Spectral extrema of graphs with bounded clique number and matching number

Hongyu Wang[a], Xinmin Hou[a,b], Yue Ma[a]

[a] School of Mathematical Sciences
University of Science and Technology of China, Hefei, Anhui 230026, China.
[b] CAS Key Laboratory of Wu Wen-Tsun Mathematics
University of Science and Technology of China, Hefei, Anhui 230026, China.
Hefei, Anhui 230026, China.

Abstract

For a set of graphs $\mathcal{F}$, let $\mathrm{ex}(n,\mathcal{F})$ and $\mathrm{spex}(n,\mathcal{F})$ denote the maximum number of edges and the maximum spectral radius of an n-vertex $\mathcal{F}$-free graph, respectively. Nikiforov (*LAA*, 2007) gave the spectral version of the Turán Theorem by showing that $\mathrm{spex}(n,K_{k+1})=\lambda(T_k(n))$, where $T_k(n)$ is the k-partite Turán graph on n vertices. In the same year, Feng, Yu and Zhang (*LAA*) determined the exact value of $\mathrm{spex}(n,M_{s+1})$, where M_{s+1} is a matching with $s+1$ edges. Recently, Alon and Frankl (arXiv2210.15076) gave the exact value of $\mathrm{ex}(n,\{K_{k+1},M_{s+1}\})$. In this article, we give the spectral version of the result of Alon and Frankl by determining the exact value of $\mathrm{spex}(n,\{K_{k+1},M_{s+1}\})$ when n is large.

1 Introduction

In this paper, we consider only simple and finte graphs. Let $G=(V,E)$ be a graph with vertex set $V=V(G)$ and edge set $E=E(G)$. We write $|G|$ for $|E(G)|$ through this paper. Let $A(G)$ be the *adjacency matrix* of G and let $\lambda(G)$ be the largest eigenvalue of $A(G)$, and call it the *spectral radius* of G.

Let $\mathcal{F}$ be a family of graphs, we say graph G is *$\mathcal{F}$-free* if G does not contain any graph in $\mathcal{F}$ as a subgraph. As the classical Turán type problem determines the maximum number of edges of an n-vertex $\mathcal{F}$-free graph, called *Turán number* and denoted by $\mathrm{ex}(n,\mathcal{F})$. Brualdi-Solheid-Turán type problems consider the maximum spectral radius of an n-vertex $\mathcal{F}$-free graph, denoted by $\mathrm{spex}(n,\mathcal{F})$, i.e.

$$\mathrm{spex}(n,\mathcal{F})=\max\{\lambda(G): G \text{ is an } n\text{-vertex } \mathcal{F}\text{-free graph}\}.$$

In the recent ten years, there are fruitful results of Brualdi-Solheid-Turán type problems, for example, in [2, 4, 9, 11, 13, 14, 16, 17, 18, 19, 21].

Let K_n and $\overline{K_n}$ denote the complete graph and the empty graph on n vertices, respectively. For any graph G and $U \subset V(G)$, write $G - U = G[V(G)\backslash U]$. Let $K_{n_1,\cdots,n_k}$ denote the complete k-partite graph with partition sets of sizes $n_1, \ldots, n_k$. A Turán graph $T_k(n)$ is the complete k-partite graph on n vertices whose partition sets have sizes as equal as possible. Define $G_k(n,s) = T_{k-1}(s) \vee \overline{K_{n-s}}$, the join of the Turán graph $T_{k-1}(s)$ and empty graph $\overline{K_{n-s}}$. Clearly, $G_k(n,s)$ is a complete k-partite graph on n vertices with one partition set of size $n-s$ and the others having sizes as equal as possible. Write M_k for a matching consisting of k edges.

A fundamental theorem (Turán Theorem) due to Turán [20] gives $\mathrm{ex}(n; K_{k+1}) = |E(T_k(n))|$ for $n > k+1 > 3$. In 2007, Nikiforov [15] gave a spectral version of the Turán Theorem by showing that $\lambda(G) \leq \lambda(T_k(n))$ for every n-vertex K_{k+1}-free graph G, with equality if and only if $G \cong T_k(n)$. When considering the bounded matching number, Feng, Yu and Zhang [8] proved that

$$\mathrm{spex}(n, M_{s+1}) = \begin{cases} \lambda(K_n), \text{ if } n = 2s \text{ or } 2s+1; \\ \lambda(K_{2s+1} \cup \overline{K_{n-2s-1}}), \text{ if } 2s+2 \leqslant n < 3s+2; \\ \lambda(K_s \vee \overline{K_{n-s}}) \text{ or } \lambda(K_{2s+1} \cup \overline{K_{n-2s-1}}), \text{ if } n = 3s+2; \\ \lambda(K_s \vee \overline{K_{n-s}}), \text{ if } n > 3s+2. \end{cases}$$

Recently, Ni, Wang and Kang [14] extended the above result by determining the exact value of $\mathrm{spex}(n, kK_{r+1})$ for $k \geq 2$, $r \geq 2$, and sufficiently large n.

Another fundamental result in graph theory is the Erdős-Gallai Theorem [6], showing that

$$\mathrm{ex}(n, M_{s+1}) = \max\left\{ |E(G_{s+1}(n,s))|, \binom{2s+1}{s+1} \right\}.$$

Recently, Alon and Frankl [1] combined the forbidden graphs of Turán Theorem and Erdős-Gallai Theorem by showing that

Theorem 1.1 ([1]). *For $n \geq 2s+1$ and $k \geq 2$,*

$$ex(n, \{K_{k+1}, M_{s+1}\}) = \max\{|T_k(2s+1)|, |G_k(n,s)|\}.$$

Observe that when n is sufficiently large,

$$\mathrm{ex}(n, \{K_{k+1}, M_{s+1}\}) = \max\{|T_k(2s+1)|, |G_k(n,s)|\} = |G_k(n,s)|.$$

In this note, we consider the Brualdi-Solheid-Turán type problem of Theorom 1.1 when n is sufficiently large. Here is our main theorem.

Theorem 1.2. *For $n \geq 4s^2 + 9s$ and $k \geq 2$,*

$$spex(n, \{K_{k+1}, M_{s+1}\}) = \lambda(G_k(n, s)).$$

The rest of the note is arranged as follows. We give some preliminaries and lemmas. The proof of Theorem 1.2 will be given in Section 2. We give some discussion in the last section.

2 Preliminaries and lemmas

The Tutte-Berge Theorem [3] (also see the Edmonds-Gallai Theorem [5]) is very useful when we cope with the problem related to matching number.

Lemma 2.1 ([3],[5]). *A graph G is M_{s+1}-free if and only if there is a set $B \subset V(G)$ such that all the components $G_1, \ldots, G_m$ of $G - B$ are odd (i.e. $|V(G_i)| \equiv 1 \pmod 2$ for $i \in [m]$), and*

$$|B| + \sum_{i=1}^{m} \frac{|V(G_i)| - 1}{2} = s.$$

The following result is due to Esser and Harary [7].

Lemma 2.2 ([7]). *For any k-partite graph $K = K_{n_1,\cdots,n_k}$ of order n, the characteristic polynomial $\Phi_K(\lambda)$ is given by*

$$\Phi_K(\lambda) = \lambda^{n-k} \left(\prod_{i=1}^{k} (\lambda + n_i) - \sum_{i=1}^{k} n_i \prod_{j=1, j \neq i}^{k} (\lambda + n_j) \right).$$

And the spectral radius of K is the largest root of $1 - \sum\limits_{i=1}^{k} \frac{n_i}{\lambda + n_i} = 0$.

The following lemma shows that for a complete multipartite graph the more balanced the graph is, the larger will the spectral radius be.

Lemma 2.3. *For any k-partite graph $K_{n_1,\cdots,n_k}$ of order n, if there exist i and j with $n_i - n_j \geq 2$, then $\lambda(K_{n_1,\cdots,n_i-1,\cdots,n_j+1,\cdots,n_k}) > \lambda(K_{n_1,\cdots,n_i,\cdots,n_j,\cdots,n_k})$.*

Proof. Let A and $\tilde{A}$ be adjacent matrices of $K = K_{n_1,\cdots,n_k}$ and $\tilde{K} = K_{n_1,\cdots,n_i-1,\cdots,n_j+1,\cdots,n_k}$, respectively, where $\tilde{K}$ is obtained from K by moving a vertex v in the i-th part V_i to the j-th part V_j. Let $\lambda = \lambda(K)$ and $\tilde{\lambda} = \lambda(\tilde{K})$. Let $\mathbf{x}$ be a unit Perron vector of A. Note that all vertices in the same part of K or $\tilde{K}$ have the same corresponding components in its unit Perron vector. Denote the components corresponding to the vertices in the ℓ-th part in $\mathbf{x}$ by x_ℓ. Let $f(x) = \sum\limits_{\ell=1}^{k} \frac{n_\ell}{x + n_\ell} - 1$. By Lemma 2.2, λ is the largest root of $f(x)$. Clearly,

$f(+\infty) = -1 < 0$ and $f(n_j) = \sum\limits_{\ell \neq i,j}^{k} \frac{n_\ell}{n_j+n_\ell} + \frac{n_i}{n_j+n_i} - \frac{1}{2} > 0$. Hence, $\lambda > n_j$. Since $A\mathbf{x} = \lambda\mathbf{x}$, we have $\lambda x_m = \sum\limits_{\ell=1}^{k} n_\ell x_\ell - n_m x_m$, i.e. $x_m = \sum\limits_{\ell=1}^{k} n_\ell x_\ell/(\lambda + n_m)$ for $m \in [k]$. Therefore,

$$\begin{aligned}
\mathbf{x}^T(\tilde{A} - A)\mathbf{x} &= \sum_{u \in V_i \setminus \{v\}} 2x_u x_v - \sum_{u \in V_j} 2x_u x_v \\
&= 2(n_i - 1)x_i^2 - 2n_j x_j x_i \\
&= 2x_i[(n_i - 1)x_i - n_j x_j] \\
&= 2x_i \sum_{\ell=1}^{k} n_\ell x_\ell \left(\frac{n_i - 1}{\lambda + n_i} - \frac{n_j}{\lambda + n_j} \right) \\
&= 2x_i \sum_{\ell=1}^{k} n_\ell x_\ell \frac{\lambda n_i - \lambda - n_j - \lambda n_j}{(\lambda + n_i)(\lambda + n_j)} \\
&> 0,
\end{aligned}$$

the last inequality holds because $n_i \geq n_j + 2$ and $\lambda > n_j$. Therefore, we have $\tilde{\lambda} > \lambda$.

$\square$

Let M be an $n \times n$ real symmetric matrix with the following block form

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & \ddots & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix}.$$

For $1 \leq i, j \leq k$, let b_{ij} denote the average row sum of M_{ij}. The matrix $B = (b_{ij})$ is called the *quotient matrix* of M. Moreover, if for each pair i, j, M_{ij} has a constant row sum, then B is called the *equitable quotient matrix* of M.

Lemma 2.4 ([10])**.** *Let M be an $n \times n$ real symmetric matrix and let B be an equitable quotient matrix of M. If M is nonnegative and irreducible, then $\lambda(M) = \lambda(B)$, where $\lambda(M)$ and $\lambda(B)$ are the largest eigenvalues of M and B, respectively.*

For two non-adjacent vertices u, v in a graph G, we define the *switching operation $u \to v$* as deleting the edges joining u to its neighbors and adding new edges connecting u to the neighborhood of v. Let $G_{u \to v}$ be the graph obtained from G by the switching operation $u \to v$, that is $V(G_{u \to v}) = V(G)$ and

$$E(G_{u \to v}) = (E(G) \setminus E_G(u, N_G(u))) \cup E_G(u, N_G(v)),$$

where $E_G(S, T)$ is the set of edges in G with one end in S and the other in T for disjoint subsets $S, T \subset V(G)$. Note that the edges between u and the common neighbors of u and v

remain unchanged by the definition of $G_{u\to v}$. For two disjoint independent sets S and T in a graph G, if all vertices in S (resp. T) have the same neighborhood $N_G(S)$ (resp. $N_G(T)$), we similarly define $G_{S\to T}$ to be the graph obtained from G by deleting the edges between S and $N_G(S)$ and adding new edges connecting S and $N_G(T)$.

Proposition 2.5. *For $r \geq 2$ and two disjoint independent sets S and T in a graph G, if all of vertices in S (resp. T) have the same neighborhood $N_G(S)$ (resp. $N_G(T)$) and $E_G(S,T) = \emptyset$, then either $G' = G_{S\to T}$ or $G' = G_{T\to S}$ has the property that $\lambda(G') \geq \lambda(G)$.*

Proof. Let S and T be two such independent sets of G. Let $\mathbf{x}$ be a unit Perron vector of $A(G)$. Without loss of generality, suppose $\sum\limits_{z\in N_G(T)} x_z \geq \sum\limits_{z\in N_G(S)} x_z$. Let $G' = G_{S\to T}$. Then

$$\begin{aligned}
\mathbf{x}^T(A(G') - A(G))\mathbf{x} &= \sum_{u\in S}\sum_{z\in N_G(T)} 2x_u x_z - \sum_{u\in S}\sum_{z\in N_G(S)} 2x_u x_z \\
&= 2\sum_{u\in S} x_u \left(\sum_{z\in N_G(u)} x_z - \sum_{z\in N_G(v)} x_z \right) \geq 0.
\end{aligned}$$

Therefore, we have $\lambda(G') \geq \lambda(G)$. ☐

3 Proof of Theorem 1.2

Now we are ready to give the proof of the main theorem.

Proof of Theorem 1.2. Suppose $n \geq 4s^2 + 9s$. Let G be an n-vertex graph with maximum spectral radius over all $\{K_{k+1}, M_{s+1}\}$-free graphs. Let $\lambda = \lambda(G)$ and $\mathbf{x}$ be a unit Perron vector of $A(G)$. We show that $\lambda(G) \leq \lambda(G_k(n,s))$.

Since G is M_{s+1}-free, by Lemma 2.1, there is a vertex set $B \subset V(G)$ such that $G - B$ consists of odd components $G_1, \dots, G_m$, and

$$|B| + \sum_{i=1}^{m} \frac{|V(G_i)| - 1}{2} = s. \tag{1}$$

Let $A_i = V(G_i)$ and $|A_i| = a_i$ for $i \in [m]$. Denote $A = \cup_{i=1}^m A_i$. Let $I_G(A) = \{i \in [m] : a_i = 1\}$. We may choose G maximizing $|I_G(A)|$ (assumption (*)). Let $|B| = b$. Then we have $b \leq s$ and $a_i \leq 2s + 1$.

Define two vertices u and v in B are equivalent if and only if $N_G(u) = N_G(v)$. Clearly, it is an equivalent relation. Therefore, the vertices of B can be partitioned into equivalent classes according to the equivalent relation defined above. We may choose G (among graphs G satisfying assumption (*)) with the minimum number of equivalent classes of B (assumption (**)). Note that each equivalent class of B is an independent set of G by the definition of the equivalent relation. We first claim that every two non-adjacent vertices of B have the same neighborhood (a spectral version of Lemma 2.1 in [1]), for completeness we include the proof.

Claim 1. *Every two non-adjacent vertices of B have the same neighborhood.*

Proof. Suppose there are two non-adjacent vertices $u, w \in B$ with $N_G(u) \neq N_G(w)$. Then u and w must be in different equivalent classes U and W by the definition of the equivalence. Since $uw \notin E(G)$, we have $E_G(U, W) = \emptyset$. Without loss of generality, suppose $\sum\limits_{z \in N_G(w)} x_z \geq \sum\limits_{z \in N_G(u)} x_z$. Let $G' = G_{U \to W}$. By Proposition 2.5, $\lambda(G') \geq \lambda(G)$. Now we show that G' is $\{K_{k+1}, M_{s+1}\}$-free too. Clearly, $G' - B$ still consists of odd components $G_1, \ldots, G_m$. Hence G' is M_{s+1}-free by Lemma 2.1. If G' contains a copy T of K_{k+1}, we must have a vertex $u' \in V(T) \cap U$. Since $N_{G'}(u') = N_{G'}(w) = N_G(w)$, $(V(T) \setminus \{u'\}) \cup \{w\}$ induces a copy of K_{k+1} in G, a contradiction. Hence, $G_{U \to W}$ is $\{K_{k+1}, M_{s+1}\}$-free. By the extremality of G, we have $\lambda(G') = \lambda(G)$. But the number of equivalent classes of G' (U and W merge into one class in G') is less than the one in G, a contradiction to the assumption (**).

□

By Claim 1 and G is K_{k+1}-free, $G[B]$ is a complete ℓ-partite graph with $\ell \leq k$. Let its partition sets be $B_1, \ldots, B_\ell$ and let $B_{\ell+1} = \cdots = B_k = \emptyset$ if $\ell < k$. Let $b_i = |B_i|$ for $i \in [k]$. Without loss of generality, assume that $\sum\limits_{v \in B_1} x_v \geq \cdots \geq \sum\limits_{v \in B_k} x_v$. By Claim 1, if there is a vertex in B_i adjacent to $v \in A_j$ then $B_i \subseteq N_G(v)$.

Claim 2. $a_2 = a_3 = \cdots = a_m = 1$.

Proof. Suppose v_1 is a vertex in A with $\sum\limits_{u \in N_G(v_1)} x_u = \max\limits_{v \in A} \sum\limits_{u \in N_G(v)} x_u$. Without loss of generality, suppose $v_1 \in A_1$. We prove by contradiction. Suppose there is an a_i with $a_i \neq 1$ for some $2 \leq i \leq m$.

If $|G[A_1]| = 0$, let G' be the resulting graph by applying the switching operations $u \to v_1$ for all vertices $u \in A \setminus \{v_1\}$ one by one. Then we have $|G'[A]| = 0$. By Proposition 2.5, $\lambda(G') \geq \lambda(G)$. With the same discussion as in the proof of Claim 1, we have that G' is still $\{K_{k+1}, M_{s+1}\}$-free. But $|I_{G'}(A)| = m > |I_G(A)|$, a contradiction to the assumption (*).

If $|G[A_1]| > 0$, i.e. $a_1 \geq 3$. Without loss of generality, assume $a_2 \geq 3$. Since $G[A_2]$ is connected, we can pick two vertices, say u_1, u_2 in A_2 such that $G[A_2 \setminus \{u_1, u_2\}]$ is still connected (u_1, u_2 exist, for example, we can pick two leaves of a spanning tree of $G[A_2]$). Let G_1 be the resulting graph by applying the switching operations $u_1 \to v_1$ and $u_2 \to v_1$ one by one. With similar discussion as in the above case, we have $\lambda(G_1) \geq \lambda(G)$ and G_1 is $\{K_{k+1}, M_{s+1}\}$-free. Continue the process after $t = \frac{a_2 - 1}{2}$ steps, we obtain a graph G_t with

$\lambda(G_t) \geq \lambda(G)$ and G_t is $\{K_{k+1}, M_{s+1}\}$-free. But $|I_{G_t}(A)| = |I_G(A)| + 1$, a contradiction to the assumption (*). $\square$

The following proof is divided into two cases according to a_1.

Case 1. $a_1 = 1$.

In this case $b = s$ by (1). By Claim 2, A is an independent set of G. Let $A = \{v_1, \ldots, v_m\}$.

If $b_k = 0$, then $G[B]$ is a complete ℓ-partite graph on s vertices with $\ell \leq k-1$. We may assume $\ell = k-1$ (Otherwise, we can add new edges in $G[B]$ to make it $(k-1)$-partite and this operation will increase the spectral radius of G by the Perron-Frobenius Theorem, a contradiction to the maximality of G). With the same reason, we can add all missing edges between sets A and B to make G a complete k-partite graph. Now by Lemma 2.3, we have $\lambda(G) \leq \lambda(G_k(n, s))$, and the equality holds if and only if $G \cong G_k(n, s)$.

If $b_k \neq 0$, then $G[B]$ is a complete k-partite graph on s vertices. Since G is K_{k+1}-free, each vertex in A is only adjacent to $k-1$ parts in B. By the assumption $\sum\limits_{v \in B_1} x_v \geq \cdots \geq \sum\limits_{v \in B_k} x_v$ and the maximality of G, we may assume every vertex of A is adjacent to $B_1, \cdots, B_{k-1}$ (the only possible exception is when $\sum\limits_{v \in B_1} x_v = \cdots = \sum\limits_{v \in B_k} x_v$, in this case, we can relabel $B_1, \ldots, B_k$ and do switching operations in vertices of A to obtain a new graph with the non-decrease spectral radius and the desired property). Now combine B_k and A as one part, we obtain that G is a complete k-partite graph. Since $\sum\limits_{i=1}^{k-1} b_i \leq s-1$, by Lemma 2.3, we have $\lambda(G) < \lambda(G_k(n, s))$. This completes the proof of the case.

Case 2. $a_1 \geq 3$

In this case $b + \frac{a_1 - 1}{2} = s$. Since G is K_{k+1}-free and has maximum spectral radius, we also can assume that $G[B]$ is a complete ℓ-partite graph with $\ell = k-1$ or k and each vertex in A is only adjacent to the first $k-1$ parts in $G[B]$. Now let $\tilde{A} = B_k \cup (A \setminus A_1)$ and $a = |\tilde{A}|$. Then $\tilde{A}$ is an independent set of G. To finish the proof, we will show that $\lambda(G) < \lambda(G_k(n, s))$. To do this, let $\tilde{G}$ be the graph obtained by adding all missing edges (if any) between the sets $\tilde{A}$ and $B \setminus B_k$, all missing edges (if any) between A_1 and $B \setminus B_k$, and all missing edges (if any) in A_1, i.e. $\tilde{G} = G[B \setminus B_k] \vee (\overline{K_a} \cup K_{a_1})$. Clearly, $G \subseteq \tilde{G}$. Hence we have $\lambda(\tilde{G}) \geq \lambda(G)$. Therefore, it is sufficient to show that $\lambda(G_k(n, s)) > \lambda(\tilde{G})$.

Claim 3. $\lambda(G_k(n, s)) > \lambda(\tilde{G})$.

Proof. The quotient matrix of $A(\tilde{G})$ according to the partition $B_1 \cup \cdots \cup B_{k-1} \cup \tilde{A} \cup A_1$ is

$$M = \begin{pmatrix} 0 & b_2 & \cdots & b_{k-1} & a & a_1 \\ b_1 & 0 & \cdots & b_{k-1} & a & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b_1 & b_2 & \cdots & 0 & a & a_1 \\ b_1 & b_2 & \cdots & b_{k-1} & 0 & 0 \\ b_1 & b_2 & \cdots & b_{k-1} & 0 & a_1 - 1 \end{pmatrix}$$

By Lemma 2.4, we have $\lambda(\tilde{G}) = \lambda(M)$, where $\lambda(M)$ is the largest eigenvalue of M. It can be calculated that the characteristic polynomial of M is

$$\Phi_M(\lambda) = (\lambda^2 + (a+1)\lambda + a(1-a_1)) \prod_{i=1}^{k-1} (-\lambda - b_i) \left(-\frac{\lambda(\lambda + 1 - a_1)}{\lambda^2 + \lambda + a(\lambda + 1 - a_1)} + \sum_{i=1}^{k-1} \frac{b_i}{b_i + \lambda} \right).$$

Thus, $\lambda(M)$ is the largest root of $\Phi_M(\lambda)$. Let

$$f_0(\lambda) = -\frac{\lambda(\lambda + 1 - a_1)}{\lambda^2 + \lambda + a(\lambda + 1 - a_1)} + \sum_{i=1}^{k-1} \frac{b_i}{b_i + \lambda}$$

and

$$h(\lambda) = \lambda^2 + (a+1)\lambda + a(1-a_1).$$

Since $a_1 \leq 2s+1$, $1 \leq b_i \leq b \leq s$ and $n \geq 4s^2 + 9s$, we have $a = n - a_1 - b \geq 4s^2 + 6s - 1$. Then we have

$$f_0(a_1) = -\frac{a_1}{a_1^2 + a_1 + a} + \sum_{i=1}^{k-1} \frac{b_i}{b_i + a_1} \geq \frac{k-1}{2s+2} - \frac{1}{2\sqrt{a} + 1} \geq \frac{1}{2s+2} - \frac{1}{4s-1} > 0$$

and

$$f_0(+\infty) = \lim_{\lambda \to +\infty} \left(-1 + \frac{(a + a_1)\lambda - a(a_1 - 1)}{\lambda^2 + \lambda + a(\lambda + 1) - aa_1} + \sum_{i=1}^{k-1} \frac{b_i}{b_i + \lambda} \right) = -1 < 0.$$

Thus the largest root of $f_0(\lambda)$ is larger than a_1. Since $h(a_1) = a_1^2 + a_1 + a > 0$ and $a + 1 > 0$, the largest root of $f_0(\lambda)$ and $\Phi_M(\lambda)$ are the same. Therefore, we have $\lambda(M) > a_1$.

Next, we will prove that $\lambda(M) < \lambda(G_k(n,s))$ by shifting vertices from A_1 to $\tilde{A}$ and some B_i for $i \in [k-1]$. Specifically, arbitrarily choose an $i \in [k-1]$, let $\tilde{G}_1$ be the graph obtained from $\tilde{G}$ by shifting one vertex from A_1 to $\tilde{A}$ and one vertex from A_1 to some B_i, where when we shift a vertex from a set X to another set Y, we delete the edges between the vertex and its neighbors and adding new edges connecting it to the neighborhood of Y. Note that $\tilde{G} = K_{b_1,\ldots,b_{k-1}} \vee (\overline{K_a} \cup K_{a_1})$. Then $\tilde{G}_1 = K_{b_1,\ldots,b_i+1,\ldots,b_{k-1}} \vee (\overline{K_{a+1}} \cup K_{a_1-2})$. Let

$$f_1(\lambda) = -\frac{\lambda(\lambda + 3 - a_1)}{\lambda^2 + \lambda + (a+1)(\lambda + 3 - a_1)} + \frac{b_i + 1}{b_i + 1 + \lambda} + \sum_{l=1, l \neq i}^{k-1} \frac{b_l}{b_l + \lambda}.$$

Then

$$\begin{aligned} f_1(\lambda) - f_0(\lambda) &= \frac{\lambda}{\lambda^2 + (2b_i + 1)\lambda + b_i^2 + b_i} \\ &\quad - \frac{\lambda(\lambda^2 + (2a_1 - 2)\lambda - a_1^2 + 4a_1 - 3)}{(\lambda^2 + (a+1)\lambda + a(1-a_1))(\lambda^2 + (a+2)\lambda + (a+1)(3-a_1))} \\ &:= \frac{\lambda}{g_1(\lambda)} - \frac{\lambda g_2(\lambda)}{g_3(\lambda)g_4(\lambda)} \\ &= \frac{\lambda g_3(\lambda)g_4(\lambda) - \lambda g_1(\lambda)g_2(\lambda)}{g_1(\lambda)g_3(\lambda)g_4(\lambda)}. \end{aligned}$$

Since $a_1 \le 2s+1$, $1 \le b_i \le s$, $a \ge 4s^2 + 6s - 1$, and $\lambda(M) > a_1$, we have

$$\begin{aligned} g_3(\lambda(M)) - g_2(\lambda(M)) &= (1 + a - 2a_1 + 2)\lambda(M) + a(1-a_1) + (a_1 - 1)(a_1 - 3) \\ &> (1 + a - 2a_1 + 2)a_1 + a(1-a_1) + (a_1 - 1)(a_1 - 3) \\ &= a - a_1^2 - a_1 + 3 \\ &\ge 4s^2 + 6s - 1 - (2s+1)^2 - (2s+1) + 3 \\ &\ge 0, \end{aligned}$$

and

$$\begin{aligned} g_4(\lambda(M)) - g_1(\lambda(M)) &= (a - 2b_i + 1)\lambda(M) + (a+1)(3-a_1) - b_i^2 - b_i \\ &> (a - 2b_i + 1)a_1 + (a+1)(3-a_1) - b_i^2 - b_i \\ &\ge 3(4s^2 + 6s) - 5s^2 - 3s \\ &> 0. \end{aligned}$$

Therefore, we have $f_1(\lambda(M)) - f_0(\lambda(M)) > 0$, which implies that the spectral radius increases after one shifting operation. Therefore, after $t = \frac{a_1 - 1}{2}$ times of shifting operations, we get a complete k-partite graph $\tilde{G}_t = K_{b'_1, \ldots, b'_{k-1}, a'}$, where $\sum_{i=1}^{k-1} b'_i = s - b_k \le s$ and $a' = n - s + b_k \ge n - s$. By Lemma 2.3, we have $\lambda(G_k(n,s)) \ge \lambda(\tilde{G}_t) > \lambda(\tilde{G})$. This completes the proof of Case 2. □

The proof of Theorem 1.2 is completed.

□

4 Concluding Remarks

In this note, we determine spex$(n, \{K_{k+1}, M_{s+1}\})$ when $n > 4s^2 + 9s$, we believe that the lower bound of n can be optimized, and when n is small, the extremal graph will be $T_k(n)$. We leave this as a problem.

Acknowledgment: The work was supported by the National Natural Science Foundation of China (No. 12071453), the National Key R and D Program of China(2020YFA0713100), the Anhui Initiative in Quantum Information Technologies (AHY150200) and the Innovation Program for Quantum Science and Technology, China (2021ZD0302904).
Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] N. Alon, P. Frankl, Turán graphs with bounded matching number, arXiv.org. (2022). https://arxiv.org/abs/2210.15076.

[2] L. Babai, B. Guiduli, Spectral extrema for graphs: The Zarankiewicz problem, The Electronic Journal of Combinatorics. 16 (2009) R123.

[3] C. Berge, Sur le couplage maximum d'un graphe, C.R. Acad. Sci. Paris Sér. I Math, 247(1958) 258-259.

[4] S. Cioabă, D.N. Desai, M. Tait, The spectral radius of graphs with no odd wheels, European Journal of Combinatorics. 99 (2022) 103420.

[5] J. Edmonds, Paths, trees, and Flowers, Canadian Journal of Mathematics. 17 (1965) 449-467.

[6] P. Erdős, T. Gallai, On maximal paths and circuits of graphs, Acta Mathematica Academiae Scientiarum Hungaricae. 10 (1959) 337-356.

[7] F. Esser, F. Harary, On the spectrum of a complete multipartite graph, European Journal of Combinatorics. 1 (1980) 211-218.

[8] L. Feng, G. Yu, X.D. Zhang, Spectral radius of graphs with given matching number, Linear Algebra and Its Applications. 422 (2007) 133-138.

[9] J. Gao, X. Hou, The spectral radius of graphs without long cycles, Linear Algebra and Its Applications. 566 (2019) 17-33.

[10] C.D. Godsil, Algebraic combinatorics, Routledge, (2017).

[11] X. Hou, B. Liu, S. Wang, J. Gao, C. Lv, The spectral radius of graphs without trees of diameter at most four, Linear and Multilinear Algebra. 69 (2019) 1407-1414.

[12] H. Lin, M. Zhai, Y. Zhao, Spectral radius, edge-disjoint cycles and cycles of the same length, The Electronic Journal of Combinatorics. 29 (2022) .

[13] X. Liu, H. Broersma, L. Wang, Spectral radius conditions for the existence of all subtrees of diameter at most four, Linear Algebra and Its Applications. 663 (2023) 80-101.

[14] Z. Ni, J. Wang, L. Kang, Spectral extremal graphs for disjoint cliques, The Electronic Journal of Combinatorics. 30 (2023).

[15] V. Nikiforov, Bounds on graph eigenvalues II, Linear Algebra and Its Applications. 427 (2007) 183-189.

[16] V. Nikiforov, Some new results in extremal graph theory, Surveys in Combinatorics 2011. (2011) 141-182.

[17] V. Nikiforov, The spectral radius of graphs without paths and cycles of specified length, Linear Algebra and Its Applications. 432 (2010) 2243-2256.

[18] D.A. Spielman, Spectral graph theory and its applications, 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS'07). (2007) 29-38.

[19] M. Tait, The colin de verdière parameter, excluded minors, and the spectral radius, Journal of Combinatorial Theory, Series A. 166 (2019) 42-58.

[20] P. Turán, On an extremal problem in graph theory, Matematikaiés Fizikai Lapok (in Hungarian). 48(1941) 436-452.

[21] M. Zhai, H. Lin, Spectral extrema of $K_{s,t}$-minor free graphs – on a conjecture of M. Tait, Journal of Combinatorial Theory, Series B. 157 (2022) 184-215.