2-LOCAL UNSTABLE HOMOTOPY GROUPS OF INDECOMPOSABLE $\mathbf{A}_3^2$ -COMPLEXES

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ABSTRACT. In this paper, we calculate the 2-local unstable homotopy groups of indecomposable $\mathbf{A}_3^2$-complexes. The main technique used is analysing the homotopy property of $J(X, A)$, defined by B. Gray for a CW-pair (X, A), which is homotopy equivalent to the homotopy fibre of the pinch map $X \cup CA \to \Sigma A$.

CONTENTS

1. INTRODUCTION

For a suspended finite CW-complex X, if $X \simeq X_1 \vee X_2$ and both X_1 and X_2 are not contractible, then X is called decomposable; otherwise X is called indecomposable. Let $\mathbf{A}_n^k$ be the homotopy category consisting of $(n-1)$-connected finite CW-complexes with dimension less than or equal to $n+k$ $(n \geq k+1)$. The objects of $\mathbf{A}_n^k$ are also called $\mathbf{A}_n^k$-complexes. In 1950, S.C.Chang classified the indecomposable homotopy types in $\mathbf{A}_n^2 (n \geq 3)$ [3], that is

(i) Spheres: S^n, S^{n+1}, S^{n+2};
(ii) Elementary Moore spaces: $M_{p^r}^n$, $M_{p^r}^{n+1}$ where p is a prime, $r \in \mathbb{Z}^+$ and $M_{p^r}^k$ denotes $M(\mathbb{Z}/p^r, k)$, whose only nontrivial reduced homology is $\tilde{H}_k(M_{p^r}^k) = \mathbb{Z}/p^r\mathbb{Z}$;

2020 *Mathematics Subject Classification.* Primary 55P15.
Key words and phrases. Homotopy groups, $\mathbf{A}_n^2$-complexes, fibration sequence, relative James construction.

(iii) Elementary Chang complexes: C^{n+2}_{η}, $C^{n+2,s}$, C^{n+2}_{r}, $C^{n+2,s}_{r}$ ($r,s\in\mathbb{Z}^+$), which are given by the mapping cones of the maps $\eta_n: S^{n+1}\to S^n$, $f^s=j_1^{n+1}(2^s\iota_{n+1})+j_2^n\eta_n: S^{n+1}\to S^{n+1}\vee S^n$, $f_r=(\eta_n,2^r\iota_n): S^{n+1}\vee S^n\to S^n$, $f_r^s=(j_1^{n+1}(2^s\iota_{n+1})+j_2^n\eta_n, j_2^n(2^r\iota_n)): S^{n+1}\vee S^n\to S^{n+1}\vee S^n$ respectively,

where $\mathbb{Z}^+$ denotes the set of positive integers; $\iota_n\in\pi_n(S^n)$ is the identity map of S^n; η_2 is the Hopf map $S^3\to S^2$ and $\eta_n=\Sigma^{n-2}\eta_2$ for $n\geq 3$; j_1^{n+1}, resp. j_2^n is the inclusion of S^{n+1}, resp. S^n, into $S^{n+1}\vee S^n$.

The suspension Σ gives us sequences of functors $\mathbf{A}_n^k\xrightarrow{\Sigma}\mathbf{A}_{n+1}^k$ for all $n\geq k+1$. The Freudenthal suspension theorem shows that these sequences stabilize in the sense that for $k+1<n$ the functor $\mathbf{A}_n^k\xrightarrow{\Sigma}\mathbf{A}_{n+1}^k$ is an equivalence of additive categories. We point out that for $k+1=n$, the suspension functor $\mathbf{A}_{k+1}^k\xrightarrow{\Sigma}\mathbf{A}_{k+2}^k$ is a full representation equivalence, i.e. it is full, dense and reflects isomorphisms [6], which implies that Σ gives a 1-1 correspondence of homotopy types. Thus we often study $\mathbf{A}_{k+1}^k$ as a beginning of the study of $\mathbf{A}_n^k$ for $n\geq k+1$. There has been a lot of research on homotopy of spheres and elementary Moore spaces, but only a few on homotopy of all indecomposable $\mathbf{A}_n^2$-complexes by taking them as a whole. In the 1950s, P.J.Hilton calculated the $n+1, n+2$-dim homotopy groups of $\mathbf{A}_n^2$-complexes [8, 9, 10]. In 1985, H.J.Baues calculated the abelian groups $[X,Y]$ and groups of homotopy equivalences $Aut(X)$ for all indecomposable $\mathbf{A}_n^2$-complexes X and Y [1]. In 2017, the authors obtained the complete wedge decomposition of smash product $X\wedge Y$ for all indecomposable $\mathbf{A}_n^2$-complexes X and Y [22], and then as an application, we prove that the stable homotopy groups of elementary Chang complexes $C_r^{n+2,r}$ are direct summands of their unstable homotopy groups[23]. In 2020, we obtained the local hyperbolicity, which is defined by R.Z.Huang and J.Wu to study the asymptotic behavior of the p-primary part of the homotopy groups of simply connected finite p-local complexes [11], of $\mathbf{A}_n^2$-complexes by an analysis of decomposition of loop suspension [24]. In recent years, the problem of realisability of groups as self-homotopy equivalences of $\mathbf{A}_n^2$-complexes are studied by C. Costoya, et al.[5]. Then D.$\acute{M}$endez study the problem of realisability of rings as the ring of stable homotopy classes of self-maps of $\mathbf{A}_n^2$-complexes [13].

Calculating the unstable homotopy groups of finite CW-complexes is a fundamental and difficult problem in algebraic topology. A lot of related work [12, 15, 16, 18, 19, 21] has been done on CW-complexes with the number of cells less than or equal to 2, such as spheres, elementary Moore spaces, projective space and so on. J.Wu calculated the homotpy groups of mod 2 Moore spaces by using the functorial decomposition [19] and recently, J.X.Yang, et al. calculate the homotopy groups of the suspended quaternionic projective plane in [21] by using the relative James construction. Although calculating the unstable homotopy groups of a CW-complex with the number of cells greater than 2 will be more complicated, we realize that it is possible to compute homotopy groups of $\mathbf{A}_n^2$-complex by similar method after reading their preprint [21]. In this paper, we will calculate the 6 and 7 dimensional unstable homotopy groups of indecomposable $\mathbf{A}_3^2$ -complexes. We should point out that for an $\mathbf{A}_3^2$ -complex X, the Freudenthal suspension theorem implies that $\pi_m(X)$ is in the stable range for $m \leq 4$, and by the calculation of [10] so is $\pi_5(X)$ when X is indecomposable. Hence the 6-dimensional homotopy group of an indecomposable $\mathbf{A}_3^2$ -complex (except $M_{p^r}^4$) is its first unstable homotopy group. As a potential application, the 7-dimensional homotopy group of $\mathbf{A}_3^2$ -complexes may be used to study the classification problem of 2-connected 8-dimensional manifolds M^8, since the homotopy class of the attaching map of its top cell is an element of $\pi_7(X)$, where X is an $\mathbf{A}_3^2$ -complex.

Theorem 1.1. *The 6, 7-homotopy groups of all 2-loacl nontrivial indecomposable $\mathbf{A}_3^2$-complexes are listed as follows:*

(1)

$$\pi_6(M_{2^r}^3) \cong \begin{cases} \mathbb{Z}_4 \oplus \mathbb{Z}_2, & r = 1; \\ \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, & r = 2; \\ \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2^r}, & r \geq 3; \end{cases}$$
$$\pi_7(M_{2^r}^3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (1-\epsilon_r)\mathbb{Z}_4;$$
$$\pi_7(M_{2^r}^4) \cong \mathbb{Z}_{2^{min\{2,r-1\}}} \oplus \mathbb{Z}_{2^{r+1}} \oplus \mathbb{Z}_2.$$

(2)

$$\pi_6(C_\eta^5) \cong \mathbb{Z}_2;$$
$$\pi_6(C_r^5) \cong \mathbb{Z}_2 \oplus (1-\epsilon_r)\mathbb{Z}_2 \oplus \mathbb{Z}_{2^{r+\epsilon_r}};$$
$$\pi_6(C^{5,s}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2^s};$$
$$\pi_6(C_r^{5,s}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (1-\epsilon_r)\mathbb{Z}_2 \oplus \mathbb{Z}_{2^{min\{r,s\}}} \oplus \mathbb{Z}_{2^{r+\epsilon_r}}.$$

(3)

$$\pi_7(C_\eta^5) \cong \mathbb{Z}_{(2)};$$
$$\pi_7(C_r^5) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{2^{r+1}};$$
$$\pi_7(C^{5,s}) \cong \mathbb{Z}_{2^{min\{s,2\}}} \oplus \mathbb{Z}_{2^{s+2}};$$
$$\pi_7(C_r^{5,s}) \cong \mathbb{Z}_{2^{min\{s-\epsilon_r,2\}}} \oplus \mathbb{Z}_{2^{min\{s+1,r+1\}}} \oplus \mathbb{Z}_{2^{s+2}} \oplus \mathbb{Z}_4,$$

where $\mathbb{Z}_{(2)}$ denotes the 2-local integers and $\mathbb{Z}_k := \mathbb{Z}/k\mathbb{Z}$.

$\epsilon_r = \begin{cases} 1, & r = 1; \\ 0, & r \geq 2 \end{cases}$ in the Theorem and we also set $\epsilon_\infty = 0$ when $r = \infty$ is allowed in the following text.

The proof of the first statement of the Theorem is given in Section 3 and the remaining proofs are given in Section 4.2 and 4.3 respectively.

2. Some notations and lemmas

In this paper, all spaces and maps are in the category of pointed CW-complexes and maps (i.e. continuous functions) preserving basepoint. And we always use $*$ and 0 to denote the basepoints and the constant maps mapping to the basepoints respectively. We denote $A \hookrightarrow X$ as an inclusion map.

Let (X, A) be a pair of spaces with base point $* \in A$, and suppose that A is closed in X. In [7], B.Gray constructed a space $(X, A)_\infty$ analogous to the James construction, which is denoted by us as $J(X, A)$ to parallel with the the absolute James construction $J(X)$. In fact, $J(X, A)$ is the subspace of $J(X)$ of words for which letters after the first are in A. Especially, $J(X, X) = J(X)$. As parallel with the familiar symbol $J_r(X)$ which is the r-th filtration of $J(X)$, we denote the r-th filtration of $J(X, A)$ by $J_n(X, A) := J(X, A) \cap J_r(X)$, which is denoted by Gray as $(X, A)_r$ in [7].

For example, $J_1(X, A) = X$, $J_2(X, A) = (X \times A)/((a, *) \sim (*, a))$ for each $a \in A$. In fact there is a pushout diagram for $r \geq 2$:

$$\begin{tikzcd} X \times A^{n-1} \arrow[r, "\Pi_r"] & J_r(X, A) \\ F \arrow[u, hook] \arrow[r] & J_{r-1}(X, A) \arrow[u, hook, "I_r"'] \end{tikzcd}$$

where $F \subset X \times A^{n-1}$ is the "fat wedge" consisting of those points in which one or more coordinates is the base-point; Π_r and I_r are the projection and the inclusion respectively and both of them are natural.

Remark 2.1. $J_n(X, A)/J_{n-1}(X, A)$ is naturally homeomorphic to $(X \times A^{n-1})/F = X \wedge A^{\wedge(n-1)}$.

It is well known that there is a natural weak homotopy equivalence $\omega : J(X) \to \Omega\Sigma X$, which is a homotopy equivalence when X is a finite CW-complex, and satisfies $\begin{tikzcd} X \arrow[r, hook] \arrow[rr, bend right, "\Omega\Sigma"'] & J(X) \arrow[r, "\omega"] & \Omega\Sigma X \end{tikzcd}$, where $X \xrightarrow{\Omega\Sigma} \Omega\Sigma X$ is the inclusion $x \mapsto \psi$ where $\psi : S^1 \to S^1 \wedge X, t \mapsto t \wedge x$.

Let $X \xrightarrow{f} Y$ be a map. We always use C_f, F_f and M_f to denote the maping cone (or say, cofibre), homotopy fibre and mapping cylinder of f, $C_f \xrightarrow{p} \Sigma X$ the pinch map and $\Omega\Sigma X \xrightarrow{\partial} F_p \to C_f \xrightarrow{p} \Sigma X$ the homotopy fibration sequence induced by p respectively. We get the relative James construction $J(M_f, X)$ (resp. r-th relative James construction $J_r(M_f, X)$) for the pair (M_f, X).

Lemma 2.2. *Let $X \xrightarrow{f} Y$ be a map. Then we have*

(i) *$F_p \simeq J(M_f, X)$;*

(ii) *$\Sigma J(M_f, X) \simeq \bigvee_{k\geq 0}(\Sigma Y \wedge X^{\wedge k})$;$\Sigma J_k(M_f, X) \simeq \bigvee_{i=0}^{k-1}(\Sigma Y \wedge X^{\wedge i})$;*

(iii) *If $Y = \Sigma Y'$, $X = \Sigma X'$, then $J_2(M_f, X) \simeq Y \cup_\gamma C(Y \wedge X')$, where $\gamma = [id_Y, f]$ is the generalized Whitehead product.*

Proof. The lemma follows from the Theorems of [7] for (M_f, X). □

Denote both the inclusion $Y \hookrightarrow J_2(M_f, X)$ and the composition of the inclusions $Y \hookrightarrow J_2(M_f, X) \hookrightarrow J(M_f, X) \simeq F_p$ by j_p without ambiguous.

Lemma 2.3. *Suppose the left diagram is commutative*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mu'"] & Y \arrow[d, "\mu"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \quad ; \quad \begin{tikzcd} F_p \simeq J(M_f, X) \arrow[r] \arrow[d, "J(\bar{\mu}, \mu')"] & M_f/X \simeq C_f \arrow[r] \arrow[d, "\bar{\mu}"] & \Sigma X \arrow[d, "\Sigma\mu'"] \\ F_{p'} \simeq J(M_{f'}, X') \arrow[r] & M_{f'}/X' \simeq C_{f'} \arrow[r] & \Sigma X' \end{tikzcd}$$

then it induces the right commutative diagrams on fibrations, where $\widehat{\mu}$ *satisfies*

$$\begin{tikzcd} Y \arrow[r, hook, "\simeq"'] \arrow[rrr, bend left, "\mu"] & M_f \arrow[r, "\widehat{\mu}"'] & M_{f'} & Y' \arrow[l, "\simeq"'] \end{tikzcd}$$

. Let

$M_f = J_1(M_f, X) \xrightarrow{J(\widehat{\mu},\mu')|_{M_f}=J_1(\widehat{\mu},\mu')=\widehat{\mu}} J_1(M_{f'}, X') = M_{f'}$,

$J_2(M_f, X) \xrightarrow{J(\widehat{\mu},\mu')|_{J_2(M_f,X)}=J_2(\widehat{\mu},\mu')} J_2(M_{f'}, X')$,

then we have the following commutative diagram

$$\begin{tikzcd} Y \wedge X \arrow[r, "\mu\wedge\mu'"] \arrow[d, "\simeq"'] & Y' \wedge X' \arrow[d, "\simeq"'] \\ J_2(M_f,X)/J_1(M_f,X) \arrow[r, "\overline{J_2(\widehat{\mu},\mu')}"] & J_2(M_{f'},X')/J_1(M_{f'},X') \end{tikzcd}$$

Proof. The above lemma is easily obtained from [7]. □

The following Lemma 2.4 to Lemma 2.8 come from [21] in original or generalized form.

Lemma 2.4. *Let* $X \xrightarrow{f} Y$ *be a map. Then the following diagram is homotopy commutative*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Omega\Sigma"'] & Y \arrow[d, hook] \\ \Omega\Sigma X \arrow[r, "\partial"] & F_p \end{tikzcd}$$

Proof. We have the following homotopy-commutative diagram

$$\begin{tikzcd} & X \arrow[r, hook] \arrow[dl, "f"'] \arrow[d, hook, "i"] \arrow[rr, bend left, "\Omega\Sigma"] & J(X) \arrow[r, "\omega"] \arrow[d, "J(i,id_X)"] & \Omega\Sigma X \arrow[d, "\partial"] \\ Y \arrow[r, hook, "\simeq"] & M_f \arrow[r, hook] & J(M_f,X) \arrow[r, "\simeq"] & F_p \end{tikzcd}$$

where the middle homotopy-commutative square comes from the naturality of the relative James construction and the right homotopy-commutative square comes from Lemma 4.1 of [7]. Thus the Lemma 2.4 is obtained. □

Lemma 2.5. *Let* $X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{p} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ *be a cofibration sequence. Then there is a homotopy commutative diagram with rows fibration sequences:*

$$\begin{tikzcd} \Omega\Sigma X \arrow[r, "\partial"] \arrow[d, equal] & J(M_f,X) \arrow[r] \arrow[d, "\phi"] & C_f \arrow[r, "p"] \arrow[d, hook] & \Sigma X \arrow[d, equal] & \\ \Omega\Sigma X \arrow[r, "\Omega(-\Sigma f)"] & \Omega\Sigma Y \arrow[r] & J(C_f,Y) \arrow[r] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

Proof. As pointed out in the proof of Lemma 4.1.of [7], there is a natural inclusion $C_f \hookrightarrow J(C_f, Y)$ lifting the inclusion $\begin{tikzcd} C_f \arrow[r, hook] \arrow[rr, bend right, "p"'] & C_f \cup_i CY \arrow[r, "\simeq"] & \Sigma X \end{tikzcd}$

The homotopy commutativity of the right square implies that there exists a map $J(M_f, X) \xrightarrow{\phi} \Omega\Sigma Y$ such that the left and the middle squares are homotopy commutative. $\square$

Similar to the James-Hopf invariant, we define the n-th relative James-Hopf invariant

$$J(X, A) \xrightarrow{H_n} J(X \wedge A^{\wedge(n-1)}),\ x_1 x_2 \dots x_t \mapsto \prod_{1 \le x_1 < x_2 < \cdots < x_t \le n} (x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_t})$$

which are natural for pairs. $H_n(x_1 x_2 \dots x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n$ implies the following lemma

Lemma 2.6. *Let $X \xrightarrow{f} Y$ be a map. Then the following diagram is homotopy commutative*

$$\begin{tikzcd}
J(M_f, X) \arrow[d, "H_n"'] & & J_n(M_f, X) \arrow[ll, hook'] \arrow[d, "pinch"] \\
J(M_f \wedge X^{\wedge(n-1)}) & M_f \wedge X^{\wedge(n-1)} \arrow[l, hook'] & = J_n(M_f, X)/J_{n-1}(M_f, X)
\end{tikzcd}$$

Remark 2.7. By abuse of notion, H_2 also denotes the composition of the maps $\Omega\Sigma X \xrightarrow{\simeq} J(X) = J(X, X) \xrightarrow{H_2} J(X \wedge X) \xrightarrow{\simeq} \Omega\Sigma(X \wedge X)$, where X is a CW-complex and let $H_2' : F_p \simeq J(M_f, X) \xrightarrow{H_2} J(M_f \wedge X) \xrightarrow{\simeq} J(Y \wedge X) \simeq \Omega\Sigma(Y \wedge X)$.

Lemma 2.8. *Let $X \xrightarrow{f} Y$ be a map. Then the following diagram is homotopy commutative*

$$\begin{tikzcd}
\Omega\Sigma X \arrow[r, "\partial"] \arrow[d, "H_2"'] & F_p \arrow[d, "H_2'"] \\
\Omega\Sigma(X \wedge X) \arrow[r, "\Omega\Sigma(f \wedge id_A)"] & \Omega\Sigma(Y \wedge X)\ .
\end{tikzcd}$$

Proof. By the Lemma 4.1 of [7] and the naturality of the 2nd relative James-Hopf invariant, we have the following homotopy commutative diagram

$$\begin{tikzcd}
\Omega\Sigma X \arrow[rrr, bend left, "\partial"] \arrow[d, dotted, "H_2"'] & J(X) = J(X, X) \arrow[l, "\simeq"] \arrow[r, "J(i)"'] \arrow[d, "H_2"'] & J(M_f, X) \arrow[r, "\simeq"'] \arrow[d, "H_2"'] & F_p \arrow[d, dotted, "H_2'"'] \\
\Omega\Sigma(X \wedge X) \arrow[rrr, bend right, "\Omega\Sigma(f \wedge id_X)"'] & J(X \wedge X) \arrow[l, "\simeq"'] \arrow[r, "J(i \wedge id_X)"] & J(M_f \wedge X) \simeq \Omega\Sigma(M_f \wedge X) \arrow[r, "\simeq"] & \Omega\Sigma(Y \wedge X)
\end{tikzcd}$$

We complete the proof. $\square$

The following lemma comes from [4]

Lemma 2.9. *If $X \xrightarrow{f} Y$ is a map, X is $n-1$ connected, C_f is $m-1$ connected, the dimension of W is less than or equal to $m+n-2$, then we have the exact sequence*

$$[W, X] \xrightarrow{f_*} [W, Y] \to [W, C_f].$$

Lemma 2.10. *Let p be a prime and suppose that there is a commutative diagrams of short exact sequences of p-torsion abelian groups with $s < r$*

$$\begin{tikzcd} 0 \arrow[r] & B_1 \arrow[r, "i_2"] & A_1 \arrow[r, "p_2"] & \mathbb{Z}_{p^r} \arrow[r] & 0\,. \\ 0 \arrow[r] & B \arrow[r, "i_1"] \arrow[u] & A \arrow[r, "p_1"] \arrow[u, "f"] & \mathbb{Z}_{p^s} \arrow[r] \arrow[u, hook, "J"'] & 0 \end{tikzcd}$$

If the characteristic $ch(B_1) \leq p^s$ and the bottom short exact sequence is split then so is the top.

Proof. It follows from an easy diagram chasing argument. $\square$

The following generators of homotopy groups of spheres after localization at 2 come from [18]. $\iota_n = [id] \in \pi_n(S^n)$; $\pi_3(S^2) = \mathbb{Z}_{(2)}\{\eta_2\}$; $\pi_{n+1}(S^n) = \mathbb{Z}_2\{\eta_n\}(n \geq 3)$; $\pi_{n+2}(S^n) = \mathbb{Z}_2\{\eta_n\eta_{n+1}\}(n \geq 3)$; $\pi_6(S^3) = \mathbb{Z}_4\{\nu'\}$; $\pi_7(S^4) = \mathbb{Z}_4\{\Sigma\nu'\}\oplus\mathbb{Z}_{(2)}\{\nu_4\}$; $\pi_{n+3}(S^n) = \mathbb{Z}_8\{\nu_n\}(n \geq 5)$; $\pi_7(S^3) = \mathbb{Z}_2\{\nu'\eta_6\}$; $\pi_8(S^4) = \mathbb{Z}_2\{\Sigma\nu'\eta_7\} \oplus \mathbb{Z}_2\{\nu_4\eta_7\}$.

Throughout the paper, we will not distinguish a map and its homotopy class in many cases.

In the following all spaces are 2-local. $2^r = 0$ is allowed, in this case we denote $r = \infty$, i.e, $2^\infty = 0$, $\mathbb{Z}_{2^\infty} = \mathbb{Z}_0 = \mathbb{Z}_{(2)}$ (after 2-localization), $\mathbb{Z}_1 = 0$ (trivial group), $min\{k, \infty\} = k$ for some integer k.

3. Elementary Moore spaces

In this section we calculate $\pi_n(M^3_{2^r})(n = 6, 7)$ and $\pi_7(M^4_{2^r})$. For $r = 1, 2, 3$, many homotopy groups of these Moore spaces have been calculated by J. Wu, J. Mukai, T. Shinpo, and X.G.Liu in [**?**],[16], and [12] respectively.

There is a canonical cofibration sequence

$$S^k \xrightarrow{2^r\iota_k} S^k \xrightarrow{i_k} M^k_{2^r} \xrightarrow{p_k} S^{k+1} \tag{1}$$

where $M^k_{2^\infty} = M^k_0 = S^k \vee S^{k+1}$.

Let $\Omega S^{k+1} \xrightarrow{\partial} F_{p_k} \to M^k_{2^r} \xrightarrow{p_k} S^{k+1}$ be the homotpy fibration sequence. By Lemma 2.2 we get $\Sigma F_{p_k} \simeq S^{k+1} \vee S^{2k+1} \vee \ldots$.

3.1. **Calculating $\pi_6(M^3_{2^r})$.** For $M^3_{2^r}$, the 8-skeleton $Sk_8(F_{p_3}) \simeq S^3 \bigcup_\gamma CS^5$ with $\Sigma\gamma = 0$ and isomorphism $[S^5, S^3] \xrightarrow{\Sigma} [S^6, S^4]$ implies that $\gamma = 0$. Thus the 8-skeleton $Sk_8(F_{p_3}) \simeq S^3 \vee S^6$. By Lemma 2.4, we have exact sequence with commutative squares

$$\begin{tikzcd} \pi_7(S^4) \arrow[r, "\partial_{6*}"] & \pi_6(F_{p_3}) \arrow[r] & \pi_6(M^3_{2^r}) \arrow[r, "q_{r*}"] & \pi_6(S^4) \arrow[r, "\partial_{5*}"] & \pi_5(F_{p_3}) \\ \pi_6(S^3) \arrow[u, "\Sigma"] \arrow[r, "(2^r\iota_3)_*"] & \pi_6(S^3) \arrow[u, hook, "\cong j_{p_3*}"'] & & \pi_5(S^3) \arrow[u, "\cong\Sigma"] \arrow[r, "(2^r\iota_3)_*=0"] & \pi_5(S^3) \arrow[u, hook, "j_{p_3*}"'] \end{tikzcd} \tag{2}$$

$\pi_5(F_{p_3}) = \mathbb{Z}_2\{j_{p_3}\eta_3\eta_4\}$; $\pi_6(F_{p_3}) = \mathbb{Z}_4\{j_{p_3}\nu'\} \oplus \mathbb{Z}_{(2)}\{j^6_{p_3}\}$, where $j^6_{p_3} : S^6 \hookrightarrow Sk_8(F_{p_3})$ is the inclusions of the wedge summand S^6.

By the right commutative square of (2), we get $Ker\partial_{5*} \cong \mathbb{Z}_2\{\eta_4\eta_5\}$.

Next calculate $Coker\partial_{6*}$ in (2).

By Lemma 4.5 of [18], $(k\iota_n)\alpha = k\alpha$ for $\alpha \in \pi_n(S^3)$, then by the left commutative square of (2)

$$\partial_{6*}(\Sigma\nu') = 2^r j_{p_3}\nu'. \tag{3}$$

Lemma 2.6, Lemma 2.8 for map $S^3 \xrightarrow{2^r\iota_3} S^3$ give the following commutative diagram

$$\begin{tikzcd}
\pi_7(S^4) \arrow[r, "\partial_{6*}"] \arrow[d, "H_2"'] & \pi_6(F_{p_3}) \arrow[d, "H_2'"'] & \pi_6(S^3\vee S^6) \arrow[l, hook', "\cong"'] \arrow[r, "Proj"] & \pi_6(S^6) \arrow[d, "\Sigma\cong"'] \\
\pi_6(\Omega\Sigma S^3\wedge S^3) \arrow[r, "(\Omega\Sigma 2^r\wedge\iota_3)_*"] & \pi_6(\Omega\Sigma S^3\wedge S^3) \arrow[rr, equal] & & \pi_7(S^7)
\end{tikzcd} \tag{4}$$

By the right commutative square, $H_2'(j^6_{p_3}) = \iota_7$ for $j^6_{p_3} \in \pi_6(F_{p_3})$. $H_2(\nu_4) = \iota_7$ by Lemma 5.4 of [18].

Thus from the left commutative square of (4), we get

$$\partial_{6*}(\nu_4) = yj_{p_3}\nu' + 2^r j^6_{p_3} \text{ for some } y \in \mathbb{Z}_4. \tag{5}$$

From Lemma 2.5, we have the following two (homotopy) commutative diagrams

$$\begin{tikzcd}
& S^3 \arrow[d, "j_{p_3}"'] \arrow[dd, bend left] & \\
\Omega S^4 \arrow[r] \arrow[d, equal] & F_{p_3} \arrow[r] \arrow[d, "\phi"'] & M^3_{2^r} \arrow[d, hook] \\
\Omega S^4 \arrow[r, "\Omega(-2^r\iota_4)"] & \Omega S^4 \arrow[r] & J(M^3_{2^r}, S^3),
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{Z}_4\{j_{p_3}\nu'\}\oplus\mathbb{Z}_{(2)}\{j^6_{p_3}\} \arrow[d, equal] \\
\pi_7(S^4) \arrow[r, "\partial_{6*}"] \arrow[dr, "P_1(-2^r\iota_4)_*"'] & \pi_6(F_{p_3}) \arrow[d, "P_1\phi_*"] \\
& \mathbb{Z}_4\{\Sigma\nu'\}
\end{tikzcd} \tag{6}$$

$P_1 : \pi_7(S^4) = \mathbb{Z}_4\{\Sigma\nu'\}\oplus\mathbb{Z}_{(2)}\{\nu_4\} \to \mathbb{Z}_4\{\Sigma\nu'\}$ is the canonical projection.

By comparing the Homology $H_3(-;\mathbb{Z})$, we get

$$\phi j_{p_3} \simeq h\Omega\Sigma : S^3 \to \Omega S^4 = \Omega\Sigma S^3,\ h \text{ is odd integer.} \tag{7}$$

$(-2^r\iota_4)_*(\nu_4) = 2^{2r}\nu_4 - 2^{r-1}(2^r+1)\Sigma\nu'$ by Lemma A.1.

$$\begin{aligned}
&P_1\phi_*\partial_{6*}(\nu_4) = P_1(y\phi_*(j_{p_3}\nu') + 2^r\phi_*(j^6_{p_3})) = hy\Sigma\nu' + 2^r P_1\phi_*(j^6_{p_3}) \text{ (By (5))}\\
&\quad = (hy + 2^r t)\Sigma\nu' \text{ (for some odd integer } t).\\
&P_1(-2^r\iota_4)_*(\nu_4) = P_1(2^{2r}\nu_4 - 2^{r-1}(2^r+1)\Sigma\nu') = -2^{r-1}(2^r+1)\Sigma\nu'.
\end{aligned}$$

From (6), we get $hy\Sigma\nu' + 2^r P_1\phi_*(j^6_{p_3}) = -2^{r-1}(2^r+1)\Sigma\nu'$, thus

$$\mathbb{Z}_4 \ni y = \begin{array}{|c|c|c|}\hline r=1 & r=2 & \infty\geq r\geq 3\\ \hline \pm 1 & 2 & 0\\ \hline\end{array} \tag{8}$$

From (3), (5), (8)

$$Coker\partial_{6*} \cong \frac{\mathbb{Z}_4\oplus\mathbb{Z}_{(2)}}{\langle(2^r,0),(y,2^r)\rangle} \cong \begin{cases} \mathbb{Z}_4, & r=1;\\ \mathbb{Z}_8\oplus\mathbb{Z}_2, & r=2;\\ \mathbb{Z}_4\oplus\mathbb{Z}_{2^r}, & r\geq 3.\end{cases} \tag{9}$$

$$0 \longrightarrow Coker\partial_{6*} \longrightarrow \pi_6(M^3_{2^r}) \xrightarrow{p_{3*}} Ker\partial_{5*} \cong \mathbb{Z}_2\{\eta_4\eta_5\} \longrightarrow 0\ . \tag{10}$$

The above short exact sequence splits for $r=1$, since $\pi_6(M^3_2) \cong \mathbb{Z}_4\oplus\mathbb{Z}_2$ by [19], i.e., there is an element $\varsigma_1 \in \pi_6(M^3_2)$ with order 2 such that $p_{3*}(\varsigma_1) =$

$\eta_4\eta_5$. Hence by the Lemma 2.5 of [16], for $r>1$, there is also an element $\varsigma_r \in \pi_6(M^3_{2^r})$ with order 2 such that $p_{3*}(\varsigma_r)=\eta_4\eta_5$. Thus short exact sequence (10) splits for $r>1$.

$$\text{So } \pi_6(M^3_{2^r}) \cong \begin{cases} \mathbb{Z}_4\oplus\mathbb{Z}_2, & r=1; \\ \mathbb{Z}_8\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2, & r=2; \\ \mathbb{Z}_4\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{2^r}, & r\geq 3. \end{cases}$$

3.2. **Calculating $\pi_7(M^3_{2^r})$.** Consider the following diagram

$$\begin{tikzcd} \pi_8(S^4) \arrow[r, "\partial_{7*}"] & \pi_7(F_{p_3}) \arrow[r] & \pi_7(M^3_{2^r}) \arrow[r, "q_{r*}"] & \pi_7(S^4) \arrow[r, "\partial_{6*}"] & \pi_6(F_{p_3}) \\ \pi_7(S^3) \arrow[u, "\Sigma"] \arrow[r, "(2^r\iota_3)_*"] & \pi_7(S^3) \arrow[u, hook, "\cong j_{p_3*}"'] & & & \end{tikzcd} \tag{11}$$

where the first row is exact sequence, and the left square follows from Lemma 2.4. $\pi_7(F_{p_3})=\mathbb{Z}_2\{j_{p_3}\nu'\eta_6\}\oplus\mathbb{Z}_2\{j^6_{p_3}\eta_6\}$.

From (3), (5), $Ker\partial_{6*}=\begin{cases}\mathbb{Z}_2\{2\Sigma\nu'\} & r=1; \\ \mathbb{Z}_4\{\Sigma\nu'\}, & r\geq 2\ .\end{cases}$

$$\partial_{7*}(\Sigma\nu'\eta_7)=j_{p_3}(2^r\iota_3)\nu'\eta_6=0 \tag{12}$$

Assume $\partial_{7*}(\nu_4\eta_7)=aj_{p_3}\nu'\eta_6+bj^6_{p_3}\eta_6$ with $a,b\in\mathbb{Z}_2$. By Lemma 2.6 and Lemma 2.8, we get the following commutative diagrams

$$\begin{tikzcd} \pi_8(S^4) \arrow[r, "\partial_{7*}"] \arrow[d, "H_2"] & \pi_7(F_{p_3}) \arrow[d, "H_2'"] & \pi_7(S^3\vee S^6) \arrow[l, hook', "\cong"'] \arrow[r, "Proj"] & \pi_7(S^6) \arrow[d, "\Sigma\cong"] \\ \pi_7(\Omega\Sigma S^3\wedge S^3) \arrow[r, "(\Omega\Sigma 2^r\wedge\iota_3)_*"] & \pi_7(\Omega\Sigma S^3\wedge S^3) \arrow[rr, equal] & & \pi_8(S^7) \end{tikzcd}$$

$$0=(\Omega\Sigma 2^r\wedge\iota_3)_*H_2(\nu_4\eta_7)=H_2'\partial_{7*}(\nu_4\eta_7)=H_2'(aj_{p_3}\nu'\eta_6+bj^6_{p_3}\eta_6)=b\eta_7,$$

which implies that $b=0$.

Diagram (6) induces the following commutative diagram

$$\begin{tikzcd} \pi_8(S^4) \arrow[r, "\partial_{7*}"] \arrow[dr, "(-2^r\iota_4)_*"'] & \pi_7(F_{p_3})=\mathbb{Z}_2\{j_{p_3}\nu'\eta_6\}\oplus\mathbb{Z}_2\{j^6_{p_3}\eta_6\} \arrow[d, "\phi_*"] & \\ & \pi_8(S^4)=\mathbb{Z}_2\{\Sigma\nu'\eta_7\}\oplus\mathbb{Z}_2\{\nu_4\eta_7\} \arrow[r, "P_1"] & \mathbb{Z}_2\{\Sigma\nu'\eta_7\} \end{tikzcd}$$

where $P_1:\mathbb{Z}_2\{\Sigma\nu'\eta_7\}\oplus\mathbb{Z}_2\{\nu_4\eta_7\}\to\mathbb{Z}_2\{\Sigma\nu'\eta_7\}$ is the canonical projection.

$$P_1(-2^r\iota_4)_*(\nu_4\eta_7)=P_1((2^{2r}\nu_4-2^{r-1}(2^r+1)\Sigma\nu')\eta_7)=\epsilon_r\Sigma\nu'\eta_7.$$
$$=P_1\phi_*\partial_{7*}(\nu_4\eta_7)=P_1\phi_*(aj_{p_3}\nu'\eta_6)=a\Sigma\nu'\eta_7,$$

Hence $a=\epsilon_r, \infty\geq r\geq 1$. Thus

$$\partial_{7*}(\nu_4\eta_7)=\epsilon_r j_{p_3}\nu'\eta_6, \tag{13}$$

The diagram (3) of [16] induces the following commutative diagram for $r>1$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2\{j_{p_3}\nu'\eta_6\}\oplus\mathbb{Z}_2\{j^6_{p_3}\eta_6\} \arrow[r] & \pi_7(M^3_{2^r}) \arrow[r] & \mathbb{Z}_4\{\Sigma\nu'\} \arrow[r] & 0\,. \\ 0 \arrow[r] & \mathbb{Z}_2\{j^6_{p_3}\eta_6\} \arrow[r] \arrow[u] & \pi_7(M^3_2) \arrow[r] \arrow[u, "c_{4*}"'] & \mathbb{Z}_2\{2\Sigma\nu'\} \arrow[r] \arrow[u, hook, "\times 2"'] & 0 \end{tikzcd} \tag{14}$$

By [19], $\pi_7(M^3_2)\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$, hence from Lemma 2.10, the top short exact sequence splits. Thus $\pi_7(M^3_{2^r})\cong\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus(1-\epsilon_r)\mathbb{Z}_4$ for $\infty\geq r\geq 1$.

3.3. **Calculating $\pi_7(M^4_{2^r})$.** For $M^4_{2^r}$, by (iii) of Lemma 2.2, the 8-skeleton $Sk_8(F_{p_4})\simeq S^4\bigcup_{\gamma=[\iota_4,2^r\iota_4]}C(S^4\wedge S^3)=S^4\bigcup_{2^r[\iota_4,\iota_4]}CS^7$. Then the cofibration sequence $S^7\xrightarrow{\gamma}S^4\overset{j_{p_4}}{\hookrightarrow}F_{p_4}\to S^8$ induces the following exact sequence by Lemma 2.9:

$$\mathbb{Z}_{(2)}\{\iota_7\}=\pi_7(S^7)\xrightarrow{\gamma_*}\pi_7(S^4)=\mathbb{Z}_4\{\Sigma\nu'\}\oplus\mathbb{Z}_{(2)}\{\nu_4\}\xrightarrow{j_{p_4*}}\pi_7(F_{p_4})\to 0$$

with $\gamma_*(\iota_7)=(2^r[\iota_4,\iota_4])\iota_7=2^{r+1}\nu_4-2^r\Sigma\nu'$.

Consider the following exact sequence with commutative squares

$$\begin{tikzcd} \pi_8(S^5) \arrow[r, "\partial_{7*}"] & \pi_7(F_{p_4}) \arrow[r] & \pi_7(M^3_{2^r}) \arrow[r] & \pi_7(S^5) \arrow[r, "\partial_{6*}"] & \pi_6(F_{p_4}) \\ \pi_7(S^4) \arrow[r, "(2^r\iota_4)_*"] \arrow[u, "\Sigma"'] & \pi_7(S^4) \arrow[u, "j_{p_4*}"'] & & \pi_6(S^4) \arrow[r, "(2^r\iota_4)_*=0"] \arrow[u, "\cong\Sigma"'] & \pi_6(S^4) \arrow[u, hook, "j_{p_4*}"'] \end{tikzcd} \tag{15}$$

The right commutative square in (15) implies $Ker\partial_6*\cong\mathbb{Z}_2$.

By the left commutative square in (15), we get

$$\partial_{7*}(\nu_5)=\partial_{7*}(\Sigma\nu_4)=j_{p_4*}[(2^r\iota_4)\nu_4]=2^{2r}j_{p_4}\nu_4-2^{r-1}(2^r-1)j_{p_4}\Sigma\nu'. \tag{16}$$

$$\begin{aligned} Coker\partial_{7*} &= \frac{\mathbb{Z}_4\{j_{p_4}\Sigma\nu'\}\oplus\mathbb{Z}_{(2)}\{j_{p_4}\nu_4\}}{\langle 2^{r+1}j_{p_4}\nu_4-2^rj_{p_4}\Sigma\nu',2^{2r}j_{p_4}\nu_4-2^{r-1}(2^r-1)j_{p_4}\Sigma\nu'\rangle} \\ &\cong \frac{\mathbb{Z}_{(2)}\{a,b\}}{\langle 2^{r+1}b-2^ra,2^{2r}b-2^{r-1}(2^r-1)a,4a\rangle}=\frac{\mathbb{Z}_{(2)}\{a,b\}}{\left\langle 2^{r+1}b,2^{min\{r-1,2\}}a\right\rangle}. \\ &= \mathbb{Z}_{2^{min\{2,r-1\}}}\{a\}\oplus\mathbb{Z}_{2^{r+1}}\{b\}. \end{aligned}$$

Since $\pi_7(M^4_2)\cong\mathbb{Z}_2\oplus\mathbb{Z}_4$ from Theorem 5.10.of [19], the same argument in dealing with diagram (14) implies that

$$\pi_7(M^4_{2^r})\cong Coker\partial_{7*}\oplus Ker\partial_{6*}\cong\mathbb{Z}_{2^{min\{2,r-1\}}}\oplus\mathbb{Z}_{2^{r+1}}\oplus\mathbb{Z}_2, r\geq 1. \tag{17}$$

4. Elementary Chang-complexes

In this section, we calculate the 6,7-dimensional unstable homotopy groups of elementary Chang-complexes in $\mathbf{A}^2_3$, i.e. $\pi_n(C^5_r)$, $\pi_n(C^{5,s})$, $\pi_n(C^{5,s}_r)$ for $n=6,7$. Note that $\pi_6(C^5_\eta)=\mathbb{Z}_6$ and $\pi_7(C^5_\eta)=\mathbb{Z}$ are given by Proposition 8.2 of [15].

In the first Section we denote j^{n+1}_1, resp. j^n_2 as the canonical inclusion of S^{n+1}, resp. S^n, into $S^{n+1}\vee S^n$. In the following, for the special case $n=3$, we simplify the notion $j_1=j^4_1$ and $j_2=j^3_2$.

4.1. **Fibration sequence and cofibration sequence.** In order to calculate homotopy groups in a unified and efficient way, we denote the space $C^{5,s}_{r,\epsilon}$ which is a mapping cone of $f^s_{r,\epsilon}$, i.e., there is a cofibration sequence

$$S^4\vee S^3 \xrightarrow{f^s_{r,\epsilon}} S^4\vee S^3 \xrightarrow{\lambda^s_{r,\epsilon}} C^{5,s}_{r,\epsilon} \xrightarrow{q^s_{r,\epsilon}} S^5\vee S^4 \tag{18}$$

where $f^s_{r,\epsilon}=(j_1(2^s\iota_4)+\epsilon j_2\eta_3, j_2(2^r\iota_3))$, $\epsilon=1$ or 0; $\infty\geq r,s>0$.

Then $C^{5,\infty}_{r,1}=C^5_r\vee S^4$; $C^{5,s}_{\infty,1}=C^{5,s}\vee S^4$; $C^{5,s}_{r,1}=C^{5,s}_r$; $C^{5,s}_{\infty,0}=M^4_{2^s}\vee S^3\vee S^4$.

Note that $f^s_{r,\epsilon}j_1=j_1(2^s\iota_4)+\epsilon j_2\eta_3$; $f^s_{r,\epsilon}j_2=j_2(2^r\iota_3)$.

Let $\Omega(S^5\vee S^4)\xrightarrow{\partial^s_{r,\epsilon}} F^s_{r,\epsilon}\to C^{5,s}_{r,\epsilon}\xrightarrow{q^s_{r,\epsilon}} S^5\vee S^4$ be the homotpy fibration sequence, where $F^s_{r,\epsilon}\simeq J(M_{f^s_{r,\epsilon}}, S^4\vee S^3)$. From Lemma 2.2, $\Sigma F^s_{r,\epsilon}\simeq S^4\vee S^5\vee S^8\vee S^8\vee S^9\vee A^s_{r,\epsilon}$ where $A^s_{r,\epsilon}$ is a wedge of spheres with dimension ≥ 10.

$$Sk_8(F^s_{r,\epsilon})\simeq J_2(M_{f^s_{r,\epsilon}}, S^4\vee S^3)=(S^4\vee S^3)\cup_{\gamma^s_{r,\epsilon}} C((S^4\vee S^3)\wedge(S^3\vee S^2))$$

where $\gamma^s_{r,\epsilon}=[id_{S^4\vee S^3}, f^s_{r,\epsilon}]$. Let

$$\gamma^s_{r,\epsilon}|_{S^3\wedge S^3}: S^3\wedge S^3\xrightarrow{j_2\wedge j^3_1}(S^4\vee S^3)\wedge(S^3\vee S^2)\xrightarrow{\gamma^s_{r,\epsilon}} S^4\vee S^3;$$

$$\begin{aligned}
\gamma^s_{r,\epsilon}|_{S^3\wedge S^3}&=\gamma^s_{r,\epsilon}(j_2\wedge j^3_1)=[id_{S^4\vee S^3}, f^s_{r,\epsilon}](j_2\wedge j^3_1)=[id_{S^4\vee S^3}j_2, f^s_{r,\epsilon}j_1]\\
&=[j_2, j_1(2^s\iota_4)+\epsilon j_2\eta_3]=[j_2, j_1(2^s\iota_4)]+\epsilon[j_2,j_2\eta_3]=2^s[j_1,j_2].
\end{aligned}$$

Similary,

$$\begin{aligned}
\gamma^s_{r,\epsilon}|_{S^4\wedge S^2}&=\gamma^s_{r,\epsilon}(j_1\wedge j^2_2)=[id_{S^4\vee S^3}, f^s_{r,\epsilon}](j_1\wedge j^2_2);\\
&=[id_{S^4\vee S^3}j_1, f^s_{r,\epsilon}j_2]=[j_1, j_2(2^r\iota_3)]=2^r[j_1,j_2];\\
\gamma^s_{r,\epsilon}|_{S^3\wedge S^2}&=\gamma^s_{r,\epsilon}(j_2\wedge j^2_2)=[id_{S^4\vee S^3}j_2, f^s_{r,\epsilon}j_2]=[j_2, j_2(2^r\iota_3)]=2^r[j_2,j_2]=0\\
\gamma^s_{r,\epsilon}|_{S^4\wedge S^3}&=\gamma^s_{r,\epsilon}(j_1\wedge j^3_1)=[id_{S^4\vee S^3}, f^s_{r,\epsilon}](j_1\wedge j^3_1)=[id_{S^4\vee S^3}j_1, f^s_{r,\epsilon}j_1]\\
&=[j_1, j_1(2^s\iota_4)+\epsilon j_2\eta_3]=[j_1, j_1(2^s\iota_4)]+[j_1,\epsilon j_2\eta_3]\\
&=2^{s+1}j_1\nu_4-2^sj_1\Sigma\nu'+\epsilon[j_1,j_2]\eta_6.
\end{aligned}$$

where $[j_1, j_1(2^s\iota_4)]=2^s[j_1,j_1]=2^sj_1[\iota_4,\iota_4]=2^sj_1(2\nu_4-\Sigma\nu')=2^{s+1}j_1\nu_4-2^sj_1\Sigma\nu'$; $[j_1,j_2\eta_3]=[j_1\Sigma\iota_3, j_2\Sigma\eta_2]=[j_1,j_2]\Sigma\iota_3\wedge\eta_2=[j_1,j_2]\eta_6$; $[j_2,j_1]=(-1)^{(3+1)(4+1)}[j_1,j_2]$ and $[j_2,j_2]=0$ by the injection $\pi_5(S^4\vee S^3)\xrightarrow{\Sigma}\pi_6(S^5\vee S^4)$. Hence there is a cofibration sequence

$$S^5\vee S^6\vee S^6\vee S^7\xrightarrow{\gamma^s_{r,\epsilon}} S^4\vee S^3\xrightarrow{j^s_{r,\epsilon}} Sk_8F^s_{r,\epsilon}\xrightarrow{p^s_{r,\epsilon}} S^6\vee S^7\vee S^7\vee S^8 \tag{19}$$

$$\begin{aligned}
Sk_8(F^s_{r,\epsilon})&\simeq(S^4\vee S^3)\cup_{\gamma^s_{r,\epsilon}=(0,2^s[j_1,j_2],2^r[j_1,j_2],\gamma^s_{r,\epsilon}|_{S^4\wedge S^3})} C(S^5\vee S^6\vee S^6\vee S^7)\\
&\simeq(S^4\vee S^3)\cup_{(\gamma^s_{r,\epsilon}|_{S^4\wedge S^3},2^s[j_1,j_2],2^r[j_1,j_2])} C(S^6\vee S^6\vee S^7)\bigvee S^6.
\end{aligned}$$

Let $j_{S^6}: S^6\to Sk_8(F^s_{r,\epsilon})$ be the canonical inclusion of the wedge summand S^6 of $Sk_8(F^s_{r,\epsilon})$. Simplify the notation $j^s_{r,\epsilon}:=j_{q^s_{r,\epsilon}}$: $S^4\vee S^3\hookrightarrow Sk_8(F^s_{r,\epsilon})$ or $S^4\vee S^3\hookrightarrow F^s_{r,\epsilon}$.

4.2. **Calculating $\pi_6(C_r^5)$, $\pi_6(C^{5,s})$ and $\pi_6(C_r^{5,s})$.** In the following, r and s cannot be equal to ∞ at the same time, unless otherwise stated..

From Lemma 2.4, we get the exact sequence with two commutative squares

$$\begin{tikzcd}
\pi_7(S^5\vee S^4) \arrow[r, "(\partial^s_{r,\epsilon})_{6*}"] & \pi_6(F^s_{r,\epsilon}) \arrow[r] & \pi_6(C^{5,s}_{r,\epsilon}) \arrow[r, "q^s_{r,\epsilon*}"] & \pi_6(S^5\vee S^4) \arrow[r, "(\partial^s_{r,\epsilon})_{5*}"] & \pi_5(F^s_{r,\epsilon}) \\
\pi_6(S^4\vee S^3) \arrow[u, "\Sigma"] \arrow[r, "f^s_{r,\epsilon*}"] & \pi_6(S^4\vee S^3) \arrow[u, "j^s_{r,\epsilon*}"] & & \pi_5(S^4\vee S^3) \arrow[u, "\cong\Sigma"] \arrow[r, "f^s_{r,\epsilon*}"] & \pi_5(S^4\vee S^3) \arrow[u, "\cong\ j^s_{r,\epsilon*}"]
\end{tikzcd} \tag{20}$$

$$\begin{aligned}
\pi_5(S^4\vee S^3) &= \mathbb{Z}_2\{j_1\eta_4\}\oplus\mathbb{Z}_2\{j_2\eta_3\eta_4\};\\
\pi_6(S^5\vee S^4) &= \mathbb{Z}_2\{\Sigma j_1\eta_5\}\oplus\mathbb{Z}_2\{\Sigma j_2\eta_4\eta_5\};\\
\pi_6(S^4\vee S^3) &= \mathbb{Z}_2\{j_1\eta_4\eta_5\}\oplus\mathbb{Z}_4\{j_2\nu'\}\oplus\mathbb{Z}_{(2)}\{[j_1,j_2]\};\\
\pi_7(S^5\vee S^4) &= \mathbb{Z}_2\{j_1^5\eta_5\eta_6\}\oplus\mathbb{Z}_4\{j_2^4\Sigma\nu'\}\oplus\mathbb{Z}_{(2)}\{j_2^4\nu_4\}.
\end{aligned}$$

By the right commutative square in (20)
$(\partial^s_{r,\epsilon})_{5*}(j_1^5\eta_5) = j^s_{r,\epsilon}f^s_{r,\epsilon}(j_1\eta_4) = j^s_{r,\epsilon}(j_1(2^s\iota_4)+\epsilon j_2\eta_3)\eta_4 = \epsilon j^s_{r,\epsilon}j_2\eta_3\eta_4$.
$(\partial^s_{r,\epsilon})_{5*}(j_2^4\eta_4\eta_5) = j^s_{r,\epsilon}f^s_{r,\epsilon}(j_2\eta_3\eta_4) = j^s_{r,\epsilon}j_2(2^r\iota_3)\eta_3\eta_4 = 0$.

Thus $Ker(\partial^s_{r,\epsilon})_{5*} = \mathbb{Z}_2\{j_2^4\eta_4\eta_5\}$ for $\epsilon = 1$.

Lemma 4.1. $\pi_6(F^s_{r,\epsilon}) = \mathbb{Z}_2\{j^s_{r,\epsilon}j_1\eta_4\eta_5\}\oplus\mathbb{Z}_4\{j^s_{r,\epsilon}j_2\nu'\}\oplus\mathbb{Z}_{2^{min\{r,s\}}}\{j^s_{r,\epsilon}[j_1,j_2]\}\oplus\mathbb{Z}_{(2)}\{j_{S^6}\iota_6\}$

Proof. From Lemma 2.9 and the section $j_{S^6}: S^6\to Sk_8(F^s_{r,\epsilon})$, the cofibration sequence (19) induces the following exact sequence

$$\begin{tikzcd}
\pi_6(S^5\vee S^6\vee S^6\vee S^7) \arrow[r, "\gamma^s_{r,\epsilon*}"] & \pi_6(S^4\vee S^3) \arrow[r, "j^s_{r,\epsilon}"] & \pi_6(F^s_{r,\epsilon}) \arrow[r, two heads] & \pi_6(S^6) = \mathbb{Z}_{(2)}\{\iota_6\} \arrow[l, bend left, "j_{S^6*}"]
\end{tikzcd}$$

where $\pi_6(S^5\vee S^6\vee S^6\vee S^7) = \mathbb{Z}_2\{j_1'\eta_5\}\oplus\mathbb{Z}_{(2)}\{j_2'\iota_6\}\oplus\mathbb{Z}_{(2)}\{j_3'\iota_6\}$, j_k' is the canonical inclusion of the k-th wedge summand of $S^5\vee S^6\vee S^6\vee S^7$; It is easy to get $\gamma^s_{r,\epsilon*}(j_3'\eta_5) = 0$;$\gamma^s_{r,\epsilon*}(j_2'\iota_6) = 2^s[j_1,j_2]$; $\gamma^s_{r,\epsilon*}(j_3'\iota_6) = 2^r[j_1,j_2]$.
Hence one gets $\pi_6(F^s_{r,\epsilon})$ by calculating $Coker\gamma^s_{r,\epsilon*}$. □

Lemma 4.2.

$$Coker(\partial^s_{r,1})_{6*}\cong\mathbb{Z}_2\oplus(1-\epsilon_r)\mathbb{Z}_2\oplus\mathbb{Z}_{2^{min\{r,s\}}}\oplus\mathbb{Z}_{2^{r+\epsilon_r}}, \infty\geq r\geq 1.$$

Proof. By the right commutative square in (20)

$$\begin{aligned}
&(\partial^s_{r,\epsilon})_{6*}(j_1^5\eta_5\eta_6) = (\partial^s_{r,\epsilon})_{6*}\Sigma(j_1\eta_4\eta_5) = j^s_{r,\epsilon}f^s_{r,\epsilon*}(j_1\eta_4\eta_5) = j^s_{r,\epsilon}(f^s_{r,\epsilon}j_1)\eta_4\eta_5\\
&= j^s_{r,\epsilon}(j_1(2^s\iota_4)+\epsilon j_2\eta_3)\eta_4\eta_5 = 2\epsilon j^s_{r,\epsilon}j_2\nu' \qquad (21)\\
&(\partial^s_{r,\epsilon})_{6*}(j_2^4\Sigma\nu') = (\partial^s_{r,\epsilon})_{6*}\Sigma(j_2\nu') = j^s_{r,\epsilon*}f^s_{r,\epsilon*}(j_2\nu') = j^s_{r,\epsilon}(f^s_{r,\epsilon}j_2)\nu'\\
&= j^s_{r,\epsilon}(j_2(2^r\iota_3))\nu' = 2^r j^s_{r,\epsilon}j_2\nu'. \qquad (22)
\end{aligned}$$

There is a map $M^3_{2^r}\xrightarrow{\bar{\theta}} C^{5,s}_{r,\epsilon}$ making the following left ladder homotopy commutative and it induces the following right homotopy commutative ladder

$$\begin{tikzcd} S^3 \arrow[r, "2^r\iota_3"] \arrow[d, "j_2"] & S^3 \arrow[r, "i_3"] \arrow[d, "j_2"] & M^3_{2^r} \arrow[r, "p_3"] \arrow[d, "\bar\theta"] & S^4 \arrow[d, "j_2^4"] & \Omega S^4 \arrow[r, "\partial"] \arrow[d, "\Omega j_2^4"] & F_{p_3} \arrow[r, "i_3"] \arrow[d, "\theta"] & M^3_{2^r} \arrow[d, "\bar\theta"] \\ S^4\vee S^3 \arrow[r, "f^s_{r,\epsilon}"] & S^4\vee S^3 \arrow[r] & C^{5,s}_{r,\epsilon} \arrow[r, "q^s_{r,\epsilon}"] & S^5\vee S^4, & \Omega(S^5\vee S^4) \arrow[r, "\partial^s_{r,\epsilon}"] & F^s_{r,\epsilon} \arrow[r] & C^{5,s}_{r,\epsilon}, \end{tikzcd} \tag{23}$$

where $\theta j_{p_3}\simeq j^s_{r,\epsilon}j_2$, i.e., $\begin{tikzcd} S^3 \arrow[r, hook, "j_{p_3}"] \arrow[rr, bend right, "j^s_{r,\epsilon}j_2"'] & F_{p_3} \arrow[r, "\theta"] & F^s_{r,\epsilon} \end{tikzcd}$ by Lemma 2.3. So we get the following commutative ladder

$$\begin{tikzcd} \pi_7(S^4) \arrow[r, "\partial_{6*}"] \arrow[d, "j^4_{2*}"] & \pi_6(F_{p_3}) \arrow[r, "Proj."] \arrow[d, "\theta_*"] & \pi_6(S^3\wedge S^3)\cong\pi_6(S^6) \arrow[d, "(j_2\wedge j_2)_*=id"] \\ \pi_7(S^5\vee S^4) \arrow[r, "(\partial^s_{r,\epsilon})_{6*}"] & \pi_6(F^s_{r,\epsilon}) \arrow[r, "p^s_{r,\epsilon*}"] & \pi_6((S^4\vee S^3)\wedge(S^4\vee S^3))\cong\pi_6(S^6) \end{tikzcd} \tag{24}$$

$$(\partial^s_{r,\epsilon})_{6*}(j_2^4\nu_4)=\theta_*\partial_{6*}(\nu_4)=\theta_*(yj_{p_3}\nu'+2^rj^6_{p_3})=(y+2^rm)j^s_{r,\epsilon}j_2\nu'\pm2^rj_{S^6}\iota_6 \tag{25}$$

where y comes from (8) and m comes from the assumption

$$\theta_*(j^6_{p_3})=\widehat{l}j^s_{r,\epsilon}j_1\eta_4\eta_5+mj^s_{r,\epsilon}j_2\nu'+\widehat{u}j^s_{r,\epsilon}[j_1,j_2]\pm j_{S^6}\iota_6, \tag{26}$$

for some $\widehat{l}\in\mathbb{Z}_2$, $m\in\mathbb{Z}_4$, $\widehat{u}\in\mathbb{Z}_{2^{min\{r,s\}}}$.

From (21),(22) and (25) , for $\epsilon=1$, we get

$$\begin{aligned} &Coker(\partial^s_{r,1})_{6*}\\ =&\frac{\mathbb{Z}_2\{j^s_{r,\epsilon}j_1\eta_4\eta_5\}\oplus\mathbb{Z}_4\{j^s_{r,\epsilon}j_2\nu'\}\oplus\mathbb{Z}_{2^{min\{r,s\}}}\{j^s_{r,\epsilon}[j_1,j_2]\}\oplus\mathbb{Z}_{(2)}\{j_{S^6}\iota_6\}}{\langle 2j^s_{r,\epsilon}j_2\nu',2^rj^s_{r,\epsilon}j_2\nu',(y+2^rm)j^s_{r,\epsilon}j_2\nu'\pm2^rj_{S^6}\iota_6\rangle}\\ \cong&\frac{\mathbb{Z}\{a,b,c,d\}}{\langle 2a,2b,2^{min\{r,s\}}c,yb\pm2^rd\rangle}\cong\mathbb{Z}_2\oplus(1-\epsilon_r)\mathbb{Z}_2\oplus\mathbb{Z}_{2^{min\{r,s\}}}\oplus\mathbb{Z}_{2^{r+\epsilon_r}},\infty\geq r\geq1. \end{aligned}$$

□

As the proof of the split of (10), there is also an element $\bar\theta\varsigma_r\in\pi_6(C^{5,s}_{r,1})$ with order 2 such that $q^s_{r,1*}(\varsigma_r)=j_2^4\eta_4\eta_5$. Hence the short exact sequence

$Coker(\partial^s_{r,1})_{6*}\hookrightarrow\pi_6(C^{5,s}_{r,1})\xrightarrow{q^s_{r,1*}}Ker(\partial^s_{r,1})_{5*}=\mathbb{Z}_2\{j_2^4\eta_4\eta_5\}$ splits.

So, $\pi_6(C^{5,s}_{r,1})\cong\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus(1-\epsilon_r)\mathbb{Z}_2\oplus\mathbb{Z}_{2^{min\{r,s\}}}\oplus\mathbb{Z}_{2^{r+\epsilon_r}},\infty\geq r\geq1$.

$\pi_6(C^{5,\infty}_{r,1})=\pi_6(C^5_r\vee S^4)\cong\pi_6(C^5_r)\oplus\pi_6(S^4)\oplus\pi_6(C^8_r)\cong\pi_6(C^5_r)\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{2^{r+\epsilon_r}}$, which implies $\pi_6(C^5_r)\cong\mathbb{Z}_2\oplus(1-\epsilon_r)\mathbb{Z}_2\oplus\mathbb{Z}_{2^{r+\epsilon_r}},r\geq1$.

$\pi_6(C^{5,s}_{\infty,1})=\pi_6(C^{5,s}\vee S^4)\cong\pi_6(C^{5,s})\oplus\pi_6(S^4)\oplus\pi_6(C^{8,s})\cong\pi_6(C^{5,s})\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{(2)}$, which implies $\pi_6(C^{5,s})\cong\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{2^s}$, $s\geq1$.

4.3. **Calculating $\pi_7(C_r^5)$, $\pi_7(C^{5,s})$ and $\pi_7(C_r^{5,s})$.** From Lemma 2.4, we get the exact sequence with two commutative squares

$$\begin{tikzcd}
\pi_8(S^5\vee S^4) \arrow[r, "(\partial^s_{r,\epsilon})_{7*}"] & \pi_7(F^s_{r,\epsilon}) \arrow[r] & \pi_7(C^{5,s}_{r,\epsilon}) \arrow[r, "q^s_{r,\epsilon*}"] & \pi_7(S^5\vee S^4) \arrow[r, "(\partial^s_{r,\epsilon})_{6*}"] & \pi_6(F^s_{r,\epsilon}) \\
\pi_7(S^4\vee S^3) \arrow[u, "\Sigma"] \arrow[r, "f^s_{r,\epsilon*}"] & \pi_7(S^4\vee S^3) \arrow[u, "j^s_{r,\epsilon*}"] & & &
\end{tikzcd} \tag{27}$$

$$\pi_7(S^4\vee S^3)=\mathbb{Z}_{(2)}\{j_1\nu_4\}\oplus\mathbb{Z}_4\{j_1\Sigma\nu'\}\oplus\mathbb{Z}_2\{j_2\nu'\eta_6\}\oplus\mathbb{Z}_2\{[j_1,j_2]\eta_6\},$$
$$\pi_8(S^5\vee S^4)=\mathbb{Z}_8\{j_1^5\nu_5\}\oplus\mathbb{Z}_2\{j_2^4\Sigma\nu'\eta_7\}\oplus\mathbb{Z}_2\{j_2^4\nu_4\eta_7\}\oplus\mathbb{Z}_{(2)}\{[j_1^5,j_2^4]\}.$$

From (8), (21), (22), (25), we get

$$Ker(\partial^s_{r,\epsilon})_{6*}=\begin{cases}\mathbb{Z}_4\{\epsilon_r j_1^5\eta_5\eta_6+j_2^4\Sigma\nu'\}, & r\geq 1,\epsilon=1;\\ \mathbb{Z}_{(2)}\{j_2^4\nu_4\}\oplus\mathbb{Z}_4\{j_2^4\Sigma\nu'\}, & r=\infty,\epsilon=1;\\ \pi_7(S^5\vee S^4), & r=\infty,\epsilon=0.\end{cases} \tag{28}$$

In the following we also allow $(r,s,\epsilon)=(0,0,0)$. Then $C^{5,0}_{0,0}\simeq(S^4\vee S^3)\bigcup_{id}C(S^4\vee S^3)\simeq *$ and all the results in Section 4.1 hold.

Lemma 4.3. *$\pi_7(F^s_{r,\epsilon})=\mathbb{Z}_2\{j_{S^6}\eta_6\}\oplus\mathbb{Z}_{(2)}\{\widetilde{\rho}^s_{r,\epsilon}\}\oplus Coker(\gamma^s_{r,\epsilon})_{7*}$, where $\widetilde{\rho}^s_{r,\epsilon}\in\pi_7(F^s_{r,\epsilon})$ is a lift of $\rho^s_{r,\epsilon}$ in (29) and $Coker(\gamma^s_{r,\epsilon})_{7*}$ is given by (30).*

Proof. There is a fibration sequence for the map $p^s_{r,\epsilon}$ in (19)

$$\Omega(S^6\vee S^7\vee S^7\vee S^8)\xrightarrow{\widehat{\partial^s_{r,\epsilon}}}F_{p^s_{r,\epsilon}}\to Sk_8F^s_{r,\epsilon}\xrightarrow{p^s_{r,\epsilon}}S^6\vee S^7\vee S^7\vee S^8.$$

where $F_{p^s_{r,\epsilon}}\simeq J(M_{\gamma^s_{r,\epsilon}},S^5\vee S^6\vee S^6\vee S^7)$, with $Sk_8F_{p^s_{r,\epsilon}}\simeq S^4\vee S^3\vee S^8$. From Lemma 2.4 and Lemma 2.9, there is an exact sequence

$$\pi_7(X)\xrightarrow{(\gamma^s_{r,\epsilon})_{7*}}\pi_7(S^4\vee S^3)\to\pi_7(F^s_{r,\epsilon})\xrightarrow{p^s_{r,\epsilon*}}\pi_7(\Sigma X)\cong\pi_6(X)\xrightarrow{(\gamma^s_{r,\epsilon})_{6*}}\pi_6(S^4\vee S^3).$$

where $X=S^5\vee S^6\vee S^6\vee S^7$ and $\pi_6(X)=\mathbb{Z}_2\{j_1'\eta_5\}\oplus\mathbb{Z}_{(2)}\{j_2'\iota_6\}\oplus\mathbb{Z}_{(2)}\{j_3'\iota_6\}$. Let $\iota_7^{34}:=\Sigma j_2'\iota_7$ and $\iota_7^{43}:=\Sigma j_3'\iota_7$.

$$Ker(\gamma^s_{r,\epsilon})_{6*}\cong\mathbb{Z}_{(2)}\{\rho^s_{r,\epsilon}\}\oplus\mathbb{Z}_2\{\Sigma j_1'\eta_6\},$$

$\rho^s_{r,\epsilon}=$ (29)

$\infty\geq s>r>0$	$\infty\geq r>s>0$	$r=s$
$2^{s-r}\iota_7^{43}-\iota_7^{34}$	$\iota_7^{43}-2^{r-s}\iota_7^{34}$	$\iota_7^{43}-\iota_7^{34}$

$$Coker(\gamma^s_{r,\epsilon})_{7*}=\frac{\mathbb{Z}_{(2)}\{j_1\nu_4\}\oplus\mathbb{Z}_4\{j_1\Sigma\nu'\}\oplus\mathbb{Z}_2\{j_2\nu'\eta_6\}\oplus\mathbb{Z}_2\{[j_1,j_2]\eta_6\}}{\langle 2^{s+1}j_1\nu_4-2^sj_1\Sigma\nu'+\epsilon[j_1,j_2]\eta_6\rangle}. \tag{30}$$

$$0\to Coker(\gamma^s_{r,\epsilon})_{7*}\xrightarrow{j^s_{r,\epsilon,*}}\pi_7(F^s_{r,\epsilon})\xrightarrow{p^s_{r,\epsilon*}}\mathbb{Z}_{(2)}\{\rho^s_{r,\epsilon}\}\oplus\mathbb{Z}_2\{\Sigma j_1'\eta_6\}\to 0.$$

Above exact sequence splits since in (19), the wedge summand S^6 of $S^6\vee S^7\vee S^7\vee S^8$ has a section $j_{S^6}:S^6\to Sk_8(F^s_{r,\epsilon})$. Thus we complete the proof of this lemma. ∎

From the commutative square in (27)

$$\begin{aligned}
&(\partial^s_{r,\epsilon})_{7*}(j_1^5\nu_5) = (\partial^s_{r,\epsilon})_{7*}\Sigma(j_1\nu_4) = j^s_{r,\epsilon,*}f^s_{r,\epsilon*}(j_1\nu_4) = j^s_{r,\epsilon}(f^s_{r,\epsilon}j_1)\nu_4\\
&= j^s_{r,\epsilon}(j_1(2^s\iota_4)+\epsilon j_2\eta_3)\nu_4 = j^s_{r,\epsilon}(j_1(2^s\iota_4)\nu_4+\epsilon j_2\eta_3\nu_4+[j_1(2^s\iota_4),\epsilon j_2\eta_3]H(\nu_4))\\
&= 2^{2s}j^s_{r,\epsilon}j_1\nu_4 - 2^{s-1}(2^s-1)j^s_{r,\epsilon}j_1\Sigma\nu' + \epsilon j^s_{r,\epsilon}j_2\nu'\eta_6 \qquad (31)\\
&(\partial^s_{r,\epsilon})_{7*}(j_2^4\Sigma\nu'\eta_7) = (\partial^s_{r,\epsilon})_{7*}\Sigma(j_2\nu'\eta_6) = j^s_{r,\epsilon}f^s_{r,\epsilon}(j_2\nu'\eta_6) = j^s_{r,\epsilon}(f^s_{r,\epsilon}j_2)\nu'\eta_6\\
&= j^s_{r,\epsilon}j_2(2^r\iota_3)\nu'\eta_6 = 0.\ (\text{Note } \eta_3\nu_4=\nu'\eta_6.) \qquad (32)
\end{aligned}$$

From (13) and the commutative diagram (23)

$$(\partial^s_{r,\epsilon})_{7*}(j_2^4\nu_4\eta_7) = \theta_*\partial_{7*}(\nu_4\eta_7) = \theta_*(\epsilon_r j_{p_3}\nu'\eta_6) = \epsilon_r\theta j_{p_3}\nu'\eta_6 = \epsilon_r j^s_{r,\epsilon}j_2\nu'\eta_6. \qquad (33)$$

It remains to compute $(\partial^s_{r,1})_{7,*}([j_1^5,j_2^4])$ the determination of which requires the computation of $(\partial^s_{\infty,0})_{7,*}([j_1^5,j_2^4])$.

Since $Coker(\gamma^s_{r,1})_{7*} = \mathbb{Z}_{2^{s+2}}\{j^s_{r,\epsilon}j_1\nu_4\}\oplus\mathbb{Z}_4\{j^s_{r,\epsilon}j_1\Sigma\nu'\}\oplus\mathbb{Z}_2\{j^s_{r,\epsilon}j_2\nu'\eta_6\}$ in (30), suppose

$$(\partial^s_{r,1})_{7*}([j_1^5,j_2^4]) = x\widetilde{\rho}^s_{r,1} + yj_{S^6}\eta_6 + kj^s_{r,1}j_1\nu_4 + lj^s_{r,1}j_1\Sigma\nu' + uj^s_{r,1}j_2\nu'\eta_6 \qquad (34)$$

where $y,u\in\mathbb{Z}_2$; $l\in\mathbb{Z}_4$; $k\in\mathbb{Z}_{2^{s+2}}$; $x\in\mathbb{Z}$ are to be determined.

By simplifying the $j^s_{\infty,0}: S^4\vee S^3\hookrightarrow Sk_8(F^s_{\infty,0})$ by j^s_0, we also suppose

$$(\partial^s_{\infty,0})_{7*}([j_1^5,j_2^4]) = t'\widetilde{\iota_7}^{43} + y'j_{S^6}\eta_6 + vj^s_0j_1\nu_4 + wj^s_0j_1\Sigma\nu' + u'j^s_0j_2\nu'\eta_6 + zj^s_0[j_1,j_2]\eta_6, \qquad (35)$$

where $t',v\in\mathbb{Z}$, $y',u',z\in\mathbb{Z}_2$, $w\in\mathbb{Z}_4$.

The determination of the first two coefficients in (34) and (34) can be done simultaneously in the following Lemma.

Lemma 4.4. *In (34), (35), $y=1, y'=0\in\mathbb{Z}_2$; $x=2^{min\{r,s\}}t$; $t'=2^st$, t is odd.*

Proof. There is a map $C^{5,0}_{0,0}\xrightarrow{\bar{\theta_0}} C^{5,s}_{r,\epsilon}$ making the following left ladder homotopy commutative and it induces the following right homotopy commutative ladder

$$\begin{tikzcd}
S^4\vee S^3 \arrow[r, "id"] \arrow[d, equal, "id"'] & S^4\vee S^3 \arrow[r] \arrow[d, "f^s_{r,\epsilon}"'] & C^{5,0}_{0,0} \arrow[r, "q^0_{0,0}"] \arrow[d, "\bar{\theta_0}"] & S^5\vee S^4 \arrow[d, equal, "id"'] \\
S^4\vee S^3 \arrow[r, "f^s_{r,\epsilon}"] & S^4\vee S^3 \arrow[r] & C^{5,s}_{r,\epsilon} \arrow[r, "q^s_{r,\epsilon}"] & S^5\vee S^4,
\end{tikzcd}
\quad
\begin{tikzcd}
\Omega(S^5\vee S^4) \arrow[r, "\partial^0_{0,0}"] \arrow[d, equal, "id"'] & F^0_{0,0} \arrow[r] \arrow[d, "\theta_0"] & C^{5,0}_{0,0} \arrow[d, "\bar{\theta_0}"] \\
\Omega(S^5\vee S^4) \arrow[r, "\partial^s_{r,\epsilon}"] & F^s_{r,\epsilon} \arrow[r] & C^{5,s}_{r,\epsilon}.
\end{tikzcd}$$

We get the following two commutative diagrams

$$\begin{tikzcd}
\pi_8(S^5\vee S^4) \arrow[r, "(\partial^0_{0,0})_{7*}"] \arrow[d, equal, "id"'] & \pi_7(F^0_{0,0}) \arrow[d, "\theta_{0*}"'] \\
\pi_8(S^5\vee S^4) \arrow[r, "(\partial^s_{r,\epsilon})_{7*}"] & \pi_7(F^s_{r,\epsilon}),
\end{tikzcd}
\quad
\begin{tikzcd}
\pi_7(S^4\vee S^3) \arrow[r, "j^0_{0,0*}"] \arrow[d, equal, "id"'] & \pi_7(F^0_{0,0}) \arrow[r, "p^0_{0,0*}"] \arrow[d, "\theta_{0*}"] & \pi_7(S^6\vee S^7\vee S^7) \arrow[d, "\widetilde{\theta_0}_*"] \\
\pi_7(S^4\vee S^3) \arrow[r, "j^s_{r,\epsilon*}"] & \pi_7(F^s_{r,\epsilon}) \arrow[r, "p^s_{r,\epsilon*}"] & \pi_7(S^6\vee S^7\vee S^7),
\end{tikzcd}$$

where $\widetilde{\theta_0} = (f^s_{r,\epsilon}\wedge id)|_{S^6\vee S^7\vee S^7}$ by Lemma 2.3. and $\Omega(S^5\vee S^4)\xrightarrow{\partial^0_{0,0}} F^0_{0,0}$ is a homotopy equivalence. Thus from $Coker(\gamma^0_{0,0})_{7*} = \mathbb{Z}_8\{j_1\nu_4\}\oplus\mathbb{Z}_2\{j_2\nu'\eta_6\}\oplus$

$\mathbb{Z}_2\{[j_1,j_2]\eta_6\}$ in (30) we get
$(\partial^0_{0,0})_{7*}([j^5_1,j^4_2]) = t\widetilde{\rho}^0_{0,0} + y_0 j_{S^6}\eta_6 + k_0 j^0_{0,0}j_1\nu_4 + u_0 j^0_{0,0}j_2\nu'\eta_6 + w_0 j^0_{0,0}[j_1,j_2]\eta_6$
where t is odd integer, $y_0, u_0, w_0 \in \mathbb{Z}_2$, $k_0 \in \mathbb{Z}_8$.

$$\begin{aligned}
&p^s_{r,\epsilon*}(\partial^s_{r,\epsilon})_{7*}([j^5_1,j^4_2]) = p^s_{r,\epsilon*}\theta_{0*}(\partial^0_{0,0})_{7*}([j^5_1,j^4_2])\\
=&\widetilde{\theta}_{0*}p^0_{0,0*}(t\widetilde{\rho}^0_{0,0} + y_0 j_{S^6}\eta_6 + k_0 j^0_{0,0}j_1\nu_4 + u_0 j^0_{0,0}j_2\nu'\eta_6 + w_0 j^0_{0,0}[j_1,j_2]\eta_6)\\
=&\widetilde{\theta}_{0*}(t(\iota^{43}_7 - \iota^{34}_7) + y_0\Sigma j'_1\eta_6) = t(2^s\iota^{43}_7 - 2^r\iota^{34}_7) + t\epsilon\Sigma j'_1\eta_6.\\
=&2^{min\{r,s\}}t\rho^s_{r,\epsilon} + t\epsilon\Sigma j'_1\eta_6
\end{aligned}$$

On the other hand, from (34) and (35)
$p^s_{r,1*}(\partial^s_{r,1})_{7*}([j^5_1,j^4_2]) = p^s_{r,1*}(x\widetilde{\rho}^s_{r,1} + y j_{S^6}\eta_6 + k j^s_{r,1}j_1\nu_4 + \dots) = x\rho^s_{r,1} + y\Sigma j'_1\eta_6$
$p^s_{\infty,0*}(\partial^s_{\infty,0})_{7*}([j^5_1,j^4_2]) = p^s_{\infty,0*}(t'\widetilde{\iota_7}^{43} + y' j_{S^6}\eta_6 + v j^s_0 j_1\nu_4 + \dots) = t'\iota^{43}_7 + y'\Sigma j'_1\eta_6$
So $y = t = 1 \in \mathbb{Z}_2$, $y' = 0 \in \mathbb{Z}_2$; $x = 2^{min\{r,s\}}t$; $t' = 2^{min\{\infty,s\}}t = 2^s t$, t is odd. $\square$

The determination of the remaining coefficients in (34) depends on the remaining coefficients in (35).

The following short exact sequence is split since $\pi_7(S^5)$ splits out of $\pi_7(M^4_{2^s})$.

$$Coker(\partial^s_{\infty,0})_{7*} \hookrightarrow \pi_7(M^4_{2^s} \vee S^3 \vee S^4) \twoheadrightarrow \pi_7(S^5 \vee S^4) \cong \pi_7(S^5) \oplus \pi_7(S^4)$$

Note that $(\partial^s_{\infty,0})_{7*}$ is given by (31) (32) (33) (35),i.e.,
$Coker(\partial^s_{\infty,0})_{7*} = \mathbb{Z}_2\{j_{S^6}\eta_6\} \oplus H^s$, where H^s is given by the following

$$\frac{\mathbb{Z}_{(2)}\{\widetilde{\iota_7}^{43}\}\oplus\mathbb{Z}_{2^{s+1}}\{j^s_0 j_1\nu_4\}\oplus\mathbb{Z}_4\{j^s_0 j_1\Sigma\nu'\}\oplus\mathbb{Z}_2\{j^s_0 j_2\nu'\eta_6\}\oplus\mathbb{Z}_2\{j^s_0[j_1,j_2]\eta_6\}}{\left\langle 2^{s+1}j^s_0 j_1\nu_4 - 2^s j^s_0 j_1\Sigma\nu',\ 2^{2s}j^s_0 j_1\nu_4 - 2^{s-1}(2^s-1)j^s_0 j_1\Sigma\nu',\ 2^s t\widetilde{\iota_7}^{43} + v j^s_0 j_1\nu_4 + w j^s_0 j_1\Sigma\nu' + u' j^s_0 j_2\nu'\eta_6 + z j^s_0[j_1,j_2]\eta_6\right\rangle}. \tag{36}$$

On the other hand, by (17), we have
$\pi_7(M^4_{2^s} \vee S^3 \vee S^4) \cong \pi_7(M^4_{2^s}) \oplus \pi_7(S^3) \oplus \pi_7(S^4) \oplus \pi_7(M^4_{2^s} \wedge S^2) \oplus \pi_7(M^4_{2^s} \wedge S^3) \oplus \pi_7(S^3 \wedge S^3) \cong \mathbb{Z}_{2^{s+1}} \oplus \mathbb{Z}_{2^{\alpha_s}} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{(2)} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2^s} \oplus \mathbb{Z}_2$,
where $\alpha_s = min\{2, s-1\}$.

$$\text{Thus } H^s \cong \mathbb{Z}_{2^{s+1}} \oplus \mathbb{Z}_{2^{\alpha_s}} \oplus \mathbb{Z}_{2^s} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2. \tag{37}$$

Lemma 4.5. *In (35), for $s \geq 1$, $2^{\alpha_s} \mid w$, $2^s \mid v$.*

The proof of the Lemma is elementary and will be postponed to the Appendix. Assuming the Lemma one gets the remaining coefficients in (34) as follows.

Lemma 4.6. *$k = 2^{min\{r,s\}}k'$ and $l = 2^{min\{s-1,1\}}l'$, for some $k', l' \in \mathbb{Z}$.*

Proof. We have following commutative diagrams by Lemma 2.3, where $F^s_{r,1} \xrightarrow{\chi} F^s_{\infty,0}$ is induced by the map $C^{5,s}_{r,1} \xrightarrow{\bar{\chi}} C^{5,s}_{\infty,0}$ in the right commutative diagrams

$$\begin{array}{ccccc}
\pi_8(S^5 \vee S^4) & \xrightarrow{(\partial^s_{r,1})_{7*}} & \pi_7(F^s_{r,1}) & \xrightarrow{p^s_{r,1*}} & \pi_7((S^4 \vee S^3)^{\wedge 2}) \\
{\scriptstyle id}\Big\| & & \Big\downarrow{\scriptstyle \chi_*} & & \Big\downarrow{\scriptstyle (\bar{j}_1\wedge id)_*} \\
\pi_8(S^5 \vee S^4) & \xrightarrow{(\partial^s_{\infty,0})_{7*}} & \pi_7(F^s_{\infty,0}) & \xrightarrow{p^s_{\infty,0*}} & \pi_7((S^4 \vee S^3)^{\wedge 2}),
\end{array}
\quad
\begin{array}{ccccc}
S^4 \vee S^3 & \xrightarrow{f^s_{r,1}} & S^4 \vee S^3 & \longrightarrow & C^{5,s}_{r,1} \\
{\scriptstyle id}\Big\| & & \Big\downarrow{\scriptstyle \bar{j}_1=(j_1,0)} & & \Big\downarrow{\scriptstyle \bar{\chi}} \\
S^4 \vee S^3 & \xrightarrow{f^s_{\infty,0}} & S^4 \vee S^3 & \longrightarrow & C^{5,s}_{\infty,0},
\end{array} \tag{38}$$

where $\chi j^s_{r,1}=j^s_0\bar{j}_1$, $$\begin{tikzcd} S^4\vee S^3 \arrow[r, "j^s_{r,1}"] \arrow[rr, bend right, "j^s_0\bar{j}_1"'] & F^s_{r,1} \arrow[r, "\chi"] & F^s_{\infty,0} \end{tikzcd}.$$

By noting that $j_{S^6}\eta_6$ is 2-torsion, for $s\geq 1$, we suppose that

$$\begin{aligned}
\chi_*(j_{S^6}\eta_6) &= 2^s t'_1 j^s_0 j_1\nu_4 + 2^{min\{s-1,1\}}t'_2 j^s_0 j_1\Sigma\nu' + t'_3 j^s_0 j_2\nu'\eta_6 + t'_4 j^s_0[j_1,j_2]\eta_6,\\
\chi_*(\widetilde{\rho}^s_{r,1}) &= x^s_r\widetilde{\iota_7}^{43} + t_1 j^s_0 j_1\nu_4 + t_2 j^s_0 j_1\Sigma\nu' + t_3 j^s_0 j_2\nu'\eta_6 + t_4 j^s_0[j_1,j_2]\eta_6.
\end{aligned}$$

where $x^s_r=\begin{cases}1, & s\leq r;\\ 2^{s-r}, & s\geq r.\end{cases}$ $t_1,t'_1\in\mathbb{Z}, t_2,t'_2\in\mathbb{Z}_4, t_3,t'_3,t_4,t'_4\in\mathbb{Z}_2$.

From (34) and Lemma 4.4

$$\begin{aligned}
&\chi_*(\partial^s_{r,1})_{7*}([j^5_1,j^4_2]) = \chi_*(2^{min\{r,s\}}t\widetilde{\rho}^s_{r,1} + j_{S^6}\eta_6 + kj^s_{r,1}j_1\nu_4 + lj^s_{r,1}j_1\Sigma\nu' + uj^s_{r,1}j_2\nu'\eta_6)\\
&= 2^s t\widetilde{\iota_7}^{43} + (2^{min\{r,s\}}tt_1 + 2^s t'_1 + k)j^s_0 j_1\nu_4 + (2^{min\{r,s\}}tt_2 + 2^{min\{s-1,1\}}t'_2 + l)j^s_0 j_1\Sigma\nu'.
\end{aligned}$$

By the left commutative diagram in (38), $\chi_*(\partial^s_{r,1})_{7*}([j^5_1,j^4_2])$ also equals to

$$(\partial^s_{\infty,0})_{7*}([j^5_1,j^4_2]) = 2^s t\widetilde{\iota_7}^{43} + vj^s_0 j_1\nu_4 + wj^s_0 j_1\Sigma\nu' + u'j^s_0 j_2\nu'\eta_6 + zj^s_0[j_1,j_2]\eta_6.$$

Thus $2^{min\{r,s\}}tt_1 + 2^s t'_1 + k = v$ and $2^{min\{r,s\}}tt_2 + 2^{min\{s-1,1\}}t'_2 + l = w$.

By Lemma 4.5, $k=2^{min\{r,s\}}k'$ and $l=2^{min\{s-1,1\}}l'$, $k',l'\in\mathbb{Z}$.

□

Remark 4.7. The case $s=\infty$ is not allowed in the above proof of Lemma 4.6, since we get the maps $F^\infty_{r,1}\xrightarrow{\chi}F^\infty_{\infty,0}$ and $C^{5,\infty}_{r,1}\xrightarrow{\bar{\chi}}C^{5,\infty}_{\infty,0}\simeq(S^4\vee S^3)\bigcup_0 C(S^4\vee S^3)$ where the targets of the maps are not covered by Lemma 4.5. However the fibration $F^\infty_{\infty,0}\to C^{5,\infty}_{\infty,0}\to S^5\vee S^4$ splits, which implies that $(\partial^\infty_{\infty,0})_{7*}=0$ in the left commutative diagram of (38). Hence it is easy to see that Lemma 4.6 is also true for $s=\infty$.

From Lemma 4.4 and Lemma 4.6, one gets $Coker(\partial^s_{r,1})_{7*}$ in the following Lemma whose proof is also postponed to the Appendix.

Lemma 4.8. $Coker(\partial^s_{r,1})_{7*}\cong\mathbb{Z}_{2^{min\{s-\epsilon_r,2\}}}\oplus\mathbb{Z}_{2^{min\{s+1,r+1\}}}\oplus\mathbb{Z}_{2^{s+2}}, \infty\geq r\geq 1$.

Lemma 4.9. *The following short exact sequence is split for* $\infty\geq r\geq 1$.

$$0\to Coker(\partial^s_{r,1})_{7*}\to\pi_7(C^{5,s}_{r,1})\to Ker(\partial^s_{r,1})_{6*}\to 0 \tag{39}$$

where $Ker(\partial^s_{r,1})_{6*}$ *and* $Coker(\partial^s_{r,1})_{7*}$ *are given by (28) and Lemma 4.8.*

Proof. For $r\geq 2$, there is $\alpha\in\pi_7(M^3_{2^r})$ with order 4, which is a lift of $\Sigma\nu'\in Ker\partial_{6*}$. By the commutative diagram (23), $\bar{\theta}\alpha\in\pi_7(C^{5,s}_{r,1})$ is a lift of $j^4_2\Sigma\nu'\in Ker(\partial^s_{r,1})_{6*}$. So the short exact sequences (39) splits for $r\geq 2$, so is for $r=\infty$.

For $r=1$, There is an induced map $M^3_2\overset{\bar{\theta}}{\hookrightarrow}C^{5,1}_{1,1}=C^{5,1}_1$ from the left commutative diagram (23). By Lemma 1.6. of [14], there is an element $\widetilde{\alpha_2}\in\pi_7(C^{5,1}_1)$ with order 4 such that $2\widetilde{\alpha_2}=\bar{\theta}\widetilde{\eta}_3\eta_5\eta_6$ where $\widetilde{\eta}_3\in\pi_5(M^3_2)$ is a lift of η_4, i.e., $p_3\widetilde{\eta}_3 = \eta_4$. We have the following commutative diagram

$$\begin{tikzcd}
\pi_7(S^5) \arrow[r, "\widetilde{\eta}_{3*}"] \arrow[rr, bend left, "\eta_{4*}\cong"] & \pi_7(M_2^3) \arrow[r, "p_{3*}"] \arrow[d, "\bar{\theta}_*"] & Ker\partial_{6*} = \mathbb{Z}_2\{\Sigma\nu'\} \arrow[d, hook, "j^4_{2*}"'] \\
& \pi_7(C_1^{5,1}) \arrow[r, "q^1_{1,1*}"] & Ker(\partial^1_{1,1})_{6*} = \mathbb{Z}_4\{j^5_1\eta_5\eta_6 + j^4_2\Sigma\nu'\},
\end{tikzcd}$$

$q^1_{1,1*}\bar{\theta}_*\widetilde{\eta}_{3*}(\eta_5\eta_6) = q^1_{1,1*}(\bar{\theta}\widetilde{\eta}_3\eta_5\eta_6) = 2q^1_{1,1*}(\widetilde{\alpha_2})$;

On the other hand

$q^1_{1,1*}\bar{\theta}_*\widetilde{\eta}_{3*}(\eta_5\eta_6) = j^4_{2*}\eta_{4*}(\eta_5\eta_6) = 2(j^5_1\eta_5\eta_6 + j^4_2\Sigma\nu')$

Hence $q^1_{1,1*}(\widetilde{\alpha_2}) = \pm(j^5_1\eta_5\eta_6 + j^4_2\Sigma\nu')$. Thus the short exact sequences (39) splits for $r = s = 1$.

For $\infty \geq s \geq 2$, there is a commutative ladder

$$\begin{tikzcd}
S^4 \vee S^3 \arrow[r, "f^1_{1,1}"] \arrow[d, equal, "id"'] & S^4 \vee S^3 \arrow[r] \arrow[d, "d_0^{s-1}"'] & C_1^{5,1} \arrow[r, "q^1_{1,1}"] \arrow[d, "\bar{\mu}"] & S^5 \vee S^4 \arrow[d, equal, "id"'] \\
S^4 \vee S^3 \arrow[r, "f^s_{1,1}"] & S^4 \vee S^3 \arrow[r] & C_1^{5,s} \arrow[r, "q^s_{1,1}"] & S^5 \vee S^4,
\end{tikzcd}$$

where $d_0^{s-1} = (j_1 2^{s-1}\iota_4, j_2\iota_3)$.

Then $q^s_{1,1*}(\bar{\mu}\widetilde{\alpha_2}) = q^1_{1,1*}(\widetilde{\alpha_2}) = \pm(j^5_1\eta_5\eta_6 + j^4_2\Sigma\nu')$. It implies the short exact sequences (39) also splits for $r = 1, \infty \geq s \geq 2$. $\square$

So

$$\pi_7(C^{5,s}_{r,1}) \cong \begin{cases} \mathbb{Z}_{2^{min\{s-\epsilon_r,2\}}} \oplus \mathbb{Z}_{2^{min\{s+1,r+1\}}} \oplus \mathbb{Z}_{2^{s+2}} \oplus \mathbb{Z}_4, & r \geq 1; \\ \mathbb{Z}_{2^{min\{s,2\}}} \oplus \mathbb{Z}_{2^{s+1}} \oplus \mathbb{Z}_{2^{s+2}} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{(2)}, & r = \infty. \end{cases}$$

From $\pi_7(C^{5,\infty}_{r,1}) = \pi_7(S^4 \vee C^5_r) \cong \pi_7(C^5_r) \oplus \pi_7(S^4) \oplus \pi_7(C^8_r)$, $\pi_7(C^{5,s}_{\infty,1}) = \pi_7(S^4 \vee C^{5,s}) \cong \pi_7(C^{5,s}) \oplus \pi_7(S^4) \oplus \pi_7(C^{8,s})$ and $\pi_7(C^8_r) = 0$, $\pi_7(C^{8,s}) \cong \mathbb{Z}_{2^{s+1}}$ (stable) in [22], we get

$\pi_7(C^5_r) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{2^{r+1}}, r \geq 1$

$\pi_7(C^{5,s}) = \pi_7(C^{5,s}_{r,1}) \cong \mathbb{Z}_{2^{min\{s,2\}}} \oplus \mathbb{Z}_{2^{s+2}}, s \geq 1.$

$\pi_7(C^{5,s}_r) = \pi_7(C^{5,s}_{r,1}) \cong \mathbb{Z}_{2^{min\{s-\epsilon_r,2\}}} \oplus \mathbb{Z}_{2^{min\{s+1,r+1\}}} \oplus \mathbb{Z}_{2^{s+2}} \oplus \mathbb{Z}_4, r \geq 1, s \geq 1.$

First author was partially supported by National Natural Science Foundation of China (Grant No. 11701430); second author were partially supported by National Natural Science Foundation of China (Grant No. 11971461).

References

[1] Baues H J. On homotopy classification problems of J.H.C. Whitehead. Lect. Notes in Math., 1985, 1172: 17–55

[2] Baues H J. Homotopy type and homology. Oxford University Press,1996

[3] Chang S C. Homotopy invariants and continuous mappings. Proc. Roy. Soc. London. Ser.A, 1950, 202: 253–263

[4] Cohen J M. Stable homotopy. Lect. Notes Math, 1970, 165

[5] Costoya C, $\acute{M}$endez D, Viruel A. The group of self-homotopy equivalences of $\mathbf{A}^2_n$-polyhedra. Journal of Group Theory, 2020, 23.4: 575–591

[6] Drozd Y A. On classification of torsion free polyhedra. Preprint series, MaxPlanckInstitut für Mathematik (Bonn) 2005,92
[7] Gray B. On the homotopy groups of mapping cones. Proc. London Math. Soc, 1973, 26.3: 497–520
[8] Hilton P J. Calculation of the homotopy groups of $\mathbf{A}_n^2$-polyhedra (i). Quarterly Journal of Mathematics, 1950, 1.1: 299–309
[9] Hilton P J. Calculation of the homotopy groups of $\mathbf{A}_n^2$-polyhedra (ii). Quarterly Journal of Mathematics, 1951, 2: 228–240
[10] Hilton P J. An Introduction To Homotopy Theory. Cambridge University Press, 1953
[11] Huang R Z, Wu J. Exponential growth of homotopy groups of suspended finite complexes. Math. Z, 2020, 295: 1301–1321
[12] Liu X G. On the Moore Space $P^n(8)$ and Its Homotopy Groups. Chinese Annals of Math, 2007, 28A.3: 305–318
[13] $\acute{M}$endez D. The ring of stable homotopy classes of self-maps of -polyhedra. Topology and its Applications, 2021, 290.1
[14] Morisugi K, Mukai J. Lifting to mod 2 Moore spaces. Journal of the Mathematical Society of Japan, 2000, 52: 515–534
[15] Mukai J. The S^1-transfer map and homotopy groups of suspended complex projective spaces. Mathematical Journal of Okayama University, 1982, 24.2: 179–200
[16] Mukai J, Shinpo T. Some homotopy groups of the mod 4 moore space. J. Fac. Sci. Shinshu Univ. 1999, 34.1: 1–14
[17] Oda N. Unstable homotopy groups of spheres. The Buletin of she Instiula for Adoaxced Research of Fukuoka Uaibersity, 1979, 44: 49–151
[18] Toda H. Composition methods in homotopy groups of spheres. Princeton University Press, 1963
[19] Wu J. Homotopy theory of the suspensions of the projective plane. Memoirs AMS, 2003, 162: no.769
[20] Whitehead G W. Elements of Homotopy Theory, Springer-Verlag, 1978
[21] Yang J X, Mukai J, Wu J. On the Homotopy Groups of the Suspended Quaternionic Projective Plane. Preprint
[22] Zhu Z J, Pan J Z. The decomposability of smash product of $\mathbf{A}_n^2$ complexes. Homology Homotopy and Applications, 2017, 19: 293–318
[23] Zhu Z J, Li P C, Pan J Z. Periodic problem on homotopy groups of Chang complexes $C_r^{n+2,r}$. Homology Homotopy and Applications, 2019, 21.2: 363–375
[24] Zhu Z J, Pan J Z. The local hyperbolicity of $\mathbf{A}_n^2$-complexes. Homology Homotopy and Applications 2021, 23.1: 367–386

Appendix A.

Lemma A.1.

$$(2^r\iota_4)\nu_4 = 2^{2r}\nu_4 - 2^{r-1}(2^r-1)\Sigma\nu';$$
$$(-2^r\iota_4)\nu_4 = 2^{2r}\nu_4 - 2^{r-1}(2^r+1)\Sigma\nu'.$$

Proof. By Proposition 2.10 of [17] and (5,8) of [18], $(2\iota_4)\nu_4 = 2\nu_4 \pm \binom{2}{2}[\iota_4,\iota_4]H(\nu_4)) = 4\nu_4 - \Sigma\nu'$ or $\Sigma\nu'$, where H is the second Hilton-Hopf invariant. If $(2\iota_4)\nu_4 =$

$\Sigma\nu'$, then $(4\iota_4)\nu_4 = (2\iota_4)(2\iota_4)\nu_4 = (2\iota_4)(\Sigma\nu') = 2\Sigma\nu'$. On the other hand, $(4\iota_4)\nu_4 = 4\nu_4 \pm \binom{4}{2}[\iota_4,\iota_4]H(\nu_4) = 16\nu_4 - 6\Sigma\nu'$ or $-8\nu_4 + 6\Sigma\nu'$, which is not equal to $2\Sigma\nu'$. Thus $(2\iota_4)\nu_4 \neq \Sigma\nu'$, i.e., $(2\iota_4)\nu_4 = 4\nu_4 - \Sigma\nu'$.

$$(2^r\iota_4)\nu_4 = (2^{r-1}\iota_4)(2\iota_4)\nu_4 = (2^{r-1}\iota_4)(4\nu_4 - \Sigma\nu') = 4(2^{r-1}\iota_4)\nu_4 - 2^{r-1}\Sigma\nu'$$
$$= 2^{2r}\nu_4 - 2^{r-1}(2^r - 1)\Sigma\nu' \text{ (by induction).}$$

$$(-2^r\iota_4)\nu_4 = (-\iota_4)(2^r\iota_4)\nu_4 = (-\iota_4)(2^{2r}\nu_4 - 2^{r-1}(2^r-1)\Sigma\nu')$$
$$= (-\iota_4)(2^{2r}\nu_4) + 2^{r-1}(2^r-1)\Sigma\nu' = -2^{2r}\nu_4 \pm \binom{2}{2}[\iota_4,\iota_4]H(2^{2r}\nu_4) + 2^{r-1}(2^r-1)\Sigma\nu'$$
$$= 2^{2r}\nu_4 - 2^{r-1}(2^r+1)\Sigma\nu' \ \textit{or} \ -3\cdot 2^{2r}\nu_4 + (3\cdot 2^{2r-1} - 2^{r-1})\Sigma\nu' \qquad (40)$$

On the other hand

$$(-2^r\iota_4)\nu_4 = -2^r\nu_4 \pm \binom{-2^r}{2}[\iota_4,\iota_4]H(\nu_4) = -2^r\nu_4 \pm \binom{2^r+1}{2}[\iota_4,\iota_4]H(\nu_4)$$
$$= 2^{2r}\nu_4 - 2^{r-1}(2^r+1)\Sigma\nu' \ \textit{or} \ -(2^{2r} + 2^{r+1})\nu_4 + 2^{r-1}(2^r+1)\Sigma\nu'. \qquad (41)$$

Compare (40) with (41), we get $(-2^r\iota_4)\nu_4 = 2^{2r}\nu_4 - 2^{r-1}(2^r+1)\Sigma\nu'$. $\square$

Proof of Lemma 4.5. Let $a = \widetilde{\iota_7}^{43}$, $b = j_0^s j_1\nu_4$, $c = j_0^s j_1\Sigma\nu'$, $d = j_0^s j_2\nu'\eta_6$, $e = j_0^s[j_1, j_2]\eta_6$. Let $w = 2^\alpha w'$, where $2 \nmid w'$; $v = 2^\beta v'$, where $2 \nmid w', 2 \nmid v'$. By (36),

$$H^s \cong \frac{\mathbb{Z}_{(2)}\{a,b,c,d,e\}}{L'_s}$$
$$L'_s = \Big\langle 2^{s+1}b - 2^s c, 2^{2s}b - 2^{s-1}(2^s-1)c, 2^s a + 2^\beta b + 2^\alpha c + u_2 d + z_2 e, 2^{s+1}b, 4c, 2d, 2e\Big\rangle$$
$$= \Big\langle 2^{s+1}b, 2^{\alpha_s}c, 2^s a + 2^\beta b + 2^\alpha c + u_2 d + z_2 e, 2d, 2e\Big\rangle$$

Note that a $\mathbb{Z}_{(2)}$-linear isomorphism of $\mathbb{Z}_{(2)}\{a,b,c\}$ dose not change the group structure of H^s.

Assume $u' = 1 \in \mathbb{Z}_2$, then

$$L'_s = \Big\langle 2^{s+1}b, 2^{\alpha_s}c, 2^s a + 2^\beta b + 2^\alpha c + d + z_2 e, 2d, 2e\Big\rangle$$
$$= \Big\langle 2^{s+1}b, 2^{\alpha_s}c, 2^s a + 2^\beta b + 2^\alpha c + d + z_2 e, 2^{s+1}a + 2^{\beta+1}b + 2^{\alpha+1}c, 2e\Big\rangle$$
$$H^s \cong \frac{\mathbb{Z}_{(2)}\{a,b,c,e\}}{\langle 2^{s+1}b, 2^{\alpha_s}c, 2^{s+1}a + 2^{\beta+1}b + 2^{\alpha+1}c, 2e\rangle}$$

which has at most four (resp. three) cyclic direct summands for $s \geq 2$ (resp. $s = 1$) and contradicts to (37). Thus $u' = 0$. By the same argument, we get $z = 0$ and $\beta \geq 1$, $\alpha \geq 1$ when $s \geq 2$. So we get

$H^s \cong \mathbb{Z}_2\{d\} \oplus \mathbb{Z}_2\{e\} \oplus \frac{\mathbb{Z}_{(2)}\{a,b,c\}}{L_s}$ with $L_s = \langle 2^{\alpha_s}c, 2^s a + 2^\beta b + 2^\alpha c, 2^{s+1}b\rangle$.

If $\alpha < \alpha_s$, then $\alpha_s = 2$, $s \geq 2$, $\alpha = 1$.

$L_s = \langle 2^{s+1}a + 2^{\beta+1}b, 2^s a + 2^\beta b + 2c, 2^{s+1}b\rangle$.

$\mathbb{Z}_{2^{s+1}} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{2^s} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong H^s \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus A$ for some group A, which is impossible.

Hence $\alpha \geq \alpha_s$, i.e., $2^{\alpha_s} \mid w$. $L_s = \langle 2^{\alpha_s}c, 2^s a + 2^\beta b, 2^{s+1}b\rangle$.

If $\beta < s$, $L_s = \left\langle 2^{\alpha_s}c, 2^s a + 2^\beta b, 2^{2s+1-\beta}a\right\rangle$. $H^s \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{\alpha_s} \oplus A$ with exact sequence $0 \to \mathbb{Z}_{2^{2s+1-\beta}} \to A \to \mathbb{Z}_{2^\beta} \to 0$. This is a contradiction since $A \cong \mathbb{Z}_{2^s} \oplus \mathbb{Z}_{2^{s+1}}$ is not a solution of above exact sequence. So $\beta \geq s$. We complete the proof of Lemma 4.5. □

Proof of Lemma 4.8. $Coker(\gamma^s_{r,1})_{7*} = \mathbb{Z}_{2^{s+2}}\{j_1\nu_4\}\oplus\mathbb{Z}_4\{j_1\Sigma\nu'\}\oplus\mathbb{Z}_2\{j_2\nu'\eta_6\}$ in (30).

Simplify $a := \widetilde{\rho^s_{r,1}}$, $b := j_{S^6}\eta_6$, $c := j^s_{r,1}j_1\nu_4$, $d := j^s_{r,1}j_1\Sigma\nu'$, $e := j^s_{r,1}j_2\nu'\eta_6$. From (31),(32),(33), (34), Lemma 4.4 and Lemma 4.6, we get

$$Coker(\partial^s_{r,1})_{7*} = \frac{\mathbb{Z}_{(2)}\{a,b,c,d,e\}}{I^s_r}.$$

$$I^s_r{=}\Big\langle 2b, 2^{s+2}c, 4d, 2e, 2^{2s}c{-}2^{s-1}(2^s{-}1)d{+}e, 2^{min\{r,s\}}ta{+}b{+}2^{min\{r,s\}}k'c{+}2^{min\{s-1,1\}}l'd{+}ue, \epsilon_r e\Big\rangle$$
$$={\langle}2^{min\{r+1,s+1\}}(ta{+}k'c){+}\ 2^{min\{s,2\}}l'd, 2^{s+2}c, 4d, 2e, 2^{2s}c{-}2^{s-1}(2^s{-}1)d{+}e,$$
$$2^{min\{r,s\}}ta{+}b{+}2^{min\{r,s\}}k'c{+}2^{min\{s-1,1\}}l'd{+}ue, \epsilon_r e\rangle$$
$$Coker(\partial^s_{r,1})_{7*} = \frac{\mathbb{Z}_{(2)}\{a,c,d,e\}}{I'^s_r}$$
$$I'^s_r = \Big\langle 2^{min\{r+1,s+1\}}(ta{+}\ k'c){+}2^{min\{s,2\}}l'd, 2^{s+2}c, 4d, 2e, 2^{2s}c{-}2^{s-1}(2^s{-}1)d{+}e, \epsilon_r e\Big\rangle$$
since $2^{min\{s,2\}}l'd \in \left\langle 2^{s+2}c, 4d, 2e, 2^{2s}c{-}2^{s-1}(2^s{-}1)d{+}e\right\rangle$,
$$I'^s_r = \Big\langle 2^{min\{r+1,s+1\}}(ta{+}k'c), 2^{s+2}c, 4d, 2e, 2^{2s}c{-}2^{s-1}(2^s{-}1)d{+}e, \epsilon_r e\Big\rangle$$
$$Coker(\partial^s_{r,1})_{7*} = \mathbb{Z}_{2^{min\{r+1,s+1\}}}\{a + \frac{k'}{t}c\} \oplus \frac{\mathbb{Z}_{(2)}\{c,d,e\}}{I''^s_r}$$
$$I''^s_r = \left\langle 2^{s+2}c, 4d, 2e, 2^{2s}c{-}2^{s-1}(2^s{-}1)d{+}e, \epsilon_r e\right\rangle.$$

For $\infty \geq r \geq 2$

$$\begin{aligned} I''^s_r &= \left\langle 2^{s+2}c, 4d, 2^{2s+1}c{-}2^s(2^s{-}1)d, 2^{2s}c{-}2^{s-1}(2^s{-}1)d{+}e\right\rangle \\ &= \Big\langle 2^{s+2}c, 2^{min\{s,2\}}d, 2^{2s}c{-}2^{s-1}(2^s{-}1)d{+}e\Big\rangle \\ Coker(\partial^s_{r,1})_{7*} &\cong \mathbb{Z}_{2^{min\{r+1,s+1\}}} \oplus \frac{\mathbb{Z}_{(2)}\{c,d\}}{\left\langle 2^{s+2}c, 2^{min\{s,2\}}d\right\rangle} \\ &\cong \mathbb{Z}_{2^{min\{r+1,s+1\}}} \oplus \mathbb{Z}_{2^{min\{s-\epsilon_r,2\}}} \oplus \mathbb{Z}_{2^{s+2}}. \end{aligned}$$

For $r = 1$,

$$I''^s_1 = \langle 2^{s+2}c, 4d, 2^{2s}c{-}2^{s-1}(2^s{-}1)d, e\rangle = \begin{cases} \langle 8c, 4c{-}d, e\rangle\,, & s = 1; \\ \left\langle 2^{s+2}c, 2^{min\{2,s-1\}}d, e\right\rangle, & \infty \geq s \geq 2. \end{cases}$$
$$\Rightarrow Coker(\partial^s_{1,1})_{7*} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{2^{s+2}} \oplus \mathbb{Z}_{2^{min\{2,s-1\}}}, \infty \geq s \geq 1.$$

We complete the proof of Lemma 4.8. □

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