Sharp Ramsey thresholds for large books

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Abstract

For graphs G and H, let $G \to H$ signify that any red/blue edge coloring of G contains a monochromatic H. Let $G(N,p)$ be the random graph of order N and edge probability p. The sharp thresholds for Ramsey properties seemed out of hand until a general technique was introduced by Friedgut (*J. AMS* 12 (1999), 1017–1054). In this paper, we obtain the sharp Ramsey threshold for the book graph $B_n^{(k)}$, which consists of n copies of K_{k+1} all sharing a common K_k. In particular, for every fixed integer $k \ge 1$ and for any real $c > 1$, let $N = c2^k n$. Then for any real $\gamma > 0$,

$$\lim_{n\to\infty} \Pr(G(N,p) \to B_n^{(k)}) = \begin{cases} 0 & \text{if } p \le \frac{1}{c^{1/k}}(1-\gamma), \\ 1 & \text{if } p \ge \frac{1}{c^{1/k}}(1+\gamma). \end{cases}$$

The sharp Ramsey threshold $\frac{1}{c^{1/k}}$ for $B_n^{(k)}$, e.g. a star, is positive although its edge density tends to zero.

Keywords: Ramsey number; Random graph; Ramsey threshold; Regularity method

1 Introduction

For graphs G and H, let $G \to H$ signify that any red/blue edge coloring of G contains a monochromatic copy of H. The Ramsey number $r(H)$ is defined as the minimum N such that $K_N \to H$. Ramsey's theory [38] guarantees that the Ramsey number $r(H)$ is finite for all H. The question of whether or not G has the Ramsey property $G \to H$ is of particular interest when G is a typical random graph from the probability space $\mathcal{G}(n,p)$, defined by Erdős-Rényi [15], where n is the number of ordered vertices and p is the probability of edge appearance. A random graph in $\mathcal{G}(n,p)$ is always denoted by $G(n,p)$.

The Ramsey threshold $p(n)$ (tending to zero) of the event $G(n,p) \to H$ is defined by

$$\lim_{n\to\infty} \Pr(G(n,p) \to H) = \begin{cases} 0 & \text{if } p \ll p(n), \\ 1 & \text{if } p \gg p(n). \end{cases}$$

We also call $p_\ell = o(p(n))$ and $p_u = \Omega(p(n))$ a lower Ramsey threshold and an upper Ramsey threshold, respectively. It is often to signify $\Pr(G(n,p) \to H) \to 1$ and $\Pr(G(n,p) \to H) \to 0$ as $n \to \infty$ by saying that *asymptotically almost surely* (a.a.s.) $G(n,p) \to H$ and a.a.s. $G(n,p) \not\to H$, respectively. If we can replace $p \ll p(n)$ and $p \gg p(n)$ in the above with $p \le (1-\gamma)p(n)$ and

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$p \geq (1+\gamma)p(n)$ for every $\gamma > 0$, respectively, then the Ramsey threshold is said to be **sharp**. For convenience, we always say such $p(n)$ is a sharp Ramsey threshold for H.

The study of Ramsey thresholds was initiated by Frankl and Rödl [16] and independently by Łuczak, Ruciński and Voigt [31], who proved that $p = 1/\sqrt{n}$ is a Ramsey threshold for triangle. In a series of papers [16, 31, 39, 40, 41], the Ramsey thresholds are determined for all graphs H. For a graph H, let $v(H)$ and $e(H)$ be the numbers of vertices and edges of H, respectively. The Ramsey threshold for a fixed graph was determined by Rödl and Ruciński [41], who proved that (except H is a path of length 3 as was pointed out in [17] or a disjoint union of stars)

$$\lim_{n\to\infty} \Pr(G(n,p) \to H) = \begin{cases} 0 & \text{if } p \ll n^{-1/m_2(H)}, \\ 1 & \text{if } p \gg n^{-1/m_2(H)}, \end{cases}$$

where $m_2(H) = \max\{\frac{e(F)-1}{v(F)-2} : \ F \subseteq H, \ v(F) \geq 3\}$. This result has a short proof from Nenadov and Steger [34].

The sharp thresholds for Ramsey properties seemed out of hand until a general technique for settling these questions was introduced by Friedgut [18]. In particular, Friedgut and Krivelevich [17] obtained all sharp thresholds for fixed trees except the star and a path of length 3. When H is a triangle, it was established by Friedgut, Rödl, Ruciński and Tetali [19].

In the following, we mainly focus on the situations when the graphs are large. A closely related problem is the size Ramsey number. For a graph H, Erdős, Faudree, Rousseau and Schelp [13] defined the size Ramsey number as $\hat{r}(H) = \min\{e(G) : G \to H\}$. Beck [3] proved $\hat{r}(P_n) = O(n)$ for path P_n of length n, who in fact showed a.a.s.

$$G(c_1 n, c_2/n) \to P_n,$$

where c_1 and c_2 are positive constants. This has been improved by Dudek and Prałat [12].

For a path P_n of length n, Gerencsér and Gyárfás [20] proved $r(P_n) = n + \lceil \frac{n}{2} \rceil$. Thus if $N < 3n/2$, $G(N,1) \to P_n$ is an impossible event. But if $c > 1$, Letzter [28] proved a.a.s.

$$G(3cn/2, p) \to P_n$$

when $pn \to \infty$, hence $p = \frac{1}{n}$ is a Ramsey threshold of $G(3cn/2, p) \to P_n$, from which Letzter [28] improved Beck's result [3] further. For more references for non-diagonal cases of the Ramsey thresholds, we refer the reader to [2, 23, 32, 33] etc.

Let $\mathcal{F}_{\Delta,n}$ be the family of graphs H with order n and maximum degrees at most Δ. Beck [4] conjectured that the size Ramsey number $\hat{r}(H_n) = O(n)$ for any $H \in \mathcal{F}_{\Delta,n}$. However, Rödl and Szemerédi [43] showed that it does not hold even for $\Delta = 3$. In 2011, Kohayakawa, Rödl, Schacht and Szemerédi [25] proved that for every fixed $\Delta \geq 2$, there exist constants $B = B(\Delta)$ and $C = C(\Delta)$ such that if $N = \lceil Bn \rceil$ and $p = C(\log N/N)^{1/\Delta}$, then for any $H \in \mathcal{F}_{\Delta,n}$,

$$\lim_{n\to\infty} \Pr\left(G(N,p) \to H\right) = 1.$$

This implies that $\hat{r}(H) = O(n^{2-1/\Delta}\log^{1/\Delta} n)$ for any $H \in \mathcal{F}_{\Delta,n}$.

Let $B_n^{(k)}$ be the book graph consisting of n copies of K_{k+1}, all sharing a common K_k. Let $K_{k,n}$ be the complete bipartite graph with two parts of sizes k and n. Clearly, both of these two special families of graphs do not belong to $\mathcal{F}_{\Delta,n}$. The corresponding Ramsey-type problems of these two families of graphs have attracted a great deal of attention. A classical result of Erdős, Faudree, Rousseau and Schelp [13] implies that

$$\hat{r}(K_{k,n}) = \Theta(n), \text{ and } \hat{r}(B_n^{(k)}) = \Theta(n^2).$$

Moreover, Li, Tang and Zang [30] proved that $r(K_{k,n}) = (2^k + o(1))n$, but $r(B_n^{(k)})$ is much harder to handle. When $k = 2$, Rousseau and Sheehan [44] showed that $r(B_n^{(2)}) \leq 4n + 2$, and the equality holds if $4n + 1$ is a prime power. After many years, Conlon [9] established that

$$r(B_n^{(k)}) = (2^k + o(1))n, \tag{1}$$

which confirms a conjecture of Thomason [47] asymptotically and also gives an answer to a problem proposed by Erdős [13]. The small term $o(1)$ in (1) is improved further by Conlon, Fox and Wigderson [10]. For more Ramsey numbers of books, the reader is referred to [6, 11, 35, 36, 37] etc.

In this paper, we obtain sharp Ramsey thresholds for $B_n^{(k)}$ and $K_{k,n}$.

Theorem 1.1 *Let $N = c2^k n$, where $k \geq 1$ is an integer and $c > 1$ is a real number. Then for any $\gamma > 0$,*

$$\lim_{n\to\infty} \Pr(G(N,p) \to B_n^{(k)}) = \begin{cases} 0 & \text{if } p \leq \frac{1}{c^{1/k}}(1-\gamma), \\ 1 & \text{if } p \geq \frac{1}{c^{1/k}}(1+\gamma). \end{cases}$$

Remark. Although the edge density of the book graph $B_n^{(k)}$ tends to zero as $n \to \infty$, the sharp Ramsey threshold for $B_n^{(k)}$ is a positive constant $c^{1/k}$.

In Theorem 1.1, if we take $c = (1 + \epsilon)$ for sufficiently small $\epsilon > 0$ and $p \to 1$, then a.a.s. $G((1 + \epsilon)2^k n, p) \to B_n^{(k)}$. Especially, (1) holds.

For $c > 1$, since the Ramsey threshold of $G(cr(P_n), p) \to P_n$ is $\frac{1}{n}$ in Letzter [28] and the sharp Ramsey threshold of $G(cr(K_{1,n}), p) \to K_{1,n}$ is $\frac{1}{c}$ from Theorem 1.1, we see that the Ramsey thresholds of $G(cr(T_n), p) \to T_n$ are so different for different types of trees T_n. Note that most Ramsey numbers $r(T_n)$ of trees T_n with n edges are unknown, ranging from $\lfloor \frac{4n+1}{3} \rfloor$ to $2n$, see Erdős, Faudree, Rousseau and Schelp [14], and Yu and Li [48]. It would be interesting to find more Ramsey thresholds for different types of trees. Moreover, it would be very interesting to determine the sharp Ramsey threshold for P_n, which seems not easy.

Combining Lemma 2.1, the following is immediate.

Corollary 1.1 *Let $N = c2^k n$, where $k \geq 1$ is an integer and $c > 1$ is a real number. Then for any $\gamma > 0$,*

$$\lim_{n\to\infty} \Pr(G(N,p) \to K_{k,n}) = \begin{cases} 0 & \text{if } p \leq \frac{1}{c^{1/k}}(1-\gamma), \\ 1 & \text{if } p \geq \frac{1}{c^{1/k}}(1+\gamma). \end{cases}$$

Notation: For a graph $G = (V, E)$ with vertex set V and edge set E, let uv denote an edge of G. For $X \subseteq V$, $e(X)$ is the number of edges in X, and $G[X]$ denotes the subgraph of G induced by X. For two disjoint subsets $X, Y \subseteq V$, $e_G(X, Y)$ denotes the number of edges between X and Y. In particular, the neighborhood of a vertex v in $U \subseteq V$ is denoted by $N_G(v, U)$, and $\deg_G(v, U) = |N_G(v, U)|$ and the degree of v in G is $\deg_G(v) = |N_G(v, V)|$. Let $X \sqcup Y$ denote the disjoint union of X and Y. We always delete the subscriptions if there is no confusion from the context. Note that we have not distinguished large x from $\lceil x \rceil$ or $\lfloor x \rfloor$ when x is supposed to be an integer since these rounding errors are negligible to the asymptotic calculations.

2 The lower Ramsey threshold

The following slightly stronger lemma implies the lower Ramsey threshold of Theorem 1.1.

Lemma 2.1 *Let $k \geq 1$ be an integer and $c > 1$ a real number. Let $N = c2^k n$. If $p \in [0, \frac{1}{c^{1/k}})$, then a.a.s. $G(N, p) \not\to K_{k,n}$.*

Proof. Let $\gamma > 0$ be sufficiently small. It suffices to show that

$$p_\ell = \frac{1}{c^{1/k}}(1-\gamma)^{1/k}$$

is a function such that a.a.s. $G(N, p_\ell) \not\to K_{k,n}$.

We first have the following claim.

Claim 2.1 *We have that a.a.s. $G(N, p_0)$ contains no $K_{k,n}$, where $p_0 = p_\ell/2$.*

Proof. All graphs in the proof are on vertex set V with $|V| = N$. Consider random graph $G(N, p_0)$. Let $U \subseteq V$ be a set with $|U| = k$ and $V \setminus U = \{v_1, v_2, \ldots, v_{N-k}\}$. For each $i = 1, 2, \ldots, N-k$, define a random variable X_i such that $X_i = 1$ if v_i is a common neighbor of U and 0 otherwise. Then $\Pr(X_i = 1) = p_0^k$.

Set $S_{N-k} = \sum_{i=1}^{N-k} X_i$ that has the binomial distribution $B(N-k, p_0^k)$. Note that the event $S_{N-k} \geq n$ means that there is a $K_{k,\,n}$ with U as the part of k vertices. Hence

$$\Pr(K_{k,n} \subseteq G(N, p_0)) \leq \binom{N}{k} \Pr(S_{N-k} \geq n).$$

We now evaluate the probability $\Pr(S_{N-k} \geq n)$. Write $n = \frac{N}{c2^k} = (p_0^k + \delta)(N-k)$, where

$$\delta = \frac{N}{c2^k(N-k)} - p_0^k = \frac{N}{c2^k(N-k)} - \frac{1}{c2^k}(1-\gamma) = \frac{1}{c2^k}\left(\gamma + \frac{k}{N-k}\right).$$

By virtue of Chernoff bound (see e.g. [1, 5, 7, 21, 29]),

$$\Pr(S_{N-k} \geq n) = \Pr\left(S_{N-k} \geq (p_0^k + \delta)(N-k)\right) \leq \exp\left\{-(N-k)\delta^2/(3p_0^k(1-p_0^k))\right\}.$$

Note that $(N-k)\delta^2 \sim N\delta^2 \sim \frac{\gamma^2}{c2^k} n$, thus we have

$$\binom{N}{k} \Pr(S_{N-k} \geq n) \lesssim N^k \exp\left\{-\frac{\gamma^2}{c2^k} n/(3p_0^k(1-p_0^k))\right\} \to 0,$$

and so a.a.s. $G(N, p_\ell/2)$ contains no $K_{k,n}$. The claim is finished. $\square$

Now we write the random variable S_{N-k} as $S_{N-k}^{p_\ell/2}(U)$ for fixed U with $|U| = k$, where the superscript $p_\ell/2$ corresponds to random graph $G(N, p_\ell/2)$. Then we have shown

$$\binom{N}{k} \Pr\left(S_{N-k}^{p_\ell/2}(U) \geq n\right) \to 0, \tag{2}$$

as $n \to \infty$. Consider an edge coloring of $G(N, p_\ell)$ with red and blue at random with probability $1/2$, independently. It is easy to see that both red graphs and blue graphs form $\mathcal{G}(N, p_\ell/2)$.

For a vertex set U of size k, let $S^{p_\ell,R}_{N-k}(U)$ and $S^{p_\ell,B}_{N-k}(U)$ be the numbers of common red and blue neighbors of U, respectively. Then

$$\Pr\left(S^{p_\ell,R}_{N-k}(U) \geq n\right) = \Pr\left(S^{p_\ell,B}_{N-k}(U) \geq n\right) = \Pr\left(S^{p_\ell/2}_{N-k} \geq n\right),$$

and thus $\Pr(S^{p_\ell,R}_{N-k}(U) \geq n \text{ or } S^{p_\ell,B}_{N-k}(U) \geq n) \leq 2\Pr(S^{p_\ell/2}_{N-k} \geq n)$. Therefore, from (2), we have

$$\binom{N}{k} \Pr\left(S^{p_\ell,R}_{N-k}(U) \geq n \text{ or } S^{p_\ell,B}_{N-k}(U) \geq n\right) \to 0$$

as $n \to \infty$, which implies that a.a.s. $G(N, p_\ell) \not\to K_{k,n}$. □

Remark. More careful calculation in the proof can yield an improvement $O\left(\sqrt{\frac{\log n}{n}}\right)$ for the small term γ.

3 The upper Ramsey threshold

The following result follows from Chernoff bound directly.

Lemma 3.1 *Let $p \in (0, 1]$ be a fixed probability. If $N \to \infty$, then a.a.s. $G \in \mathcal{G}(N, p)$ with vertex set V satisfies the following properties:*

(i) For any vertex $v \in V$ and subset $U \subseteq V$, $\deg(v, U) = p|U| + o(N)$;
(ii) For any pair of distinct vertices u and v, $|N(u) \cap N(v)| = p^2N + o(N)$;
(iii) For any subsets $U \subseteq V$, $e(U) = p\binom{|U|}{2} + o(N^2)$;
(vi) For any disjoint vertex sets U and W, $e(U, W) = p|U||W| + o(N^2)$.

The assertion is clear when $k = 1$, so we may assume $k \geq 2$.

3.1 The first case for $k = 2$

In this subsection, we include a short proof for the case when $k = 2$ of Theorem 1.1. Denote B_n instead of $B^{(2)}_n$. The upper Ramsey threshold for $k = 2$ follows from the following lemma.

Lemma 3.2 *Let $c > 1$ and $p = \frac{1+\gamma}{\sqrt{c}}$, where $\gamma \in (0, \sqrt{c}-1]$. Let G be a graph of order $N = \lfloor 4cn \rfloor$ that satisfies properties in Lemma 3.1. Then $G \to B_n$ for all large n.*

Proof. Suppose that there is an edge-coloring of G by red and blue that contains no monochromatic B_n. We shall show this assumption would lead to a contradiction.

Let V be the vertex set of G. Let R and B denote the red and blue subgraphs, respectively. Let M_r and M_b be the number of monochromatic triangles in red and blue, respectively. Let M_{rb} be the numbers of non-monochromatic triangles. Denote by $M = M_r + M_b$ the number of monochromatic triangles. Let T be the number of triangles in G. Clearly, $M = T - M_{rb}$.

Note from Lemma 3.1 that $e(G) \sim \frac{1}{2}pN^2$, and $|N(u) \cap N(v)| \sim p^2N$, we have

$$T = \frac{1}{3} \sum_{uv \in E(G)} |N(u) \cap N(v)| \sim \frac{1}{6}p^3N^3, \tag{3}$$

where coefficient $\frac{1}{3}$ of the sum follows from that each triangle is counted triply in the sum.

Since a red edge uv and n red common neighbors of u and v yield a red B_n, we have $|N_R(u) \cap N_R(v)| \le n-1$. Hence

$$M_r = \frac{1}{3} \sum_{uv \in E(R)} |N_R(u) \cap N_R(v)| \le \frac{1}{3}(n-1)e(R).$$

Similarly, $M_b \le \frac{1}{3}(n-1)e(B)$, and thus

$$M \le \frac{1}{3}(n-1)e(G) \sim \frac{1}{6}pnN^2. \tag{4}$$

For any $v \in V$, each edge between $N_R(v)$ and $N_B(v)$ is contained in a non-monochromatic triangle, and thus

$$M_{rb} = \frac{1}{2} \sum_{v \in V} e(N_R(v), N_B(v)) = \frac{1}{2} \sum_{v \in V} p \deg_R(v) \deg_B(v) + o(N^3),$$

where $\frac{1}{2}$ comes from that each such triangle is counted by its two vertices and the term $o(N^3)$ comes from the third property in Lemma 3.1. Since $\deg_R(v) + \deg_B(v) = \deg(v)$, we have $\deg_R(v) \deg_B(v) \le \frac{1}{4}[\deg(v)]^2$. Therefore,

$$M_{rb} \le \frac{1}{8}p \sum_{v \in V} [\deg(v)]^2 + o(N^3) \sim \frac{1}{8}p^3N^3. \tag{5}$$

Recall $M = T - M_{rb}$, which and (3), (4) and (5) yield

$$\frac{1}{6}pnN^2 \ge (1-o(1))\left(\frac{1}{6}p^3N^3 - \frac{1}{8}p^3N^3\right) = \left(\frac{1}{24} - o(1)\right)p^3N^3,$$

which implies that $p^2 \le (1+o(1))\frac{4n}{N} = (1+o(1))\frac{1}{c}$, contradicting to the assumption $p = \frac{1+\gamma}{\sqrt{c}}$ and the proof is completed. $\square$

3.2 The regularity method and useful lemmas

Szemerédi regularity lemma [45, 46] is a powerful tool in extremal graph theory. There are many important applications of the regularity lemma. We refer the reader to nice surveys [26, 27, 42] and other related references. The proof for the upper Ramsey thresholds of Theorem 1.1 for general $k \ge 3$ mainly relies on the regularity method.

Given $p \in (0,1]$ and $\varepsilon > 0$, the p-density of a pair (U, W) of disjoint sets of vertices in a graph G is defined as

$$d_{G,p}(U,W) = \frac{e_G(U,W)}{p|U||W|}.$$

We say that the pair (U,W) is *(ε, p)-regular* in G if $|d_{G,p}(U,W) - d_{G,p}(U',W')| \le \varepsilon$ for all $U' \subset U$ and $W' \subset W$ with $|U'| \ge \varepsilon|U|$ and $|W'| \ge \varepsilon|W|$. When $p=1$, it is the usual edge density, denoted by $d_G(U,W)$, between U and W. If $U \cap W \ne \emptyset$, then the edges in $U \cap W$ will be counted twice. Given $0 < \eta, p \le 1$, $D \ge 1$, a graph G is called (η, p, D)-upper-uniform if, for all disjoint sets of vertices U, W of size at least $\eta|V(G)|$, the density $d_{G,p}(U,W)$ is at most D.

Given a red-blue coloring of the edges of G, we write R and B for the graphs on $V(G)$ induced by the red and blue edges, respectively. We say that $V(G) = \sqcup_{i=1}^m V_i$ is an *equitable partition* for the coloring (R,B) of G if $\big||V_i| - |V_j|\big| \le 1$ for all $1 \le i < j \le m$.

We will use the following regularity lemma for random graphs.

Lemma 3.3 *For any $\varepsilon > 0$ and integer $M_0 \ge 1$, there exists $M = M(\varepsilon, M_0) > M_0$ such that the following holds. If $p \in (0,1]$ is fixed, then* **a.a.s.** *every 2-coloring of the edges of $G \in \mathcal{G}(N,p)$ has an (ε,p)-regular equitable partition $V(G) = \sqcup_{i=1}^m V_i$ where $M_0 \le m \le M$ such that*

(i) each part V_i is (ε,p)-regular;
(ii) for each V_i, all but at most εm parts V_j such that (V_i,V_j) are (ε,p)-regular;
(iii) for any vertex $v \in V$ and for $1 \le i \le m$, $\deg(v,V_i) = p|V_i| + o(N)$;
(iv) for $1 \le i \le m$, $e(V_i) = p\binom{|V_i|}{2} + o(N^2)$;
(v) for $1 \le i < j \le m$, $e(V_i,V_j) = p|V_i||V_j| + o(N^2)$.

Proof. We only sketch the proof of Lemma 3.3 as follows. From Lemma 3.1, a.a.s. $G \in \mathcal{G}(N,p)$ satisfies that (1) for any vertex $v \in V$ and subset $U \subseteq V$, $\deg(v,U) = p|U| + o(N)$; (2) for any subsets $U \subseteq V$, $e(U) = p\binom{|U|}{2} + o(N^2)$; (3) for any disjoint vertex sets U and W, $e(U,W) = p|U||W| + o(N^2)$. Therefore, the random graph G and hence the red subgraph R and the blue subgraph B are a.a.s. upper uniform (with suitable parameters). Let $\varepsilon_1 = \varepsilon/2$, $\varepsilon_2 = \varepsilon^2/128$, $K_1 = K(\varepsilon_1) \le 2^{(1/\varepsilon_1)^{(10/\varepsilon_1)^{15}}}$, and let $\eta = \min\{\varepsilon_1/K_1, \varepsilon^3/256\}$ as in Conlon, Fox and Wigderson [10, Lemma 2.1]. We can first apply the colored version of Letzter [28, Theorem 5.2] (from an original version by Kohayakawa and Rödl [22, 24]) to obtain that there exists $L = L(\eta, M_0) > M_0$ such that the following holds. If $p \in (0,1]$ is fixed, then we have that a.a.s. every 2-coloring of the edges of $G \in \mathcal{G}(N,p)$ has an equitable partition $V(G) = \sqcup_{i=1}^\ell W_i$ with $\max\{M_0, 1/\eta\} \le \ell \le L$ such that all but at most $\varepsilon\binom{m}{2}$ pairs (W_i,W_j) are (η,p)-regular. Then we apply [10, Lemma 2.4] to each W_i to get an equitable partition $W_i = U_{i1} \sqcup \cdots \sqcup U_{iK_1}$ such that each U_{ij} for $1 \le j \le K_1$ is ε_1-regular. Subsequently, by a similar argument as that in [10, Lemma 2.1], we can obtain an (ε,p)-regular equitable partition $V(G) = \sqcup_{i=1}^m V_i$ satisfying the conditions from the above equitable partition as desired. $\square$

The following is a counting lemma by Conlon [9, Lemma 5], which will be used to find a large monochromatic book.

Lemma 3.4 (Conlon [9]) *For any $\delta > 0$ and any integer $k \ge 1$, there is $\varepsilon > 0$ such that if $V_1, \ldots, V_k, V_{k+1}, \ldots, V_{k+\ell}$, are (not necessarily distinct) vertex with $(V_i, V_{i'})$ ε-regular of density $d_{i,i'}$ for all $1 \le i < i' \le k$ and $1 \le i \le k < i' \le k+\ell$ and $d_{i,i'} \ge \delta$ for all $1 \le i < i' \le k$, then there is a copy of K_k with the ith vertex in V_i for each $1 \le i \le k$ which is contained in at least*

$$\sum_{j=1}^{\ell}\left(\prod_{i=1}^{k} d_{i,k+j} - \delta\right)|V_{k+j}|$$

copies of K_{k+1} with the $(k+1)$-th vertex in $\cup_{j=1}^{\ell} V_{k+j}$.

We also need the following standard counting lemma, one can see Conlon, Fox and Wigderson [10, Lemma 2.5], or see Zhao [49, Theorem 3.27] for a detailed proof.

Lemma 3.5 *Suppose that $V_1, \dots, V_k$ are (not necessarily distinct) subsets of a graph G such that all pairs (V_i, V_j) are ε-regular. Then the number of labeled copies of K_k whose ith vertex is in V_i for all i is at least*

$$\left(\prod_{1\le i<j\le k} d(V_i,V_j) - \varepsilon\binom{k}{2}\right)\prod_{i=1}^{k}|V_i|.$$

We have the following corollary by Conlon, Fox and Wigderson [10, Corollary 2.6] from Lemma 3.5, which counts the monochromatic extensions of cliques.

Corollary 3.1 (Conlon, Fox and Wigderson [10]) *Let $\varepsilon, \delta \in (0,1)$ and $\varepsilon \le \delta^3/k^2$. Suppose $U_1, \dots, U_k$ are (not necessarily distinct) vertex sets in a graph G and all pairs (U_i, U_j) are ε-regular with $\prod_{1\le i<j\le k} d(U_i, U_j) \ge \delta$. Let Q be a randomly chosen copy of K_k with one vertex in each U_i with $1 \le i \le k$ and say that a vertex u extends Q if u is adjacent to every vertex of Q. Then, for any u,*

$$\Pr(u \text{ extends } Q) \ge \prod_{i=1}^{k} d(u, U_i) - 4\delta. \tag{6}$$

3.3 General case for $k \ge 3$

Now we give a proof for the upper Ramsey threshold of Theorem 1.1 for $k \ge 3$. For any $c > 1$ and $k \ge 3$, let $N = c2^k n$ and $p = \frac{1}{c^{1/k}}(1+\gamma)$, where $\gamma > 0$ is sufficiently small and n is sufficiently large. Set

$$p_0 = \frac{1}{c^{1/k}}\left(1 + \frac{\gamma}{2}\right).$$

Let δ and ε be sufficiently small positive reals such that

$$\delta = \min\left\{\frac{\gamma}{4c},\ \frac{p_0^k}{2^{k+5}}\gamma\right\}, \quad \text{and} \quad \varepsilon = \min\left\{\frac{1}{k^2}(\delta p)^k,\ \frac{1}{k^2}(p_0/2)^{\binom{k}{2}}\right\}. \tag{7}$$

We begin by applying Lemma 3.3 to the graph $G \in \mathcal{G}(N,p)$ with ε and $M_0 = 1/\varepsilon$ to obtain a constant $M = M(\varepsilon)$ such that **a.a.s.** every 2-coloring of edges of $G \in \mathcal{G}(N,p)$ has an (ε,p)-regular equitable partition $V(G) = \sqcup_{i=1}^{m} V_i$ where $M_0 \le m \le M$ satisfying

(i) each part V_i is (ε, p)-regular;
(ii) for each V_i, all but at most εm parts V_j such that (V_i, V_j) are (ε,p)-regular;
(iii) for any vertex $v \in V$ and for $1 \le i \le m$, $\deg_G(v, V_i) \ge p_0|V_i|$;
(iv) for $1 \le i \le m$, $e(V_i) \ge p_0\binom{|V_i|}{2}$;
(v) for $1 \le i < j \le m$, $d_G(V_i, V_j) \ge p_0$.

Let R and B be the subgraphs of G induced by all red and blue edges, respectively. Without loss of generality, we may assume that there are at least $m' \ge m/2$ of the parts, say $V_1, \dots, V_{m'}$, have internal **red** p-density at least $\frac{1}{2}$. Let Γ_B be the subgraph of the reduced graph Γ defined on $\{v_1, \dots, v_m\}$ in which $v_iv_j \in E(\Gamma_B)$ if (V_i, V_j) is (ε,p)-regular and $d_{B,p}(V_i, V_j) \ge \delta$. Let Γ'_B be the subgraph of Γ_B induced by the "red" vertices v_i for $1 \le i \le m'$.

Suppose that, in Γ'_B, some vertex v_i has at least $(2^{1-k} + 2\varepsilon)m'$ non-neighbors. Then, since v_i has at most $\varepsilon m \le 2\varepsilon m'$ non-neighbors, we have that there are at least $2^{1-k}m'$ parts V_j with

$1 \le j \le m'$ such that (V_i, V_j) is (ε, p)-regular. Let J be the set of all these indices j such that v_j is the non-neighbor of v_i and (V_i, V_j) is (ε, p)-regular. Then $|J| \ge m/2^k$. Note that

$$d_{B,p}(V_i, V_j) + d_{R,p}(V_i, V_j) = \frac{e_B(V_i, V_j) + e_R(V_i, V_j)}{p|V_i||V_j|} \ge \frac{p_0}{p},$$

thus if $v_i v_j \notin E(\Gamma_B)$, then we have $d_{R,p}(V_i, V_j) \ge \frac{p_0}{p} - \delta$ and so the edge density between V_i and V_j satisfies $d_R(V_i, V_j) \ge p_0 - p\delta$. Since the red p-density is at least $1/2$, from Lemma 3.4, there exists a red K_k which is contained in at least

$$\begin{aligned}\sum_{j\in J}\left((p_0 - p\delta)^k - \delta\right)|V_j| &\ge \left(\left(\frac{1}{c^{1/k}}\left(1+\frac{\gamma}{2}\right) - \delta\right)^k - \delta\right)|J|\frac{N}{m} \\ &\ge \left(\frac{1}{c}\left(1 + ck\delta\right) - \delta\right)|J|\frac{N}{m} \ge n\end{aligned}$$

red K_{k+1} by noting (7) that $\delta \le \frac{\gamma}{4c}$. Thus, we obtain a red $B_n^{(k)}$ as desired.

Therefore, we may assume that every vertex in Γ'_B has degree at least $(1 - 2^{1-k} - 2\varepsilon)m'$. Since $2^{1-k} + 2\varepsilon < \frac{1}{k-1}$ for $k \ge 2$, it follows from Turán's theorem that Γ'_B contains a K_k on vertices $v_{i_1}, \dots, v_{i_k}$. Let $W_j = V_{i_j}$ for $1 \le j \le k$. Then every pair (W_i, W_j) with $i \le j$ is (ε, p)-regular and $d_{B,p}(W_i, W_j) \ge \delta$ for $i \ne j$, and each W_i has red p-density at least $\frac{1}{2}$.

From Lemma 3.5 and (7), the number of blue K_k with the ith vertex in W_i is at least

$$\left(\prod_{1\le i<j\le k}[p \cdot d_{B,p}(W_i, W_j)] - \varepsilon\binom{k}{2}\right)\prod_{i=1}^{k}|W_i| \ge \left(\delta^k p^k - \varepsilon\binom{k}{2}\right)\prod_{i=1}^{k}|W_i| > 0.$$

Similarly, the number of red K_k in any W_i is at least

$$\left([p_0 \cdot d_{R,p}(W_i)]^{\binom{k}{2}} - \varepsilon\binom{k}{2}\right)|W_i|^k \ge \left((p_0/2)^{\binom{k}{2}} - \varepsilon\binom{k}{2}\right)|W_i|^k > 0.$$

For any vertex v, define

$$d_{B,p}(v, W_i) := \frac{\deg_B(v, W_i)}{p_0|W_i|}.$$

Similarly, we define $d_{R,p}(v, W_i)$. From the assumption that $\deg_G(v, W_i) \ge p_0|W_i|$, we have

$$d_{R,p}(v, W_i) + d_{B,p}(v, W_i) \ge 1. \tag{8}$$

Now, for any vertex v and for $1 \le i \le k$, let $x_i(v) := d_{B,p}(v, W_i)$. Then $d_{R,p}(v, W_i) \ge 1 - x_i(v)$. From a technical analytic inequality by Conlon [9, Lemma 8], we know that

$$\prod_{i=1}^{k} x_i(v) + \frac{1}{k}\sum_{i=1}^{k}(1 - x_i(v))^k \ge 2^{1-k}.$$

Therefore, we have either $\prod_{i=1}^{k} x_i(v) \ge 2^{-k}$ or $\frac{1}{k}\sum_{i=1}^{k}(1 - x_i(v))^k \ge 2^{-k}$. There are two cases as follows.

Case 1. $\prod_{i=1}^k x_i(v) \geq 2^{-k}$.

For a given vertex v, if we pick $w_i \in W_i$ with $1 \leq i \leq k$ uniformly and independently at random, then the probability that all the edges (v, w_i) are blue is roughly $\prod_{i=1}^k [px_i(v)]$. Together with the regularity of the pairs (W_i, W_j), a random blue K_k spanned by $(W_1, \ldots, W_k)$ will also have probability close to $\prod_{i=1}^k [px_i(v)]$ of being in the blue neighborhood of a random chosen v. Indeed, from Corollary 3.1, the expected number of blue extensions of a randomly chosen blue K_k spanned by $(W_1, \ldots, W_k)$ is at least

$$\begin{aligned}\sum_{v\in V}\left(\prod_{i=1}^k [p_0 \cdot d_{B,p}(v, W_i)] - 4\delta\right) &= \sum_{v\in V}\left(\prod_{i=1}^k [p_0 x_i(v)] - 4\delta\right) \geq \left(2^{-k} - \frac{4\delta}{p_0^k}\right) p_0^k N \\ &= \left(2^{-k} - \frac{4\delta}{p_0^k}\right) \frac{1}{c}\left(1 + \frac{\gamma}{2}\right)^k \cdot c2^k n \geq n\end{aligned}$$

by noting $\delta \leq \frac{p_0^k}{2^{k+5}}\gamma$ from (7). Therefore, a randomly chosen blue K_k spanned by $(W_1, \ldots, W_k)$ will have at least n blue extensions in expectation, giving us a blue $B_n^{(k)}$.

Case 2. $\frac{1}{k}\sum_{i=1}^k (1 - x_i(v))^k \geq 2^{-k}$.

For this case, we have

$$\frac{1}{k}\sum_{i=1}^k \sum_{v\in V} (1 - x_i(v))^k = \frac{1}{k}\sum_{v\in V}\sum_{i=1}^k (1 - x_i(v))^k \geq 2^{-k} N.$$

Thus there must exist some $1 \leq i \leq k$ for which $\sum_{v\in V}(1 - x_i(v))^k \geq 2^{-k}N$. Similarly, from the regularity of W_i, for a random red K_k in W_i and for a random $v \in V$, v will form a red extension of the K_k with probability close to $p^k(1 - x_i(v))^{-k}$. Indeed, we can apply Corollary 3.1 again to obtain that the expected number of extensions of a random red K_k in W_i is at least

$$\sum_{v\in V}\left([p_0(1 - x_i(v))]^k - 4\delta\right) \geq (2^{-k} - 4\delta/p_0^k) p_0^k N \geq n,$$

yielding a red $B_n^{(k)}$ as desired. Theorem 1.1 is proved. $\square$

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