4, it suffices to show that the image of this embedding is contained in $\mathsf{D}(\mathcal{G}_Y) = \big[\mathsf{F}(\mathcal{G}_Y), \mathsf{F}(\mathcal{G}_Y)\big]$. Since group homomorphisms map commutators to commutators, and since V is equal to its own commutator (because it is simple), it follows that the image of the embedding $V \hookrightarrow \mathsf{F}(\mathcal{G}|_Y)$ is contained in $\mathsf{D}(\mathcal{G}_Y) = \mathsf{A}(\mathcal{G}_Y)$. □

The last statement in Proposition 3.7 is not equivalent to the remaining ones, that is, having sufficiently many embeddings of V into the topological full group does not imply pure infiniteness of the groupoid. For example, if we let Thompson's group V act trivially on the Cantor space and denote by $\mathcal{G}$ the associated transformation groupoid, then $\mathcal{G}$ is not purely infinite even though V embeds into $\mathsf{F}(\mathcal{G}) = V$ with full support.

Also, we do not seem to be able to obtain much information about the way that V acts on $\mathcal{G}^{(0)}$ through these embeddings. The next example, which was provided to us by the referee, shows that V does not act vigorously on any parts of $\mathcal{G}^{(0)}$.

Example 3.8. Let the trivial group $\mathbf{1}$ act (trivially) on the natural numbers $\mathbb{N}$, and let $\mathbb{N}V_{\mathbf{1}}$ denote the associated twisted higher-dimensional Brin-Thompson group defined in [2, Section 1]. Writing $\mathcal{C}$ for the Cantor space, we consider the natural action of $\mathbb{N}V_{\mathbf{1}}$ on $\mathcal{C}^{\mathbb{N}}$. By [2, Theorem A], this group is not finitely generated. It is clear from the definition of $\mathbb{N}V_{\mathbf{1}}$ that this group agrees with the ascending union $\bigcup_{n\in\mathbb{N}} nV$ of the higher-dimensional Thompson-Brin groups. For every $n \in \mathbb{N}$, it is easy to see that the restriction of the action $\mathbb{N}V_{\mathbf{1}} \curvearrowright \mathcal{C}^{\mathbb{N}}$ to nV is the product of the canonical action of nV on $\mathcal{C}^{\{1,\dots,n\}}$ and the trivial action on the remaining coordinates. In particular, this action is not vigorous. Since every finitely generated subgroup of $\mathbb{N}V_{\mathbf{1}}$ must be contained in nV for some $n \in \mathbb{N}$, we deduce that no finitely generated subgroup of $\mathbb{N}V_{\mathbf{1}}$ acts vigorously on $\mathcal{C}^{\mathbb{N}}$. By letting $\mathcal{G}$ denote the groupoid of germs of the action $\mathbb{N}V_{\mathbf{1}} \curvearrowright \mathcal{C}^{\mathbb{N}}$, we conclude that $\mathcal{G}$ satisfies the assumptions of Proposition 3.7 but the embeddings of V into its topological full group provided by said proposition do not act vigorously on their supports.

The following is a curious feature of the example described above, which suggests that it is challenging to detect pure infiniteness and minimality of a groupoid by looking at finitely generated subgroups of its topological full group.

Remark 3.9. Let the notation be as in Example 3.8, and write $\mathcal{G}$ for the transformation groupoid of the action $\mathbb{N}V_{\mathbf{1}} \curvearrowright \mathcal{C}^{\mathbb{N}}$. Then $\mathcal{G}$ is purely infinite and minimal. On the other hand, if $G \leq \mathbb{N}V_{\mathbf{1}}$ is any finitely generated subgroup, then the transformation groupoid associated to the restriction to G of $\mathbb{N}V_{\mathbf{1}} \curvearrowright \mathcal{C}^{\mathbb{N}}$ is no longer purely infinite and minimal.

We now derive some conclusions regarding the (generalised) word problem for topological full groups.

Definition 3.10. Let Γ be a finitely generated group with finite generating set S. We say that Γ has

(1) *solvable word problem* (with respect to S), if given a finite word $w \in S^n$ there exists an algorithm that determines, in finitely many steps, whether or not w is the identity.
(2) *solvable generalised word problem* (with respect to S), if for all subgroups $\Lambda \subseteq \Gamma$, given a finite word $w \in S^n$, there exists an algorithm that determines, in finitely many steps, whether w belongs to Λ.

Knowing that topological full groups contain V allows us to deduce that they have unsolvable generalised word problem:

Corollary 3.11. Let $\mathcal{G}$ be a minimal, essentially principal, ample, purely infinite groupoid. Then $\mathsf{F}(\mathcal{G})$ and $\mathsf{A}(\mathcal{G})$ do not have a solvable generalised word problem.

Proof. Note that unsolvable generalised word problems are inherited by containing groups. Thus, it suffices to show the statement for the alternating group. In turn, since $\mathsf{A}(\mathcal{G})$ contains V by Proposition 3.7, it suffices to argue that V has an unsolvable generalized word problem; this follows from [29]. $\square$

In contrast to the above corollary, in many cases $\mathsf{F}(\mathcal{G})$ *does* have a solvable word problem, for example the topological full groups of many groupoids that arise as left regular representations of Garside Categories (see [21, Corollary C]) are known to have solvable word problem. Therefore, we obtain infinitely many nonisomorphic groups spanning many families that demonstrate this boundary between the word problem and the generalised word problem.

3.2. **Vigor.** Note that $\mathsf{A}(\mathcal{G})$ is vigorous whenever $\mathcal{G}$ is an étal, Cantor, purely infinite minimal groupoid, by Proposition 3.7. In fact, vigor of the derived subgroup is *equivalent* to the groupoid being purely infinite and minimal even in the effective case.

Proposition 3.12. Let $\mathcal{G}$ be an étale, effective Cantor groupoid. Then the following are equivalent:

(1) $\mathcal{G}$ is purely infinite and minimal
(2) One (equivalently, both) of $\mathsf{A}(\mathcal{G})$ or $\mathsf{F}(\mathcal{G})$ is vigorous.

Proof. Note that vigor of $\mathsf{A}(\mathcal{G})$ implies vigor of $\mathsf{F}(\mathcal{G})$. Thus, it suffices to show that (1) is equivalent to vigor of $\mathsf{A}(\mathcal{G}) \leq \mathrm{Homeo}(\mathcal{G}^{(0)})$.

(1) implies (2). This proof is entirely constructive. Let $X, Y_1, Y_2 \subseteq \mathcal{G}^{(0)}$ be compact and open sets with $Y_1, Y_2 \subseteq X$, $Y_1 \neq X$ and $Y_2 \neq \emptyset$. We aim to find a multisection $\alpha \in \mathsf{F}(\mathcal{G})$ of order 3 such that α is the identity outside of X and takes Y_1 inside Y_2. Write $Y_2 \setminus Y_1$ as a nontrivial disjoint union $Y_2 \setminus Y_1 = Z_{2,1} \sqcup Z_{2,2}$ of compact and open sets. Since $\mathcal{G}$ is minimal and purely infinite, part (2) of Lemma 3.3 provides us with a compact open bisection $B_1 \subseteq \mathcal{G}$ with $s(B_1) = Y_1$ and $r(B_1) \subseteq Z_{2,1}$. Use part (2) of Lemma 3.3 again to find a compact open bisection B_2 with $s(B_2) = r(B_1) \subseteq Z_{2,1}$ and $r(B_2) \subseteq Z_{2,2}$. Note that $(B_2B_1)^{-1}$ is also a bisection. Set

$$\alpha = \big((B_2B_1)^{-1} \cup B_1 \cup B_2\big) \cup \big(\mathcal{G}^{(0)} \setminus s(B_1) \cup r(B_1) \cup r(B_2)\big).$$

One readily checks that α is a full bisection, so it defines an element of $\mathsf{F}(\mathcal{G})$. It is also clear that it defines a multisection in the sense of Definition 2.8, since it satisfies

$$\alpha(Y_1) = Z_{2,1}, \quad \alpha(Z_{2,1}) = Z_{2,2}, \quad \text{and} \quad \alpha(Z_{2,2}) = Y_1,$$

while α acts trivially on the rest of $\mathcal{G}^{(0)}$. Since α has order 3, it follows that $\alpha \in \mathsf{A}(\mathcal{G})$. Note that α is the identity outside of X, and that $\alpha(Y_1) = r(B_1) \subseteq Z_{2,1} \subseteq Y_2$. This shows that $\mathsf{A}(\mathcal{G})$ is vigorous.

(2) implies (1). We check condition (2) of Lemma 3.3. Let Y_1, Y_2 be nonempty compact and open subsets of $\mathcal{G}^{(0)}$ with $Y_1 \neq \mathcal{C}$. Use vigor with $X = \mathcal{C}$ to find $g \in \mathsf{A}(\mathcal{G})$ with $\mathrm{supp}(g) = Y_1 \cup Y_2$ and $g(Y_1) \subseteq Y_2$. If $B \subseteq \mathcal{G}$ denotes the compact open full bisection determining g, then $B|_{Y_1}$ is a compact open bisection satisfying $s(B|_{Y_1}) = Y_1$ and $r(B|_{Y_1}) \subseteq Y_2$, as desired. □

As an immediate consequence, we show that $\mathsf{F}(\mathcal{G})$ enjoys a strengthening of the ICC property with respect to $\mathsf{D}(\mathcal{G})$; this will be needed later.

Corollary 3.13. Let $\mathcal{G}$ be a minimal, purely infinite, étale Cantor groupoid. Then the $\mathsf{D}(\mathcal{G})$-conjugacy class of every nontrivial element of $\mathsf{F}(\mathcal{G})$ is infinite.

Proof. This proof is similar to that of Proposition 3.6. Let $g \in \mathsf{F}(\mathcal{G}) \setminus \{1\}$. Find a compact and open subset $Y_1 \subseteq \mathcal{G}^{(0)}$ such that $g(Y_1) \cap Y_1 = \emptyset$, and find an infinite family $\{Y_2^{(n)} : n \in \mathbb{N}\}$ of pairwise disjoint compact and open subsets of Y_1. Use vigor of $\mathsf{A}(\mathcal{G})$ to find, for every $n \in \mathbb{N}$, a group element $h_n \in \mathsf{A}(\mathcal{G})$ with support contained in $\mathcal{G}^{(0)} \setminus Y_1$, such that $h_n(g(Y_1)) \subseteq Y_2^{(n)}$. One readily checks that $h_n^{-1} g h_n \neq h_m^{-1} g h_m$ if $n \neq m$, as they map Y_1 to different subsets (namely $Y_2^{(n)}$ and $Y_2^{(m)}$, respectively). This finishes the proof. □

Expansivity is a condition for étale groupoids which was introduced by Nekrashevych in [31] in order to show that certain alternating groups in topological full groups are 2-generated. We recall his notion below:

Definition 3.14. Let $\mathcal{G}$ be an étale groupoid with Cantor unit space.

(i) A compact set $K \subseteq \mathcal{G}$ is said to be *generating* if $\mathcal{G} = \bigcup_{n\in\mathbb{N}}(K \cup K^{-1})^n$.
(ii) A finite collection $\mathcal{B}$ of compact open bisections in $\mathcal{G}$ is said to be *expansive* if $\bigcup_{n\in\mathbb{N}}(\mathcal{B} \cup \mathcal{B}^{-1})^n$ is a basis for the topology of $\mathcal{G}$.

We say that $\mathcal{G}$ is *expansive* if there are a compact generating subset $K \subseteq \mathcal{G}$ and a finite cover $\mathcal{B}$ of K which is expansive.

We obtain the following consequence of our results in combination with [31, 3].

Corollary 3.15. Let $\mathcal{G}$ be an expansive, ample, essentially principal, minimal, purely infinite groupoid. Given $n > 2$, there exist $g_1, g_2 \in \mathsf{A}(\mathcal{G})$ such that g_1 has order n, the order of g_2 is finite, and $\mathsf{A}(\mathcal{G})$ is generated by $\{g_1, g_2\}$.

Proof. Since $\mathcal{G}$ is minimal and essentially principal, it follows from [31, Theorem 1.1] that $\mathsf{A}(\mathcal{G})$ is simple. Moreover, since $\mathcal{G}$ is expansive, [31, Theorem 5.6] implies that $\mathsf{A}(\mathcal{G}) = \mathsf{D}(\mathcal{G})$ is finitely generated. By Proposition 3.12, $\mathsf{A}(\mathcal{G})$ is vigorous, so the result follows from [3, Theorem 1.12]. □

3.3. **Compressibility.** The existence of compressible actions was shown to have relevance to the representation theory of Thompson-like groups in [13]. We recall the definition below:

Definition 3.16. An action of a discrete group Γ on a locally compact Hausdorff space X is said to be *compressible*, if there exists a subbase $\mathcal{U}$ for the topology on X such that:

(1) for all $g \in \Gamma$, there exists $U \in \mathcal{U}$ such that $\mathrm{supp}(g) \subseteq U$;
(2) for all $U_1, U_2 \in \mathcal{U}$, there exists $g \in \Gamma$ such that $g(U_1) \subseteq U_2$;
(3) for all $U_1, U_2, U_3 \in \mathcal{U}$ with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$, there exists $g \in \Gamma$ such that $g(U_1) \cap U_3 = \emptyset$ and $\mathrm{supp}(g) \cap U_2 = \emptyset$;
(4) for all $U_1, U_2 \in \mathcal{U}$, there exists $U_3 \in \mathcal{U}$ such that $U_1 \cup U_2 \subseteq U_3$.

Remark 3.17. Note that if $\Gamma \curvearrowright X$ is compressible, then X cannot be compact. This is because X cannot belong to $\mathcal{U}$ by (2), and at the same time X cannot be written as a finite union of elements of $\mathcal{U}$ by (4). Therefore often when dealing with a group acting by homeomorphisms on the Cantor space, we consider point-stabiliser subgroups.

Compressibility and vigor are closely related notions. For example, the following is essentially a generalisation of the discussion in [13, Subection 3.2].

Lemma 3.18. Let $\mathcal{C}$ be a Cantor space and let $D \leq \mathrm{Homeo}(\mathcal{C})$ be a vigorous subgroup. For $x_0 \in \mathcal{C}$, set

$$D_{x_0} := \{g \in D\colon \text{there exists a neighbourhood } Y \text{ of } x_0 \text{ such that } g|_Y = \mathrm{id}_Y\}.$$

Then $D_{x_0} \curvearrowright \mathcal{C} \setminus \{x_0\}$ is compressible.

Proof. For an open set $Y \subseteq \mathcal{C}$, we set $D_Y = \{g \in D\colon g|_Y = \mathrm{id}_Y\}$. Fix $x_0 \in \mathcal{C}$, and let $(Y_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of compact and open subsets of $\mathcal{C}$ such that $\bigcap_{n=1}^{\infty} Y_n = \{x_0\}$. Then $D_{x_0} = \bigcup_{n=1}^{\infty} D_{Y_n}$.

Let $\mathcal{U}$ be any basis of compact open subsets for the topology on $\mathcal{C} \setminus \{x_0\}$ which is closed under finite unions. We verify properties (1) through (4) in Definition 3.16 for $D_{x_0} \curvearrowright \mathcal{C} \setminus \{x_0\}$ below.

(1). Let $g \in D_{x_0}$, and find a neighbourhood Y of x_0 such that $g|_Y = \mathrm{id}_Y$. Find a basic open set $U \in \mathcal{U}$ such that $U \subseteq Y \setminus \{x_0\}$. Since $\mathcal{C} \setminus Y$ is compact, there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that the set $U = U_1 \cup \ldots \cup U_n$ contains $\mathcal{C} \setminus Y$. Since $\mathcal{U}$ is closed under finite unions, it follows that U belongs to $\mathcal{U}$. Since g is supported on U, this shows (1).

(2). Let $U_1, U_2 \in \mathcal{U}$ be given. Since $x_0 \notin U_1, U_2$, this follows immediately by using vigor of $D \curvearrowright \mathcal{C}$.

(3). Let $U_1, U_2, U_3 \in \mathcal{U}$ satisfy $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Since $U_1 \cup U_2 \cup U_3 \subseteq \mathcal{C} \setminus \{x_0\}$, we may find $n \in \mathbb{N}$ large enough so that $U_j \cap Y_n = \emptyset$ for $j = 1, 2, 3$. Set $X = \overline{U_1} \cup Y_n \setminus Y_{n+1}$. Noting that $U_2 \cap X = \emptyset$ and that X^c contains the neighbourhood Y_{n+1} of x_0, we use vigor to find an element $g \in D_{x_0}$ such that

$$g(U_1) \subseteq g(\overline{U_1}) \subseteq Y_n \setminus Y_{n+1} \subseteq Y_n \subseteq (U_1 \cup U_2 \cup U_3)^c \subseteq U_3^c.$$

Moreover, the support of g, which is a subset of X, contains a neighbourhood of x_0 and is contained in U_2^c. Thus $g \in D_{x_0}$ satisfies the required properties.

(4). This follows by construction, as $\mathcal{U}$ is closed under finite unions. $\square$

We now obtain further characterisations of pure infiniteness for minimal groupoids:

Lemma 3.19. Let $\mathcal{G}$ be an essentially principal, étale Cantor groupoid. Then, the following are equivalent:

(1) $\mathcal{G}$ is purely infinite and minimal.
(2) For all $x_0 \in \mathcal{G}^{(0)}$, the action $\mathsf{A}(\mathcal{G})_{x_0} \curvearrowright \mathcal{G}^{(0)} \setminus \{x_0\}$ is compressible

Proof. If $\mathcal{G}$ is purely infinite and minimal, then $\mathsf{A}(\mathcal{G})$ is vigorous by Proposition 3.12, and thus $\mathsf{A}(\mathcal{G})_{x_0} \curvearrowright \mathcal{G}^{(0)} \setminus \{x_0\}$ is compressible for all $x_0 \in \mathcal{G}^{(0)}$ by Lemma 3.18. Conversely, suppose that $\mathsf{A}(\mathcal{G})_{x_0} \curvearrowright \mathcal{G}^{(0)} \setminus \{x_0\}$ is compressible for all $x_0 \in \mathcal{G}^{(0)}$. Let $X, Y \subseteq \mathcal{G}^{(0)}$ be compact and open subsets. Shrinking Y if necessary, we may assume that $X \cup Y$ is strictly smaller than $\mathcal{G}^{(0)}$. Fix $x_0 \in \mathcal{G}^{(0)} \setminus (X \cup Y)$, and use compressibility of $\mathsf{A}(\mathcal{G})_{x_0} \curvearrowright \mathcal{G}^{(0)} \setminus \{x_0\}$ to find a subbasis $\mathcal{U}$ for the topology of $\mathcal{G}^{(0)} \setminus \{x_0\}$ satisfying the conditions in Definition 3.16. By condition (4) in Definition 3.16, there exists $U_1 \in \mathcal{U}$ with $X \subseteq U_1$. Let $U_2 \in \mathcal{U}$ satisfy $U_2 \subseteq Y$, and use (2) in Definition 3.16 to find $g \in \mathsf{A}(\mathcal{G})_{x_0}$ such that $g(X) \subseteq g(U_1) \subseteq U_2 \subseteq Y$. Restricting the source of the bisection corresponding to g to X, this shows that there is a compact and open bisection $B \subseteq \mathcal{G}^{(0)}$ satisfying $\alpha_B(X) \subseteq Y$. Since X and Y are arbitrary, Lemma 3.3 shows that $\mathcal{G}^{(0)}$ is purely infinite and minimal, as desired. $\square$

We now turn to the definition of proper characters in a group. We warn the reader that these are not characters in the sense of Pontryagin duality, that is, they are not group homomorphisms to the unit circle, and they are only assumed to be invariant under conjugation.

Definition 3.20. Let Γ be a group. A *character* on Γ is a map $\chi\colon \Gamma \to \mathbb{C}$ satisfying the following conditions:

(a) $\chi(gh) = \chi(hg)$ for all $g, h \in \Gamma$;
(b) $\chi(1) = 1$;
(c) for every finite collection $\{g_1, \ldots, g_n\}$ of elements in Γ, the $n \times n$ matrix with (i, j)-th entry $(\chi(g_i g_j^{-1}))$, is non-negatively definite.

A character χ is called *decomposable* if there exist characters χ_1, χ_2 and a real number $\lambda \in (0, 1)$ such that $\chi = \lambda\chi_1 + (1 - \lambda)\chi_2$. Otherwise, χ is called indecomposable.

The *regular character* is the indecomposable character given by $\chi(g) = 0$ whenever $g \neq 1$. The *identity character* is the indecomposable character given by $\chi(g) = 1$ for all $g \in \Gamma$. We say that Γ *has no proper characters* if the only indecomposable characters are the identity character and the regular character.

Using this definition, we deduce the following useful fact about derived subgroups in purely infinite groupoids.

Corollary 3.21. Let $\mathcal{G}$ be a purely infinite and minimal Cantor groupoid. Then $\mathsf{A}(\mathcal{G})$ has no proper characters.

Proof. Note that $\mathsf{A}(\mathcal{G})$ is vigorous by Proposition 3.12. Fix $x_0 \in \mathcal{G}^{(0)}$. It follows from Lemma 3.18 that $\mathsf{A}(\mathcal{G})_{x_0} \curvearrowright \mathcal{G}^{(0)} \setminus \{x_0\}$ is compressible.

We claim that $\mathsf{A}(\mathcal{G})_{x_0}$ is simple. Let $(Z_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact and open subsets of $\mathcal{G}^{(0)} \setminus \{x_0\}$ such that $\bigcup_{n\in\mathbb{N}} Z_n = \mathcal{G}^{(0)} \setminus \{x_0\}$. Note that $\mathcal{G}|_{Z_n}$ is a purely infinite minimal Cantor groupoid for all $n \in \mathbb{N}$, and thus $\mathsf{A}(\mathcal{G}|_{Z_n})$ is simple by Proposition 3.4. Therefore $\mathsf{A}(\mathcal{G})_{x_0} = \bigcup_{n\in\mathbb{N}} \mathsf{A}(\mathcal{G}|_{Z_n})$ is simple as well.

Applying [13, Theorem 2.9], we deduce that $\mathsf{A}(\mathcal{G})_{x_0}$ has no proper characters. This implies that $\mathsf{A}(\mathcal{G})$ has no proper characters: if χ was a character for $\mathsf{A}(\mathcal{G})$, then it restricts to a character on $\mathsf{A}(\mathcal{G})_{x_0}$, therefore $\mathsf{A}(\mathcal{G})_{x_0} \subseteq \ker(\chi)$, and by simplicity we then get $\ker(\chi) = \mathsf{A}(\mathcal{G})$, as desired. $\square$

We can also obtain some information about the finite factor representations of topological full groups. Before we define these representations, we recall that for a subset $S \subseteq \mathsf{B}(H)$ of the bounded operators on a Hilbert space H, the commutant S' is the weak-$*$ closed subalgebra of $\mathsf{B}(H)$ given by

$$S' = \{a \in \mathsf{B}(H) \colon as = sa \text{ for all } s \in S\}.$$

Definition 3.22. Let $\pi\colon \Gamma \to \mathsf{B}(H)$ be a unitary representation of a group Γ on a Hilbert space H. We say that π is a *finite factor representation* if $\mathcal{M}_\pi = \pi(\Gamma)''$ is a factor, that is, $\mathcal{M}_\pi' \cap \mathcal{M}_\pi = \mathbb{C}\mathrm{id}_H$.

It is well known that finite factor representations are in one-to-one correspondence with proper characters. For a groupoid $\mathcal{G}$, we write

$$\pi_{\mathrm{ab}}^{\mathcal{G}}\colon \mathsf{F}(\mathcal{G}) \to \mathsf{F}(\mathcal{G})_{\mathrm{ab}} \cong \mathsf{F}(\mathcal{G})/\mathsf{D}(\mathcal{G})$$

for the canonical quotient map (the abelianisation).

Proposition 3.23. Let $\mathcal{G}$ be a purely infinite, minimal Cantor groupoid, and let $\chi\colon \mathsf{F}(\mathcal{G}) \to \mathbb{T}$ be an indecomposable character. Then χ is either regular, or has the form $\chi(g) = \rho \circ \pi_{\mathrm{ab}}^{\mathcal{G}}$ for some group homomorphism $\rho\colon \mathsf{F}(\mathcal{G})_{\mathrm{ab}} \to \mathbb{T}$. In particular, the finite factor representations of $\mathsf{F}(\mathcal{G})$ are all of the form $g \mapsto \rho(\pi_{\mathrm{ab}}^{\mathcal{G}}(g))\mathrm{id}_H$, where ρ is a character on $\mathsf{F}(\mathcal{G})_{\mathrm{ab}}$ and H is a Hilbert space.

Proof. We verify the assumptions of [13, Theorem 2.11] for $G = \mathsf{F}(\mathcal{G})$ and $R = \mathsf{A}(\mathcal{G})$. Note that $\mathsf{A}(\mathcal{G})$ is ICC by the combination of Proposition 3.12 and part (2) of Proposition 3.6; has no proper characters by Corollary 3.21; and is normal in $\mathsf{F}(\mathcal{G})$.

Let $g \in \mathsf{F}(\mathcal{G}) \setminus \{1\}$. Find a compact and open subset $Y_1 \subseteq \mathcal{G}^{(0)}$ such that $g(Y_1) \cap Y_1 = \emptyset$. Find an infinite family $\{Y_2^{(n)} \colon n \in \mathbb{N}\}$ of nonempty pairwise disjoint compact and open subsets of Y_1. Since the canonical action of $\mathsf{A}(\mathcal{G})$ on $\mathcal{G}^{(0)}$ is vigorous by Proposition 3.12, for every $n \in \mathbb{N}$ there exists $h_n \in \mathsf{A}(\mathcal{G})$ with support contained in $\mathcal{G}^{(0)} \setminus Y_1$, such that $h_n(g(Y_1)) \subseteq Y_2^{(n)}$. For $n, m \in \mathbb{N}$ distinct, it follows that $h_n^{-1} g h_n$ is different from $h_m^{-1} g h_m$. Moreover, we have

$$(h_n g^{-1} h_n^{-1})(h_m g h_m^{-1}) = (h_n g^{-1} h_n^{-1} g)(g^{-1} h_m g h_m^{-1}) = \left[h_n, g^{-1}\right]\left[g^{-1}, h_m\right],$$

which therefore belongs to $\mathsf{D}(\mathcal{G}) = \mathsf{A}(\mathcal{G})$. The result now follows from [13, Theorem 2.11]. $\square$

3.4. **Proof of Main Theorems.** In this subsection, we aim to complete the proofs of Theorems A and F. We begin with Theorem A, whose statement we reproduce:

Theorem 3.24. Let $\mathcal{G}$ be an essentially principal, ample groupoid. Then the following are equivalent:

(1) $\mathcal{G}$ is purely infinite and minimal.
(2) $\mathsf{A}(\mathcal{G}) \leq \mathrm{Homeo}(\mathcal{G}^{(0)})$ is vigorous.
(3) For every $x_0 \in \mathcal{G}^{(0)}$, the subgroup

$$\mathsf{A}(\mathcal{G})_{x_0} = \{g \in \mathsf{A}(\mathcal{G})\colon \text{there is a neighbourhood } Y \text{ of } x_0 \text{ such that } g|_Y = \mathrm{id}_Y\}$$

acts compressibly on $\mathcal{G}^{(0)} \setminus \{x_0\}$.

If any of the above conditions are satisfied, then for every compact open subset $X \subsetneq \mathcal{G}^{(0)}$, there exists an embedding

$$\phi_X \colon V \to \mathsf{A}(\mathcal{G})$$

such that $X \subseteq \mathrm{supp}(\phi_X(V))$.

Proof. What we have shown in the previous sections proves the result assuming that $\mathcal{G}^{(0)}$ is a Cantor set. Indeed, the equivalence between (1) and (2) is Proposition 3.12; the equivalence between (1) and (3) is Lemma 3.19; and the fact that (1) implies the last part of the statement is Proposition 3.7.

We now explain how to obtain the result in full generality from the Cantor case. Note that the equivalence between (1) and (2) is automatic, since being purely infinite and minimal is invariant under restrictions to compact subsets $X \subseteq \mathcal{G}^{(0)}$, and the property in (2) is local. For the same reason, (1) implies the last part of the statement.

It thus remains to show that (1) is equivalent (3). Assuming (1), let $x_0 \in \mathcal{G}^{(0)}$ be given, and find a compact open cover $(X_i)_{i\in I}$ of $\mathcal{G}^{(0)} \setminus \{x_0\}$. For all $i \in I$, it follows from the compact case that the action

$$\mathsf{A}(\mathcal{G})_{X_i \cup \{x_0\}} \curvearrowright X_i \cup \{x_0\} \setminus \{x_0\}$$

is compressible; compressibility of the overall action then follows easily. Conversely, assuming (3), we will show that (1) holds. Let $X, Y \subseteq \mathcal{G}^{(0)}$ be nonempty compact open subsets, and assume without loss of generality that $X \cup Y \neq \mathcal{G}^{(0)}$. Let $x_0 \notin X \cup Y$, and let $\mathcal{U}$ be a compressible cover associated with the action of $\mathsf{F}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)} \setminus \{x_0\}$. Let $U_1, U_2 \in \mathcal{U}$ with $X \subseteq U_1$ and $Y \subseteq U_2$ be given. (Such U_1 exists because of condition (4) of compressibility.) Using condition (2) of compressibility, find $g \in \mathsf{F}(\mathcal{G})$ such that $g(U_1) \subset U_2$. Then $g(X) \subseteq q(U_1) \subseteq U_2 \subseteq Y$, and the restriction $g|_X$ gives a compact open bisection B with $BX \subseteq Y$. $\square$

We stress the fact that for the groupoids covered by the above theorem, the alternating and derived subgroups always agree by Proposition 3.4. In particular, the alternating group is always perfect (meaning that it equals its commutator subgroup).

Remark 3.25. For Hausdorff groupoids, essential principality is equivalent to effectiveness. On the other hand, effectiveness is the right notion to consider in the non-Hausdorff setting. We expect the above theorem to hold in this case; however, this involves obtaining (mostly routine) generalisations of many results from the Hausdorff to the non-Hausdorff case. Since for most applications the Hausdorff case is sufficient, we focus on essentially principal groupoids.

We now aim to complete our proof of Theorem F. For this, we need to use that the canonical action $\mathsf{F}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is rigid in the sense of Rubin [37], so we define this notion first.

Definition 3.26. Let Γ be a discrete group and let C be a locally compact Hausdorff space without isolated points. We say that an action $\Gamma \curvearrowright C$ is a *Rubin action* if for every open set $U \subseteq C$ and every $x \in U$, the closure of

$$\{g \cdot x \colon g \in \Gamma \text{ and } \mathrm{supp}(g) \subseteq U\}$$

contains an open neighbourhood of x.

Rubin's reconstruction Theorem from [37] is extremely useful when working with Rubin actions.

Theorem 3.27 (Rubin)**.** Let $\Gamma_1 \curvearrowright X_1$ and $\Gamma_2 \curvearrowright X_2$ be Rubin actions of discrete groups on compact Hausdorff spaces X_1 and X_2 without isolated points. If $\rho\colon \Gamma_1 \to \Gamma_2$ is a group isomorphism, then there exists a homeomorphism $\Phi\colon X_1 \to X_2$ satisfying $\Phi(g \cdot x) = \rho(g) \cdot \Phi(x)$ for all $x \in X_1$ and all $g \in \Gamma_1$.

We will the notion of a *locally closed* subgroup of homeomorphisms, due to Nyland-Ortega; see [32, Definition 4.4].

Definition 3.28. Let X be a locally compact Hausdorff space and let $\Gamma \leq \mathrm{Homeo}(X)$ let be a subgroup. We say that a homeomorphism $\phi \in \mathrm{Homeo}(X)$ is *locally in* Γ if for all $x \in X$, there exist a neighbourhood U of x and an element $g \in \Gamma$ such that $\phi|_U = g|_U$. We say that Γ is *locally closed* if any homeomorphism of X which is locally in Γ is automatically in Γ.

In other words, Γ is locally closed if homeomorphisms from it can be "glued" to get another homeomorphism of Γ. It is not difficult to see that topological full groups satisfy this property, essentially by construction; see [32, Theorem 4.5].

Proposition 3.29. Let $\mathcal{G}$ be an ample essentially principal groupoid. Then $\mathsf{F}(\mathcal{G})$ is locally closed.

Next, we completely characterise the subgroups of $\mathrm{Homeo}(\mathcal{C})$ that arise as full groups of minimal, purely infinite groupoids: they are precisely the vigorous groups which are also locally closed. We can also abstractly characterise the derived subgroups of such groupoids: they are the simple, vigorous groups.

The following is an existence and uniqueness theorem. While the existence part is completely new, the uniqueness part can be deduced from Matui's spatial isomorphism theorem. We nevertheless give a shorter proof using Rubin's Theorem and Proposition 3.12, since it illustrates the scope of our results. It should be pointed out that isomorphism arguments of this nature have also been used in [34].

Theorem 3.30. Let $\mathcal{C}$ denote the Cantor space, and let $F \leq \mathrm{Homeo}(\mathcal{C})$ be a subgroup. Then the following are equivalent:

- (F.1) There exists a minimal, purely infinite, essentially principal, Cantor étale groupoid $\mathcal{G}_F$ with $\mathsf{F}(\mathcal{G}_F) \cong F$, and
- (F.2) F is vigorous and locally closed.

For a subgroup $A \leq \mathrm{Homeo}(\mathcal{C})$, the following are equivalent:

- (A.1) There exists a minimal, purely infinite, essentially principal, Cantor étale groupoid $\mathcal{G}_A$ with $\mathsf{A}(\mathcal{G}_A) \cong A$, and
- (A.2) A is vigorous and simple.

Moreover, the groupoids $\mathcal{G}_F$ and $\mathcal{G}_A$ as in (F.1) and (A.1) above are unique up to groupoid isomorphism.

Proof. (F.1) implies (F.2). Let $\mathcal{G}$ be a minimal, purely infinite, essentially principal, Cantor étale groupoid. Then $\mathsf{F}(\mathcal{G})$ is locally closed by Proposition 3.29, and it is vigorous by Proposition 3.12.

(F.2) implies (F.1). Let $\mathcal{G}$ be the groupoid of germs of the canonical action $F \curvearrowright \mathcal{C}$. We claim that $F = \mathsf{F}(\mathcal{G})$. It is immediate that $F \subseteq \mathsf{F}(\mathcal{G})$, so we only prove the converse inclusion. Given $g \in \mathsf{F}(\mathcal{G})$, we use the description of the topological full group of a germ groupoid given in Example 2.7 to find a partition $\mathcal{C} = X_1 \sqcup \cdots \sqcup X_n$ of $\mathcal{C}$ into compact and open sets, and group elements $f_1, \ldots, f_n \in F$ such that

$g|_{X_j} = f_j|_{X_j}$ for all $j = 1, \ldots, n$. Since F is locally closed, it follows immediately that g belongs to F, as desired.

The fact that $\mathcal{G}_F = \mathrm{Germ}(F \curvearrowright \mathcal{C})$ is minimal and purely infinite follows from Proposition 3.12, since its topological full group is vigorous.

(A.1) implies (A.2). Similarly to the first part of this proof, vigor of A follows from Proposition 3.12, while simplicity follows from Proposition 3.4.

(A.2) implies (A.1). Let $F_A \leq \mathrm{Homeo}(\mathcal{C})$ denote the smallest subgroup of $\mathrm{Homeo}(\mathcal{C})$ containing A which is locally closed. Since A is vigorous, it is immediate to see that so is F_A. By the first part of this theorem, there is a minimal, purely infinite, essentially principal, Cantor étale groupoid $\mathcal{G}_A$ satisfying $\mathsf{F}(\mathcal{G}_A) \cong F_A$. By the equivalence between (1) and (6) in [3, Theorem 1.11], it follows that A is the commutator subgroup of F_A, and in combination with Proposition 3.4 we get $A = [F_A, F_A] \cong \mathsf{D}(\mathcal{G}_A) = \mathsf{A}(\mathcal{G}_A)$, as desired.

We now show the uniqueness part of the statement. We claim that if $\mathcal{G}$ is a purely infinite, minimal Cantor groupoid, then $\mathsf{A}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is a Rubin action. (This implies that $\mathsf{F}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is also a Rubin action, since $\mathsf{F}(\mathcal{G})$ contains $\mathsf{A}(\mathcal{G})$.) To show this, let $U \subseteq \mathcal{G}^{(0)}$ be a compact and open set and let $x, y \in U$. Write U as a disjoint union $U = X \sqcup Y$ such that $x \in X$ and $y \in Y$. Let $Y_n \subseteq Y$ be a decreasing sequence of compact and open neighbourhoods of y such that $\bigcap_{n\in\mathbb{N}} Y_n = \{y\}$. Since $\mathsf{A}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is vigorous by Proposition 3.12, for every $n \in \mathbb{N}$ there exists $g_n \in \mathsf{A}(\mathcal{G})$ with $\mathrm{supp}(g_n) \subseteq U$ such that $g_n(X) \subseteq Y_n$. Therefore $g_n \cdot x \in Y_n$ for all $n \in \mathbb{N}$, and thus $y = \lim\limits_{n\to\infty} g_n \cdot x$ belongs to the closure of

$$\big\{g \cdot x \colon g \in \mathsf{A}(\mathcal{G}) \text{ and } g|_{\mathcal{G}^{(0)}\setminus U} = \mathrm{id}\big\}.$$

Since $y \in U$ is arbitrary, this shows that

$$U = \overline{\big\{g \cdot x \colon g \in \mathsf{A}(\mathcal{G}) \text{ and } g|_{\mathcal{G}^{(0)}\setminus U} = \mathrm{id}\big\}},$$

and hence $\mathsf{A}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is a Rubin action. This proves the claim.

We now show uniqueness of $\mathcal{G}_F$ in (F.1). Note that, since $\mathcal{G}_F$ is essentially principal, it is isomorphic to the groupoid of germs of its canonical action $\mathsf{F}(\mathcal{G}_F) \curvearrowright \mathcal{G}_F^{(0)}$. Now, assume that $\mathcal{H}$ is another minimal, purely infinite, essentially principal, étale groupoid with $\mathsf{F}(\mathcal{H}) \cong F$. Since $\mathsf{F}(\mathcal{G}_F)$ and $\mathsf{F}(\mathcal{H})$ act in a Rubin manner on the Cantor space by the previous paragraph, Theorem 3.27 implies that $\mathsf{F}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is conjugate to $\mathsf{F}(\mathcal{H}) \curvearrowright \mathcal{H}^{(0)}$. In particular, their groupoids of germs are clearly isomorphic, and thus $\mathcal{G} \cong \mathcal{H}$.

Finally, we turn to uniqueness of $\mathcal{G}_A$ in (A.1), so let $\mathcal{H}$ be a minimal, purely infinite, essentially principal, étale groupoid with $\mathsf{A}(\mathcal{H}) \cong A$. Since $\mathsf{A}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ and $\mathsf{A}(\mathcal{H}) \curvearrowright \mathcal{H}^{(0)}$ are Rubin actions, it follows from Theorem 3.27 that they are conjugate. Thus, if we denote by $F_{\mathsf{A}(\mathcal{H})}$ the smallest subgroup of $\mathrm{Homeo}(\mathcal{H}^{(0)})$ which is locally closed and contains $\mathsf{A}(\mathcal{H})$, it follows that $F_A \cong F_{\mathsf{A}(\mathcal{H})}$. On the other hand, it is routine to check that $F_{\mathsf{A}(\mathcal{H})}$ is precisely $\mathsf{F}(\mathcal{H})$. Putting these things together, we conclude that

$$\mathsf{F}(\mathcal{G}_A) \cong F_A \cong F_{\mathsf{A}(\mathcal{H})} \cong \mathsf{F}(\mathcal{H}).$$

Thus $\mathcal{H} \cong \mathcal{G}_A$ by uniqueness in (F.1). □

Remark 3.31. The proof above also shows that if $A \leq F \leq \mathrm{Homeo}(\mathcal{C})$ are nested groups such that A is vigorous and simple and F is locally closed, then there exists a unique minimal, purely infinite, topologically principal, étale Cantor groupoid $\mathcal{G}$ such that $A \cong \mathsf{A}(\mathcal{G})$ and $F \cong \mathsf{F}(\mathcal{G})$. In particular, normality of A in F follows from the remaining assumptions.

As a corollary, we can deduce that simple, vigorous subgroups of $\mathrm{Homeo}(\mathcal{C})$ are C*-simple. More precisely:

Corollary 3.32. Let $A \leq F \leq \mathrm{Homeo}(\mathcal{C})$ be nested groups such that A is vigorous and simple, and F is locally closed. Then any intermediate group $A \leq H \leq F$ is C*-simple.

Proof. By Theorem 3.30 and the remark after it, there exists a minimal, purely infinite, topologically principal, étale Cantor groupoid $\mathcal{G}$ such that $A \cong \mathsf{A}(\mathcal{G})$ and $F \cong \mathsf{F}(\mathcal{G})$. Using vigor of $\mathsf{A}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$, and in particular the absence of invariant probability measures (Proposition 3.6), the arguments in [38] can be adapted in a routine way to show that both $\mathsf{A}(\mathcal{G})$ and $\mathsf{F}(\mathcal{G})$ are C*-simple. Instead of doing this, we give a more direct proof using the recent results in [1]; we thank Eduardo Scarparo for sharing this argument with us and for allowing us to include it here.

By Corollary 3.13, the $\mathsf{D}(\mathcal{G})$-conjugacy classes of $\mathsf{F}(\mathcal{G})$ are infinite. Hence, it follows from [1, Theorem 6.2], that every intermediate C*-algebra $C^*_r(\mathsf{A}(\mathcal{G})) \leq A \leq C^*_r(\mathsf{F}(\mathcal{G}))$ is simple. Since $C^*_r(H)$ is of that form, the result follows. □

Here is another direct application of Theorem 3.30.

Corollary 3.33. Vigorous simple groups have no proper characters.

One can combine Theorem 3.30 with Theorem 3.24 in order to obtain further equivalent conditions for a group to be realized as the alternating group of a purely infinite, minimal groupoid. More specifically:

Remark 3.34. Let $\mathcal{C}$ denote the Cantor space and let $A \leq \mathrm{Homeo}(\mathcal{C})$ be a subgroup. Then the following are equivalent:

(A.1) There exists a minimal, purely infinite, essentially principal, Cantor étale groupoid $\mathcal{G}_A$ with $\mathsf{A}(\mathcal{G}_A) \cong A$, and
(A.2) A is vigorous and simple.
(A.3) A is simple and for all $x \in \mathcal{C}$, the subgroup

$$A_x = \{g \in A \colon \text{there exists a neighbourhood } Y \text{ of } x \text{ such that } g|_Y = \mathrm{id}_Y\}$$

acts compressibly on $\mathcal{C} \setminus \{x\}$.

We close this section by pointing out that Theorem 3.30 can be generalized to non-compact unit spaces. Indeed, the only step in our arguments where compactness is needed is in the uniqueness part, since we use Rubin's theorem which is valid only for compact spaces. The way to deal with non-compact spaces is to use the generalisation of Matui's spatial isomorphism proved by Nyland-Ortega in [32, Theorem 7.2] in the setting where we replace $\mathcal{C}$ with an arbitrary totally disconnected locally compact Hausdorff space X. In this context, one must also replace $\mathrm{Homeo}(\mathcal{C})$ with the group of compactly supported homeomorphisms:

$$\mathrm{Homeo}_{\mathrm{c}}(X) = \{f \in \mathrm{Homeo}(X) \colon \mathrm{supp}(f) \text{ is compact}\}.$$

We omit the details, but stress the fact that these arguments also allow one to deal with non-Hausdorff groupoids since they only depend on effectiveness.

4. Examples and applications

For the purposes of this section, we use groupoid homology, primarily as a way to compute the quotients $\mathsf{F}(\mathcal{G})/\mathsf{D}(\mathcal{G})$ for key examples. Fortunately, groupoid homology has already been computed in many interesting examples. For more on groupoid homology, we refer the reader to [26]. Thanks to Li's proof of the AH-conjecture for a large class of ample groupoids [22, Corollary E], we now have very concrete pictures of the abelianisation $\mathsf{F}(\mathcal{G})_{\mathrm{ab}}$ of the full groups of these groupoids in terms of their groupoid homology:

Theorem 4.1. ([22, Corollary E]). Let $\mathcal{G}$ be an ample, minimal groupoid with comparison[2] and such that $\mathcal{G}^{(0)}$. Then there exists a short exact sequence:

$$H_2(\mathcal{G}) \longrightarrow H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{j} \mathsf{F}(\mathcal{G})_{\mathrm{ab}} \xrightarrow{I} H_1(\mathcal{G}) \longrightarrow 0.$$

Our first application giving us restrictions on what kinds of actions are possible for $\mathsf{F}(\mathcal{G})$.

Proposition 4.2. Let $\mathcal{G}$ be a purely infinite minimal groupoid such that $\mathsf{F}(\mathcal{G})_{\mathrm{ab}}$ is finite (for example, if $H_1(\mathcal{G})$ is finite and $H_0(\mathcal{G})$ has finite rank). Then every faithful ergodic measure-preserving action of $\mathsf{F}(\mathcal{G})$ is essentially free.

Proof. We saw in Proposition 3.23 that every finite factor representation factors through the abelianisation $\mathsf{F}(\mathcal{G})_{\mathrm{ab}}$. Therefore, whenever $\mathsf{F}(\mathcal{G})_{\mathrm{ab}}$ is finite, there are finitely many factor representations. We may then apply [13, Theorem 2.11] to get the conclusion. □

We can be even more concrete in some cases where homology has been computed.

Corollary 4.3. For each of the following groups, every faithful ergodic measure-preserving action is essentially free:

(1) The Brin-Higman-Thompson groups $nV_{k,r}$.
(2) More generally, topological full groups arising from products of shift of finite type groupoids.

Proof. It is shown in [28] that the abelianisations of the topological full groups of such groupoids are finite. Therefore the result follows from Proposition 4.2. □

As another application, we confirm Conjecture B for many natural examples including the class of groups above.

Example 4.4. The following groups, which are already known to be finitely presented and simple, are 2-generated.

(1) Derived subgroups of the topological full groups of Graph groupoids .
(2) Derived subgroups of the topological full groups of Katsura-Exel-Pardo groupoids.
(3) Simple subgroups arising as subgroups of topological full groups arising from products of shifts of finite type.
(4) Derived subgroups of the topological full groups of Beta expansion groupoids.
(5) Certain simple groups arising from groupoids that are left regular representations of Garside categories.

In particular, Conjecture B holds true for these examples.

Proof. The groups listed above are known to be the topological full groups of purely infinite, minimal, ample, and effective groupoids: for (1), this is shown in [32]; for (2), this is shown in [33]; for (3), this is shown in [28]; for (4), this is shown in [25]; and for (5) this is shown in [21, Theorem C] and [23, Theorem A]. By Theorem A, these groups act vigorously on the Cantor set. It thus follows from [3, Theorem 1.12] that these groups are all 2-generated. □

Next, we show C*-simplicity for many examples of Thompson-like groups.

Example 4.5. The following Thompson-like groups are C*-simple:

(1) The Higman-Thompson groups $V_{k,r}$, for $k \geq 2 \in \mathbb{N}$ and $r \geq 1$.

[2]Comparison is a notion for groupoids defined in the second paragraph of page 6 of [22]. For our purposes, it suffices to mention that comparison is automatic for purely infinite, minimal groupoids; see the comments at the top of page 5 of [22].

(2) More generally, the Brin-Higman-Thompson groups $nV_{k,r}$, for $r, n \in \mathbb{N}$ and $k \geq 2$.
(3) Irrational slope Thompson groups.

Proof. All these groups act vigorously on the Cantor set and are locally closed; in other words, they are topological full groups of minimal, purely infinite Cantor groupoids by Theorem 3.30. The result then follows from Corollary 3.32. $\square$

Using Theorem F, we can bring new groups into our framework. As an example, we discuss the twisted Brin-Higman-Thompson groups introduced by Belk and Zaremsky [2], which were previously not known to be topological full groups of purely infinite, minimal groupoids. We thank James Belk for bringing this example class to our attention and for an outline of this proof.

Example 4.6 (Twisted Brin-Higman-Thompson groups)**.** The twisted Brin-Higman-Thompson groups SV were introduced in [2]. Already in their construction, we see that they are topological full groups, and hence they are locally closed. In the case where S is finite, say $|S| = n$, it follows that nV embeds into SV with full support, and therefore SV is vigorous. Similarly, if S is countable, we have that nV embeds into SV for all n in a way that $\bigcup_{n\in\mathbb{N}} \operatorname{supp}(nV)$ covers the whole of the Cantor space. Therefore, SV is again vigorous. We conclude that SV is always a locally closed, vigorous subgroup of $\operatorname{Homeo}(\mathcal{C})$. Applying Theorem F, we deduce that SV is the topological full group of an essentially principal, purely infinite, minimal Cantor étale groupoid.

As a further application, we can describe all proper characters and finite factor representations of the Brin-Higman-Thompson groups $nV_{k,r}$. Our discussion parallels and extends that of Matui in [28, Section 5.5], and we take this opportunity to rectify a minor typo in [28, Theorem 5.20]: in the case where $n = 1$, we have that $(V_{k,r})_{\mathrm{ab}}$ vanishes when k is even and is $\mathbb{Z}_2$ for k odd, and not the other way around.

Example 4.7. Let $n, k, r \in \mathbb{N}$, and consider the groupoid $\mathcal{R}_r \times \mathcal{E}_k^n$ consisting of the r-stabilisation of the n-fold product of the shift of finite type groupoid $\mathcal{E}_k$ associated to the single vertex graph with k loops. Then $\mathsf{F}(\mathcal{R}_r \times \mathcal{E}_k^n)$ has been explicitly identified with the Brin-Higman-Thompson group $nV_{k,r}$ in [28]. By [22, Theorem F], groupoid homology is independent of r, and it is computable via the Künneth formula, yielding:

$$H_i(\mathcal{R}_r \times \mathcal{E}_k^n) \cong \mathbb{Z}_{k-1}^{\,{}^{n-1}C_i}$$

for all $i \geq 0$, where ${}^nC_i$ is the binomial coefficient ${}^nC_i = \frac{n!}{(n-i)!i!}$. Plugging the homology of the groupoid for the Brin-Higman-Thompson group $nV_{k,r}$ into Theorem 4.1, we obtain:

$$\begin{tikzcd} \mathbb{Z}_{k-1}^{\,{}^{n-1}C_2} \arrow[r, "h"] & \mathbb{Z}_{k-1} \otimes \mathbb{Z}_2 \arrow[r, "j"] & (nV_{k,r})_{\mathrm{ab}} \arrow[l, bend left, dashed, "\rho"] \arrow[r, "I"] & \mathbb{Z}_{k-1}^{n-1} \arrow[r] & 0. \end{tikzcd}$$

We divide the discussion into cases:

(1) If k is even, then $\mathbb{Z}_{k-1} \otimes \mathbb{Z}_2 = 0$ and hence I is an isomorphism. In this case, we get $(nV_{k,r})_{\mathrm{ab}} = \mathbb{Z}_{k-1}^{n-1}$.
(2) If $k \in 4\mathbb{Z} + 1$, say $k = 4m + 1$ we can see that the map

$$j\colon \mathbb{Z}_2 \cong \mathbb{Z}_{4m} \otimes \mathbb{Z}_2 \to (nV_{4m+1,r})_{\mathrm{ab}}$$

is injective. Thus there is a homomorphism $\rho\colon (nV_{k,r})_{\mathrm{ab}} \to \mathbb{Z}_2$ such that $j \circ \rho = \mathrm{id}_{(nV_{k,r})_{\mathrm{ab}}}$, so the short exact sequence splits. In this case we get $(nV_{k,r})_{\mathrm{ab}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{k-1}^{n-1}$.

(3) If $k \in 4\mathbb{Z}+3$, say $k = 4m+3$, then we need to consider the cases $n = 1, n = 2$ and $n > 2$ separately. When
   (a) $n = 1$: then j is necessarily injective on $\mathbb{Z}_{k-1} \otimes \mathbb{Z}_2$ since $H_2(\mathcal{R}_r \times \mathcal{E}_k) = 0$. Moreover, j is an isomorphism since $H_1(\mathcal{G})$ also vanishes.
   (b) $n = 2$: again j is necessarily injective on $\mathbb{Z}_{k-1} \otimes \mathbb{Z}_2$ since $H_2(\mathcal{R}_r \times \mathcal{E}_k) = 0$. Additionally, one can show as in the proof of [28, Lemma 5.19(3)] that there exists a surjection $2V_{k,r} \to \mathbb{Z}_{2m-2}$, so that the abelianisation must be $\mathbb{Z}_{2m-2}$.
   (c) $n > 2$: then $j = 0$ because $h\colon \mathbb{Z}_{4m+1}^{{}^{n-1}C_2} \to \mathbb{Z}_2$ is a surjection.

In summary, for arbitrary $n, r, k \in \mathbb{N}$ with $k \geq 2$ and $r, n \geq 1$, the abelianisation of the Brin-Higman-Thompson groups is given by:

$$(nV_{k,r})_{\mathrm{ab}} = \begin{cases} 0 & k = 2, \text{ and } n > 1; \\ 0 & k > 2 \text{ is even and } n = 1; \\ \mathbb{Z}_{k-1}^{n-1} & k \text{ even and } n > 1 \\ \mathbb{Z}_2 & k \in 2\mathbb{Z}+1 \text{ and } n = 1; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_{k-1}^{n-1} & k \in 4\mathbb{Z}+1 \text{ and } n > 1; \\ \mathbb{Z}_{2k-2} & k \in 4\mathbb{Z}+3 \text{ and } n = 2; \\ \mathbb{Z}_{k-1}^{n-1} & k \in 4\mathbb{Z}+3 \text{ and } n > 2. \end{cases}$$

Therefore, using Proposition 3.23 we can concretely describe all the proper characters, which are in one-to-one correspondence with finite factor representations of $nV_{k,r}$. The outcome is the following:

(i) $nV_{2,r}$ has no proper characters, and $V_{k,r}$ has no proper characters if k is even.
(ii) If $k > 2$ is even and $n > 1$, then $nV_{k,r}$ has $n-1$ characters, all of order $k-1$. Also, if $k \in 4\mathbb{Z}+3$ and $n > 2$, then $nV_{k,r}$ has $n-1$ characters, all of order $k-1$.
(iii) If k is odd and $n = 1$, then $V_{k,r}$ has one proper character of order 2.
(iv) If $k \in 4\mathbb{Z}+1$ and $n > 1$, then $nV_{k,r}$ has n proper characters, one of order 2 and the others of order $k-1$.
(v) If $k \in 4\mathbb{Z}+1$ and $n = 2$, then $2V_{k,r}$ has one proper character of order $2k-2$.

We end with an example that generalises the above, advertising the flexibility that comes with working with topological full groups of étale groupoids. These groups have recently turned out to be very relevant in the study of embeddings of tensor products of L^p-Cuntz algebras; see [16].

Example 4.8 (Perfect Brin-Higman-Thompson like groups)**.** Consider the variation of the Brin-Higman-Thompson groups on r^n cubes such that on each dimension $j = 1, \ldots, n$ we have potentially different (integrally generated) slope sets generated by integers $k_j \geq 2$. Write $\overline{k}$ for the n-tuple $\overline{k} = (k_1, \ldots, k_n)$, and denote the group described above by $V_{\overline{k},r}$. Note that, if $k_1 = \cdots = k_n =: k$, then $V_{\overline{k},r}$ is just the Brin-Higman-Thompson group $nV_{k,r}$.

It is not difficult to see that $V_{\overline{k},r}$ is the topological full group of the groupoid:

$$\mathcal{G}_{\overline{k},r} := \mathcal{R}_r \times \prod_{j=1}^{n} \mathcal{E}_{k_j}.$$

This subclass of products of shifts of finite type groupoids fits into the framework of [28], and it is not difficult to see that if we set $g = \gcd(k_1 - 1, \ldots, k_n - 1)$, then for $j \geq 0$ we have:

$$H_j(\mathcal{G}_{\overline{k},r}) = (\mathbb{Z}/g\mathbb{Z})^{{}^{(n-1)}C_j}.$$

In particular, if in the discussion above we have $g = 1$, for example if $k_j = 2$ for some j, then the homology vanishes.

Proposition 4.9. Let $k_1, \ldots, k_n \geq 2$ satisfy $\gcd(k_1 - 1, \ldots, k_n - 1) = 1$. Then $V_{\overline{k},r}$ has the following properties:

(1) It is acyclic, so in particular perfect and simple.
(2) It is F_∞; in particular it is finitely presented.
(3) It is 2-generated and C*-simple.
(4) It has no proper characters.
(5) Every faithful ergodic measure preserving action of $V_{\overline{k},r}$ is essentially free.

Proof. We observed above that $\gcd(k_1-1, \ldots, k_n-1) = 1$ implies that the homology of $\mathcal{G}_{\overline{k},r}$ vanishes. Inserting this computation into Theorem 4.1, it is easily seen that $V_{\overline{k},r} = \mathsf{F}(\mathcal{G}_{\overline{k},r})$ is perfect in this case. Moreover, by [22, Corollary D], it follows that $V_{\overline{k},r}$ is acyclic. The rest of the claims follow from our results. □

5. Outlook

Our first two questions address obvious 'gaps' in Table 1. In [7], the so-called braided Higman-Thompson groups were introduced; these groups exhibit many shared properties with V and are often type F_∞. It would be interesting to understand if these groups fit into our framework.

Question 5.1. Can the braided Higman-Thompson groups be realised as topological full groups of some purely infinite minimal groupoids?

Another class which has recently been of interest is certain full automorphism groups $V_r(\Sigma)$ of Cantor algebras, as described in [24], which have been shown to be type F_∞ whenever the underlying Cantor algebras $U_r(\Sigma)$ are *valid, bounded* and *complete*. In a similar spirit to the question above, we ask:

Question 5.2. Can the family of groups $V_r(\Sigma)$, for $U_r(\Sigma)$ valid, bounded and complete, be described as the topological full groups of purely infinite minimal groupoids?

In view of Theorem F, Questions 5.1 and 5.2 are equivalent to asking whether the (simple) derived subgroups of the groups described admit vigorous actions on the Cantor space. The difficulty in both cases above is that the groups in question are not constructed as subgroups of homeomorphisms of the Cantor set.

The next question relates to Conjecture B in the introduction. Topological full groups provide interesting examples of simple, finitely generated groups outside of the purely infinite setting, but it is currently not known if any example outside of the purely infinite setting is finitely generated. We therefore ask:

Question 5.3. Are there finitely presented, minimal topological full groups outside of the purely infinite setting?

Our final question addresses the scope of Corollary D. In recent work in preparation by Dudko-Medynets, they show that for AF-groupoids, there is a one-to-one correspondence between proper characters of the topological full group and invariant probability measures on the unit space. Corollary D shows that the same is true for purely infinite minimal groupoids (since these do not admit invariant probability measures). It seems reasonable to believe that this correspondence may exist in full generality, so we ask:

Question 5.4. Let $\mathcal{G}$ be an étale, essentially principal Cantor groupoid. Is there a one-to-one correspondence between invariant probability measures on $\mathcal{G}$ and a proper characters on $\mathsf{A}(\mathcal{G})$?

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