REPRESENTATION TYPE OF BLOCKS OF CYCLOTOMIC HECKE ALGEBRAS OF TYPE $G(r,1,n)$

YANBO LI AND XIANGYU QI

Abstract. Let K be an algebraically closed field with $CharK \neq 2$ and $(s_1, s_2, \cdots, s_r) \in \mathbb{Z}^r$ a multicharge with $r > 2$. Let $\mathcal{H}_n(q,Q)$ be a cyclotomic Hecke algebra of type $G(r,1,n)$, where $q \neq 0,1$ and $Q = (q^{s_1}, q^{s_2}, \cdots, q^{s_r})$. For each block B of $\mathcal{H}_n(q,Q)$, we introduce a new invariant, called block move vector, which can be considered as a generalization of the weight $w(B)$. We prove by using block move vector that block B has finite representation type if and only if $w(B) < 2$, or B is Morita equivalent to $K[x]/x^{w(B)+1}$. Blocks of finite representation type with weight more than one are determined completely by block move vectors. This result implies that some blocks of finite type are Brauer tree algebras whose Brauer trees have exceptional vertex. We also determine representation type for all the blocks of cyclotomic q-Schur algebras. Moreover, by using our result, we construct examples of blocks with the same weight that are not derived equivalent. Examples of derived equivalent blocks being in different orbits under the adjoint action of the affine Weyl group are also given.

Contents

2010 *Mathematics Subject Classification.* 16G60, 20C08, 20G43, 05E10, 06A11.

Key words and phrases. blocks of cyclotomic Hecke algebra; representation type; weight; abacus; cellular algebra.

The work is supported by the Natural Science Foundation of Hebei Province, China (A2021501002); China Scholarship Council (202008130184) and Natural Science Foundation of China (11871107).

1. Introduction

The cyclotomic Hecke algebras of type $G(r, 1, n)$ (Ariki-Koike algebra) were introduced in [8], [14] and [17] which include Hecke algebras of types A and B as special cases. They play an important role in modular representation theory of finite groups of Lie type, and are in relation with various significant objects, such as quantum groups and rational Cherednik algebras. Consequently, these algebras have received intensively and continuously study since their appearance. The interest on these algebras has been strengthened by a result of Brundan and Kleshchev [15], in which an explicit isomorphism between blocks of cyclotomic Hecke algebras and type A cyclotomic Khovanov-Lauda-Rouquier algebras were constructed, and many profound results emerged. For example, Jun Hu and Lei Shi proved the famous center conjectures in [31] recently. However, it is still an open problem to determine the representation type of blocks of a cyclotomic Hecke algebra.

In general, it is an essential and difficult problem in representation theory to determine the representation type of a finite dimensional algebra. Recall that a finite dimensional algebra A has finite representation type if the number of indecomposable A-modules up to isomorphism is finite. Otherwise, A has infinite representation type. If A has infinite type, then Drozd [21] proved that A is either tame or wild and not both (we refer the reader to [21] for relevant definitions). Back to Hecke algebras, the representation type of blocks of a Hecke algebra of type A was determined by Erdmann and Nakano in [23]. Then Ariki [5] determined representation type for all the block algebras of Hecke algebras of other classical types by using a result obtained in [7] and the techniques developed in a series of papers [6, 10, 11, 12]. In 2010, Wada [50] gave a necessary and sufficient condition on parameters for a type $G(r, 1, n)$ cyclotomic Hecke algebra to have finite representation type. Regarding to the blocks, as far as we know, there are two kinds of relevant results, which are both given by Ariki and his collaborators. On one hand, the representation type of some special cases are determined in [6, 10], in which the main tool is cyclotomic Khovanov-Lauda-Rouquier algebras. On the other hand, Ariki et al proved that an indecomposable self-injective cellular algebra A has finite representation type if and only if A is isomorphic to a Brauer tree algebra whose Brauer tree is a straight line with at most one exceptional vertex [7, Theorem 6.8]. However, given a finite dimensional algebra, it is in general difficult to determine the structures of all its indecomposable projective modules and so one can not determine the representation type of a block directly by the result aforementioned. That is, it is still an open problem to determine when a block of a cyclotomic Hecke algebra of type $G(r, 1, n)$ has finite representation type. Our main purpose in this paper is to study this problem.

In order to describe the main result precisely, we need to recall some definitions and fix some notations, which are also useful in other sections. Let K be an algebraically closed field, $1 \neq q \in K^{\times}$. Define the quantum characteristic of q to be the positive integer e which is minimal such that $1 + q + \cdots + q^{e-1} = 0$. If no such e exists, we set $e = \infty$. Fix positive integers n and r. Let $(s_1, s_2, \cdots, s_r) \in \mathbb{Z}^r$ be a multicharge and define $Q = (Q_1, Q_2, \cdots, Q_r) \in K^r$ with $Q_i = q^{s_i}$. Then $\mathcal{H}_n(q, Q)$, the cyclotomic algebra of type $G(r, 1, n)$ with parameters q and Q, is the unital associative K-algebra with generators T_0, T_1, $\cdots$, T_{n-1} subject to the following relations:

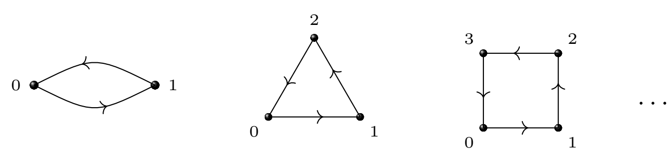
(H1) $(T_0 - Q_1)(T_0 - Q_2) \cdots (T_0 - Q_r) = 0$;
(H2) $T_0T_1T_0T_1 = T_1T_0T_1T_0$;
(H3) $(T_i + 1)(T_i - q) = 0$ for $1 \leq i \leq n - 1$;
(H4) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ for $1 \leq i \leq n - 2$;
(H5) $T_iT_j = T_jT_i$ for $0 \leq i < j - 1 \leq n - 2$.

The cyclotomic q-Schur algebra $\mathcal{S}_{n,r}(q, Q_1, Q_2, \cdots, Q_r)$ associated to $\mathcal{H}_n(q, Q)$ is the endomorphism algebra $\mathcal{S}_{n,r} = \mathrm{End}_{\mathcal{H}_n(q,Q)}(\bigoplus_{\boldsymbol{\mu}} M^{\boldsymbol{\mu}})$, where $\boldsymbol{\mu}$ runs all over the r-multipartitions of n and where $M_{\boldsymbol{\mu}}$ is a certain $\mathcal{H}_n(q, Q)$-module (see [19] and [44] for more details).

Now let us connect $\mathcal{H}_n(q, Q)$ to Lie theory. Let Γ_e be the oriented quiver with vertex set $I = \mathbb{Z}/e\mathbb{Z}$ and with directed edges $i \longrightarrow i + 1$, for all $i \in I$. Note that $I = \mathbb{Z}$ if $e = \infty$. Thus, Γ_e is the quiver of type A_∞ if $e = \infty$, and if $e \geq 2$ then it is a cyclic quiver of type $A^{(1)}_{e-1}$:

$$A_\infty : \cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

A^1_{e-1} :



$e = 2$ $\qquad$ $e = 3$ $\qquad$ $e = 4$

Let $(a_{i,j})_{i,j \in I}$ be the symmetric Cartan matrix associated with Γ_e, so that

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \neq j \pm 1, \\ -1 & \text{if } e \neq 2 \text{ and } i = j \pm 1, \\ -2 & \text{if } e = 2 \text{ and } i = j + 1. \end{cases}$$

Following Kac [35, Chapt. 1], let $(\mathfrak{h}, \Pi, \check{\Pi})$ be a realization of the Cartan matrix, and $\Pi = \{\boldsymbol{\alpha}_i \mid i \in I\} \subset \mathfrak{h}^*$ the associated set of simple roots, $\{\boldsymbol{\Lambda}_i \mid i \in I\} \subset \mathfrak{h}^*$ the fundamental dominant weights, and $(\cdot, \cdot)$ the bilinear form determined by

$$(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j) = a_{i,j} \qquad \text{and} \qquad (\boldsymbol{\Lambda}_i, \boldsymbol{\alpha}_j) = \delta_{ij}, \qquad \text{for } i, j \in I.$$

Denote by $P = \bigoplus_{i \in I} \mathbb{Z}\boldsymbol{\Lambda}_i$ and $P_+ = \bigoplus_{i \in I} \mathbb{N}\boldsymbol{\Lambda}_i$ the weight lattice and the dominant weight lattice of $(\mathfrak{h}, \Pi, \check{\Pi})$, $Q_+ = \bigoplus_{i \in I} \mathbb{N}\boldsymbol{\alpha}_i$ the positive root lattice, and W the affine Weyl group, which is generated by $\{\sigma_i\}_{i \in I}$, the fundamental reflections of the space $\mathfrak{h}^*$. Note that $\sigma_i\mathbf{\Lambda} = \mathbf{\Lambda} - (\mathbf{\Lambda},\, \boldsymbol{\alpha}_i)\boldsymbol{\alpha}_i$ for $\mathbf{\Lambda} \in P$ and $\sigma_i\boldsymbol{\alpha}_j = \boldsymbol{\alpha}_j - a_{ji}\boldsymbol{\alpha}_i$ for arbitrary $j \in I$. The Kac-Moody Lie algebra corresponding to this data is $\widehat{\mathfrak{sl}}_e$.

Given a multicharge $\mathbf{s} = (s_1, s_2, \cdots, s_r)$, one can associate it with a dominant weight $\mathbf{\Lambda} = \sum_{i\in I} k_i\mathbf{\Lambda}_i$, where

$$k_i = \begin{cases} \sharp\{1 \le j \le r \mid s_j \equiv i \,(\mathrm{mod}\, e)\} & \text{if } e < \infty, \\ \sharp\{1 \le j \le r \mid s_j = i\} & \text{if } e = \infty, \end{cases}$$

and the algebra $\mathcal{H}_n(q, Q)$ will be denoted by $\mathcal{H}_n^{\mathbf{\Lambda}}$ if needed. By [15, Section 4.1], there is a natural system $\{e(\mathbf{i}) \mid \mathbf{i} \in I^n\}$ (some could be zero) of pairwise orthogonal idempotents in $\mathcal{H}_n^{\mathbf{\Lambda}}$. Take $\boldsymbol{\beta} \in Q_+$ with $\sum_{i\in I}(\mathbf{\Lambda}_i, \boldsymbol{\beta}) = n$ and let

$$I^{\boldsymbol{\beta}} = \{\mathbf{i} = (i_1, i_2, \cdots, i_n) \in I^n \mid \boldsymbol{\alpha}_{i_1} + \cdots + \boldsymbol{\alpha}_{i_n} = \boldsymbol{\beta}\}.$$

If $I^{\boldsymbol{\beta}} \neq \varnothing$, then by [44, Theorem 2.11], $e_{\boldsymbol{\beta}} = \sum_{\mathbf{i}\in I^{\boldsymbol{\beta}}} e(\mathbf{i})$ is a primitive central idempotent of $\mathcal{H}_n^{\Lambda}$. Write by $\mathcal{H}_{\boldsymbol{\beta}}^{\mathbf{\Lambda}}$ the block algebra $e_{\boldsymbol{\beta}}\mathcal{H}_n^{\mathbf{\Lambda}}$. Then

$$\mathcal{H}_n^{\mathbf{\Lambda}} = \bigoplus_{\boldsymbol{\beta}\in Q_+,\, I^{\boldsymbol{\beta}}\neq\emptyset} \mathcal{H}_{\boldsymbol{\beta}}^{\mathbf{\Lambda}}$$

is the decomposition of $\mathcal{H}_n^{\mathbf{\Lambda}}$ into a direct sum of blocks.

In [16, (3.4)], Brundan, Kleshchev and Wang defined the defect of $\boldsymbol{\beta} \in Q_+$ to be $(\mathbf{\Lambda}, \boldsymbol{\beta}) - \frac{1}{2}(\boldsymbol{\beta}, \boldsymbol{\beta})$. It coincides with Fayers [24] definition of weight for the block algebras $\mathcal{H}_{\boldsymbol{\beta}}^{\mathbf{\Lambda}}$ and will be denoted by $w(\mathcal{H}_{\boldsymbol{\beta}}^{\mathbf{\Lambda}})$.

In addition, we emphasize that using the derived equivalence given by Chuang and Rouquier [18], which lifts Weyl group action, two blocks $\mathcal{H}_{\boldsymbol{\alpha}}^{\mathbf{\Lambda}}$ and $\mathcal{H}_{\boldsymbol{\beta}}^{\mathbf{\Lambda}}$ of cyclotomic Hecke algebras are derived equivalent if $\mathbf{\Lambda} - \boldsymbol{\alpha}$ and $\mathbf{\Lambda} - \boldsymbol{\beta}$ are in the same W-orbit.

The following is the main result of this paper.

Main Theorem. *Let K be an algebraically closed field with $CharK \neq 2$ and $q \in K^\times$, $q \neq 1$. Let $\mathcal{H}_{\boldsymbol{\beta}}^{\mathbf{\Lambda}}$ be a block of $\mathcal{H}_n(q, Q)$, that is a cyclotomic Hecke algebra of type $G(r, 1, n)$ with $r > 2$. Then the following are equivalent.*

(1) *Block $\mathcal{H}_{\boldsymbol{\beta}}^{\mathbf{\Lambda}}$ has finite representation type.*

(2) *The weight of $\mathcal{H}_{\boldsymbol{\beta}}^{\mathbf{\Lambda}}$ is less than 2, or $\mathcal{H}_{\boldsymbol{\beta}}^{\mathbf{\Lambda}}$ is Morita equivalent to $K[x]/x^{w(\mathcal{H}_{\boldsymbol{\beta}}^{\mathbf{\Lambda}})+1}$.*

Let us describe in some details our approach to the proof of the above theorem. It is generally believed that the weight can measure how complicated a block is. We have no doubt on it in the case of type A because some well-known evidences, for example, a block has finite representation if and only if the weight is less than 2. However, this is no longer true for $\mathcal{H}_n(q, Q)$. A block with a small weight is relatively simple, but the opposite is not correct in general. As a result, most results of type A that are related with wights have not cyclotomic versions, including representation type of blocks. In fact, the representation type of a block of $\mathcal{H}_n(q, Q)$ can not be determined by its weight completely. Consequently we need to look for a generalization of the weight of a block. Thanks to [7, Theorem 6.8] aforementioned, we first studied the structure of a finite representation type cellular Brauer tree algebra, and found that the so-called multiplication poset has to be a totally ordered set. This simple observation leaded us to picking out the blocks whose multiplication posets are not totally ordered sets. The main method is to construct incomparable abaci. During the study, we gradually realized that the structure of a multiplication poset, to some extent, is controlled by the process of an arbitrary abacus in the block moving to its core through elementary operations, which were introduced in [33] by Jacon and Lecouvey. Accordingly, we defined a new invariant for each block, which is called block move vector (see Definition 3.4.5 for details). It is a key concept in this paper. Roughly speaking, just as its name implies, a block move vector is a vector that records the process of an abacus in the block moving to its core. Note that the sum of components of a block move vector is just the weight of the corresponding block. This is just the reason why we say that block move vectors generalize weights. Fortunately, one can determine all of the blocks of finite representation type with weight more than one by a criterion about block move vectors.

As by-products, our results can be used to determine the representation type of blocks of a cyclotomic q-Schur algebra. We can also study the derived equivalence among blocks of $\mathcal{H}_n(q, Q)$. It is well-known that two blocks of a Hecke algebra of type A are derived equivalent if and only if they have the same weight, and if and only if they are in the same orbit under the adjoint action of the affine Weyl group. This is no longer right in cyclotomic case. We will construct examples of blocks with the same weight that are not derived equivalent. Moreover, examples of derived equivalent blocks in different orbits will also be given.

The paper is organized as follows. We begin our study with the multiplication poset of an indecomposable selfinjective cellular algebra A in Section 2. The main result is that if A is of finite representation type, then the multiplication poset must be a totally ordered set. We emphasize that the result can help us determine the representation type of most blocks in a cyclotomic algebra and may be used to consider blocks of other algebras. Note that the cell modules of a cyclotomic Hecke algebra are indexed by multipartitions and each multipartition can be represented by an abaci. Then in Section 3, after give some preliminaries on blocks and abaci, we study properties of abaci in details. Firstly, we give a description of the action of an affine Weyl group on blocks by the language of abaci. Secondly, we define the so-called incomparable abaci and prove that given a block, the existence of a pair of incomparable abaci implies the existence of a pair of incomparable multipartitions with respect to dominance order, which leads to infinity of the representation type by Section 2. Thirdly, we define the (block) move vector and conduct some preliminary study. Finally, we classify blocks and provide a framework of the proof of Main theorem by block move vectors. Based on the preparation given in Section 2 and Section 3, We begin the proof in Section 4, which handles the case that at least one component of block move vector is at least 2. In Section 5, we deal with the case of all components of the block move vector being equal to 1 and Section 6 is devoted to the case left. In Section 7, we apply our results obtained to study derived equivalence and in Section 8, we determine the representation type of all blocks of a cyclotomic q-Schur algebra.

2. Poset and representation type

Let A be an indecomposable selfinjective cellular K-algebra with cell datum $(\Lambda, M, C, *)$, where K is a field with the characteristic different from two. We prove in this section that if A is of finite type, then the poset Λ must be a totally ordered set. This result will play a very important role in the whole paper.

We begin with the definitions and some well-known results of cellular algebras and Brauer tree algebras. The main references are [1, 13, 28]. The main result of this section is given in Subsection 2.3.

2.1. **Cellular algebras.** Cellular algebras were introduced by Graham and Lehrer in [28] in 1996. Cellular theory provides a systematic framework for studying the representation theory of many interesting and important algebras coming from mathematics and physics, including Ariki-Koike algebras, the main research object of this paper. In particular, an Ariki-Koike algebra satisfying assumptions of Main theorem is symmetric. The reader can find more information about symmetric cellular algebras in [41, 42, 43].

Definition 2.1.1 ([28, Definition 1.1]). *Let R be a noetherian commutative integral domain. An associative unital R-algebra A is called a cellular algebra with cell datum $(\Gamma, M, C, *)$ if the following conditions are satisfied:*

- *(C1) The finite set Γ is a poset with order relation $\geq$. Associated with each $\lambda \in \Gamma$, there is a finite set $M(\lambda)$. The algebra A has an R-basis $\{C^{\lambda}_{S,T} \mid \lambda \in \Gamma,\ S, T \in M(\lambda)\}$.*
- *(C2) The map $*$ is an R-linear anti-automorphism of A such that $(C^{\lambda}_{S,T})^* = C^{\lambda}_{T,S}$ for all $\lambda \in \Lambda$ and $S, T \in M(\lambda)$.*
- *(C3) Let $\lambda \in \Lambda$ and $S, T \in M(\lambda)$. For any element $a \in A$, we have*

$$aC^{\lambda}_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^{\lambda}_{S',T} \mod \mathrm{A}(> \lambda),$$

where $r_a(S^{'}, S) \in R$ is independent of T and $A(> \lambda)$ is the R-submodule of A generated by $\{C^{\mu}_{U,V} \mid U,\ V \in M(\mu),\ \mu > \lambda\}$.

Before giving a rather lengthy list of definitions associated with cellular algebras, let us give a remark about the poset Γ in Definition 2.1.1. Firstly, we define another poset Λ, which is equal to Γ as a set, and a partial ordering is as follows. For arbitrary two elements $\lambda, \mu \in \Lambda$, we say $\lambda \geq \mu$ if and only if there exist some $C^{\lambda}_{S,T}$ and $a \in A$ such that certain $C^{\mu}_{U,V}$ appears in the linear expansion of $aC^{\lambda}_{S,T}$ with nonzero coefficient. Clearly, Γ is a refinement of Λ. It is worthwhile to note that Λ is more essential than Γ for our study and will be called the multiplication poset of A. It is easy to check that $(\Lambda, M, C, *)$ is a cell datum of A too, and we will always instead poset Γ by Λ throughout this section unless otherwise specified.

We now turn to give a number of basic definitions and results connected with cellular algebras. As a natural consequence of the axioms, the cell module is defined as follows.

Definition 2.1.2 ([28, Definition 2.1]). *Let A be a cellular algebra with cell datum $(\Lambda, M, C, *)$. For each $\lambda \in \Lambda$, the cell module W_{λ} is an R-module with basis $\{C_S \mid S \in M(\lambda)\}$ and the left A-action defined by*

$$aC_S = \sum_{S' \in M(\lambda)} r_a(S^{'}, S) C_{S'} \quad (a \in A,\ S \in M(\lambda)),$$

where $r_a(S^{'}, S)$ is the element of R defined in Definition 2.1.1 (C3).

Let A be a cellular algebra with cell datum $(\Lambda, M, C, *)$. For arbitrary elements $S, T, U, V \in M(\lambda)$, Definition 2.1.1 implies that

$$C^{\lambda}_{S,T} C^{\lambda}_{U,V} \equiv \Phi(T,U) C^{\lambda}_{S,V} \mod \mathrm{A}(> \lambda),$$

where $\Phi(T,U) \in R$ depends only on T and U. It is easy to check that $\Phi(T,U) = \Phi(U,T)$ for arbitrary $T, U \in M(\lambda)$. By using these $\Phi(T,U)$, one can define a bilinear form for cell module W_λ introduced in Definition 2.1.2:

$$\begin{aligned} \Phi_\lambda : \ W_\lambda \times W_\lambda &\longrightarrow R \\ (C_S,\, C_T) &\longmapsto \Phi(S,T) \end{aligned}$$

Define

$$\operatorname{rad} \lambda := \{x \in W_\lambda \mid \Phi_\lambda(x,y) = 0 \ \text{for all} \ y \in W_\lambda\}.$$

If $\Phi_\lambda \neq 0$, then $\operatorname{rad}\lambda$ is the radical of the A-module W_λ.

When R is a field, Graham and Lehrer [28] proved the following result.

Lemma 2.1.3. [28, Theorem 3.4] *For any $\lambda \in \Lambda$, denote the A-module $W_\lambda / \operatorname{rad}\lambda$ by L_λ. Let $\Lambda_0 = \{\lambda \in \Lambda \mid \Phi_\lambda \neq 0\}$. Then $\{L_\lambda \mid \lambda \in \Lambda_0\}$ is a complete set of (representative of equivalence classes of) absolutely simple A-modules.*

For $\lambda \in \Lambda$ and $\mu \in \Lambda_0$, let $d_{\lambda\mu}$ be the multiplicity of L_μ in W_λ. Sometimes, we write $d_{\lambda\mu}$ as $[W_\lambda : L_\mu]$. Denote the matrix $(d_{\lambda\mu})_{\lambda\in\Lambda,\, \mu\in\Lambda_0}$ by D, which will be called the decomposition matrix of A.

Lemma 2.1.4. [28, Proposition 3.6] *Let $\lambda \in \Lambda$ and $\mu \in \Lambda_0$. Then $d_{\mu\mu} = 1$. Moreover, if $d_{\lambda\mu} \neq 0$, then $\lambda \geq \mu$.*

An equivalent basis-free definition of a cellular algebra was given by Koenig and Xi [36], which is useful in dealing with structural problems.

Definition 2.1.5. [36, Definition 3.2] *Let A be an algebra over a noetherian commutative integral domain R with an R-involution $*$. A two-sided ideal J in A is called a cell ideal if and only if the following data are given and the following conditions are satisfied:*

(1) The ideal J is fixed by $*$: $(J)^* = J$.

(2) There exists a free R-module $\Delta \subset J$ of finite rank, such that there is an isomorphism of A-bimodules $\alpha : J \simeq \Delta \otimes_R \Delta^*$ ($\Delta^* \subset J$ is the $*$-image of Δ) making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\ \alpha\ } & \Delta \otimes_R \Delta^* \\ {\scriptstyle *}\big\downarrow & & \big\downarrow{\scriptstyle v_1 \otimes v_2 \mapsto v_2^* \otimes v_1^*} \\ J & \xrightarrow{\ \alpha\ } & \Delta \otimes_R \Delta^* \end{array}$$

The algebra A with R-involution $*$ is called cellular if and only if there is an R-module decomposition $A = J'_{\mu_1} \oplus J'_{\mu_2} \oplus \cdots J'_{\mu_m}$ (for some m) with $(J'_{\mu_j})^* = J'_{\mu_j}$ for each j ($j = 1, \ldots, m$) and such that setting $J_{\mu_j} := \bigoplus_{l=1}^{j} J'_{\mu_l}$ gives a chain of two-sided ideals of A:

$$0 = J_{\mu_{m+1}} \subset J_{\mu_m} \subset J_{\mu_{m-1}} \subset J_{\mu_{m-2}} \subset \cdots \subset J_{\mu_1} = A$$

each of them fixed by $*$, and each $J'_{\mu_j} = J_{\mu_j}/J_{\mu_{j+1}}$ is a cell ideal of $A/J_{\mu_{j+1}}$ (with respect to the involution induced by $*$ on the quotient).

For $\lambda \in \Lambda_0$, let P_λ be the projective cover of L_λ. By applying the functor $-\otimes_A P_\lambda$ to the cell chain of A in Definition 2.1.5 gives a sequence of A-submodules

$$0 = P_\lambda(\mu_{m+1}) \subset P_\lambda(\mu_m) \subset P_\lambda(\mu_{m-1}) \subset P_\lambda(\mu_{m-2}) \subset \cdots \subset P_\lambda(\mu_1) = P_\lambda,$$

which is in fact a cell filtration of P_λ.

Lemma 2.1.6. [28, Theorem 3.7] *For $\lambda \in \Lambda_0$, denote the multiplicity of a cell module W_μ in P_λ by $[P_\lambda : W_\mu]$. Then $[P_\lambda : W_\mu] = [W_\mu : L_\lambda]$. This implies that a simple module L_λ is a composition factor of a cell module W_μ if and only if W_μ is a factor of P_λ.*

We should note that given a cellular algebra with cell datum $(\Lambda, M, C, *)$, the set $\{\mu_1, \mu_2, \cdots, \mu_m\}$ in Definition 2.1.5 is in fact a linear extension of Λ. However, we can find some more subtle structure of P_λ if we work with the original poset Λ. The following lemma is a simple corollary of Lemma 2.1.6.

Lemma 2.1.7. *Let $\lambda \in \Lambda_0$, $\mu, \nu \in \Lambda$ and μ, ν incomparable, denoted by $\mu \parallel \nu$. Then $W_\nu^{\oplus d_{\nu\lambda}} \oplus W_\mu^{\oplus d_{\mu\lambda}}$ is a subquotient of P_λ.*

Proof. Clearly, $\overline{A} := A/(A(> \nu) + A(> \mu))$ is a cellular algebra. By abusing the notations, we still denote the cell modules corresponding to ν and μ by W_ν and W_μ, respectively. Since $\mu \parallel \nu$, $J'_\nu \oplus J'_\mu$ is an ideal of $\overline{A}$, and thus W_ν and W_μ are both submodules of a quotient of P_λ. Now the lemma is clear by Lemma 2.1.6. □

2.2. **Brauer tree algebras.** The main references of this subsection are [1, 13]. A Brauer tree is a finite tree together with two additional structures on each vertex i:

(1) a circular ordering of the edges adjacent to i;
(2) a positive integer $m(i)$, called the multiplicity satisfying at most one $m(i)$ is larger than one.

The vertex with multiplicity more than one is called the exceptional vertex, which is drawn customarily as a blackened circle. A finite dimensional algebra A is called a Brauer tree algebra if the structure of the indecomposable projective modules can be described in terms of a Brauer tree in the following way.

(1) There is a one to one correspondence between the edges α of the tree and the indecomposable projective A-modules P_α and hence the corresponding simple A-modules L_α.
(2) For an edge α connecting vertices i and j, let $(\alpha = \alpha_1, \cdots, \alpha_t)$ and $(\alpha = \beta_1, \cdots, \beta_r)$ be the circular orderings of edges adjacent to i and j, respectively. Then $\operatorname{rad} P_\alpha = U_\alpha + V_\alpha$, where $U_\alpha \cap V_\alpha = L_\alpha$, U_α is uniserial with composition factors $L_{\alpha_2}, \cdots, L_{\alpha_t}, L_{\alpha_1}$, $m(i)$ times from top to bottom, and V_α is uniserial with composition factors $L_{\beta_2}, \cdots, L_{\beta_r}, L_{\beta_1}$, $m(j)$ times from top to bottom.

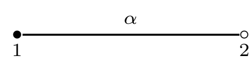
Given a Brauer tree T, let Γ be the quiver whose vertices are in one to one correspondence with the edges of T. For a vertex i in T, the circular ordering $(\alpha_1, \cdots, \alpha_t)$ give rise to an oriented cycle C_i. Then the vertex v_α in Γ corresponding to the edge α in T connecting vertices i and j belongs exactly two oriented cycles C_i and C_j. Denote the path in C_i without repeated arrows starting and ending at α_k by $p^{(i)}_{\alpha_t}$. Define relations ρ to be $(p^{(i)}_\alpha)^{m(i)} - (p^{(j)}_\alpha)^{m(j)}, uv$, where u is the arrow in C_i with $e(u) = \alpha$ and v the arrow in C_j with $s(v) = \alpha$, or u is the arrow in C_j

with $e(u) = \alpha$ and v the arrow in C_i with $s(v) = \alpha$. Then the algebra $K(\Gamma, \rho)$ is a Brauer tree algebra given by T. It is worth to note that a Brauer tree determines a unique Brauer tree algebra up to Morita equivalence (see [38, Corollary 4.3.3] for details). Furthermore, the cellularity of a Brauer tree algebra was studied by Koenig and Xi in [36].

Lemma 2.2.1. [36, Proposition 5.3] *A Brauer tree algebra is cellular if and only if the Brauer tree is a straight line.*

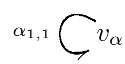
Let us illustrate some examples for later use.

Example 2.2.2. (1) *Let A be a Brauer tree algebra for the following Brauer tree T_1.*



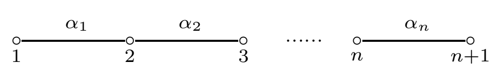
Then the indecomposable projective module is uniserial with composition factor L_α, $m(1)+1$ times from top to bottom.

Let Γ_{I} be the quiver



and let ρ be $(\alpha_{1,1})^{m(1)+1}$. Then $K(\Gamma_{\mathrm{I}}, \rho)$ is a Brauer tree algebra given by T_1.

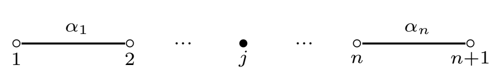
(2) *Let A be a Brauer tree algebra for the following Brauer tree T_2 with $n > 1$.*



Then the structures of indecomposable projective A-modules are illustrated as follows.

$$P_{\alpha_1} = \begin{pmatrix} L_{\alpha_1} \\ L_{\alpha_2} \\ L_{\alpha_1} \end{pmatrix}; \quad P_{\alpha_2} = \begin{pmatrix} & L_{\alpha_2} & \\ L_{\alpha_1} & & L_{\alpha_3} \\ & L_{\alpha_2} & \end{pmatrix}; \quad \cdots \quad P_{\alpha_n} = \begin{pmatrix} L_{\alpha_n} \\ L_{\alpha_{n-1}} \\ L_{\alpha_n} \end{pmatrix}.$$

(3) *Let A be a Brauer tree algebra for the following Brauer tree T_3 with $n > 1$.*



The indecomposable projective A-modules are illustrated as follows.

$$P_{\alpha_1} = \begin{pmatrix} L_{\alpha_1} \\ L_{\alpha_2} \\ L_{\alpha_1} \end{pmatrix}, \ j \neq 1, 2; \qquad P_{\alpha_{1=j}} = \begin{pmatrix} & L_{\alpha_1} & \\ L_{\alpha_1} & & L_{\alpha_2} \\ \cdots & & \\ L_{\alpha_1} & & \\ & L_{\alpha_1} & \end{pmatrix}; \quad \cdots;$$

$$P_{\alpha_{j-1}} = \begin{pmatrix} & L_{\alpha_{j-1}} & \\ & & L_{\alpha_j} \\ L_{\alpha_{j-2}} & & L_{\alpha_{j-1}} \\ & & \vdots \\ & & L_{\alpha_j} \\ & L_{\alpha_{j-1}} & \end{pmatrix}; \qquad P_{\alpha_j} = \begin{pmatrix} & L_{\alpha_j} & \\ L_{\alpha_{j-1}} & & \\ L_{\alpha_j} & & L_{\alpha_{j+1}} \\ \vdots & & \\ L_{\alpha_{j-1}} & & \\ & L_{\alpha_j} & \end{pmatrix}; \quad \cdots;$$

$$P_{\alpha_n} = \begin{pmatrix} L_{\alpha_n} \\ L_{\alpha_{n-1}} \\ L_{\alpha_n} \end{pmatrix}, \ j \neq n+1; \qquad P_{\alpha_n} = \begin{pmatrix} & L_{\alpha_n} & \\ L_{\alpha_{n-1}} & & L_{\alpha_n} \\ & & \cdots \\ & & L_{\alpha_n} \\ & L_{\alpha_n} & \end{pmatrix}, \ j = n+1.$$

2.3. **Posets of finite type self-injective cellular algebras.** Let A be a Brauer tree algebra whose Brauer tree is a straight line. Then A is a cellular algebra by Lemma 2.2.1. Assume that $(\Lambda, M, C, *)$ is a cell datum of A. Then by Lemma 2.1.3, there is a one to one correspondence between the set of edges α_i and Λ_0. For simplicity, we still denote the image of α_i in Λ_0 by α_i. Let us determine the form of a cell module of A.

Lemma 2.3.1. *Keep notations as above and let W_λ be a cell module. We have*

(1) $d_{\lambda\alpha_i} \leq 1$ *for all* i.

(2) *Neither* $\begin{pmatrix} L_{\alpha_{i-1}} & & L_{\alpha_{i+1}} \\ & L_{\alpha_i} & \end{pmatrix}$ *nor* $\begin{pmatrix} L_{\alpha_1} & & L_{\alpha_2} \\ & L_{\alpha_1} & \end{pmatrix}$ *is a submodule of a cell module* W.

(3) *Neither* $\begin{pmatrix} & L_{\alpha_i} & \\ L_{\alpha_{i-1}} & & L_{\alpha_{i+1}} \end{pmatrix}$ *nor* $\begin{pmatrix} & L_{\alpha_n} & \\ L_{\alpha_{i-1}} & & L_{\alpha_n} \end{pmatrix}$ *is a quotient module of a cell module* W.

Proof. (1) If $d_{\lambda\alpha_i} > 1$ for some α_i, then by Lemma 2.1.7, $W_\lambda^{\oplus d_{\lambda\alpha_i}}$ is a subquotient of P_{α_i}. This is impossible for a Brauer tree algebra whose Brauer tree is a straight line. We refer the reader to Example 2.2.2 for structures of P_{α_i}.

(2) If there exists a cell module W containing a submodule, which is of the form $\begin{pmatrix} L_{\alpha_{i-1}} & & L_{\alpha_{i+1}} \\ & L_{\alpha_i} & \end{pmatrix}$ $(i > 1)$, then we conclude from Lemma 2.1.6 that $L_{\alpha_{i+1}}$ is a composition factor of $P_{\alpha_{i-1}}$. However, according to the definition of a Brauer tree algebra, the possible composition factors of $P_{\alpha_{i-1}}$ are $L_{\alpha_{i-2}}$, $L_{\alpha_{i-1}}$ and L_{α_i}. It is a contradiction.

Moreover, if $j \neq 1$, then clearly, $\begin{pmatrix} L_{\alpha_1} & & L_{\alpha_2} \\ & L_{\alpha_1} & \end{pmatrix}$ is not a submodule of any cell module W. If $j = 1$ and $\begin{pmatrix} L_{\alpha_1} & & L_{\alpha_2} \\ & L_{\alpha_1} & \end{pmatrix}$ is a submodule of some cell module W, then the multiplicity of L_{α_1} in P_{α_2} is at least 2. This is impossible.

(3) is proved similarly as (2). $\square$

Corollary 2.3.2. *For a cellular Brauer tree algebra, each cell module W_μ is of the form L_{α_1}, L_{α_n}, $\begin{pmatrix} L_{\alpha_{i-1}} \\ L_{\alpha_i} \end{pmatrix}$ or $\begin{pmatrix} L_{\alpha_{i+1}} \\ L_{\alpha_i} \end{pmatrix}$*

Proof. An immediate corollary of Lemma 2.1.6 is that each cell module must be a cell factor of an indecomposable projective module. Apply Lemma 2.3.1 to the structures of indecomposable projective modules of a cellular Brauer tree algebra A. We get that a cell module W_λ must be of the form L_{α_i}, $\begin{pmatrix} L_{\alpha_{i-1}} \\ L_{\alpha_i} \end{pmatrix}$ or $\begin{pmatrix} L_{\alpha_{i+1}} \\ L_{\alpha_i} \end{pmatrix}$. Hence we only need to prove each L_{α_i} with $1 < i < n$ is not a cell module. In fact, L_{α_i} with $1 < i < n$ is a composition factor of $P_{\alpha_{i-1}}$. If L_{α_i} a cell module, then by Lemma 2.1.6, $[L_{\alpha_i} : L_{\alpha_{i-1}}] = [P_{\alpha_{i-1}} : L_{\alpha_i}] \geq 1$. It is a contradiction. $\square$

Definition 2.3.3. *We call the cell modules $\begin{pmatrix} L_{\alpha_{i-1}} \\ L_{\alpha_i} \end{pmatrix}$ are of type I and $\begin{pmatrix} L_{\alpha_{i+1}} \\ L_{\alpha_i} \end{pmatrix}$ are of type II.*

Let us investigate the cell filtration of indecomposable projective modules P_{α_i}. We begin with P_{α_1}.

Lemma 2.3.4. *Let A be a cellular Brauer tree algebra with cell datum $(\Lambda, M, C, *)$. We have*

(1) If vertex 1 is not exceptional, then a cell filtration of P_{α_1} is either of the form $L_{\alpha_1} \subset \begin{pmatrix} L_{\alpha_1} \\ L_{\alpha_2} \\ L_{\alpha_1} \end{pmatrix} \subset \cdots \subset P_{\alpha_1}$, or of the form $\begin{pmatrix} L_{\alpha_2} \\ L_{\alpha_1} \end{pmatrix} \subset \begin{pmatrix} L_{\alpha_2} \\ L_{\alpha_1} \\ L_{\alpha_2} \\ L_{\alpha_1} \end{pmatrix} \subset \cdots \subset P_{\alpha_1}$.

(2) If vertex 1 is exceptional, then cell filtration of P_{α_1} is either of the form $L_{\alpha_1} \subset \begin{pmatrix} L_{\alpha_1} \\ L_{\alpha_1} \end{pmatrix} \subset \cdots \subset P_{\alpha_1}$, or of the form $\begin{pmatrix} L_{\alpha_2} \\ L_{\alpha_1} \end{pmatrix} \subset \begin{pmatrix} L_{\alpha_1} \\ L_{\alpha_2} \\ L_{\alpha_1} \end{pmatrix} \subset \cdots \subset P_{\alpha_1}$.

Proof. It follows from Corollary 2.3.2 that each L_{α_2} is either a submodule of a type I cell module or a quotient module of a type II cell module defined in Definition 2.3.3.

(1) If vertex 1 is not exceptional, then P_{α_1} is uniserial and $\operatorname{rad} P_{\alpha_1}$ has composition factors $(L_{\alpha_2}, L_{\alpha_1})$, $m(2)$ times from top to bottom. A consequence of Lemma 2.3.2 is that each L_{α_2} has to combine with an L_{α_1} to form a cell module. This forces the top or the socle of P_{α_1} to be a cell module. If the socle of P_{α_1} is a cell module, then all the other $m(2)$ cell modules have to be of type I, and if the top is a cell module, then all the others have to be of type II.

(2) When vertex 1 is exceptional, only one L_{α_2} appears in P_{α_1}, and this L_{α_2} combining with the top or the socle of P_{α_1} forms a cell module. Now by Corollary 2.3.2, each L_{α_1} left is a cell module. $\square$

A interesting result about the cellular structure of a cellular Brauer tree algebra is that all the cell factors of P_{α_i} will be determined once the type of cell factors of P_{α_1} is fixed. We describe it as a lemma as follows for later use.

Lemma 2.3.5. *Let A be a cellular Brauer tree algebra with cell datum $(\Lambda, M, C, *)$. Then all cell modules with two composition factors are of the same type.*

Proof. Assume that the cell factors with two composition factors of P_{α_1} are of type I. By Lemma 2.1.6, all of these cell modules are cell factors of P_{α_2}. Then the structure of P_{α_2} forces $\operatorname{top} P_{\alpha_2}$ to be a quotient module of a cell module $\begin{pmatrix} L_{\alpha_2} \\ L_{\alpha_3} \end{pmatrix}$, which is of type I. As a result, all cell factors with two composition factors of P_{α_2} are of type I. Continuing this analysis for P_{α_i} one by one, we can deduce all cell modules with two composition factor are of type I.

A similar analysis is efficient too when the cell factors with two composition factors of P_{α_1} are of type II, and then we have completed the proof. $\square$

Based on the above preparation, we can study the poset of a cellular Brauer tree algebra.

Lemma 2.3.6. *Let A be a cellular Brauer tree algebra with cell datum $(\Lambda, M, C, *)$. Then Λ is a totally ordered set.*

Proof. Denote the set of indexes of cell factors of P_{α_i} by Λ_{α_i}. We claim that Λ_{α_i} is a totally ordered set. In fact, let W_{α_i} be the cell module with $\operatorname{top} W_{\alpha_i} = \operatorname{top} P_{\alpha_i}$. Then according to Lemma 2.3.2, 2.3.4 and 2.3.5, $\operatorname{top} W_{\alpha_i}$ is a composition factor of each cell factor of P_{α_i}. By Lemma 2.1.4 this implies that α_i is minimal in Λ_{α_i}. Moreover, assume that there exist $\mu, \nu \in \Lambda_{\alpha_i}$ with $\mu \,||\, \nu$. Then $W_\mu \oplus W_\nu$ is a subquotient of P_{α_i} due to Lemma 2.1.7. It is in conflict with the structures of cell factors of P_{α_i} determined by Lemma 2.3.4 and 2.3.5.

On the other hand, all cell modules with two composition factors are of the same type by Lemma 2.3.5. If a type I cell module appears, Lemma 2.1.4 implies that $\alpha_i \geq \alpha_{i+1}$ for $1 \leq i < n$. Combine it with the above claim makes Λ to be a totally ordered set, the minimal element of which is α_n. For the case of type II, we can deduce from Lemma 2.1.4 that $\alpha_{i+1} \geq \alpha_i$, and this also forces Λ to be a totally ordered set, in which the α_1 is the minimal element. $\square$

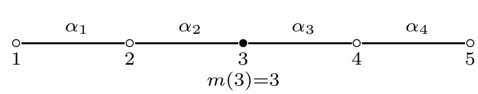
Now we are ready to give the main result of this section.

Proposition 2.3.7. *Let A be an indecomposable self-injective cellular K-algebra with cell datum $(\Lambda, M, C, *)$, where K is an algebraically closed field with the characteristic different from two. If A is of finite type, then the poset Λ must be a totally ordered set.*

Proof. By [7, Theorem 6.8] and [37, Theorem 8.1], if A is an indecomposable self-injective cellular K-algebra of finite type, then A is Morita equivalent to a cellular Brauer tree algebra. It is well known that if two algebras are Morita equivalent, then the lattices of ideals are isomorphic. The proposition then follows from Lemma 2.3.6. $\square$

According to the results obtained above, for an indecomposable self-injective cellular K-algebra of finite type, we can determine its the cellular structures completely. Let us illustrate an example.

Example 2.3.8. *Let A be a Brauer tree algebra for the following Brauer tree.*



Fix a cellular structure of A. Then the cell modules in sequence have to be one of the following:

$$L_{\alpha_1} < \begin{pmatrix} L_{\alpha_2} \\ L_{\alpha_1} \end{pmatrix} < \begin{pmatrix} L_{\alpha_3} \\ L_{\alpha_2} \end{pmatrix} < \begin{pmatrix} L_{\alpha_3} \\ L_{\alpha_2} \end{pmatrix} < \begin{pmatrix} L_{\alpha_3} \\ L_{\alpha_2} \end{pmatrix} < \begin{pmatrix} L_{\alpha_4} \\ L_{\alpha_3} \end{pmatrix} < L_{\alpha_4}$$

$$L_{\alpha_4} < \begin{pmatrix} L_{\alpha_3} \\ L_{\alpha_4} \end{pmatrix} < \begin{pmatrix} L_{\alpha_2} \\ L_{\alpha_3} \end{pmatrix} < \begin{pmatrix} L_{\alpha_2} \\ L_{\alpha_3} \end{pmatrix} < \begin{pmatrix} L_{\alpha_2} \\ L_{\alpha_3} \end{pmatrix} < \begin{pmatrix} L_{\alpha_1} \\ L_{\alpha_2} \end{pmatrix} < L_{\alpha_1}$$

The following result is a simple corollary of the cellular structure of an indecomposable self-injective cellular algebra of finite type. We write it here for later use. The proof is left to the reader as an exercise.

Corollary 2.3.9. *Let A be the same as in Proposition 2.3.7 and $\lambda, \mu \in \Lambda$. Assume that*

(1) neither λ nor μ is maximal;

(2) $\lambda, \mu \notin \Lambda_0$.

Then $\dim W_\lambda = \dim W_\mu$.

3. Abaci orbit, incomparable abaci and move vector

It is well-known that a block of a cyclotomic Hecke algebra of type $G(r, 1, n)$ satisfying the assumptions of Main Theorem is symmetric. In the light of the result obtained in Section 2, in order to determine the representation type of an indecomposable symmetric cellular algebra, one can analyze the poset first. We will use this idea to determine the representation type of the blocks of a cyclotomic Hecke algebra. The first important thing is to find a property which can help to study the relevant posets. We emphasize that it is also needed to transform a block into an easy-to-study form by derived equivalence. The aim of this section is to make all necessary preparations, which includes the following. We begin with some preliminaries about blocks and abaci. Then we translate the action of an affine Weyl group on blocks into the language of abaci for later use. Next we define and study the so-called incomparable abaci, which is one of key notions in this paper. Finally, we introduce move vector and give a framework of the proof of Main Theorem.

3.1. **Preliminaries on blocks and abaci.** We begin with some combinatorics. Let n be a positive integer. A partition λ of n is a non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \cdots, \lambda_s)$ such that $\sum_{i=1}^{s} \lambda_i = n$ and we write $|\lambda| = n$. The Young diagram of a partition λ is the set of nodes $[\lambda] = \{(i, j) \mid 1 \leq i, 1 \leq j \leq \lambda_i\}$. The conjugate of λ is defined to be a partition $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$, where λ'_j is equal to the number of nodes in column j of $[\lambda]$ for $j = 1, 2, \cdots$. A rim e-hook (or simply an e-hook) of $[\lambda]$ is a connected subset of the rim of $[\lambda]$ with exactly e nodes, which can be removed from $[\lambda]$ to obtain another Young diagram $[\mu]$. Given a partition λ, unwrapping e-rim hook of $[\lambda]$ one by one until none can be unwrapped, the partition obtained is called the e-core of $[\lambda]$ and the number of e-rim hook unwrapped is called the e-weight of λ.

An r-partition of n is an r-tuple $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(1)}, \cdots, \boldsymbol{\lambda}^{(r)})$ of partitions such that $|\boldsymbol{\lambda}| = \sum_{i=1}^{r} |\boldsymbol{\lambda}^{(i)}| = n$. The partitions $\boldsymbol{\lambda}^{(1)}, \cdots, \boldsymbol{\lambda}^{(r)}$ are the components of $\boldsymbol{\lambda}$. The conjugate of an r-partition $\boldsymbol{\lambda}$ is defined to be $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}^{(r)'}, \cdots, \boldsymbol{\lambda}^{(1)'})$. For $\sigma \in \mathfrak{S}$, $\boldsymbol{\lambda}^\sigma$

is defined to be $(\boldsymbol{\lambda}^{(\sigma(1))}, \cdots, \boldsymbol{\lambda}^{(\sigma(r))})$. Denote the set of r-partitions of n by $\mathscr{P}_{r,n}$. Then for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{r,n}$, we write $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ (or $\boldsymbol{\mu} \trianglelefteq \boldsymbol{\lambda}$) if

$$\sum_{t=1}^{s-1} |\boldsymbol{\lambda}^{(t)}| + \sum_{i=1}^{j} \boldsymbol{\lambda}_i^{(s)} \geq \sum_{t=1}^{s-1} |\boldsymbol{\mu}^{(t)}| + \sum_{i=1}^{j} \boldsymbol{\mu}_i^{(s)}$$

for all $1 \leq s \leq r$ and all $j \geq 1$. Write $\boldsymbol{\lambda} \triangleright \boldsymbol{\mu}$ (or $\boldsymbol{\mu} \triangleleft \boldsymbol{\lambda}$) if $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$. The Young diagram of an r-partition $\boldsymbol{\lambda}$ is the set of nodes

$$[\boldsymbol{\lambda}] = \{(i, j, k) \mid 1 \leq i, 1 \leq j \leq \boldsymbol{\lambda}_i^{(k)}, 1 \leq k \leq r\}.$$

Define the residue of node $(i, j, k) \in [\boldsymbol{\lambda}]$ to be q^{j-i+s_k} and define $c_f(\boldsymbol{\lambda})$ to be the number of nodes in $[\boldsymbol{\lambda}]$ of residue f. By using the residues, Fayers defined the weight for a multipartition, which coincides with defect given in Section 1.

Definition 3.1.1. [24, (2.1)] *Let $\boldsymbol{\lambda}$ be an r-partition of n. The weight of $\boldsymbol{\lambda}$ is defined to be the integer*

$$w(\boldsymbol{\lambda}) = (\sum_{i=1}^{r} c_{Q_i}(\boldsymbol{\lambda})) - \frac{1}{2} \sum_{f \in K^*} (c_f(\boldsymbol{\lambda}) - c_{qf}(\boldsymbol{\lambda}))^2.$$

A $\boldsymbol{\lambda}$-tableau is a bijective map $\mathfrak{t} : [\boldsymbol{\lambda}] \to \{1, 2, \cdots, n\}$. A $\boldsymbol{\lambda}$-tableau $\mathfrak{t}$ is called standard if the entries increase along each row and down each column in each component. The set of standard $\boldsymbol{\lambda}$-tableaux is denoted by $\mathrm{Std}(\boldsymbol{\lambda})$. The residue sequence of $\mathfrak{t}$ is defined to be the sequence $(\operatorname{res}\mathfrak{t}^{-1}(1), \cdots, (\operatorname{res}\mathfrak{t}^{-1}(n))$. An r-partition $\boldsymbol{\lambda}$ is said to be (Q, e)-restricted if there exists $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ such that the residue sequence of any standard tableau of shape $\boldsymbol{\mu} \triangleleft \boldsymbol{\lambda}$ is not the same as the residue sequence of $\mathfrak{t}$.

As far as we know, algebra $\mathcal{H}_n(q, Q)$ has three cellular bases. The first one was given by Graham and Lehrer in [28] using Kazhdan-Lusztig basis of $\mathcal{H}(S_n)$, the Hecke algebra of S_n. The second one is standard basis, which was constructed in [19] by Dipper, James and Mathas. It is similar to the basis of $\mathcal{H}(S_n)$ introduced by Murphy [47]. Then one has cell (Specht) modules $W(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda}$ are r-partitions. Moreover, Ariki [3] proved that if $\boldsymbol{\lambda}$ is a Kleshchev r-partition, then $L(\boldsymbol{\lambda})$, the top of $W(\boldsymbol{\lambda})$ is simple, and $\{L(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \text{ Kleshchev}\}$ provide a complete set of simple $\mathcal{H}_n(q, Q)$-modules up to isomorphism.

In [32], Jacon proved the generalized DJM conjecture about Kleshchev r-partitions. The reader can find more details about this conjecture in [29]. The following lemma if a special case of [34, Theorem 5.1.1].

Lemma 3.1.2. *An r-partition $\boldsymbol{\lambda}$ is Kleshchev if and only if $\boldsymbol{\lambda}$ is (Q, e)-restricted.*

Recall that given a finite dimensional algebra A, two A-modules belong to the same block if all of their composition factors belong to the same block. For a cellualr algebra, according to [28, 3.9.8], all composition factors of a cell module belong to the same block. Therefore, we can say in this sense block $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ of $\mathcal{H}_n(q, Q)$ contains cell module $W(\boldsymbol{\lambda})$ and we abuse notation to say that $\boldsymbol{\lambda}$ lies in block $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. More exactly, we often write pair $(\boldsymbol{\lambda}, \mathbf{s})$ in $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ because the multicharge $\mathbf{s}$ cannot be omitted in the circumstances of this paper. It is helpful to point out here that $\boldsymbol{\beta}$ is uniquely determined by $\boldsymbol{\lambda}$ with $\boldsymbol{\beta} = \sum_{i \in I} c_{q^i}(\boldsymbol{\lambda}) \boldsymbol{\alpha}_i$.

Given two pairs $(\boldsymbol{\lambda}, \mathbf{s})$ and $(\boldsymbol{\mu}, \mathbf{s})$, we have the following lemma.

Lemma 3.1.3. [28, Proposition 5.9 (ii)][44, Theorem 2.11] *Pairs* $(\boldsymbol{\lambda}, \mathbf{s})$ *and* $(\boldsymbol{\mu}, \mathbf{s})$ *belong to the same block if and only of* $c_f(\boldsymbol{\lambda}) = c_f(\boldsymbol{\mu})$ *for all* $f \in K$.

The third cellular basis of $\mathcal{H}_n(q,Q)$ that we want to introduce was given by Hu and Mathas in [30], which is compatible with the block decompose of $\mathcal{H}_n(q,Q)$.

Lemma 3.1.4. [30, Lemma 5.4 and Corollary 5.5, Theorem 5.8, Corollary 5.12] *Let* $\mathcal{H}_n(q,Q)$ *be a cyclotomic Hecke algebra of type* $G(r,1,n)$. *Then*

(1) *The algebra* $\mathcal{H}_n(q,Q)$ *is a graded cellular algebra with poset* $(\mathscr{P}_{r,n}, \trianglerighteq)$ *and graded cellular basis* $\{\psi^{\boldsymbol{\lambda}}_{\mathfrak{s},\mathfrak{t}} \mid \boldsymbol{\lambda} \in \mathscr{P}_{r,n}, \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})\}$ (HM basis).
(2) *Let* $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ *be a block of* $\mathcal{H}_n(q,Q)$. *Then there exists* $\mathscr{P}^{\mathcal{H}_{\boldsymbol{\beta}}}_{r,n} \subseteq \mathscr{P}_{r,n}$ *such that* $\{\psi^{\boldsymbol{\lambda}}_{\mathfrak{s},\mathfrak{t}} \mid \boldsymbol{\lambda} \in \mathscr{P}^{\mathcal{H}_{\boldsymbol{\beta}}}_{r,n}, \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})\}$ *is a graded cellular basis of* $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$.
(3) *The corresponding ungraded cell modules coincide with the cell modules determined by standard basis, respectively.*

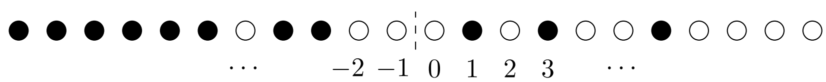
We also need the following easy results about $\mathcal{H}_n(q,Q)$.

Lemma 3.1.5. [50, Remarks 2.4 (iii)] *For any* $0 \neq c \in K$, *we have an isomorphism* $\mathcal{H}_n(q, Q_1, \cdots, Q_r) \cong \mathcal{H}_n(q, cQ_1, \cdots, cQ_r)$. *For any permutation* σ *of* r *letters,* $\mathcal{H}_n(q, Q_1, \cdots, Q_r) = \mathcal{H}_n(q, Q_{\sigma(1)}, \cdots, Q_{\sigma(r)})$.

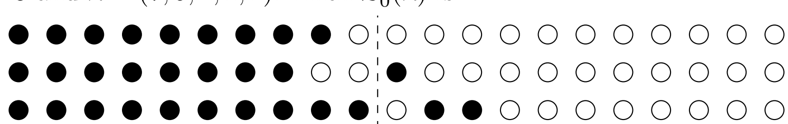
Based on Lemma 3.1.5 and 3.1.3, pairs $(\boldsymbol{\lambda}^\sigma, \mathbf{s}^\sigma)$ and $(\boldsymbol{\lambda}, \mathbf{s})$ are in the same block for $\sigma \in \mathfrak{S}_r$.

We now begin to recall some result about r-abaci. The main reference is [33]. Abaci first appeared in the work of Gordon James [34]. Given a partition λ and $s \in \mathbb{Z}$, one can associate it to a set of integers $L_s(\lambda) = \{\lambda_j - j + s \mid j \in \mathbb{N}^+\}$. Note that we assume that λ has an infinite number of zero parts. As is well-known, the set $L_s(\lambda)$ can be expressed by an abacus. Let us illustrate an example.

Example 3.1.6. *Let* $\lambda = (7,5,4,1,1)$ *and* $s = 0$. *For each* $i \in L_s(\lambda)$, *we set a bead at* i*-th position on the horizontal abacus. Then* $L_s(\lambda)$ *is expressed as below. A position without a bead will be called an empty position. We usually omit the labels.*



Moreover, an abacus $L_s(\lambda)$ can also be represented by an e-tuple of abacus, in which the abaci are labeled by $L_0, L_1, \cdots, L_{e-1}$ from bottom to top. For each $k \in L_s(\lambda)$, if $k = ye + x$ with $x, y \in \mathbb{Z}$ and $0 \leq x < e$, then place a bead in position (x, y), which means the y-th position of L_x. We will denote this r-tuple by $\mathcal{L}^e_s(\lambda)$. Let $e = 3$ and $\lambda = (7,5,4,1,1)$. Then $\mathcal{L}^3_0(\lambda)$ is



Let $\mathbf{s} = (s_1, s_2, \cdots, s_r) \in \mathbb{Z}^r$ be a multicharge and $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(1)}, \cdots, \boldsymbol{\lambda}^{(r)})$ an r-partition. Then the pair $(\boldsymbol{\lambda}, \mathbf{s})$ can be associated with an r-abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$ by setting abaci $L_{s_i}(\boldsymbol{\lambda}^{(i)})$, $i = 1, \cdots, r$, from bottom to top so that all the beads in position 0 of each abacus appear in the same vertical line, which will be called $(e, \mathbf{s})$-abacus of pair $(\boldsymbol{\lambda}, \mathbf{s})$. Note that each r-abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$ can be mapped to a abacus by Uglov map, whose definition is as follows.

Definition 3.1.7. [49, Section 4.1], [33, Section 2.4] *Let $\boldsymbol{\lambda}$ be an r-partition. Then the image of or pair $(\boldsymbol{\lambda}, \mathbf{s})$ under Uglov map $\tau_{e,r}$ is (λ, s), which is defined as follows. For each bead at position (x, y) in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, let $y = k.e + c$ with $k \in \mathbb{Z}$ and $c \in \{0, ..., e-1\}$. Then we set a bead in our new 1-abacus $L_s(\lambda)$ in position $(r-x)e + ker + c$.*

It is not difficult to check the Uglov map is a bijection (see [49, Section 4.1]) and $s = \sum_i s_i$.

In an r-abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$, the number of beads in column k is denoted by $c_k(L_{\mathbf{s}}(\boldsymbol{\lambda}))$, or c_k if there is no danger of confusion. Denote by $\bullet^i_j(\boldsymbol{\lambda}, \mathbf{s})$ the j-th bead from right to left of $L_{s_i}(\boldsymbol{\lambda}^{(i)})$. If there is no dangerous of confusion, we write it simply as $\bullet^i_j$. The following lemma is easy to check and we omit the proof.

Lemma 3.1.8. *For the (e, s)-abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$ of a pair $(\boldsymbol{\lambda}, \mathbf{s})$, we have*

(1) The number of empty positions between $\bullet^i_x$ and $\bullet^i_{x+1}$ is equal to $\boldsymbol{\lambda}^{(i)}_x - \boldsymbol{\lambda}^{(i)}_{x+1}$.
(2) The number of empty positions that are to the left of $\bullet^i_j$ is equal to $\boldsymbol{\lambda}^{(i)}_j$.
(3) Assume that the number of beads in $L_{s_i}(\boldsymbol{\lambda}^{(i)})$ on the right of the dashed vertical line is n_{i1} and that the number of empty positions on the left of the dashed vertical line is n_{i2}. Then $n_{i1} - n_{i2} = s_i$.

By using the above lemma, we can study the relationship between the numbers of certain positions in two r-abaci. Denote by $\mathfrak{n}^h_i(L_{\mathbf{s}}(\boldsymbol{\lambda}))$ the number of beads on the right side of the h-th position in runner i. If there is no dangerous of confusion, we write it simply as $\mathfrak{n}^k_x$.

Lemma 3.1.9. *Let $L_{\mathbf{s}}(\boldsymbol{\lambda})$ and $L_{\mathbf{v}}(\boldsymbol{\mu})$ be two r-abaci and $1 \leq i, j \leq r$. Let h be an integer such that all positions (i, l) in $L_{s_i}(\boldsymbol{\lambda}^{(i)})$ and positions (j, l) in $L_{v_i}(\boldsymbol{\mu}^{(j)})$, where $l \leq h$, are occupied by beads. Then $\mathfrak{n}^h_j(L_{\mathbf{v}}(\boldsymbol{\mu})) - \mathfrak{n}^h_i(L_{\mathbf{s}}(\boldsymbol{\lambda})) = v_j - s_i$.*

Proof. Without loss of generality, we can assume $h < 0$. By dividing the $\mathfrak{n}^h_j(L_{\mathbf{v}}(\boldsymbol{\mu}))$ beads into two parts with the dashed vertical line, one can obtained the following equality: $\mathfrak{n}^h_j(L_{\mathbf{v}}(\boldsymbol{\mu})) = -h - 1 - n_{j2} + n_{j1}$. Note that by Lemma 3.1.8 (3), $-n_{j2} + n_{j1} = v_j$ and hence $\mathfrak{n}^h_j(L_{\mathbf{v}}(\boldsymbol{\mu})) = -h - 1 + v_j$. Similarly, we have $\mathfrak{n}^h_i(L_{\mathbf{s}}(\boldsymbol{\lambda})) = -h - 1 + s_i$. Combining the two equalities above, we complete the proof. $\square$

Before we introduce more results on r-abaci, we need some additional notations defined by Jacon and Lecouvey in [33]. Given two abaci L and L', we write $L \subseteq L'$ if for each bead in position i of L, there is a bead in position i in L'. We also need to recall two subsets of $\mathbb{Z}^r$.

$$\overline{\mathcal{A}}^r_e := \{(s_1, \cdots, s_r) \in \mathbb{Z}^r \mid \forall\, i, j \in \{1, \cdots, r\}, i < j,\, 0 \leq s_j - s_i \leq e\},$$
$$\mathcal{A}^r_e := \{(s_1, \cdots, s_r) \in \mathbb{Z}^r \mid \forall\, i, j \in \{1, \cdots, r\}, i < j,\, 0 \leq s_j - s_i < e\}.$$

Clearly, $\mathcal{A}^r_e \subset \overline{\mathcal{A}}^r_e$.

Definition 3.1.10. [33, Definition 2.8, 2.10] *An r-abacus is called $(e, \mathbf{s})$-complete if*

$$L_{s_1}(\boldsymbol{\lambda}^{(1)}) \subset L_{s_2}(\boldsymbol{\lambda}^{(2)}) \subset \cdots \subset L_{s_r}(\boldsymbol{\lambda}^{(r)}) \subset L_{s_1+e}(\boldsymbol{\lambda}^{(1)}).$$

A pair $(\boldsymbol{\lambda}, \mathbf{s})$ is said to be a reduce $(e, \mathbf{s})$-core if its $(e, \mathbf{s})$-abacus is $(e, \mathbf{s})$-complete.

By an easy observation, Jacon and Lecouvey revealed a relation between reduced $(e, \mathbf{s})$-cores and $\overline{\mathcal{A}}_e^r$.

Lemma 3.1.11. [33, Proposition 2.11] *Given a pair $(\boldsymbol{\lambda}, \mathbf{s})$, if $\boldsymbol{\lambda}$ is a reduced $(e, \mathbf{s})$-core, then $\mathbf{s} \in \overline{\mathcal{A}}_e^r$.*

For convenience we agree the nonexistent positions $(r+j, h)$ to be $(j, h-e)$ for $1 \leq j < r$ when $e \neq \infty$. Let us collect some simple properties about the $(e, \mathbf{s})$-abacus of a pair $(\boldsymbol{\lambda}, \mathbf{s})$.

Lemma 3.1.12. *Given a pair $(\boldsymbol{\lambda}, \mathbf{s})$ and its $(e, \mathbf{s})$-abacus, we have*

(1) Let $1 \leq i \leq r$ and $e < \infty$. If position (i, l) is empty, then there exists some $k \in \mathbb{N}$, such that the bead at position $(i, l-(k+1)e)$ is black and position $(i, l-ke)$ is empty. If position (i, l) has a bead, then there exists some k such that position $(i, l+ke)$ has a bead and position $(i, l+(k+1)e)$ is empty.

(2) Let $\mathbf{s} \in \overline{\mathcal{A}}_e^r$. If in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, positions (i, l_t) have beads and positions $(i+j, l_t)$ are empty, where $(t = 1, \cdots, m)$, $1 \leq i \leq r$ and $1 \leq j < r$, then there exist $h_1, \cdots, h_m$ such that positions (i, h_x) are empty and positions $(i+j, h_x)$ have beads for $x = 1, \cdots, m$.

(3) Let $e \neq \infty$ and $L_{\mathbf{s}}(\boldsymbol{\lambda})$ be $(e, \mathbf{s})$-complete. If there exists a bead in column l of $L_{\mathbf{s}}(\boldsymbol{\lambda})$, then all positions of column $l-ke$ have bead for each $k \in \mathbb{N}^+$.

(4) Assume that $s_i + k \leq s_j$, for $1 \leq i, j \leq r$ and $k \in \mathbb{N}^+$. Then there exist $h_1, \cdots, h_k \in \mathbb{Z}$ such that in $(\boldsymbol{\lambda}, \mathbf{s})$, positions (i, h_t) are empty and positions (j, h_t) have beads, where $1 \leq t \leq k$. In particular, if $L_{\mathbf{s}}(\boldsymbol{\lambda})$ is complete, then $i < j$ and there exist $h_1, \cdots, h_k \in \mathbb{Z}$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, positions (x, h_t) are empty and positions (y, h_t) have beads, where $1 \leq x \leq i$, $j \leq y \leq r$ and $1 \leq t \leq k$.

Proof. (1), (2) and (3) are easy. We only prove (4). Let l be an integer such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, all positions (x, y) have beads, where $1 \leq x \leq r$ and $y \leq l$. Since $s_i + k \leq s_j$, by Lemma 3.1.8, $\mathfrak{n}_j^l - \mathfrak{n}_i^l = s_j - s_i \geq k$. We have proved the first half of the lemma.

Moreover, if $L_{\mathbf{s}}(\boldsymbol{\lambda})$ is complete, then in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ position (i, h) being empty forces positions (x, h) to be empty where $1 \leq x \leq i$, h, and position (j, h) being occupied by a bead implies that all positions (y, h) have beads, where $j \leq y \leq r$. Consequently, the second half of the lemma holds. $\square$

3.2. **Orbits of abaci.** In this subsection, we give an explanation by abaci for the action of an affine Weyl group on blocks. Let us first fix some notations. Given a pair $(\boldsymbol{\lambda}, \mathbf{s})$, it clearly belongs a unique block. Denote by $\boldsymbol{\Lambda}_{\boldsymbol{\lambda}, \mathbf{s}}$ and $\boldsymbol{\beta}_{\boldsymbol{\lambda}, \mathbf{s}}$ the corresponding elements in dominant weight lattice and positive root lattice, respectively. In addition, if $e = \infty$, then both $j \pmod e$ and $j + ke$, $k \in \mathbb{Z}$ means j, and $0 \leq j \leq e-1$ means $j \in \mathbb{Z}$. For $j \in I$ and pair (λ, s), define an action of σ_j on $(\boldsymbol{\lambda}, \mathbf{s})$ by $\sigma_j(\boldsymbol{\lambda}, \mathbf{s}) = (\sigma_j(\boldsymbol{\lambda}), \sigma_j(\mathbf{s}))$, where the abacus of $(\sigma_j(\boldsymbol{\lambda}), \sigma_j(\mathbf{s}))$ is obtained by interchanging columns $j-1+ke$ and $j+ke$ in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ for all $k \in \mathbb{Z}$. A simple result about the action is that the charge is invariant. Let $0 \leq i < e$. The diagram obtained by putting all j-th columns with $j \equiv i \pmod e$ together in the original order and with the original labels is called the i-th subabacus of $L_{\mathbf{s}}(\boldsymbol{\lambda})$. For arbitrary $x \in \mathbb{Z}$, we sometimes say x-th subabacus, which means the i-the one, $x \equiv i \pmod e$ and $0 \leq i < e$. Clearly, for a given abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$, both the number of the beads in $j-1$-th and j-th subabaci are infinite. However, if we ignore positions $j-1+ke$ and $j+ke$, $k\in\mathbb{Z}$, as long as both them are occupying by beads, then in a natural way, we can say the difference between the number of beads in $j-1$-th subabacus and that in j-th one. For convenience, denote the difference by $\mathfrak{m}_j^{j-1}(L_{\mathbf{s}}(\boldsymbol{\lambda}))=\sum_{k\in\mathbb{Z}}(c_{j-1+ke}-c_{j+ke})$.

Lemma 3.2.1. *For $j\in I$, $\sigma_j(\mathbf{s})=\mathbf{s}$.*

Proof. It is a direct corollary of the definition of the action of σ_j on a pair and Lemma 3.1.8 (3). □

To achieve the target of this subsection, we first do some work on 1-abaci. Given an abacus $L_s(\lambda)$, let $x\geq 1$ and $\bullet_x$ at the h-th position. If the $h+1$-th position is empty, then slide $\bullet_x$ to the $h+1$-th position and denote by (μ,u) the new abacus. By Lemma 3.1.8 (2), $\mu_x=\lambda_x+1$, and $\lambda_i=\mu_i$ for each $i\neq x$. Moreover, we have from Lemma 3.1.8 (3) that $u=s$. Using this method, each abacus $L_s(\lambda)$ can be obtained from $L_s(\varnothing)$ by finite steps. Clearly, this process is invertible.

Lemma 3.2.2. *If $\bullet_i$ is at $j+ke$-th position in $L_s(\lambda)$, where $0\leq j\leq e-1$, then the residue of node (i,λ_i) in $[\lambda]$ is j.*

Proof. By the definition of abacus$L_s(\lambda)$, we have $s+\lambda_i-i=j+ke$, that is, the residue of node (i,λ_i) is j. □

Lemma 3.2.3. *Assume that in abacus $L_s(\lambda)$, position $j+ke$ is empty and there is a bead at position $j-1+ke$, where $0\leq j\leq e-1$. Denote by $L_s(\mu)$ the abacus obtained by sliding the bead at position $j-1+ke$ to position $j+ke$. Then $\boldsymbol{\beta}_{\mu,s}=\boldsymbol{\beta}_{\lambda,s}+\boldsymbol{\alpha}_j$.*

Proof. Suppose that the bead at position $j-1+ke$ is $\bullet_i$. Then $\mu_i=\lambda_i+1$ and $\mu_x=\lambda_x$ for all $x\neq i$. We can deduce from Lemma 3.2.2 that the residue of node (i,μ_i) in $[\mu]$ is j. That is, $\boldsymbol{\beta}_{\mu,s}=\boldsymbol{\beta}_{\lambda,s}+\boldsymbol{\alpha}_j$. □

Lemma 3.2.4. *Given an abacus $L_s(\lambda)$, then*

$$\boldsymbol{\beta}_{\sigma_j(\lambda),s}=\boldsymbol{\beta}_{\lambda,s}+\mathfrak{m}_j^{j-1}\boldsymbol{\alpha}_j$$

for $j\in I$.

Proof. Firstly, let us find all pairs of positions that need to interchange under σ_j. Assume $X=\{b_i\mid i=1,\cdots,l\}$ are all integers such that position $j-1+b_ie$ is empty and position $j+b_ie$ has a bead. Note that X may be empty. Therefore there is a set $Y=\{a_h\in\mathbb{Z}\mid h=1,\cdots,l+\mathfrak{m}_j^{j-1}\}$ such that position $j-1+a_he$ has a bead and position $j+ke$ is empty. By Lemma 3.2.3,

$$\boldsymbol{\beta}_{\sigma_j(\lambda,s)}=\boldsymbol{\beta}_{\lambda,s}+(\mathfrak{m}_j^{j-1}+l)\boldsymbol{\alpha}_j-l\boldsymbol{\alpha}_j=\boldsymbol{\beta}_{\lambda,s}+\mathfrak{m}_j^{j-1}\boldsymbol{\alpha}_j.$$

□

Lemma 3.2.5. *For $j\in I$ and a pair λ,s, we have $\sigma_j(\boldsymbol{\Lambda}_{\lambda,s}-\boldsymbol{\beta}_{\lambda,s})=\boldsymbol{\Lambda}_{\lambda,s}-\boldsymbol{\beta}_{\sigma_j(\lambda),s}$.*

Proof. Without loss of generality, we assume $0\leq s\leq e-1$. Since arbitrary abacus $L_s(\lambda)$ can be obtained by a process aforementioned in this subsection, the lemma can be proved **by induction**.

We start from $\lambda = \varnothing$. Obviously, $\boldsymbol{\beta}_{\varnothing,s} = 0$. If $j \neq s$, we can deduce from the shape of $L_s(\lambda)$ that $\sigma_j(\varnothing, s) = (\varnothing, s)$, that is, $\boldsymbol{\beta}_{\sigma_j(\varnothing),s} = \boldsymbol{\beta}_{\varnothing,s} = 0$.

$$\begin{aligned}\sigma_j(\boldsymbol{\Lambda}_{\varnothing,s} - \boldsymbol{\beta}_{\varnothing,s}) &= \sigma_j \boldsymbol{\Lambda}_s \\ &= \boldsymbol{\Lambda}_s - \delta_{js}\alpha_j \\ &= \boldsymbol{\Lambda}_s \\ &= \boldsymbol{\Lambda}_{\lambda,s} - \boldsymbol{\beta}_{\sigma_j(\varnothing),s}.\end{aligned}$$

If $j = s$, according to the shape of $L_s(\varnothing)$ and the definition of the action of σ_j on pairs, $L_s(\sigma_s(\varnothing))$ is obtained from $L_s(\lambda)$ by sliding the bead at position $s-1$ to position s, and consequently, $\sigma_s(\varnothing) = (1)$. This implies that $\boldsymbol{\Lambda}_{(1),s} = \boldsymbol{\Lambda}_s$, $\boldsymbol{\beta}_{(1),s} = \boldsymbol{\alpha}_s$ and thus

$$\sigma_s(\boldsymbol{\Lambda}_{\varnothing,s} - \boldsymbol{\beta}_{\varnothing,s}) = \sigma_s \boldsymbol{\Lambda}_s = \boldsymbol{\Lambda}_s - \boldsymbol{\alpha}_s = \boldsymbol{\Lambda}_{\sigma_s(\varnothing),s} - \boldsymbol{\beta}_{\sigma_s(\varnothing),s}$$

Assume the lemma holds for pair (λ, s) and abacus $L_s(\mu)$ is obtained from $L_s(\lambda)$ by sliding the bead at position $l-1+he$ to position $l+he$, where $0 \leq l \leq e-1$. We now prove the lemma holds for pair (μ, s). By Lemma 3.2.3,

$$\boldsymbol{\beta}_{\mu,s} = \boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l. \tag{3.2.1}$$

Since for arbitrary pair (λ, s), the dominant weight $\boldsymbol{\Lambda}_{\lambda,s}$ is uniquely determined by s and independent of λ, we denote it by $\boldsymbol{\Lambda}$ for simplicity during the rest of this proof. Clearly $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_s$.

It is easy to check that $\mathfrak{m}_l^{l-1}(L_s(\lambda)) = \mathfrak{m}_l^{l-1}(L_s(\mu)) + 2$.

For $j = l$, by Lemma 3.2.4, we have

$$\boldsymbol{\beta}_{\sigma_j(\lambda),s} = \boldsymbol{\beta}_{\lambda,s} + \mathfrak{m}_j^{j-1}(L_s(\lambda))\boldsymbol{\alpha}_j \tag{3.2.2}$$

and

$$\boldsymbol{\beta}_{\sigma_j(\mu),s} = \boldsymbol{\beta}_{\mu,s} + (\mathfrak{m}_j^{j-1}(L_s(\lambda)) - 2)\boldsymbol{\alpha}_j \tag{3.2.3}$$

Substituting (3.2.1) into (3.2.3), we obtain

$$\boldsymbol{\beta}_{\sigma_j(\mu),s} = \boldsymbol{\beta}_{\lambda,s} + (\mathfrak{m}_j^{j-1}(L_s(\lambda)) - 1)\boldsymbol{\alpha}_j. \tag{3.2.4}$$

Then the lemma can be derived from the above preparation as follows.

$$\begin{aligned}
&\sigma_j(\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\mu,s}) \\
= \ &\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\mu,s} - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\mu,s})\boldsymbol{\alpha}_j \ \ \text{(Defnition of } \sigma_j\text{)} \\
= \ &\boldsymbol{\Lambda} - (\boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l) - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - (\boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l))\boldsymbol{\alpha}_j \ \ \text{(Substituting (3.2.1))} \\
= \ &\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s} - \boldsymbol{\alpha}_l - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s})\boldsymbol{\alpha}_j + (\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_l)\boldsymbol{\alpha}_j \\
= \ &(\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s} - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s})\boldsymbol{\alpha}_j) + \boldsymbol{\alpha}_l \ \ \text{(Definition of bilinear form(,))} \\
= \ &\sigma_j(\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s}) + \boldsymbol{\alpha}_l \ \ \text{(Defnition of } \sigma_j\text{)} \\
= \ &\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\sigma_j(\lambda),s} + \boldsymbol{\alpha}_l \ \ \text{(Induction hypothesis)} \\
= \ &\boldsymbol{\Lambda} - (\boldsymbol{\beta}_{\lambda,s} + (\mathfrak{m}_j^{j-1}(L_s(\lambda)) - 1)\boldsymbol{\alpha}_j) \ \ \text{(Substituting (3.2.2))} \\
= \ &\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\sigma_j(\mu),s}. \ \ \text{(Substituting (3.2.4))}
\end{aligned}$$

For $j \neq l$, we consider two cases according to whether or not $e = 2$.

Case 1. $e = 2$. In this case, $j = l - 1$. It follows from Lemma 3.2.4 that

$$\boldsymbol{\beta}_{\sigma_j(\lambda),s} = \boldsymbol{\beta}_{\lambda,s} - \mathfrak{m}_l^{l-1}(L_s(\lambda))\boldsymbol{\alpha}_j \tag{3.2.5}$$

and

$$\boldsymbol{\beta}_{\sigma_j(\mu),s} = \boldsymbol{\beta}_{\mu,s} + (2 - \mathfrak{m}_l^{l-1}(L_s(\lambda)))\boldsymbol{\alpha}_j \tag{3.2.6}$$

Substituting (3.2.1) into (3.2.6), we get

$$\boldsymbol{\beta}_{\sigma_j(\mu),s} = \boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l + (2 - \mathfrak{m}_l^{l-1}(L_s(\lambda)))\boldsymbol{\alpha}_j. \tag{3.2.7}$$

With the above preparation, we can start our computation.

$$\begin{aligned}
& \sigma_j(\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\mu,s}) \\
= \;& \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\mu,s} - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\mu,s})\boldsymbol{\alpha}_j \;\; \text{(Defnition of } \sigma_j\text{)} \\
= \;& \boldsymbol{\Lambda} - (\boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l) - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - (\boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l))\boldsymbol{\alpha}_j \;\; \text{(Substituting (3.2.1))} \\
= \;& \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s} - \boldsymbol{\alpha}_l - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s})\alpha_j + (\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_l)\boldsymbol{\alpha}_j \\
= \;& (\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s} - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s})\boldsymbol{\alpha}_j) - \boldsymbol{\alpha}_l - 2\boldsymbol{\alpha}_j \;\; \text{(Definition of bilinear form(,))} \\
= \;& \sigma_j(\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s}) - \boldsymbol{\alpha}_l + 2\boldsymbol{\alpha}_j \;\; \text{(Defnition of } \sigma_j\text{)} \\
= \;& \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\sigma_j(\lambda),s} - \boldsymbol{\alpha}_l - 2\boldsymbol{\alpha}_j \;\; \text{(Induction hypothesis)} \\
= \;& \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s} + \mathfrak{m}_l^{l-1}(L_s(\lambda))\boldsymbol{\alpha}_j - \boldsymbol{\alpha}_l - 2\boldsymbol{\alpha}_j \;\; \text{(Substituting (3.2.5))} \\
= \;& \boldsymbol{\Lambda} - (\boldsymbol{\beta}_{\lambda,s} + (2 - \mathfrak{m}_l^{l-1}(L_s(\lambda)))\boldsymbol{\alpha}_j + \boldsymbol{\alpha}_l) \\
= \;& \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\sigma_j(\mu),s} \;\; \text{(Substituting (3.2.7))}
\end{aligned}$$

Case 2. $e \neq 2$**.** There are three subcases need to handle.

Subcase 1. $j = l + 1$. Clearly, $\mathfrak{m}_{l+1}^l(L_s(\lambda)) = \mathfrak{m}_{l+1}^l(L_s(\mu)) - 1$ We have from Lemma 3.2.4 that

$$\boldsymbol{\beta}_{\sigma_j(\lambda),s} = \boldsymbol{\beta}_{\lambda,s} + \mathfrak{m}_{l+1}^l(L_s(\lambda))\boldsymbol{\alpha}_j \tag{3.2.8}$$

and

$$\boldsymbol{\beta}_{\sigma_j(\mu),s} = \boldsymbol{\beta}_{\mu,s} + (\mathfrak{m}_{l+1}^l(L_s(\lambda)) + 1)\boldsymbol{\alpha}_j \tag{3.2.9}$$

Substituting (3.2.1) into (3.2.9), we obtain

$$\boldsymbol{\beta}_{\sigma_j(\mu),s} = \boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l + (\mathfrak{m}_{l+1}^l(L_s(\lambda)) + 1)\boldsymbol{\alpha}_j. \tag{3.2.10}$$

Now we can check the lemma.

$$\begin{aligned}
& \sigma_j(\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\mu,s}) \\
= \;& \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\mu,s} - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\mu,s})\boldsymbol{\alpha}_j \\
= \;& \boldsymbol{\Lambda} - (\boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l) - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - (\boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l))\boldsymbol{\alpha}_j \;\; \text{(Substituting (3.2.1))} \\
= \;& \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s} - \boldsymbol{\alpha}_l - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s})\boldsymbol{\alpha}_j + (\boldsymbol{\alpha}_j, \alpha_l)\boldsymbol{\alpha}_j \\
= \;& (\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s} - (\alpha_j, \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s})\boldsymbol{\alpha}_j) - \boldsymbol{\alpha}_l - \boldsymbol{\alpha}_j \;\; \text{(Definition of bilinear form(,))} \\
= \;& \sigma_j(\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s}) - \boldsymbol{\alpha}_l - \boldsymbol{\alpha}_j \\
= \;& \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\sigma_j(\lambda),s} - \boldsymbol{\alpha}_l - \boldsymbol{\alpha}_j \;\; \text{(Induction hypothesis)} \\
= \;& \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s} - \mathfrak{m}_{l+1}^l(L_s(\lambda))\boldsymbol{\alpha}_j - \boldsymbol{\alpha}_l - \boldsymbol{\alpha}_j \;\; \text{(Substituting (3.2.8))} \\
= \;& \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\sigma_j(\mu),s} \;\; \text{(Substituting (3.2.10))}
\end{aligned}$$

Subcase 2. $j = l - 1$. It is proved similarly as Subcase 1 by consider $l - 2$-th subabacus and $l - 1$-th one.

Subcase 3. $j \neq l, l \pm 1$. Under this condition, it is clear that $\mathfrak{m}_j^{j-1}(L_s(\lambda)) = \mathfrak{m}_j^{j-1}(L_s(\mu))$. By Lemma 3.2.4, we have

$$\boldsymbol{\beta}_{\sigma_j(\lambda),s} = \boldsymbol{\beta}_{\lambda,s} + \mathfrak{m}_j^{j-1}(L_s(\lambda))\boldsymbol{\alpha}_j \tag{3.2.11}$$

and

$$\boldsymbol{\beta}_{\sigma_j(\mu),s} = \boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l + \mathfrak{m}_j^{j-1}(L_s(\lambda))\boldsymbol{\alpha}_j \tag{3.2.12}$$

. By using (3.2.11) and (3.2.12),

$$\begin{aligned}
\sigma_j(\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\mu,s}) &= \boldsymbol{\Lambda} - (\boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l) - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - (\boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l))\boldsymbol{\alpha}_j \\
&= (\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s} - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s})\boldsymbol{\alpha}_j) - \boldsymbol{\alpha}_l + (\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_l)\boldsymbol{\alpha}_j \\
&= \sigma_j(\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s}) - \boldsymbol{\alpha}_l \\
&= \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\lambda,s} - \mathfrak{m}_j^{j-1}(L_s(\lambda))\boldsymbol{\alpha}_j - \boldsymbol{\alpha}_l \\
&= \boldsymbol{\Lambda} - (\boldsymbol{\beta}_{\lambda,s} + \boldsymbol{\alpha}_l + \mathfrak{m}_j^{j-1}(L_s(\lambda))\boldsymbol{\alpha}_j) \\
&= \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\sigma_j(\mu),s}
\end{aligned}$$

To sum up, the lemma holds for pair (μ, s).

According to induction, we have completed the proof. ◻

Now we are in a position to give the main result of this subsection.

Proposition 3.2.6. *For $0 \leq j \leq e-1$ and pair $(\boldsymbol{\lambda}, \mathbf{s})$, we have*

$$\sigma_j(\boldsymbol{\Lambda}_{\boldsymbol{\lambda},\mathbf{s}} - \boldsymbol{\beta}_{\boldsymbol{\lambda},\mathbf{s}}) = \boldsymbol{\Lambda}_{\boldsymbol{\lambda},\mathbf{s}} - \boldsymbol{\beta}_{\sigma(\boldsymbol{\lambda}),\mathbf{s}}.$$

Proof. It follows from Lemma 3.2.5 that

$$\begin{aligned}
\sigma_j(\boldsymbol{\Lambda}_{\boldsymbol{\lambda},\mathbf{s}} - \boldsymbol{\beta}_{\boldsymbol{\lambda},\mathbf{s}}) &= \sigma_j \sum_{i=1}^{r} (\boldsymbol{\Lambda}_{\boldsymbol{\lambda}^{(i)},\mathbf{s}_i} - \boldsymbol{\beta}_{\boldsymbol{\lambda}^{(i)},\mathbf{s}_i}) \\
&= \sum_{i=1}^{r} \sigma_j(\boldsymbol{\Lambda}_{\boldsymbol{\lambda}^{(i)},\mathbf{s}_i} - \boldsymbol{\beta}_{\boldsymbol{\lambda}^{(i)},\mathbf{s}_i}) \\
&= \sum_{i=1}^{r} (\boldsymbol{\Lambda}_{\boldsymbol{\lambda}^{(i)},\mathbf{s}_i} - \boldsymbol{\beta}_{\sigma_j(\boldsymbol{\lambda}^{(i)}),\mathbf{s}_i}) \\
&= \boldsymbol{\Lambda}_{\boldsymbol{\lambda},\mathbf{s}} - \boldsymbol{\beta}_{\sigma_j(\boldsymbol{\lambda}),\mathbf{s}}
\end{aligned}$$

The proof is completed. ◻

3.3. **Incomparable abaci.** In this subsection we introduce the so-called incomparable abaci, which is an very important concept in our study. The main result is that from incomparable abaci we can construct incomparable multipartitions with respect to the dominance order.

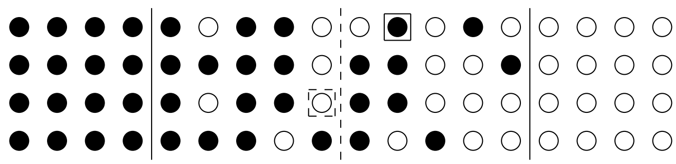
Definition 3.3.1. *Given two r-abaci $L_{\mathbf{s}}(\boldsymbol{\lambda})$ and $L_{\mathbf{s}}(\boldsymbol{\mu})$ with $|\boldsymbol{\lambda}| = |\boldsymbol{\mu}|$. Assume that there exist $\iota_1, \iota_2 \in \mathbb{Z}$ and $\kappa_1 \neq \kappa_2$, $1 \leq \kappa_1, \kappa_2 \leq r$ such that*

(1) In $L_{\mathbf{s}}(\boldsymbol{\lambda})$, there is a bead at position (κ_1, ι_1) and position (κ_2, ι_2) is empty, and in $L_{\mathbf{s}}(\boldsymbol{\mu})$, position (κ_1, ι_1) is empty and there is a bead at position (κ_2, ι_2).

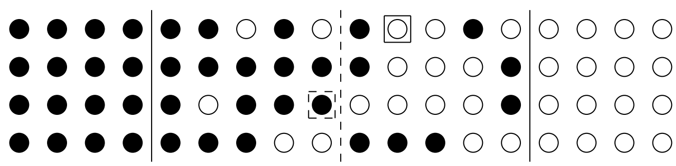
(2) *The beads on the right side of ι_1-th position in $L_{\mathbf{s}}(\boldsymbol{\lambda}^{(\kappa_1)})$ are the same as those in $L_{\mathbf{s}}(\boldsymbol{\mu}^{(\kappa_1)})$, and the beads on the left side of ι_2-th position in $L_{\mathbf{s}}(\boldsymbol{\lambda}^{(\kappa_2)})$ are the same as those in $L_{\mathbf{s}}(\boldsymbol{\mu}^{(\kappa_2)})$.*

Then we say $L_{\mathbf{s}}(\boldsymbol{\lambda})$ and $L_{\mathbf{s}}(\boldsymbol{\mu})$ are incomparable, which will be denoted by $L_{\mathbf{s}}(\boldsymbol{\lambda}) \parallel L_{\mathbf{s}}(\boldsymbol{\mu})$, or $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\lambda})$.

Example 3.3.2. *Let $e = 5$, $\boldsymbol{\lambda} = ((2,1,1),(2,2,1,1),(2,2,1),(4,3,1,1))$, and $\mathbf{s} = (1,0,2,0)$. Then $L_{\mathbf{s}}(\boldsymbol{\lambda})$ is*



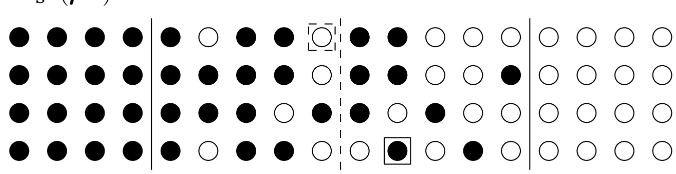
Let $\boldsymbol{\mu} = ((2,2,2),(5,1,1,1),(3),(4,2,1))$ and $\mathbf{s} = (1,0,2,0)$. Then $L_{\mathbf{s}}(\boldsymbol{\mu})$ is



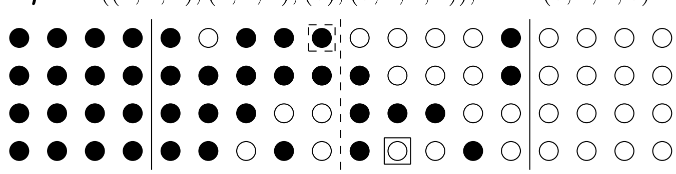
Take $\kappa_1 = 4$, $\kappa_2 = 2$, $\iota_1 = 1$, $\iota_2 = -1$. It is easy to check that the conditions (1) *and* (2) *of Definition 3.3.1 are satisfied. Then $L_{\mathbf{s}}(\boldsymbol{\lambda})$ and $L_{\mathbf{s}}(\boldsymbol{\mu})$ are incomparable abaci.*

The original intention of introducing the concept of incomparable abaci is to find incomparable multi-partitions that are in the same block. An easy computation gives $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$ in Example 3.3.2. This implies that we can not conclude from two abaci $L_{\mathbf{s}}(\boldsymbol{\lambda})$ and $L_{\mathbf{s}}(\boldsymbol{\mu})$ being incomparable that the corresponding multi-partitions are incomparable with respect to the dominance order $\trianglerighteq$.

A simple observation tells us that if $L_{\mathbf{s}}(\boldsymbol{\lambda})$ and $L_{\mathbf{s}}(\boldsymbol{\mu})$ are incomparable abaci, then for arbitrary $\sigma \in \mathfrak{S}_r$, $L_{\mathbf{s}^\sigma}(\boldsymbol{\lambda}^\sigma)$ and $L_{\mathbf{s}^\sigma}(\boldsymbol{\mu}^\sigma)$ are incomparable too, where $\mathbf{s}^\sigma$ is defined to be $(\mathbf{s}_{\sigma(1)}, \cdots, \mathbf{s}_{\sigma(r)})$. If we choose $\sigma = (124)$ in Example 3.3.2, then $L_{\mathbf{s}^\sigma}(\boldsymbol{\lambda}^\sigma)$ and $L_{\mathbf{s}^\sigma}(\boldsymbol{\mu}^\sigma)$ are as follows.



$$\boldsymbol{\mu}^\sigma = ((4,2,1),(2,2,2),(3),(5,1,1,1)), \mathbf{s}^\sigma = (0,1,2,0)$$



Now let us consider $\boldsymbol{\lambda}^\sigma = ((4,3,1,1),(2,1,1),(3,1,1),(2,2,1,1))$ and $\boldsymbol{\mu}^\sigma = ((4,2,1),(2,2,2),(3),(5,1,1,1))$. It is easy to check $\boldsymbol{\lambda}^\sigma \parallel \boldsymbol{\mu}^\sigma$. It is worth to point out that this has general significance. That is, we can prove that if $L_{\mathbf{s}}(\boldsymbol{\lambda}) \parallel L_{\mathbf{s}}(\boldsymbol{\mu})$, then there exists some $\sigma \in \mathfrak{S}_r$ such that $\boldsymbol{\lambda}^\sigma$ and $\boldsymbol{\mu}^\sigma$ are incomparable. To this aim, let us first give some characterizations of incomparable abaci.

Lemma 3.3.3. *Let $L_{\mathbf{s}}(\boldsymbol{\lambda}) \parallel L_{\mathbf{s}}(\boldsymbol{\mu})$. Then*

(1) The number of empty positions on the left side of position (κ_1, ι_1+1) in $L_{\mathbf{s}}(\boldsymbol{\lambda}^{(\kappa_1)})$ is the same as that in $L_{\mathbf{s}}(\boldsymbol{\mu}^{(\kappa_1)})$.

(2) The number of the beads on the right side of position (κ_2, ι_2-1) in $L_{\mathbf{s}}(\boldsymbol{\lambda}^{(\kappa_2)})$ is the same as that in $L_{\mathbf{s}}(\boldsymbol{\mu}^{(\kappa_2)})$.

Proof. (1) Choose $k \in \mathbb{Z}$ such that all positions those are to the left of $k+1$-th positions of both $L_{\mathbf{s}}($
blam) and $L_{\mathbf{s}}(\boldsymbol{\mu})$ are occupied by beads. According to Definition 3.3.1 (1), $k < \iota_1$. On the other hand, we have from Lemma 3.1.8(2) that $\mathfrak{n}^k_{\kappa_1}(L_{\mathbf{s}}(\boldsymbol{\lambda})) = \mathfrak{n}^k_{\kappa_1}(L_{\mathbf{s}}(\boldsymbol{\mu}))$. Then the result follows from Definition 3.3.1 (2).

(2) is proved similarly as (1). □

With the above result, we can prove a special case.

Lemma 3.3.4. *Keep notations as in Definition 3.3.1. Let $L_{\mathbf{s}}(\boldsymbol{\lambda}) \parallel L_{\mathbf{s}}(\boldsymbol{\mu})$ and $\kappa_1 < \kappa_2$. If $c \in \{1, \cdots, \kappa_1 - 1\} \cup \{\kappa_2+1, \cdots, r\}$ implies that $L_{\mathbf{s}}(\boldsymbol{\lambda}^{(c)})$ and $L_{\mathbf{s}}(\boldsymbol{\mu}^{(c)})$ are the same, then $\boldsymbol{\lambda} \parallel \boldsymbol{\mu}$.*

Proof. Suppose that the bead at position (κ_1, ι_1) in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ is $\bullet^{\kappa_1}_k(\boldsymbol{\lambda}, \mathbf{s})$. Since $L_{\mathbf{s}}(\boldsymbol{\lambda}) \parallel L_{\mathbf{s}}(\boldsymbol{\mu})$, by Definition 3.3.1, the beads on the right side of ι_1-th position in $L_{\mathbf{s}}(\boldsymbol{\lambda}^{(\kappa_1)})$ are the same as those in $L_{\mathbf{s}}(\boldsymbol{\mu}^{(\kappa_1)})$. This implies that for each $1 \le a < k$, if $\bullet^{\kappa_1}_a(\boldsymbol{\lambda}, \mathbf{s})$ is in position b, then so is $\bullet^{\kappa_1}_a(\boldsymbol{\mu}, \mathbf{s})$. Moreover, we have from Lemma 3.3.3 that the number of empty positions on the left side of position (κ_1, ι_1+1) in $L_{\mathbf{s}}(\boldsymbol{\lambda}^{(\kappa_1)})$ is the same as that in $L_{\mathbf{s}}(\boldsymbol{\mu}^{(\kappa_1)})$. Consequently, the number of empty positions on the left side of $\bullet^{\kappa_1}_a(\boldsymbol{\lambda}, \mathbf{s})$ in $L_{\mathbf{s}}(\boldsymbol{\lambda}^{(\kappa_1)})$ is the same as that on the left side of $\bullet^{\kappa_1}_a(\boldsymbol{\mu}, \mathbf{s})$ in in $L_{\mathbf{s}}(\boldsymbol{\mu}^{(\kappa_1)})$ for $1 \le a < k$. Then we obtain by Lemma 3.1.8 (2) that

$$\boldsymbol{\lambda}^{(\kappa_1)}_a = \boldsymbol{\mu}^{(\kappa_1)}_a. \tag{3.3.1}$$

Furthermore, since the position (κ_1, ι_1) in $L_{\mathbf{s}}(\boldsymbol{\mu})$ is empty, it is clear that $\bullet^{\kappa_1}_k(\boldsymbol{\mu}, \mathbf{s})$ is on the left side of position ι_1. Using Lemma 3.3.3 (1) again, there are fewer empty positions on the left side of $\bullet^{\kappa_1}_k(\boldsymbol{\mu}, \mathbf{s})$ than on the left side of $\bullet^{\kappa_1}_k(\boldsymbol{\lambda}, \mathbf{s})$. We deduce from Lemma 3.1.8 (2) that

$$\boldsymbol{\mu}^{(\kappa_1)}_k < \boldsymbol{\lambda}^{(\kappa_1)}_k. \tag{3.3.2}$$

Combining (3.3.1) with (3.3.2) yields

$$\sum_{a=1}^{k} \boldsymbol{\mu}^{(\kappa_1)}_a < \sum_{a=1}^{k} \boldsymbol{\lambda}^{(\kappa_1)}_a. \tag{3.3.3}$$

Note that the assumption implies that the rows below κ_1-th row in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ are the same as that in $L_{\mathbf{s}}(\boldsymbol{\mu})$, and thus for each $1 \le c < \kappa_1$, $\boldsymbol{\lambda}^{(c)} = \boldsymbol{\mu}^{(c)}$. This gives

$$\sum_{c=1}^{\kappa_1-1} |\boldsymbol{\mu}^c| = \sum_{c=1}^{\kappa_1-1} |\boldsymbol{\lambda}^c|. \tag{3.3.4}$$

By combining (3.3.3) with (3.3.4), we get

$$\sum_{c=1}^{\kappa_1-1}|\boldsymbol{\mu}^{(c)}|+\sum_{a=1}^{k}\boldsymbol{\mu}_a^{(\kappa_1)}<\sum_{c=1}^{\kappa_1-1}|\boldsymbol{\lambda}^{(c)}|+\sum_{a=1}^{k}\boldsymbol{\lambda}_a^{(\kappa_1)}. \tag{3.3.5}$$

On the other hand, let the bead at position (κ_2,ι_2) in $L_{\mathbf{s}}(\boldsymbol{\mu})$ be $\bullet_t^{\kappa_2}(\boldsymbol{\mu},\mathbf{s})$. Then by analyzing the number of beads as above, we can obtain the following two formulas:

(1) $\boldsymbol{\lambda}_x^{(\kappa_2)}=\boldsymbol{\mu}_x^{(\kappa_2)}$ for arbitrary $x>t$.

(2) $\boldsymbol{\mu}_t^{(\kappa_2)}<\boldsymbol{\lambda}_t^{(\kappa_2)}$.

This implies that

$$\sum_{x\geq t}\boldsymbol{\mu}_x^{(\kappa_2)}<\sum_{x\geq t}\boldsymbol{\lambda}_x^{(\kappa_2)}.$$

Combing this formula with

$$\sum_{c=\kappa_2+1}^{r}|\boldsymbol{\mu}^{(c)}|=\sum_{c=\kappa_2+1}^{r}|\boldsymbol{\lambda}^{(c)}|,$$

which is an easy corollary of the assumption, leads to

$$\sum_{x\geq t}\boldsymbol{\mu}_x^{(\kappa_2)}+\sum_{c=\kappa_2+1}^{r}|\boldsymbol{\mu}^{(c)}|<\sum_{x\geq t}\boldsymbol{\lambda}_x^{(\kappa_2)}+\sum_{c=\kappa_2+1}^{r}|\boldsymbol{\lambda}^{(c)}|. \tag{3.3.6}$$

Moreover, it follows from $|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|$ that

$$\sum_{c=1}^{r}|\boldsymbol{\mu}^{(c)}|=\sum_{c=1}^{r}|\boldsymbol{\lambda}^{(c)}|. \tag{3.3.7}$$

Then (3.3.7) − (3.3.6) is

$$\sum_{c=1}^{r}|\boldsymbol{\mu}^{(c)}|-(\sum_{x\geq t}\boldsymbol{\mu}_x^{(\kappa_2)}+\sum_{c=\kappa_2+1}^{r}|\boldsymbol{\mu}^{(c)}|)>\sum_{c=1}^{r}|\boldsymbol{\lambda}^{c}|-(\sum_{x\geq t}\boldsymbol{\lambda}_x^{(\kappa_2)}+\sum_{c=\kappa_2+1}^{r}|\boldsymbol{\lambda}^{(c)}|).$$

Note that

$$\sum_{c=1}^{r}|\boldsymbol{\mu}^{(c)}|-(\sum_{x\geq t}\boldsymbol{\mu}_x^{(\kappa_2)}+\sum_{c=\kappa_2+1}^{r}|\boldsymbol{\mu}^{(c)}|)=\sum_{c=1}^{\kappa_2-1}|\boldsymbol{\mu}^{(c)}|+\sum_{x=1}^{t-1}\boldsymbol{\mu}_x^{(\kappa_2)}$$

and

$$\sum_{c=1}^{r}|\boldsymbol{\lambda}^{c}|-(\sum_{x\geq t}\boldsymbol{\lambda}_x^{(\kappa_2)}+\sum_{c=\kappa_2+1}^{r}|\boldsymbol{\lambda}^{(c)}|)=\sum_{c=1}^{\kappa_2-1}|\boldsymbol{\lambda}^{(c)}|+\sum_{x=1}^{t-1}\boldsymbol{\lambda}_x^{(\kappa_2)}.$$

We arrive at

$$\sum_{c=1}^{\kappa_2-1}|\boldsymbol{\mu}^{(c)}|+\sum_{x=1}^{t-1}\boldsymbol{\mu}_x^{(\kappa_2)}>\sum_{c=1}^{\kappa_2-1}|\boldsymbol{\lambda}^{(c)}|+\sum_{x=1}^{t-1}\boldsymbol{\lambda}_x^{(\kappa_2)}. \tag{3.3.8}$$

Now we come to the conclusion $\boldsymbol{\lambda}\parallel\boldsymbol{\mu}$ by combining (3.3.5) and (3.3.8) together. □

We can deduce a simple fact immediately from Lemma 3.3.4, that is, two incomparable abaci $L_{\mathbf{s}}(\boldsymbol{\lambda})\parallel L_{\mathbf{s}}(\boldsymbol{\mu})$ with $\kappa_1=1$ and $\kappa_2=r$ give a pair of incomparable multi-partitions $\boldsymbol{\lambda}\parallel\boldsymbol{\mu}$. Clearly, for $L_{\mathbf{s}}(\boldsymbol{\lambda})\parallel L_{\mathbf{s}}(\boldsymbol{\mu})$, we can always find $\sigma\in\mathfrak{S}_r$ such that $\sigma(\kappa_1)=1$ and $\sigma(\kappa_2)=r$. As a result, $L_{\mathbf{s}^\sigma}(\boldsymbol{\lambda}^\sigma)\parallel L_{\mathbf{s}^\sigma}(\boldsymbol{\mu}^\sigma)$.

Proposition 3.3.5. *Suppose that $L_{\mathbf{s}}(\boldsymbol{\lambda}) \parallel L_{\mathbf{s}}(\boldsymbol{\mu})$. Then there exists $\sigma \in \mathfrak{S}_r$ such that $\boldsymbol{\lambda}^\sigma \parallel \boldsymbol{\mu}^\sigma$ with respect to the dominance order.*

3.4. **Move vector.** It is well-known that for an abacus $\mathcal{L}_s^e(\lambda)$ of a partition λ, sliding a bead from position (x, y) to $(x, y-1)$ is equivalent to unwrapping an e-rim hook from $[\lambda]$. An r-abacus version of this operation was introduced by Jacon and Lecouvey.

Definition 3.4.1. [33, Section 4.1] *Let $\boldsymbol{\lambda}$ be an r-partition of rank n. An elementary operation in runner x on the $(e, \mathbf{s})$-abacus of $L_{\mathbf{s}}(\boldsymbol{\lambda})$ is a move of one bead from runner x to another.*

1. *First kind: if the bead at position (x, y) with $1 \le x < r$ is black and $(x+1, y)$ is empty, then slide the bead to position $(x+1, y)$ (note that the resulting r-abacus corresponds to an r-partition of rank $n - s_{x+1} + s_x - 1$).*
2. *Second kind: if $e \neq \infty$ and the bead at position (r, y) is black and $(1, y-e)$ is empty, then slide the bead to position $(1, y-e)$ (note that the resulting r-abacus corresponds to an r-partition of rank $n - s_1 + s_l - e - 1$).*

To continue our study, we need to define concepts "before", "after" and "between", which are about relationship between positions in an abacus.

Definition 3.4.2. *Given an r-partition $\boldsymbol{\lambda}$ and a multicharge $\mathbf{s}$, let $L_{\mathbf{s}}(\boldsymbol{\lambda})$ be the associated (e, s)-abacus. We say position (j, h) is before position (i, l) if one of the conditions below is satisfied*

1. *$h = l$ and $i < j \le r$.*
2. *$h = l - (k+1)e$, where $k \in \mathbb{N}$.*

If position (j, h) is before position (i, l), then we also say position (i, l) is after position (j, h). Let position (j, h) be before position (i, l). We say position (x, y) is between positions (j, h) and (i, l) if it is after (j, h) and before (i, l).

Clearly, in an abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$, there exist two positions (i, l) and (j, h) such that (i, l) is neither before nor after (j, h). However, this phenomenon will not happen in a subabacus of $L_{\mathbf{s}}(\boldsymbol{\lambda})$. Let us list some observations about subabaci for later use.

1. None of the positions of $L_{\mathbf{s}}(\boldsymbol{\lambda})$ belong to two different subabaci simultaneously.
2. We can not move a bead from one subabacus to another by elementary operations.
3. Index a bead in a subabacus by x if there are exactly $x-1$ beads after it. Then elementary operations do not change the index of a bead.

Based on the above observations, elementary operations from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{v}}(\boldsymbol{\mu})$ is a definite set, which is independent of the choice of the procedure of move. We often call this set operation set and denote it by $\mathcal{F}$. Moreover, an elementary operation happened in a subabacus does not affect other subabaci. Therefore, each elementary operation of an abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$ can be naturally viewed as an elementary operation of a subabacus. Given an elementary operation that slides the number x bead at position (i, l), we record it by a triple $[(i, l), x]$. When the index need not to be pointed out, the triple will be written as $[(i, l), *]$. The following is an obvious result about operation sets.

Lemma 3.4.3. *Let $\mathcal{F}$ be the operation set from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to its core with $[(i, h), *] \in \mathcal{F}$, where $1 \le i \le r$ (if $e = \infty$, $i \neq r$) and $h \in \mathbb{Z}$. We have in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,*

(1) if position $(i+1,h)$ has a bead, then there is an empty position before it;
(2) if position (i,h) is empty, then there is a position with a bead after it.

An interesting fact about elementary operations is that they are "commute" with Uglov map.

Lemma 3.4.4. *Let $\mathcal{F}$ be the operation set from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{v}}(\boldsymbol{\mu})$. Then there exists a definite operation set $\tau_{\mathcal{F}}$ from $L_s(\lambda)$ to $L_v(\mu)$, where (λ, s) and (μ, v) are images of $(\boldsymbol{\lambda}, \mathbf{s})$ and $(\boldsymbol{\mu}, \mathbf{v})$ under Uglov map, respectively, such that the following diagram commute.*

$$\begin{tikzcd} L_{\mathbf{s}}(\boldsymbol{\lambda}) \arrow[r, "\tau_{e,r}"] \arrow[d, "\mathcal{F}"'] & L_s(\lambda) \arrow[d, "\tau_{\mathcal{F}}"] \\ L_{\mathbf{v}}(\boldsymbol{\mu}) \arrow[r, "\tau_{e,r}"] & L_v(\mu) \end{tikzcd}$$

Proof. (Sketch) We only need to consider the case of $\mathcal{F}$ containing only one elementary operation $[(x,y),*]$. Let $y = ke + c$. It is not difficult to check that if $y < r$, the corresponding elementary operation is moving in $\mathcal{L}_s^e(\lambda)$ the bead in position $(c, r-x+kr)$ to $(c, r-x-1+kr)$. If $x = r$, the corresponding elementary operation is moving in $\mathcal{L}_s^e(\lambda)$ the bead in position (c, kr) to $(c, kr-1)$. Then it is a routine to check the diagram commute. □

Operation sets give rise to a key notion in this paper, move vector, whose definition is as follows.

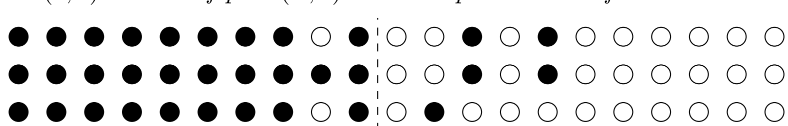
Definition 3.4.5. *Let $\mathcal{F}$ be the operation set from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{v}}(\boldsymbol{\mu})$. Define*

$$m_i = \sharp\{[(i,h),x] \in \mathcal{F} \mid h \in \mathbb{Z}, x \in \mathbb{N}^+\}.$$

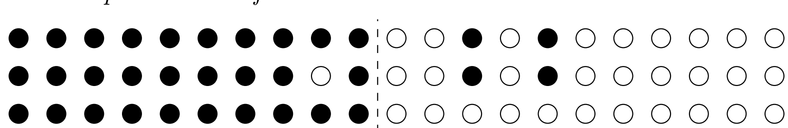
Then $\mathcal{M} = (m_1, m_2, \cdots, m_r)$ is called the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{v}}(\boldsymbol{\mu})$.

For the reason of importance, let us give an example of operation sets and move vectors.

Example 3.4.6. *Let $e = 3$, $\mathbf{s} = (0,2,1)$ and $\boldsymbol{\lambda} = ((2,1),(3,2),(4,3,1))$. Then the associated $(e,\mathbf{s})$-abacus of pair $(\boldsymbol{\lambda}, \mathbf{s})$ can be represented as follows.*



Let $\boldsymbol{\mu} = (\emptyset, (4,3,1), (3,2))$ and $\mathbf{v} = (0,1,2)$. Then the associated $(e,\mathbf{s})$-abacus $L_{\mathbf{v}}(\boldsymbol{\mu})$ can be represented as follows.



The operation set from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{v}}(\boldsymbol{\mu})$ is

$$\mathcal{F} = \{[(2,-2),4],[(1,1),3],[(2,1),3],[(3,1),3]\},$$

and consequently, $\mathcal{M} = (1,2,1)$.

The following lemma reveals that the move vector from abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to another one can reflect variations of multicharge $\mathbf{s}$ and the shape of $L_{\mathbf{s}}(\boldsymbol{\lambda})$. It is a simple corollary of Definition 3.4.5 and Lemma 3.1.9, and we omit the proof and leave it as an exercise.

Lemma 3.4.7. *Let $\mathcal{M} = (m_1, m_2, \cdots, m_r)$ be the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{u}}(\boldsymbol{\mu})$. Then for $1 \le i \le r$, we have $\mathfrak{n}_i^h(L_{\mathbf{s}}(\boldsymbol{\lambda})) - \mathfrak{n}_i^h(L_{\mathbf{u}}(\boldsymbol{\mu})) = m_i - m_{i-1} = s_i - u_i$ (the meaning of m_0 is m_r).*

The purpose of introducing concept "move vector" is to study blocks of cyclotomic Hecke algebras. Let us recall the definition of the core of a pair $(\boldsymbol{\lambda}, \mathbf{s})$ which introduced by Jacon and Lecouvey.

Definition 3.4.8. [33, Definition 4.2] *Let $(\boldsymbol{\lambda}, \mathbf{s})$ be a pair with $\mathbf{s} \in \overline{\mathcal{A}}_e^r$. The core of $(\boldsymbol{\lambda}, \mathbf{s})$ is a pair $(\boldsymbol{\lambda}^*, \mathbf{s}^*)$, whose abacus is complete and is obtained from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ by elementary operations.*

Lemma 3.4.9. *Let $\mathbf{s} \in \overline{\mathcal{A}}_e^r$ and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^r$. If the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{v}}(\boldsymbol{\nu})$ is equal to that from $L_{\mathbf{u}}(\boldsymbol{\mu})$ to $L_{\mathbf{v}}(\boldsymbol{\nu})$, then $\mathbf{s} = \mathbf{u}$ and $(\boldsymbol{\lambda}, \mathbf{s})$ and $(\boldsymbol{\mu}, \mathbf{s})$ belong to the same block.*

Proof. Let $(m_1, \cdots, m_r)$ be the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{v}}(\boldsymbol{\nu})$ with $m_1 + \cdots + m_r = l$. Then by Lemma 3.4.7 $v_i - s_i = m_i - m_{i-1} = v_i - u_i$ for $1 \le i \le r$, in which m_0 means m_r. This implies $\mathbf{s} = \mathbf{u}$.

Let $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda}) = \lambda$, $\tau_{e,\mathbf{s}}(\boldsymbol{\mu}) = \mu$, $\tau_{e,\mathbf{v}}(\boldsymbol{\nu}) = \nu$ and $s = \sum_{i=1}^r s_i$, $v = \sum_i^r v_i$. If we need k elementary operations from $\mathcal{L}_v^e(\nu)$ to its e-core, then by Lemma 3.4.4 we can obtain this e-core from both $\mathcal{L}_s^e(\lambda)$ and $\mathcal{L}_s^e(\mu)$ by $k + l$ elementary operations, respectively. Therefore, (λ, s) and (μ, s) belong to the same block. This is equivalent to $\mathcal{C}_{e,s}(\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})) = \mathcal{C}_{e,s}(\tau_{e,\mathbf{s}}(\boldsymbol{\mu})$. By [33, Corollary 2.23], $\mathcal{C}_{e,\mathbf{s}}(\boldsymbol{\lambda}) = \mathcal{C}_{e,\mathbf{s}}(\boldsymbol{\mu})$. We deduce from Lemma 3.1.3 that $(\boldsymbol{\lambda}, \mathbf{s})$ and $(\boldsymbol{\mu}, \mathbf{s})$ belong to the same block. □

The most important value of move vector is that they induce a new block invariant, which can play a greater role than weights. Let us prove a lemma for giving the definition.

Lemma 3.4.10. *Let $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $\mathbf{s} \in \overline{\mathcal{A}}_e^r$ and $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ its core. If $(\boldsymbol{\mu}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$, then its core is also $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ and the move vector from $L_{\mathbf{s}}(\boldsymbol{\mu})$ to $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ is equal to that from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$.*

Proof. Since $(\boldsymbol{\lambda}, \mathbf{s})$ and $(\boldsymbol{\mu}, \mathbf{s})$ are in the same block, we have from [33, Lemma 2.23] that (λ, s) and (μ, s) are in the same block and therefore (λ, s) and (μ, s) have the same e-core (λ^*, s^*). Note that Uglov map is injective. This implies that (λ^*, s^*) has a unique preimage, which is a reduced $(e, \mathbf{s})$-core by Lemma 3.4.4. Consequently, the preimage has to be $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$.

Suppose that the weight of $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ is w, the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ is $(m_1, \cdots, m_r)$, and the move vector from $L_{\mathbf{s}}(\boldsymbol{\mu})$ to $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ is $(m_1', \cdots, m_r')$. Then Lemma 3.4.7 implies $m_i - m_{i-1} = s_i^* - s_i = m_i' - m_{i-1}'$ for $1 \le i \le r$, in which m_0 means m_r. Then the result follows from $m_1 + \cdots + m_r = w = m_1' + \cdots + m_r'$. □

In the light of Lemma 3.4.10, we also say $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ is the core of block $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$.

Definition 3.4.11. *Let $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $\boldsymbol{\lambda} \in \mathcal{A}_e^r$. Then the block move vector of $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ is defined to be the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to the core of the block.*

In order to simplify the descriptions of some lemmas' proofs in subsequent sections, let us do some work that is related to operation sets and move vectors beforehand.

Firstly, we study the move vector related to a simple deformation of abaci. Given an abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$ in $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $\mathbf{s} \in \overline{\mathcal{A}}_e^r$ and $e < \infty$, denote by $L_{\boldsymbol{\mu}}(\boldsymbol{\mu})$ the abacus obtained by deleting the the first i runners and putting $L_{s_x+e}(\boldsymbol{\lambda}^{(x)})$, $1 \leq x \leq i$ on the top in $L_{\mathbf{s}}(\boldsymbol{\lambda})$. Let $\mathcal{M} = (m_1, \cdots, m_r)$ and $\mathcal{M}' = (m_1', \cdots, m_r')$ be the move vectors and $\mathcal{F}$ and $\mathcal{F}'$ the operation sets from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ and $L_{\mathbf{u}}(\boldsymbol{\mu})$ to their cores, respectively. Then we have some easy results as follows.

Lemma 3.4.12. *The followings hold.*

(1) $m_j' = m_{i+j}$ *for* $1 \leq j \leq r - i$.
(2) $m_j' = m_{i+j-r}$ *for* $r - i + 1 \leq j \leq r$.
(3) $[(j, h), *] \in \mathcal{F}'$ *if and only if* $[(i + j, h), *] \in \mathcal{F}$ *for* $1 \leq j \leq r$.
(4) $L_{\mathbf{u}^*}(\boldsymbol{\mu}^*)$ *can be obtained from* $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ *by deleting the the first i runners and putting* $L_{s_x^*+e}(\boldsymbol{\lambda}^{*(x)})$, $1 \leq x \leq i$ *in the original order on the top in* $L_{\mathbf{s}^*}(\boldsymbol{\lambda})$.
(5) $\mathbf{u} \in \overline{\mathcal{A}}_e^r$ *and* $(\boldsymbol{\mu}, \mathbf{u}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$.

Secondly, we consider the composition of certain elementary operations. The following lemma is a generalization of [33, Remark 4.3 (1)]. It is easy and we omit its proof.

Lemma 3.4.13. *Let $L_{\mathbf{s}}(\boldsymbol{\lambda})$ be an abacus with position (i, l) having a bead and position $(j, l-ke)$ is empty and before (i, l), where $1 \leq i, j \leq r$, $l \in \mathbb{Z}$, $k \in \mathbb{N}$ and $k = 0$ when $e = \infty$. Move in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ the bead at position (i, l) to position $(j, l - ke)$ and denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the new abacus obtained. Then $L_{\mathbf{u}}(\boldsymbol{\mu})$ can be obtained from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ by elementary operations and the operation set is $\{[(i, l), *], \cdots, [(r, l), *], [(1, l - e), *], \cdots, [(r, l - e), *], \cdots, [(r, l - ke + e), *], [(1, l - ke), *], \cdots, [(j, l - ke), *]\}$.*

The following two results are direct corollaries of Lemma 3.4.13. We list them below without proofs.

Lemma 3.4.14. *Let $\mathcal{F}$ be the operation set from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to its core $(\boldsymbol{\lambda}^*, \mathbf{s}^*)$. If in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, position (i, h) has a bead and position $(j, h - ke)$ is empty and before (i, h), where $k \in \mathbb{N}$ and $k = 0$ if $e = \infty$. Then the operations moving the bead at position (i, h) to $(j, h - ke)$ are contained in $\mathcal{F}$.*

Lemma 3.4.15. *Let $L_{\mathbf{s}}(\boldsymbol{\lambda})$ be an abacus with position (i, l) having a bead and position (j, l) being empty, where $1 \leq i < j \leq r$. Let $L_{\mathbf{u}}(\boldsymbol{\mu})$ be the abacus obtained by moving in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ the bead at position (i, l) to position (j, l). Then $L_{\mathbf{u}}(\boldsymbol{\mu})$ can be obtained from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ by elementary operations with move vector $\mathcal{M} = (m_1, \cdots, m_r)$, where $m_t = 1$ if $i \leq t \leq j - 1$ and $m_t = 0$ otherwise.*

Let us consider a special case of Lemma 3.4.13. We can give more details in this case.

Lemma 3.4.16. *Let $L_{\mathbf{s}}(\boldsymbol{\lambda})$ be an abacus.*

(1) Assume that position (i, l) is empty and position $(i, l+e)$ has a bead. Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the abacus obtained from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ by moving the bead at position $(i, l + e)$ to position (i, l). Then $\mathbf{u} = \mathbf{s}$ and the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{u}}(\boldsymbol{\mu})$ is $(1, \cdots, 1)$. Moreover, $[\boldsymbol{\mu}]$ can be obtained by deleting a rim e-hook from $[\boldsymbol{\lambda}^{(i)}]$.

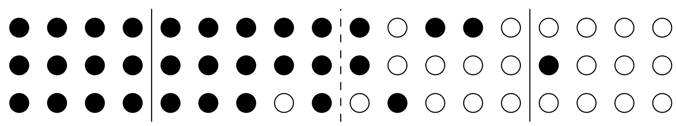
(2) If $[\boldsymbol{\mu}]$ can be obtained by deleting a rim e-hook from $[\boldsymbol{\lambda}^{(i)}]$, then there exists $l \in \mathbb{Z}$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, position (i, l) is empty and position $(i, l+e)$ has a bead.

Proof. It is an evident corollary of Lemma 3.4.7, 3.4.13 and [33, Remark 4.3(1)]. □

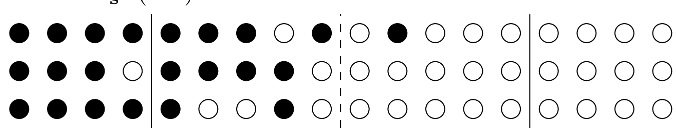
Thirdly, we introduce a new concept, dual abacus.

Definition 3.4.17. *Given an abaci $L_{\mathbf{s}}(\boldsymbol{\lambda})$, the dual of $L_{\mathbf{s}}(\boldsymbol{\lambda})$, denoted by $L_{\mathbf{s}^D}(\boldsymbol{\lambda}^D)$, is an abacus, in which position (i, h) is empty if and only if in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, there is a bead at position $(r-i+1, -h-1)$. Furthermore, $(\boldsymbol{\lambda}^D, \mathbf{s}^D)$ is called the dual pair of $(\boldsymbol{\lambda}, \mathbf{s})$.*

Example 3.4.18. *Let $\boldsymbol{\lambda} = ((2,1),(4),(1,1))$ and $\mathbf{s} = (0,2,3)$. Then $L_{\mathbf{s}}(\boldsymbol{\lambda})$ is*



The dual abacus $L_{\mathbf{s}^D}(\boldsymbol{\lambda}^D)$ is



It is easy to check that $\boldsymbol{\lambda}^D = ((2),(1,1,1,1),(2,1))$ and $\mathbf{s}^D = (-3,-2,0)$.

By an easy observation on Example 3.4.18, we have $\boldsymbol{\lambda}^D$ is in fact the conjugate of $\boldsymbol{\lambda}$ and $s_i^D = -s_{r-i+1}$. We emphasize that this has general significance. Let us summarize into a lemma for later use.

Lemma 3.4.19. *Let $L_{\mathbf{s}^D}(\boldsymbol{\lambda}^D)$ be the dual of $L_{\mathbf{s}}(\boldsymbol{\lambda})$. Then $\boldsymbol{\lambda}^D = \boldsymbol{\lambda}'$, $s_i^D = -s_{r-i+1}$ for $1 \le i \le r$ and consequently, if $\mathbf{s} \in \overline{\mathcal{A}}_e^r$, then $\mathbf{s}^D \in \overline{\mathcal{A}}_e^r$ and if $\mathbf{s} \in \mathcal{A}_e^r$, then $\mathbf{s}^D \in \mathcal{A}_e^r$.*

About dual abaci, we have the following lemmas, which will be used later.

Lemma 3.4.20. *Let $(\boldsymbol{\lambda}, \mathbf{s})$ and $(\boldsymbol{\mu}, \mathbf{s})$ be two pairs. Then*

(1) $(\boldsymbol{\lambda}, \mathbf{s})$ and $(\boldsymbol{\mu}, \mathbf{s})$ belong to the same block if and only if $(\boldsymbol{\lambda}^D, \mathbf{s}^D)$ and $(\boldsymbol{\mu}^D, \mathbf{s}^D)$ belong to the same block.

(2) $L_{\mathbf{s}}(\boldsymbol{\lambda}) \parallel L_{\mathbf{s}}(\boldsymbol{\mu})$ if and only if $L_{\mathbf{s}^D}(\boldsymbol{\lambda}^D) \parallel L_{\mathbf{s}^D}(\boldsymbol{\mu}^D)$.

Lemma 3.4.21. *Let $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{F}'$, $\mathcal{M}'$ be the operation sets and move vectors from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ and $L_{\mathbf{s}^D}(\boldsymbol{\lambda}^D)$ to their cores, respectively. Then $[(i,h),*] \in \mathcal{F}$ if and only if $[(r-i,-h-1),*] \in \mathcal{F}'$. Moreover, $m_i = m'_{r-i+1}$ for all $1 \le i \le r$.*

Proof. We first consider operation sets. It is clear by Definition 3.4.17 that we only need to prove if $[(i,h),*] \in \mathcal{F}$, then $[(r-i,-h-1),*] \in \mathcal{F}'$. In fact, take an element $[(i,h),*] \in \mathcal{F}$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, position is empty and position (i,h) has a bead. Then in $L_{\mathbf{s}^D}(\boldsymbol{\lambda}^D)$, position $(r-i,-h-1)$ has a bead and position $(r-i+1,-h-1)$ is empty. This implies $[(r-i,-h-1),*] \in \mathcal{F}'$. By sliding the bead at position (i,h) to $(i+1,h)$ in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, and sliding bead at position $(r-i,-h-1)$ to $(r-i+1,-h-1)$ in $L_{\mathbf{s}^D}(\boldsymbol{\lambda}^D)$, we clearly get two dual abaci. Since $\mathcal{F}$ is a finite set, by repeating this process finite many times, abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$ arrives to its core.

Furthermore, as a direct corollary of the above, we can obtain the result on move vectors and we complete the proof. □

3.5. **Frame of the proof of Main Theorem.** We conclude this section by provide a frame for the proof of the Main Theorem.

Let K be an algebraically closed field with $CharK \neq 2$ and $q \in K^{\times}$, $q \neq 1$ with quantum characteristic e. Let $\mathbf{s} = (s_1, s_2, \cdots, s_r) \in \mathbb{Z}^r$ be a multicharge. Define $\widetilde{\mathbf{s}} := (s'_1, \cdots, s'_r) \in \{0, \cdots, e-1\}^r$ such that $s'_i \equiv s_i \,(\mathrm{mod}\, e)$. Then there exists a unique $\sigma_{\mathbf{s}} \in \mathfrak{S}_r$ such that

(1) $s'_{\sigma_{\mathbf{s}}(1)} \leq s'_{\sigma_{\mathbf{s}}(2)} \leq \cdots \leq s'_{\sigma_{\mathbf{s}}(r)}$

(2) If $s_{\sigma_{\mathbf{s}}(i)} = s_{\sigma_{\mathbf{s}}(i+1)}$ for $i \in \{1, \cdots, r-1\}$, then $\sigma_{\mathbf{s}}(i) < \sigma_s(i+1)$.

Set $\widetilde{\mathbf{s}}^{\sigma_{\mathbf{s}}} := (s'_{\sigma_s(1)}, s'_{\sigma_{\mathbf{s}}(2)}, \cdots, s'_{\sigma_{\mathbf{s}}(r)})$. Then clearly, $\widetilde{\mathbf{s}}^{\sigma_{\mathbf{s}}} \in \mathcal{A}_e^r$. Since $(\boldsymbol{\lambda}, \mathbf{s})$ and $(\boldsymbol{\lambda}^{\sigma_{\mathbf{s}}}, \widetilde{\mathbf{s}}^{\sigma_{\mathbf{s}}})$ belong to two isomorphic blocks, without loss of generality, we can assume $\mathbf{s} \in \mathcal{A}_e^r$. Because the representation type of blocks of a Hecke algebra of type A or B have been determined by Erdmann and Nakano in [23] and by Ariki in [5], we always assume in this paper that $r > 2$.

Given a cyclotomic Hecke algebra $\mathcal{H}_n(q, Q)$, let $\mathcal{S}_{n,r}(q, Q_1, Q_2, \cdots, Q_r)$ be the cyclotomic q-Schur algebra associated to it. Take $\boldsymbol{\nu} = (\varnothing, \cdots, \varnothing, (1^n))$. Let $\varphi_{\boldsymbol{\nu}} \in \mathcal{S}_{n,r}$ be the identity map on $M^{\boldsymbol{\nu}}$ and zero on others. Then $\varphi_{\boldsymbol{\nu}}$ is an idempotent of $\mathcal{S}_{n,r}$ and $\varphi_{\boldsymbol{\nu}} \mathcal{S}_{n,r} \varphi_{\boldsymbol{\nu}}$ is isomorphic to $\mathcal{H}_n(q, Q)$. If $\mathcal{B}$ is a block of $\mathcal{S}_{n,r}$, then clearly, $\varphi_{\boldsymbol{\nu}} \mathcal{B} \varphi_{\boldsymbol{\nu}}$ is isomorphic to a block of $\mathcal{H}_n(q, Q)$. Moreover, it is well-known ([22]) that if $\varphi_{\boldsymbol{\nu}} \mathcal{B} \varphi_{\boldsymbol{\nu}}$ has infinite type, then so is $\mathcal{B}$. If the weight of a block of $\mathcal{S}_{n,r}$ is 1, then by [24, Theorem 4.12] and [50, Proposition 1.7] the block has finite representation type. Consequently, the weight one blocks of $\mathcal{H}_n(q, Q)$ has finite representation type. As a result, we only need to handle the blocks of weight more than one.

Based on the above analysis, the proof of **Main Theorem** is divided into three parts according to the characteristic of the block move vector $\mathcal{M} = (m_1, m_2, \cdots, m_r)$:

Part I. There exists some $m_i \geq 2$.

Part II. All m_i are equal to 1, for $i = 1, \cdots, r$.

Part III. All m_i are less or equal to 1, and there exist at least one $m_j = 0$.

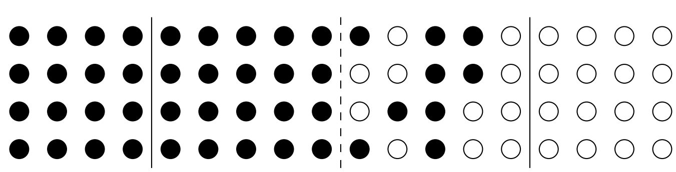
4. Proof of Main Theorem: Part I

Given a pair $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $s \in \overline{\mathcal{A}}_e^r$ and $r > 2$, let $\mathcal{F}$ be the operation set from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to its core and $(m_1, m_2, \cdots, m_r)$ the block move vector. In this section, we prove that if there exists $1 \leq i \leq r$ such that $m_i \geq 2$, then $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ has infinite representation type. To this aim, in the light of Proposition 2.3.7 and 3.3.5, we can form incomparable abaci in $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. The method is to analyzing the property of the associated $\mathcal{F}$ of an arbitrary pair $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. It is easy to know that each $\mathcal{F}$ must be one of the following two types:

Type I. There exists $1 \leq i \leq r$ such that $[(i, h_1), *]$, $[(i, h_2), *] \in \mathcal{F}$ with $h_1 \neq h_2$;

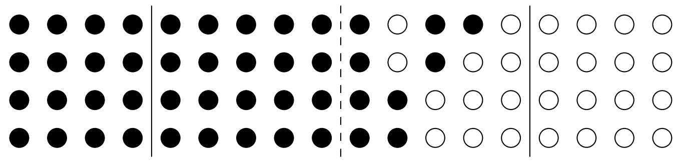
Type II. $[(i, h_1), *]$, $[(i, h_2), *] \in \mathcal{F}$ implies $h_1 = h_2$ for all $1 \leq i \leq r$.

For example, let $e = 5$, $\mathbf{s} = (2, 2, 2, 3)$ and $\boldsymbol{\lambda} = ((1), (1, 1), (2, 2)(1, 1))$. Then $L_{\mathbf{s}}(\boldsymbol{\lambda})$ is



Clearly, $[(2,0),*]$, $[(2,1),*]\in\mathcal{F}$, $0\neq 1$, that is, pair $(\boldsymbol{\lambda},\mathbf{s})$ is of Type I.

If we take $\boldsymbol{\mu}=(\varnothing,\varnothing,(1),(1,1))$, then $L_{\mathbf{s}}(\boldsymbol{\mu})$ is



Clearly, $\mathcal{F}=\{[(2,1),2],\ [(3,1),2],\ [(1,1),1],\ [(2,1),1]\}$, and thus pair $(\boldsymbol{\mu},\mathbf{s})$ if of Type II.

We will begin with some preparation in Subsection 1., and then prove the result of this section in Subsection 4.2 and 4.3.

4.1. **Some existence conditions of incomparable abaci.** This subsection is devoted to provide some circumstances, under which one can construct incomparable abaci.

Lemma 4.1.1. *Let* $(\boldsymbol{\lambda},\mathbf{s})\in\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ *with* $s\in\overline{\mathcal{A}}^{r}_{e}$. *If there exist* $1\leq j\leq r$, $h_1,\ h_2\in\mathbb{Z}$ *with* $h_1\neq h_2$ *such that in* $L_s(\boldsymbol{\lambda})$:

(1) there is a bead at (j,h_1) *and position* $(j+1,h_1)$ *is empty;*

(2) there is a bead at (j,h_2) *and position* $(j+1,h_2)$ *is empty,*

then there exist r*-partitions* $\boldsymbol{\mu},\boldsymbol{\nu}$ *such that* $(\boldsymbol{\mu},\mathbf{s})$, $(\boldsymbol{\nu},\mathbf{s})\in\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ *and* $L_{\mathbf{s}}(\boldsymbol{\mu})\parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

Proof. Slide the beads at positions (j,h_1) and (j,h_2) in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to positions $(j+1,h_1)$ and $(j+1,h_2)$, respectively. Denote by $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the new abacus obtained. Then the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ is $\mathcal{M}=(m_1,m_2,\cdots,m_r)$, where

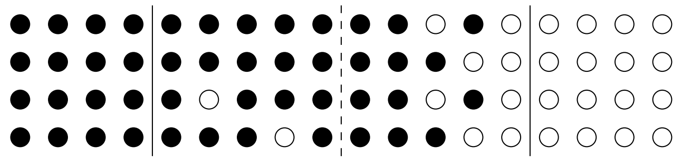
$$m_i=\begin{cases}2, & \text{if } i=j;\\ 0, & \text{if } i\neq j.\end{cases}$$

It follows from Lemma 3.1.12 (2) that there exist h_3,h_4 such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ the positions (j,h_3) and (j,h_4) are empty and positions $(j+1,h_3)$ and $(j+1,h_4)$ are occupied by beads. Then in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$, positions $(j,h_1),(j,h_2),(j,h_3)$ and (j,h_4) are empty and there are beads at positions $(j+1,h_1)$, $(j+1,h_2)$, $(j+1,h_3)$ and $(j+1,h_4)$. Define $\{l_1,l_2,l_3,l_4\}$ to be equal to $\{h_1,h_2,h_3,h_4\}$ as a set satisfying $l_1<l_2<l_3<l_4$.

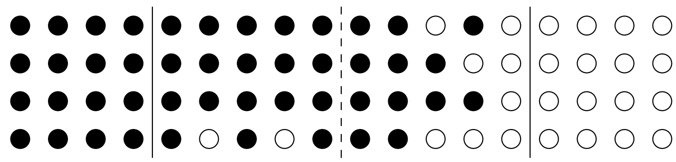
Slide the beads at positions $(j+1,l_1)$ and $(j+1,l_4)$ to positions (j,l_1) and (j,l_4) in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$, respectively. Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the new abacus. Slide the beads at positions $(j+1,l_2)$ and $(j+1,l_3)$ to positions (j,l_2) and (j,l_3) in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$, respectively. Denote the new abacus by $L_{\mathbf{v}}(\boldsymbol{\nu})$. Clearly, the move vectors from $L_{\mathbf{u}}(\boldsymbol{\mu})$ and $L_{\mathbf{v}}(\boldsymbol{\nu})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ are both equal to $\mathcal{M}$. Hence we can deduce from Lemma 3.4.9 that $\mathbf{s}=\mathbf{u}=\mathbf{v}$ and $(\boldsymbol{\mu},\mathbf{s}),(\boldsymbol{\nu},\mathbf{s})\in\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$. Moreover, take $(\kappa_1,\iota_1)=(j,\ l_4)$ and $(\kappa_2,\iota_2)=(j+1,\ l_1)$. It is easy to check that the conditions (1) and (2) of Definition 3.3.1 are satisfied. Then $L_{\mathbf{s}}(\boldsymbol{\lambda})\parallel L_{\mathbf{s}}(\boldsymbol{\mu})$. □

The way to construct two incomparable abaci in Lemma 4.1.1 will be used over and over again. In order to help the reader understand the construction process, let us give an example below.

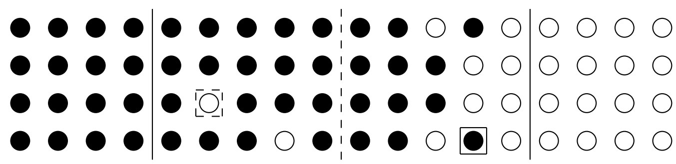
Example 4.1.2. *Let* $e=5$, $\boldsymbol{\lambda}=((1^4),(2,1^5),\varnothing,(1)$ *and* $\mathbf{s}=(2,2,3,3)$. *Then* $L_{\mathbf{s}}(\boldsymbol{\lambda})$ *is*

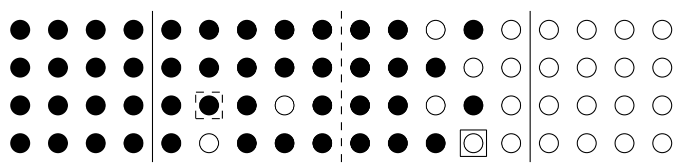


Taking $j = 1, h_1 = -4$ and $h_2 = 2$ and sliding the beads according to Lemma 4.1.1, we get $L_{\overline{\mathbf{s}}}(\overline{\boldsymbol{\lambda}})$ as follows, where $\overline{\boldsymbol{\lambda}} = ((2^3, 1), \varnothing, \varnothing, (1))$ and $\overline{\mathbf{s}} = (0, 4, 3, 3)$.



Choose $h_3 = -2$, $h_4 = 3$. Then $l_1 = -4$, $l_2 = -2$, $l_3 = 2$ and $l_4 = 3$ and the abaci $L_{\mathbf{s}}(\boldsymbol{\mu})$ and $L_{\mathbf{s}}(\boldsymbol{\nu})$ constructed by using Lemma 4.1.1 are as follows, where $\boldsymbol{\mu} = ((2, 1^3), (1^6), \varnothing, (1))$ and $\boldsymbol{\nu} = ((1^6), (2, 1^3), \varnothing, (1))$.





The following lemma will be used only if the number of runners in an abacus is not less than four.

Lemma 4.1.3. *Let $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ with $s \in \overline{\mathcal{A}}^r_e$, where $r \geq 4$. Assume that in $L_s(\lambda)$,*

(1) there is a bead at position (i, l) and $(i+1, l)$ is an empty position;
(2) there is a bead at position (j, h) and $(j+1, h)$ is an empty position,

where $l, h \in \mathbb{Z}$, $1 \leq i, j \leq r$ with $i + 1 < j$ and $i \neq 1$ if $j = r$. Then there exist r-partitions $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ such that $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ and $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

Proof. Slide the beads at positions (i, l) and (j, h) in $L_s(\lambda)$ to positions $(i + 1, l)$ and $(j + 1, h)$, respectively. Denote by $L_{\bar{s}}(\bar{\lambda})$ the new abacus obtained. Then the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ is $\mathcal{M} = (m_1, m_2, \cdots, m_r)$, where

$$m_k = \begin{cases} 1, & \text{if } k = i; \\ 1, & \text{if } k = j; \\ 0, & \text{others.} \end{cases}$$

It follows from Lemma 3.1.12 (2) that there exist $l', h' \in \mathbb{Z}$ such that in $L_s(\lambda)$ the positions (i, l') and (j, h') are empty and there are beads at positions $(i + 1, l')$ and $(j + 1, h')$. Then in $L_{\bar{s}}(\bar{\lambda})$, the positions (i, l), (i, l'), (j, h) and (j, h') are empty and there are beads at positions $(i+1, l)$, $(i+1, l')$, $(j+1, h)$ and $(j+1, h')$. Define $\{h_1, h_2\}$ to be equal to $\{h, h'\}$ as a set satisfying $h_1 < h_2$. Define $\{l_1, l_2\}$ to be equal to $\{l, l'\}$ as a set satisfying $l_1 < l_2$. Hence we can deduce from Lemma 3.4.9 that $\mathbf{s} = \mathbf{u} = \mathbf{v}$ and $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}^{\mathbf{\Lambda}}_{\boldsymbol{\beta}}$.

Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the abacus obtained by sliding the beads at positions $(i+1, l_2)$ and $(j+1, h_1)$ in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ to positions (i, l_2) and (j, h_1), respectively and denote by $L_{\mathbf{v}}(\boldsymbol{\nu})$ the abacus obtained by sliding the beads at positions $(i+1, l_1)$ and $(j+1, h_2)$ in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ to positions (i, l_1) and (j, h_2), respectively. Clearly, the move vectors from $L_{\mathbf{u}}(\boldsymbol{\mu})$ and $L_{\mathbf{v}}(\boldsymbol{\nu})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ are both equal to $\mathcal{M}$.

If we take $(\kappa_1, \iota_1) = (i, l_2)$, $(\kappa_2, \iota_2) = (j+1, h_1)$, then $L_{\mathbf{s}}(\boldsymbol{\mu})$ and $L_{\mathbf{s}}(\boldsymbol{\nu})$ satisfy the conditions (1) and (2) of Definition 3.3.1 and consequently, $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$. □

Lemma 4.1.4. *Let $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}^{\mathbf{\Lambda}}_{\boldsymbol{\beta}}$ and $\mathbf{s} \in \overline{\mathcal{A}}^{r}_{e}$. Assume there exists $1 \le i \le r$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,*

(1) there is a bead at position (i, l_1) and position $(i+1, l_1)$ is empty;
(2) position (i, l_2) is empty and there is a bead at position $(i+1, l_2)$;
(3) there is s a bead at position $(i+1, l_3)$ and position $(i+2, l_3)$ is empty;
(4) position $(i+1, l_4)$ is empty and there is a bead at position $(i+2, l_4)$,

where $l_1, l_2, l_3, l_4 \in \mathbb{Z}$ such that $l_1 \neq l_4$ or $l_2 \neq l_3$ holds. Then there exist r-partitions $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ with $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}^{\mathbf{\Lambda}}_{\boldsymbol{\beta}}$ and $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}\boldsymbol{\nu})$.

Proof. According to the relationship among l_1, l_2, l_3, l_4, we divide the proof into the following three cases.

Case 1. *$l_1 \neq l_4$ and $l_2 \neq l_3$.*

In $L_{\mathbf{s}}(\boldsymbol{\lambda})$, slide the bead at position (i, l_1) to position $(i+1, l_1)$ and slide the bead at position $(i+1, l_3)$ to position $(i+2, l_3)$. Denote by $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the new abacus obtained. Then the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ is $\mathcal{M} = (m_1, m_2, \cdots, m_r)$, where

$$m_k = \begin{cases} 1, & \text{if } k = i, i+1; \\ 0, & \text{others.} \end{cases}$$

In $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$, the positions (i, l_1), (i, l_2), $(i+1, l_2)$ and $(i+1, l_3)$ are empty and there are beads at positions $(i+1, l_1)$, $(i+1, l_2)$, $(i+2, l_3)$ and $(i+2, l_4)$. Define $\{h_1, h_2\}$ to be equal to $\{l_1, l_2\}$ as a set satisfying $h_1 < h_2$ and define $\{h_3, h_4\}$ to be equal to $\{l_3, l_4\}$ as a set satisfying $h_3 < h_4$.

Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the abacus obtained by sliding the beads at positions $(i+1, h_2)$ and $(i+2, h_3)$ to positions (i, h_2) and $(i+2, h_3)$ in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ and denote $L_{\mathbf{v}}(\boldsymbol{\nu})$ the abacus obtained by sliding the beads at positions $(i+1, h_1)$ and $(i+2, h_4)$ to (i, h_1) and $(i+1, h_4)$ in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$. Clearly, both the moving vectors from $L_{\mathbf{u}}(\boldsymbol{\mu})$ and $L_{\mathbf{v}}(\boldsymbol{\nu})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ are equal to $\mathcal{M}$. We can deduce from Lemma 3.4.9 that $\mathbf{s} = \mathbf{u} = \mathbf{v}$ and $(\boldsymbol{\mu}, \mathbf{s})$, $(\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}^{\mathbf{\Lambda}}_{\boldsymbol{\beta}}$. Furthermore, it is easy to know $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$ by taking $(\kappa_1, \iota_1) = (i, h_2)$ and $(\kappa_2, \iota_2) = (i+2, h_3)$.

Case 2. *$l_1 = l_4$ and $l_2 \neq l_3$*

Since there are beads at positions $(i+1, l_2)$ and $(i+1, l_3)$ and position $(i+2, l_3)$ is empty, if position $(i+2, l_2)$ is empty, then by Lemma 4.1.1, the result follows. Now we assume that there is a bead at position $(i+1, l_2)$.

Slide in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ the beads at positions (i, l_1) and $(i+1, l_3)$ to positions $(i+1, l_1)$ and $(i+2, l_3)$, respectively. Denote by $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the new abacus obtained. The move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ is $\mathcal{M}=(m_1,m_2,\cdots,m_r)$, where

$$m_k=\begin{cases}1, & \text{if } k=i,i+1;\\ 0, & \text{others.}\end{cases}$$

Let us take a look at $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$. Positions (i,l_1), (i,l_2) and $(i+1,l_3)$ are empty, and there are beads at positions $(i+1,l_1)$, $(i+1,l_2)$, $(i+2,l_1)$, $(i+2,l_2)$ and $(i+2,l_3)$. Define $\{h_1,h_2\}$ to be equal to $\{l_1,l_2\}$ as a set satisfying $h_1<h_2$. The rest of our discussion is divided into the following two subcases.

Subcase 1. $l_3>h_2$ or $l_3<h_1$. Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the abacus obtained by sliding in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the beads at positions $(i+1,h_2)$ and $(i+2,h_2)$ to (i,h_2) and $(i+1,h_2)$, respectively, and denote by $L_{\mathbf{v}}(\boldsymbol{\nu})$ the abacus obtained by sliding in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the beads at positions $(i+1,h_1)$ and $(i+2,l_3)$ to positions (i,h_1) and $(i+1,l_3)$, respectively. Clearly, both the move vectors from $L_{\mathbf{u}}(\boldsymbol{\mu})$ and $L_{\mathbf{v}}(\boldsymbol{\nu})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ are equal to $\mathcal{M}$. Hence we can deduce from Lemma 3.4.9 that $\mathbf{s}=\mathbf{u}=\mathbf{v}$ and $(\boldsymbol{\mu},\mathbf{s}),(\boldsymbol{\nu},\mathbf{s})\in\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. It is easy to check $L_{\mathbf{s}}(\boldsymbol{\mu})\parallel L_{\mathbf{s}}(\boldsymbol{\nu})$ by taking $(\kappa_1,\iota_1)=(i,h_2)$, $(\kappa_2,\iota_2)=(i+2,h_2)$.

Subcase 2. $h_1<l_3<h_2$. We construct two incomparable abaci as follows. In $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$, sliding the beads at positions $(i+1,h_1)$ and $(i+2,h_1)$ to positions (i,h_1) and $(i+1,h_1)$, respectively, gives an abacus $L_{\mathbf{s}}(\boldsymbol{\mu})$ and sliding the beads at positions $(i+2,l_3)$ and $(i+1,h_2)$ to positions (i,h_1) and (i,h_2), respectively, gives another one, $L_{\mathbf{s}}(\boldsymbol{\nu})$. Then by taking $(\kappa_1,\iota_1)=(i+1,h_2)$ and $(\kappa_2,\iota_2)=(i+2,h_1)$, we know $L_{\mathbf{s}}(\boldsymbol{\mu})\parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

Case 3. *$l_1\neq l_4$ and $l_2=l_3$.*

It is a dual case of Case 2. □

Corollary 4.1.5. *Let $(\boldsymbol{\lambda},\mathbf{s})\in\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ and $\mathbf{s}\in\overline{\mathcal{A}}_e^r$. Assume there exists $1\leq i\leq r$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,*

(1) position (i,h_1) has a bead and position $(i+1,h_1)$ is empty;
(2) position $(i+1,h_2)$ has a bead and position $(i+2,h_2)$ is empty;
(3) position $(i+2,h_1)$ is empty or position (i,h_2) has a bead.

Then there exist $(\boldsymbol{\mu},\mathbf{s}),(\boldsymbol{\nu},\mathbf{s})\in\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $L_{\mathbf{s}}(\boldsymbol{\mu})\parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

Proof. By Lemma 3.1.12, there exist $h_3,h_4\in\mathbb{Z}$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$

- position (i,h_3) is empty and position $(i+1,h_3)$ has a bead;
- position $(i+1,h_4)$ is empty and position $(i+2,h_4)$ has a bead.

Clearly, if position (i,h_2) has bead, then $h_3\neq h_2$, and if position $(i+2,h_1)$ is empty, then $h_4\neq h_1$. By Lemma 4.1.4 the proof is completed. □

Let us do some work under the condition $e<\infty$.

Lemma 4.1.6. *Let $e\neq\infty$ and $(\boldsymbol{\lambda},\mathbf{s})\in\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $s\in\overline{\mathcal{A}}_e^r$. Assume that there exist $1\leq i\leq r$, $l_1,l_2\in\mathbb{Z}$ with $l_1+e\neq l_2$, such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$*

(1) there is a bead at position (i,l_1) and position $(i+1,l_1)$ is empty;
(2) there is a bead at position $(i+1,l_2)$ and $(i+2,l_2)$ is empty,

and there is a bead at position $(i+1,l_1+e)$ or position $(i+1,l_2-e)$ is empty. Then there exist $(\boldsymbol{\mu},\mathbf{s}),(\boldsymbol{\nu},\mathbf{s})\in\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ such that $L_{\mathbf{s}}(\boldsymbol{\mu})\parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

Proof. It follows from Lemma 3.1.12 (2) that there exist $h_1,h_2\in\mathbb{Z}$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, there are beads at positions $(i+1,h_1)$ and $(i+2,h_2)$ and positions (i,h_1)

$(i+1,h_2)$ are empty. If $h_1 \neq l_2$ or $h_2 \neq l_1$, then we are in the circumstances of Lemma 4.1.4. Hence we only need to consider $h_1 = l_2$ and $h_2 = l_1$.

We first assume that there is a bead at position $(i+1, l_1+e)$. If there exists $l_3 \in \mathbb{Z}$ with $l_3 \neq l_1$, $l_3 \neq l_2$ such that there is a bead at position $(i+1,l_3)$, and position (i,l_3) or $(i+2,l_3)$ is empty, then by Lemma 4.1.1 and Lemma 4.1.4 the result follows. Now we suppose that both positions (i,l_3) and $(i+2,l_3)$ are occupied by beads whenever there is a bead at position $(i+1,l_3)$ for each $l_3 \in \mathbb{Z}$ with $l_3 \neq l_1$ and $l_3 \neq l_2$. There are three possible cases need to consider.

Case 1. $l_2 < l_1$

There exists $k \in \mathbb{N}$ by Lemma 3.1.12(1) such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ position $(i, l_2 - ke)$ is empty and position $(i, l_2-(k+1)e)$ is occupied by a bead. Move the beads at positions $(i+1, l_1+e)$ and $(i+1,l_2)$ in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to positions $(i+1,l_1)$ and $(i+2,l_2)$, respectively. Denote by $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the new abacus obtained. Then by Lemma 3.4.16 the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ is $\mathcal{M} = (m_1, m_2, \cdots, m_r)$, where

$$m_t = \begin{cases} 2, & \text{if } t = i+1; \\ 1, & \text{others.} \end{cases}$$

Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the abacus obtained by moving in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the beads at positions $(i, l_2-(k+1)e)$ and $(i+2, l_2)$ to positions $(i, l_1 - ke)$ and $(i+1, l_2)$, respectively and denote by $L_{\mathbf{v}}(\boldsymbol{\nu})$ the abacus obtained by moving in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the beads at positions $(i+1, l_1)$ and $(i+2, l_1)$ to positions (i, l_1+e) and $(i+1, l_1)$, respectively. Clearly, the move vectors from $L_{\mathbf{u}}(\boldsymbol{\mu})$ and $L_{\mathbf{v}}(\boldsymbol{\nu})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ are both equal to $\mathcal{M}$. We can deduce by a routine verification process that $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$ with $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$.

Case 2. $l_1 < l_2 < l_1 + e$

Move the beads at positions $(i+1, l_2)$, (i, l_1) and (i, l_1+e) in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to positions $(i+2, l_2)$, $(i+1, l_1)$ and (i, l_1), respectively. Denote by $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the new abacus obtained. Then by Lemma 3.4.16 the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ is $\mathcal{M} = (m_1, m_2, \cdots, m_r)$, where

$$m_t = \begin{cases} 2, & \text{if } t = i, i+1; \\ 1, & \text{others.} \end{cases}$$

Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the abacus obtained by moving in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the beads at positions $(i+2, l_2)$, $(i+1, l_2)$ and (i, l_1) to positions $(i+1, l_2)$, (i, l_2) and $(i+1, l_1+e)$ respectively, and denote by $L_{\mathbf{u}}(\boldsymbol{\nu})$ the abacus obtained by moving in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the beads at positions $(i+1, l_1+e)$, $(i+2, l_1+e)$ and $(i+1, l_2-(k+1)e)$ to positions (i, l_1+e), $(i+1, l_1+e)$ and $(i+1, l_2-ke)$, respectively. Clearly, the move vectors from $L_{\mathbf{u}}(\boldsymbol{\mu})$ and $L_{\mathbf{v}}(\boldsymbol{\nu})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ are both equal to $\mathcal{M}$. Hence we can deduce from Lemma 3.4.9 that $\mathbf{s} = \mathbf{u} = \mathbf{v}$ and $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$ with $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$.

Case 3. $l_1 + e < l_2$

We have from Lemma 3.1.12 (1) that there exists $k \in \mathbb{N}$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ position $(i, l_2 - ke)$ is empty and there is a bead at position $(i, l_2-(k+1)e)$. In $L_{\mathbf{s}}(\boldsymbol{\lambda})$, move the bead at position $(i+1, l_2)$ to position $(i+2, l_2)$ and move the bead at position $(i+1, l_1+e)$ to position $(i+1, l_1)$. Denote by $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the new abacus obtained. Then by Lemma 3.4.13 the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ is

$\mathcal{M} = (m_1, m_2, \cdots, m_r)$, where

$$m_t = \begin{cases} 2, & \text{if } t = i+1; \\ 1, & \text{others.} \end{cases}$$

Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the abacus obtained by moving the beads at position $(i+2, l_1+e)$ to $(i+1, l_1+e)$ and moving the bead at position $(i, l_2-(k+1)e)$ to position (i, l_2-ke) in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$. The move vector from $L_{\mathbf{u}}(\boldsymbol{\mu})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ is equal to $\mathcal{M}$, and by Lemma 3.4.9 $\mathbf{s} = \mathbf{u}$. Then $L_{\mathbf{s}}(\boldsymbol{\lambda})$ and $L_{\mathbf{s}}(\boldsymbol{\mu})$ are incomparable abaci that we need to construct. □

Lemma 4.1.7. *Let $e \neq \infty$ and $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathscr{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $\mathbf{s} \in \overline{\mathcal{A}}_e^r$. Suppose there exist $1 \leq i \leq r$ and $l_1, l_2 \in \mathbb{Z}$ such that*

(1) there are beads at positions (i, l_1), (i, l_1+e) and $(i-1, l_1)$;
(2) position $(i+1, l_1)$ is empty.

Then there exist $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ such that $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

Proof. Since in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, position (i, l_1) is occupied by a bead and position $(i+1, l_1)$ is empty, we have from Lemma 3.1.12 (2) that there exist $l_2 \in \mathbb{Z}$ such that position (i, l_2) is empty and there is a bead at position $(i+1, l_2)$. Clearly, $l_2 \neq l_1 + e$. We consider two cases.

Case 1. *Position $(i-1, l_2)$ is occupied by a bead.*

By Lemma 3.1.12 (2), the bead at $(i-1, l_2)$ and empty position (i, l_2) ensure the existence of $h_1 \in \mathbb{Z}$, such that position $(i-1, h_1)$ is empty and position (i, h_1) is occupied by a bead. Moreover, $h_1 \neq l_1$ and we have in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,

- position $(i-1, l_2)$ is occupied by a bead and position (i, l_2) is empty;
- position $(i-1, h_1)$ is empty and position (i, h_1) is occupied by a bead;
- position (i, l_1) is occupied by a bead and position $(i+1, l_1)$ is empty;
- position (i, l_2) is empty and position $(i+1, l_2)$ is occupied by a bead.

That is, we are in the circumstances of Lemma 4.1.4 and thus the result follows.

Case 2. *Position $(i-1, l_2)$ is empty.*

Slide in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ the bead at position (i, l_1) to position $(i+1, l_1)$ and denote the new abacus by $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$. The move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ is $\mathcal{M} = (m_1, \cdots, m_r)$, where

$$m_t = \begin{cases} 1, & \text{if } t = i; \\ 0, & \text{others.} \end{cases}$$

Slide in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the bead at position $(i+1, l_2)$ to position (i, l_2) and denote by $L_{\mathbf{s}''}(\boldsymbol{\xi})$ the new abacus obtained. It is easy to check that the move vector from $L_{\mathbf{s}''}(\boldsymbol{\xi})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ is equal to $\mathcal{M}$. By Lemma 3.4.9, we have $\mathbf{s}'' = \mathbf{s}$ and $(\boldsymbol{\xi}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. Now in $L_{\mathbf{s}}(\boldsymbol{\xi})$, we have

- position $(i-1, l_1)$ is occupied by a bead and position (i, l_1) is empty;
- position (i, l_2) is occupied by a bead and position $(i+1, l_2)$ is empty;
- position (i, l_1+e) is occupied by a bead and $l_1 + e \neq l_2$,

which are requirements of Lemma 4.1.6 . This complete the proof. □

Lemma 4.1.8. *Let $e \neq \infty$ and $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathscr{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $\mathbf{s} \in \overline{\mathcal{A}}_e^r$. Assume that there exist $h_1, h_2 \in Z$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,*

(1) position $(r-1, h_1)$ is occupied by a bead and position (r, h_1) is empty;

(2) position $(r-1,h_2)$ is empty and position (r,h_2) is occupied by a bead.

If the first empty position before (r,h_2) is in runner 1 and the first bead after $(r-1,h_2)$ is not in runner r, then there exist $(\boldsymbol{\mu},\mathbf{s}),(\boldsymbol{\nu},\mathbf{s})\in\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ such that $L_{\mathbf{s}}(\boldsymbol{\mu})\parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

Proof. Suppose that the first empty position before (r,h_2) in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ is $(1,h_2-(k_1+1)e)$, where $k_1\in\mathbb{N}$. Then the positions between $(1,h_2-(k_1+1)e)$ and $(r-1,h_2)$ are occupied by beads. Particularly, there is a bead at position (r,h_2-k_1e). Suppose that the first bead after $(r-1,h_2)$ is (l,h_2+k_2e), where $k_2\in\mathbb{N}$. Then the positions between (r,h_2) and (l,h_2+k_2e) are empty. Particularly, $(l+1,h_2+k_2e)$ is empty.

If $k_1\neq 0$, by direct observation of abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$, we have

- position $(r-1,h_1)$ is occupied by a bead and position (r,h_1) is empty;
- position (r,h_2-k_1e) is occupied by a bead and position $(1,h_2-(k_1+1)e)$ is empty;
- position $(r-1,h_2)$ is empty and position (r,h_2) is occupied by a bead, $h_2\neq h_2-k_1e$,

and the result follows from Lemma 4.1.4.

Now we assume $k_1=0$ and consider three cases as follows.

Case 1. $1<l$ *and* $l+1<r$.

We have in thus case that positions $(1,h_2-e)$ and $(l+1,h_2+k_2e)$ are empty and there are beads at positions (l,h_2+k_2e) and (r,h_2), which are conditions required by Lemma 4.1.3.

Case 2. $l+1=r$. We need to consider two subcases.

Subcase 1. $h_2+k_2e\neq h_1$. In $L_{\mathbf{s}}(\boldsymbol{\lambda})$, positions (r,h_2+k_2e) and (r,h_1) are empty and there are beads at positions $(r-1,h_2+k_2e)$ and $(r-1,h_1)$. Then the result holds by Lemma 4.1.1.

Subcase 2. $h_2+k_2e=h_1$. It is clear $k_2\neq 0$ because $h_2\neq h_1$. Note that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, position (r,h_2) is occupied by a bead and position $(1,h_2-e)$ is empty. By Lemma 3.1.12 (2) there exists $h_3\in\mathbb{Z}$ such that position (r,h_3) is empty and there is a bead at position $(1,h_3-e)$. Moreover, position $(1,h_2+k_2e-e)$ is empty since it is between positions $(r-1,h_2)$ and $(r-1,h_2+k_2e)$, and consequently, $h_3\neq h_2+k_2e$. To sum up, we have in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,

- position $(r-1,h_2+k_2e)$ is occupied by a bead and position (r,h_2+k_2e) is empty;
- position (r,h_2) is occupied by a bead and position $(1,h_2-e)$ is empty;
- position (r,h_3) is empty and position $(1,h_3-e)$ is occupied by a bead,

which are conditions of Lemma 4.1.4.

Case 3. $l=1$.

If $r>3$, then $l+1<r-1$ and the conditions of Lemma 4.1.3 have been satisfied. That is, positions $(l+1,h_2+k_2e)$ and (r,h_1) are empty and there are beads at positions (l,h_2+k_2e) and $(r-1,h_1)$.

So we now assume $r=3$ and consider two subcases.

Subcase 1. $k_2\neq 0$. Under this condition, we have in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,

- position $(1,h_2+k_2e)$ is occupied by a bead and position $(2,h_2+k_2e)$ is empty;
- position $(2,h_1)$ is is occupied by a bead and position $(3,h_1)$ is empty;

- position $(2, h_2)$ is empty and position $(3, h_2)$ is occupied by a bead, $h_2 \neq h_2 + k_2 e$.

Then by Lemma 4.1.4 we get the result.

Subcase 2. $k_2 = 0$. The following is observation of abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$:

- position $(2, h_1)$ is occupied by a bead and position $(3, h_1)$ is empty;
- position $(1, h_2)$ is occupied by a bead and position $(2, h_2)$ is empty;
- position $(3, h_2)$ is occupied by a bead and $(1, h_2 - e)$ is empty.

Slide the bead at position $(2, h_1)$ to position $(3, h_1)$ and denote by $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$ the new abacus obtained. The move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$ is $\mathcal{M} = (0, 1, 0)$. Define $\{l_1, l_2\}$ to be equal to $\{h_1, h_2\}$ as a set satisfying $l_1 < l_2$.

In $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$, slide the bead at position $(3, l_1)$ to position $(2, l_1)$ and denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the new abacus obtained. Slide the bead at position $(3, l_2)$ to position $(2, l_2)$ and denote by $L_{\hat{\mathbf{u}}}(\hat{\boldsymbol{\mu}})$ the new abacus obtained. Clearly, both the move vectors from $L_{\mathbf{u}}(\boldsymbol{\mu})$ and $L_{\hat{\mathbf{u}}}(\hat{\boldsymbol{\mu}})$ to $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$ are equal to $\mathcal{M}$, and we have by Lemma 3.4.9 that $\mathbf{s} = \mathbf{u} = \hat{\mathbf{u}}$, $(\boldsymbol{\mu}, \mathbf{s}), (\hat{\boldsymbol{\mu}}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$.

Notice that position $(2, l_2)$ in $L_{\mathbf{s}}(\hat{\boldsymbol{\mu}})$ is occupied by a bead. By Lemma 3.1.12 (1), there exists $k \in \mathbb{N}$ such that position $(2, l_2 + ke)$ is occupied by a bead and position $(2, l_2 + (k+1)e)$ is empty. Move in $L_{\mathbf{s}}(\hat{\boldsymbol{\mu}})$ the bead at position $(1, h_2)$ to position $(1, h_2 - e)$ and denote by $L_{\mathbf{s}''}(\boldsymbol{\xi})$ the new abacus obtained. Then by Lemma 3.4.16 the move vector from $L_{\mathbf{s}}(\hat{\boldsymbol{\mu}})$ to $L_{s''}(\boldsymbol{\xi})$ is $\mathcal{M}' = (1, 1, 1)$.

In $L_{\mathbf{s}''}(\boldsymbol{\xi})$, move the bead at position $(2, l_2 + ke)$ to position $(2, l_2 + (k+1)e)$. Denote the new abacus obtained by $L_{\mathbf{v}}(\boldsymbol{\nu})$. Clearly, the move vector from $L_{\mathbf{v}}(\boldsymbol{\nu})$ to $L_{\mathbf{s}''}(\boldsymbol{\xi})$ is equal to $\mathcal{M}'$. For this reason, we can deduce from Lemma 3.4.9 that $\mathbf{v} = \mathbf{s}$ and $(\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$. Then the proof is completed by taking $(\kappa_1, \iota_1) = (1, h_2)$ and $(\kappa_2, \iota_2) = (3, l_1)$. □

Lemma 4.1.9. *Let $e \neq \infty$ and $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathscr{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ with $\mathbf{s} \in \overline{\mathcal{A}}^{r}_{e}$. If there exist $1 \leq j < i < r$, $l_1, l_2 \in \mathbb{Z}$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,*

(1) positions (i, l_1) and $(i+1, l_1)$ are empty;
(2) there are beads at positions (i, l_2) and $(i+1, l_2)$;
(3) there is a bead at (j, l_1) and position $(j+1, l_1)$ is empty;
(4) there is a bead at $(j-1, l_2 - e)$ and position at $(j, l_2 - e)$ is empty,

then there exist $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ with $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

Proof. Let us consider position (j, l_2). If there is a bead at (j, l_2), then it is easy to check that we are in the circumstances of Lemma 4.1.6. So we assume from now on that position (j, l_2) is empty. We consider two cases.

Case 1. $i + 1 \neq r$ *or* $j \neq 1$.

By moving the bead at position (j, l_1) to position (i, l_1), we obtain a new abacus $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$. By Lemma 3.4.13 the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$ is $\mathcal{M} = (m_1, \cdots, m_r)$, where

$$m_t = \begin{cases} 1, & \text{if } j \leq t < i; \\ 0, & \text{others.} \end{cases}$$

In $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$, move the bead at position (i, l_2) to position (j, l_2). Denote this new abacus by $L_{\mathbf{u}}(\boldsymbol{\mu})$. Clearly, the move vector from $L_{\mathbf{u}}(\boldsymbol{\mu})$ to $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$ is equal to $\mathcal{M}$ and by Lemma 3.4.9, $\mathbf{s} = \mathbf{u}$, $(\boldsymbol{\mu}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$. Observing abacus $L_{\mathbf{s}}(\boldsymbol{\mu})$ gives

- position $(j-1, l_2 - e)$ is occupied by a bead and position $(j, l_2 - e)$ is empty;

- position (i, l_1) is occupied by a bead and position $(i+1, l_1)$ is empty.

If $j \neq 1$, $j < i$, the result follows from Lemma 4.1.3. If $j = 1$, then $i + 1 < r$. Furthermore, according to the previous engagement, position $(j-1, l_2-e)$ is (r, l_2). Then Lemma 4.1.3 leads to the result.

Case 2. *$i + 1 = r$ and $j = 1$.*

We first rewrite the known conditions on abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$ in this case.

(1) Positions $(r-1, l_1)$ and (r, l_1) are empty.
(2) There are beads at position $(r-1, l_2)$ and (r, l_2).
(3) There is a bead at position $(1, l_1)$.
(4) Positions $(1, l_2-e)$ and $(1, l_2)$ are empty.

In $L_{\mathbf{s}}(\boldsymbol{\lambda})$, slide the bead at position (r, l_2) to position $(1, l_2-e)$ and slide the bead at position $(r-1, l_2)$ to position (r, l_2). Then move the bead at position $(1, l_1)$ to position (r, l_1). Denote by $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$ the new abacus obtained. The move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$ is $\mathcal{M} = (m_1, m_2, \cdots, m_r)$, where

$$m_t = \begin{cases} r-1, & \text{if } t = i; \\ 1, & \text{others.} \end{cases}$$

Define $\{h_1, h_2\}$ to be equal to $\{l_1, l_2\}$ as a set satisfying $h_1 < h_2$. Note that in $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$, there is a bead at position (r, h_2). By Lemma 3.1.12 (2), there exists $k \in \mathbb{N}$ such that position $(r, h_2 + ke)$ is occupied by a bead and position $(r, h_2 + ke + e)$ is empty.

Let us construct two incomparable abaci from $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$ by elementary operations. The first one, $L_{\mathbf{u}}(\boldsymbol{\mu})$ is obtained by moving the bead at position $(1, l_2-e)$ to position $(1, l_2)$, and sliding the bead at position (r, h_2) to position $(r-1, h_2)$. The second one, $L_{\mathbf{v}}(\boldsymbol{\nu})$ is obtained by moving the bead at position $(r, h_2 + ke)$ to position $(r, h_2 + (k+1)e)$, and sliding the bead at position (r, h_1) to position $(r-1, h_1)$. It is easy to check that both the move vector from $L_{\mathbf{u}}(\boldsymbol{\mu})$ and $L_{\mathbf{v}}(\boldsymbol{\nu})$ to $L_{\mathbf{s}'}(\bar{\boldsymbol{\lambda}})$ are equal to $\mathcal{M}$. This implies by lemma 3.4.9 that $\mathbf{u} = \mathbf{v} = \mathbf{s}$ and $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$.

To know $L_{\mathbf{u}}(\boldsymbol{\mu}) \parallel L_{\mathbf{v}}(\boldsymbol{\nu})$, one only need to take $(\kappa_1, \iota_1) = (1, l_2), (\kappa_2, \iota_2) = (r-1, h_1)$. $\square$

Lemma 4.1.10. *Let $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ with $\mathbf{s} \in \overline{\mathcal{A}}^r_e$, where $e \neq \infty$. If there exist $1 \le i_1, i_2 \le r$ with $i_1 \neq i_2$ and $h_1, h_2 \in \mathbb{Z}$ such that in abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$*

(1) position (i_1, h_1) is empty and position $(i_1, h_1 + e)$ has a bead;
(2) position (i_2, h_2) is empty and position $(i_2, h_2 + e)$ has a bead,

then there exist $(\boldsymbol{\mu}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ such that $L_{\mathbf{s}}(\boldsymbol{\lambda}) \parallel L_{\mathbf{s}}(\boldsymbol{\mu})$.

Proof. Let $j \neq i_1$ and $j \neq i_2$. There exist h_3, $h_4 \in \mathbb{Z}$ with $e \nmid h_3 - h_4$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,

- position (j, h_3) has a bead and position $(j, h_3 + e)$ is empty;
- position (j, h_4) has a bead and position $(j, h_4 + e)$ is empty.

Move in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ the bead at positions $(i_1, h_1 + e)$ and $(i_2, h_2 + e)$ to positions (i_1, h_1) and (i_2, h_2), respectively. Denote by $L_{\hat{\mathbf{s}}}(\hat{\boldsymbol{\lambda}})$ the new abacus. We have from Lemma 3.4.16 that the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\hat{\mathbf{s}}}(\hat{\boldsymbol{\lambda}})$ is $\mathcal{M} = (2, \cdots, 2)$. Next, move in $L_{\hat{\boldsymbol{\lambda}}}(\hat{\boldsymbol{\lambda}})$ the beads at positions (j, h_3) and (j, h_4) to positions $(j, h_3 + e)$ and position $(j, h_4 + e)$, respectively. Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the new abacus obtained. It is easy to check that the move vector from $L_{\mathbf{u}}(\boldsymbol{\mu})$ to $L_{\hat{\mathbf{s}}}(\hat{\boldsymbol{\lambda}})$ is $\mathcal{M}$. We can deduce from Lemma 3.4.9 that $\mathbf{u} = \mathbf{s}$ and $(\boldsymbol{\mu}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. In order to prove $L_{\mathbf{s}}(\boldsymbol{\lambda}) \parallel L_{\mathbf{s}}(\boldsymbol{\mu})$, one only need to take $(\kappa_1, \iota_1) = (i_1, h_1 + e)$. $(\kappa_2, \iota_2) = (i_2, h_2)$. $\square$

Remark 4.1.11. *The above lemma still holds without condition $i_1 \neq i_2$. We omit the details here.*

Corollary 4.1.12. *Let $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $\mathbf{s} \in \overline{\mathcal{A}}_e^r$, where $e \neq \infty$. If there exist $1 \leq i_1, i_2 \leq r$, $i_1 \neq i_2$, $h_1, h_2 \in \mathbb{Z}$, $k_1, k_2 \in \mathbb{N}^+$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,*

(1) position (i_1, h_1) is empty and position $(i_1, h_1 + k_1 e)$ has a bead;

(2) position (i_2, h_2) is empty and position $(i_2, h_2 + k_2 e)$ has a bead,

then there exists $(\boldsymbol{\mu}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ such that $L_{\mathbf{s}}(\boldsymbol{\lambda}) \parallel L_{\mathbf{s}}(\boldsymbol{\mu})$.

Proof. It is easy to check there exist $l_1, l_2 \in \mathbb{N}$ such that positions $(i_1, h_1 + l_1 e)$ and $(i_2, h_2 + l_2 e)$ are empty and positions $(i_1, h_1 + (l_1 + 1)e)$ and $(i_2, h_2 + (l_2 + 1)e)$ have beads. Then the result follows from Lemma 4.1.10. $\square$

The following lemmas are mainly used in the proof of Type II.

Lemma 4.1.13. *Let $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $\mathbf{s} \in \overline{\mathcal{A}}_e^r$. If there exist $1 \leq i_1 < i_2 < i_3 < i_4 \leq r$ such that*

(1) position (i_1, h) is occupied by a bead and position (i_3, h) is empty;

(2) position (i_2, h) is occupied by a bead and position (i_4, h) is empty,

then there exist $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ such that $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

Proof. Move in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ the beads at positions (i_1, h) and (i_2, h) to positions (i_3, h) and (i_4, h), respectively. Denote by $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the new abacus obtained. The move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ is $\mathcal{M} = (m_1, m_2, \cdots, m_r)$, where

$$m_t = \begin{cases} 1, & \text{if } i_3 \leq t \leq i_4 - 1; \\ 2, & \text{if } i_2 \leq t \leq i_3 - 1; \\ 1, & \text{if } i_1 \leq t \leq i_2 - 1; \\ 0, & \text{if others.} \end{cases}$$

It follows from Lemma 3.1.12 (2) that there exist $h_1 \neq h, h_2 \neq h$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, and consequently in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$

- position (i_1, h_1) is empty and position (i_3, h_1) is occupied by a bead;
- position (i_2, h_2) is empty and position (i_4, h_2) is occupied by a bead.

Moreover, in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$,

- position (i_1, h) is empty and position (i_3, h) is occupied by a bead;
- position (i_2, h) is empty and position (i_4, h) is occupied by a bead.

Define $\{l_1, l_3\}$ to be equal to $\{h, h_1\}$ as a set satisfying $l_1 < l_3$. and define $\{l_2, l_4\}$ to be equal to $\{h, h_2\}$ as a set satisfying $l_2 < l_4$. Denote by $L_{s'}(\boldsymbol{\mu})$ the abacus obtained by moving in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the beads at positions (i_3, l_3) and (i_4, l_2) to positions (i_1, l_3) and (i_2, l_2), respectively and denote by $L_{s''}(\boldsymbol{\nu})$ the abacus obtained by moving in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the beads at positions (i_3, l_4) and (i_4, l_1) to positions (i_1, l_4) and (i_2, l_1). It is easy to check that both the move vectors from $L_{\mathbf{s}'}(\boldsymbol{\mu})$ and $L_{\mathbf{s}''}(\boldsymbol{\nu})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ are equal to $\mathcal{M}$. Hence we can deduce from Lemma 3.4.9 that $\mathbf{s} = \mathbf{s}' = \mathbf{s}''$ and $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. Finally, it is a routine task to check that $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$ by taking $(\kappa_1, \iota_2) = (i_1, l_3)$ and $(\kappa_2, \iota_2) = (i_4, l_2)$. $\square$

Lemma 4.1.14. *Let $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $\mathbf{s} \in \overline{\mathcal{A}}_e^r$. Assume that there exist $1 \leq i_1 < i_2 < i_3 \leq r$ such that*

(1) position (i_1, h_1) is occupied by a bead and position (i_2, h_1) is empty;

(2) position (i_3, h_1) is empty and position $(i_3, h_1 + e)$ is occupied by a bead.

Then there exist $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ such that $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

Proof. Move the bead at position (i_1, h_1) to position (i_2, h_1) in $L_{\mathbf{s}}(\boldsymbol{\lambda})$. Denote by $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ the new abacus obtained. Then by Lemma 3.4.13 the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ is $\mathcal{M} = (m_1, m_2, \cdots, m_r)$, where

$$m_t = \begin{cases} 1, & \text{if } i_1 \leq t \leq i_2 - 1; \\ 0, & \text{if others.} \end{cases}$$

It is ensured by Lemma 3.1.12 the existence of $h_2 \neq h_1$, such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ position (i_2, h_2) is occupied by a bead and position (i_1, h_2) is empty.

Define $\{l_1, l_2\}$ to be equal to $\{h_1, h_2\}$ as a set satisfying $l_1 < l_2$. Denote by $L_{\mathbf{s}'}(\boldsymbol{\mu})$ the abacus obtained by moving the bead at position (i_2, l_2) to position (i_1, l_2) in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$, and denote by $L_{\mathbf{s}''}(\boldsymbol{\xi})$ the abacus obtained by moving the bead at position (i_2, l_1) to position (i_1, l_1) in $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$. Clearly, both the move vectors from $L_{\mathbf{s}'}(\boldsymbol{\mu})$ and $L_{\mathbf{s}''}(\boldsymbol{\xi})$ to $L_{\bar{\mathbf{s}}}(\bar{\boldsymbol{\lambda}})$ are equal to $\mathcal{M}$. By Lemma 3.4.9, $\mathbf{s} = \mathbf{s}' = \mathbf{s}''$ and $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\xi}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$.

Let us construct a new abacus from $L_{\mathbf{s}}(\boldsymbol{\xi})$. Since position (i_2, l_1) is empty, we have from Lemma 3.1.12 (2) that there exists $k \in \mathbb{N}$ such that position $(i_2, l_1 - (k+1)e)$ is occupied by a bead and position $(i_2, l_1 - ke)$ is empty. We first slide the bead at position $(i_3, h_1 + e)$ to position (i_3, h_1) in $L_{\mathbf{s}}(\boldsymbol{\xi})$ and denote by $L_{\hat{\mathbf{s}}}(\hat{\boldsymbol{\xi}})$ the new abacus obtained. Then by Lemma 3.4.16 the move vector from $L_{\mathbf{s}}(\boldsymbol{\xi})$ to $L_{\hat{\mathbf{s}}}(\hat{\boldsymbol{\xi}})$ is $\mathcal{M}' = (m'_1, m'_2, \cdots, m'_r)$, where $m'_t = 1$ for all $1 \leq t \leq r$. Then in $L_{\hat{\mathbf{s}}}(\hat{\boldsymbol{\xi}})$, move the bead at position $(i_2, l_1 - (k+1)e)$ to position $(i_1, l_1 - ke)$, and denote by $L_{\mathbf{s}'''}(\boldsymbol{\nu})$ the abacus obtained. Clearly, the move vector from $L_{\mathbf{s}'''}(\boldsymbol{\nu})$ to $L_{\hat{\mathbf{s}}}(\hat{\boldsymbol{\xi}})$ is equal to $\mathcal{M}'$. Therefore, $\mathbf{s} = \mathbf{s}'''$ and $(\boldsymbol{\nu}, \mathbf{s})$ and $(\boldsymbol{\xi}, \mathbf{s})$ are in the same block $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$.

By taking $(\kappa_1, \iota_2) = (i_1, l_2)$ and $(\kappa_2, \iota_2) = (i_3, h_1)$, we arrive the result $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$. ☐

A dual lemma of Lemma 4.1.14 is as follows. Its correctness is ensured by Lemma 3.4.19 and Lemma 3.4.20.

Lemma 4.1.15. *Let $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ and $\mathbf{s} \in \overline{\mathcal{A}}_e^r$. If*

(1) position (i_2, h_1) is occupied by a bead and position (i_3, h_1) is empty;

(2) position $(i_1, h_1 - e)$ is empty and position (i_i, h_1) is occupied by a bead,

where $1 \leq i_1 < i_2 < i_3 \leq r$, then there exist $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}_{\beta}^{\Lambda}$ such that $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

4.2. **Proof of Type I.** In this subsection, we will prove the result of this section as long as in block $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$, there is a pair $(\boldsymbol{\lambda}, \mathbf{s})$ such that the operation set $\mathcal{F}$ from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to its core is of Type I. We divide the proof into **7 cases** according to the possible shape of abaci.

Case 1. *Positions $(i+1, h_1)$ and $(i+1, h_2)$ are empty and there are beads at positions (i, h_1) and (i, h_2).*

This is just the result of Lemma 4.1.1.

Case 2. *Position $(i+1, h_1)$ is empty and there are beads at positions (i, h_1), (i, h_2) and $(i+1, h_2)$.*

Note that the case of $i = r$ can be transformed into the case $i < r$ by Lemma 3.4.12. So we only need to prove for the case of $i \neq r$. Firstly, let $e \neq \infty$. By lemma 3.4.3, there exist empty positions before $(i+1, h_2)$. Let $(j+1, h_2 - ke)$ be the first one, where $1 \leq j \leq r$, $k \in \mathbb{N}$. Then all the positions between $(j+1, h_2 - ke)$ and $(i-1, h_2)$ are not empty. Particularly, position $(j, h_2 - ke)$ is occupied by a bead. We discuss according to whether or not $j = r$.

Subcase 1. $j \neq r$. There are two possibilities to consider.

(1) $|i - j| > 1$. Then in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, positions $(j+1, h_2 - ke)$ and $(i+1, h_1)$ are empty and there are beads at positions $(j, h_2 - ke)$ and (i, h_1). These are just the requirements of Lemma 4.1.3.

(2) $|i - j| \leq 1$. This implies that $j + 1 = i$, or $i + 1 = j$ or $j = i$.

(i) If $j + 1 = i$, then $k \neq 0$ because position $(i, h_2 - ke)$ is before $(i+1, h_2)$. As a result, there are beads at positions $(i, h_2 - ke + e)$ and $(i+1, h_2 - ke + e)$ since they are between positions $(i, h_2 - ke)$ and $(i-1, h_2)$. Moreover, it is clear that $h_2 - ke + e \neq h_1$. Now we have possessed all the conditions required by Lemma 4.1.6. For the convenience of the reader, we illustrate them as follows.

- Position $(i-1, h_2 - ke)$ has a bead and position $(i, h_2 - ke)$ is empty.
- Position (i, h_1) has a bead and position $(i+1, h_1)$ is empty.
- Position $(i, h_2 - ke + e)$ has a bead, $h_2 - ke + e \neq h_1$.

(ii) If $i + 1 = j$, then there is a bead at position $(i, h_2 - ke)$ since it is between positions $(i+2, h_2 - ke)$ and $(i-1, h_2)$. By combining this fact with facts

- there is a bead at position (i, h_1) and position $(i+1, h_1)$ is empty;
- there is a bead at position $(i+1, h_2 - ke)$ and position $(i+2, h_2 - ke)$ is empty,
- there is bead at position $(i, h_2 - ke)$.

we get all the conditions required by Corollary 4.1.5.

(iii) If $i = j$, we consider the relationship between h_1 and $h_2 - ke$.

If $h_1 \neq h_2 - ke$, then in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ we have

- position (i, h_1) is occupied by a bead and position $(i+1, h_1)$ is empty;
- position $(i, h_2 - ke)$ is bead and $(i+1, h_2 - ke)$ is empty position.

Then the result follows from Lemma 4.1.1.

Conversely, we assume $h_1 = h_2 - ke$. Note that position $(i+1, h_2 - ke)$ is before $(i+1, h_2)$. This implies $k \neq 0$ and therefore, both positions $(i, h_2 - ke + e)$ and $(i-1, h_2 - ke)$ are between positions $(i+1, h_2 - ke)$ and $(i-1, h_2)$. Consequently, these two positions are occupied by beads. In addition, there is a bead at position $(i, h_2 - ke)$ and position $(i+1, h_2 - ke)$ is empty. By Lemma 4.1.7, we get the result.

Subcase 2. $j = r$. We need to analyze the following three possibilities.

(1) $1 < i$ and $i + 1 < r$. Then in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, we have

- position (i, h_1) is occupied by a bead and position $(i+1, h_1)$ is empty;
- position $(r, h_2 - ke)$ is occupied by a bead and position $(1, h_2 - ke - e)$ is empty.

They are conditions of Lemma 4.1.3.

(2) $1 = i$. The proof is similar to Subcase 1. (2) (i). We omit the details and leave it as an exercise.

(3) $i + 1 = r$. The proof is similar to Subcase 1. (2) (ii). We leave it as an exercise.

A final remark is for the case of $e = \infty$. The proof is a copy of Subcase 1, in which k need to be replaced by 0. Then we have completed the proof.

Case 3. *Positions (i, h_1), $(i + 1, h_1)$ and $(i + 1, h_2)$ are empty and there is a bead at position (i, h_2).*

It is dual to Case 2.

Case 4. *Positions $(i+1, h_1)$ and (i, h_2) are empty and there are beads at positions (i, h_1) and $(i + 1, h_2)$.*

By Lemma 3.4.12, we can transform the case $i = r$ into the case $i < r$. So we only need to handle the case $i < r$. Let us first give the proof under the assumption $e \neq \infty$. Note that the existence of empty positions before position $(i + 1, h_2)$ is ensured by Lemma 3.4.3, and thus we suppose that position $(j + 1, h_2 - k_1 e)$ is the first one, where $k_1 \in \mathbb{N}$, $1 \leq j \leq r$. Then there is a bead at position $(j, h_2 - k_1 e)$. Furthermore, Lemma 3.4.3 also tells us there are positions occupied by blacks beads after position (i, h_2). Let $(l, h_2 + k_2 e)$ be the first one, where $k_2 \in \mathbb{N}$, $1 \leq l \leq r$. Note that then position $(l + 1, h_2 + k_2 e)$ is empty. We divide the proof into four parts according to the values of j and l.

Subcase 1. $j = l = r$. Under this condition, position $(r, h_2 + k_2 e)$ is after (i, h_2). This implies $k_2 \neq 0$ and therefore, $h_2 - k_1 e \neq h_2 + k_2 e$. Now in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,

- position $(r, h_2 - k_1 e)$ is occupied by a bead and position $(1, h_2 - (k_1 + 1)e)$ is empty;
- position $(r, h_2 + k_2 e)$ is occupied by a bead and position $(1, h_2 - k_2 e - e)$ is empty;

We complete the proof of this subcase by Lemma 4.1.1.

Subcase 2. $j = r$ and $l \neq r$. We consider three possibilities.

(1) $1 < i$ and $i + 1 < r$. Now the conditions of Lemma 4.1.3 are all in readiness.

- Position $(r, h_2 - k_1 e)$ is occupied by a bead and position $(1, h_2 - (k_1 + 1)e)$ is empty.
- Position (i, h_1) is occupied by a bead and position $(i + 1, h_1)$ is empty.

(2) $1 = i$ and $i + 1 < r$. We have owned all conditions required by Lemma 4.1.4. Let us illustrate them as follows.

- Position $(r, h_2 - k_1 e)$ has a bead and position $(1, h_2 - (k_1 + 1)e)$ is empty.
- Position $(1, h_1)$ has a bead and position $(2, h_1)$ is empty.
- Position $(1, h_2)$ is empty and position $(2, h_2)$ has a bead.
- $h_2 \neq h_2 - (k_1 + 1)e$.

(3) $1 < i$ and $i + 1 = r$. We are in the circumstances of Lemma 4.1.4.

Subcase 3. $j \neq r$ and $l = r$.

It is dual to (2).

Subcase 4. $j \neq r$ and $l \neq r$. We consider two possibilities according to the relationship among i, j and l.

(1) $|i - j| > 1$ or $|i - l| > 1$.

If $|i - j| > 1$, by combining it with facts that

- position (i, h_1) is occupied by a bead and position $(i+1, h_1)$ is empty;
- position $(j, h_2 - k_1 e)$ is occupied by a bead and position $(j+1, h_2 - k_1 e)$ is empty,

we have gathered all the conditions required by Lemma 4.1.3.

On the other hand, if $|i - l| > 1$, we can obtain the result by the following information of $L_{\mathbf{s}}(\boldsymbol{\lambda})$, which is required by Lemma 4.1.3.

- Position (i, h_1) has a bead and position $(i+1, h_1)$ is empty.
- Position $(l, h_2 + k_2 e)$ has a bead and position $(l+1, h_2 + k_2 e)$ is empty.

(2) $|i - j| \leq 1$ and $|i - l| \leq 1$. We consider 7 possibilities.

(i) If $j = i+1$ and $l = i-1$, then $l+1 < j$ and

- position $(l, h_2 + k_2 e)$ has a bead and position $(l+1, h_2 + k_2 e)$ is empty;
- position $(j, h_2 - k_1 e)$ has a bead and position $(j+1, h_2 - k_1 e)$ is empty,

which are requirements of Lemma 4.1.3.

(ii) If $j = i+1$ and $l = i$, then position $(i, h_2 + k_2 e)$ is after position (i, h_2) and hence $k_2 \neq 0$. Since all positions between (i, h_2) and $(i, h_2 + k_2 e)$ are empty, we have position $(i+2, h_2 + k_2 e)$ is empty. Note that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, position $(i+1, h_2 - k_1 e)$ has a bead and position $(i+2, h_2 - k_1)$ is empty. We have from Lemma 3.1.12 that there exists $h_3 \in \mathbb{Z}$ such position $(i+1, h_3)$ is empty and position $(i+2, h_3)$ has a bead with $h_3 \neq h_2 + k_2 e$. Here is a summary of useful information in abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$.

- Position $(i, h_2 + k_2 e)$ has a bead and position $(i+1, h_2 + k_2 e)$ is empty.
- Position $(i+1, h_2 - k_1 e)$ has a bead and position $(i+2, h_2 - k_1 e)$ is empty.
- Position $(i+1, h_3)$ is empty and there is a bead at position $(i+2, h_3)$.

which suit the conditions of Lemma 4.1.4.

(iii) If $j = i+1$ and $l = i+1$, then $(i+1, h_2 + k_2 e)$ is after (i, h_2) and this leads to $k_2 \neq 0$ and $h_2 - k_1 e \neq h_2 + k_2 e$. Note that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,

- position $(i+1, h_2 - k_1 e)$ has a bead and position $(i+2, h_2 - k_1 e)$ is empty;
- position $(i+1, h_2 + k_2 e)$ has a bead and position $(i+2, h_2 + k_2 e)$ is empty.

It is proper to use Lemma 4.1.1.

(iv) If $j = i-1$ and $|l - i| \leq 1$, then position $(i, h_2 - k_1 e)$ is before position $(i+1, h_2)$ and so $k_1 \neq 0$. By observing abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$, we have

- position $(i-1, h_2 - k_1 e)$ has a bead and position $(i, h_2 - k_1 e)$ is empty;
- position (i, h_1) has a bead and position $(i+1, h_1)$ is empty;
- position (i, h_2) is empty and position $(i+1, h_2)$ has a bead.

Clearly, $h_1 \neq h_2$. Then by Lemma 4.1.4, we get the result.

(v) If $j = i+1$ and $l = i-1$, then all positions between position $(i+1, h_2 - k_1 e)$ and position (i, h_2) are occupied by beads, which include position $(i-1, h_2 - k_1 e)$. Notice that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, position $(i-1, h_2 + k_2 e)$ has a bead and position $(i, h_2 + k_2 e)$ is empty. By Lemma 3.1.12, there exists $h_3 \in \mathbb{Z}$ with $h_3 \neq h_2 - k_1 e$ such that position $(i-1, h_3)$ is empty and position (i, h_3) has a bead. We have gathered all conditions of Lemma 4.1.4 Let us make a summary as follows.

- Position $(i-1, h_2 + k_2 e)$ has a bead and position $(i, h_2 + k_2 e)$ is empty.
- Position $(i, h_2 - k_1 e)$ has a bead and position $(i+1, h_2 - k_1 e)$ is empty.
- Position $(i-1, h_3)$ is empty and position (i, h_3) has a bead.

(vi) If $j = i+1$ and $l = i$, then we have $k_1 \neq 0$ from that position $(i+1, h_2 - k_1 e)$ is before position $(i+1, h_2)$. As a result, $h_2 - k_1 e \neq h_2 + k_2 e$. Then in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, we have

- position $(i, h_2 - k_1 e)$ has a bead and position $(i+1, h_2 - k_1 e)$ is empty;
- position $(i, h_2 + k_2 e)$ has a bead and position $(i+1, h_2 + k_2 e)$ is empty.

This fits Lemma 4.1.1.

(vii) If $j = l = i+1$, then $k_2 \neq 0$ because position $(i+1, h_2 + k_2 e)$ is after position (i, h_2). This implies that $h_2 \neq h_2 + k_2 e$. Now in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,

- position $(i, h_2 - k_1 e)$ has a bead and position $(i+1, h_2 - k_1 e)$ is empty;
- position $(i+1, h_2 + k_2 e)$ has a bead and position $(i+2, h_2 + k_2 e)$ is empty;
- position (i, h_2) is empty and position $(i+1, h_2)$ has a bead.

The conditions of Lemma 4.1.4 have been satisfied.

Let us give a final remark to complete the proof. If $e = \infty$, then the proof is a copy of case $j \neq r$, $l \neq r$, in which both k_1 and $=k_2$ are replaced by 0.

Case 5. *Positions (i, h_1) and $(i+1, h_1)$ are empty and there are beads at positions (i, h_2) and $(i+1, h_2)$.*

For the same reason as in the cases above, we assume $i < r$. Firstly, let $e \neq \infty$. By Lemma 3.4.3, there exist beads after position (i, h_1). Let the bead at position $(l, h_1 + k_1 e)$ be the first one, where $1 \leq l \leq r$, $k_1 \in \mathbb{N}$. Note that position $(i+1, h_1)$ is empty. Then all the positions between $(i+2, h_1)$ and $(l, h_1 + k_1 e)$ are empty. Particularly, position $(l+1, h_1 + k_1 e)$ is empty. Moreover, we have from Lemma 3.4.3 that there exist empty positions before position $(i+1, h_2)$. Let position $(j+1, h_2 - k_2 e)$ be the first one, where $1 \leq j \leq r$, $k_2 \in \mathbb{N}$. Note there is a bead at position (i, h_2). Then all positions between $(j+1, h_2 - k_2 e)$ and $(i-1, h_2)$ are occupied by beads, including position $(j, h_2 - k_2 e)$. We consider four subcases.

Subcase 1. $l = j = r$. There are two possibilities.

(1) If $h_1 + k_1 e \neq h_2 - k_2 e$, then in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,

- there is a bead at position $(r, h_1 + k_1 e)$ and position $(1, h_1 + k_1 e - e)$ is empty.
- there is a bead at position $(r, h_2 - k_2 e)$ and position $(1, h_2 - k_2 e - e)$ is empty.

By Lemma 4.1.1, the result follows.

(2) If $h_1 + k_1 e = h_2 - k_2 e$, then positions (i, h_1) and $(i+1, h_1)$ are empty and there are beads at positions $(i, h_1 + (k_1 + k_2)e)$ and $(i+1, (k_1 + k_2)e)$, which are just conditions of Corollary 4.1.12.

Subcase 2. $l < r$ and $j = r$. We consider three situations.

(1) If $1 < l$ and $l + 1 < r$, then in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,

- there is a bead at $(l, h_1 + k_1 e)$ and empty at $(l+1, h_1 + k_1 e)$ is empty;
- there is a bead at $(j, h_2 - k_2 e)$ and empty at $(j+1, h_2 - k_2 e)$ is empty.

It is clear that we are in the circumstances of Lemma 4.1.3.

(2) If $l+1 = r$, then there is a bead at position $(r-1, h_1 + k_1 e)$. Since $(l, h_1 + k_1 e)$ is after position (i, h_1), position $(l+2, h_1 + k_1 e)$ is between positions $(i+2, h_1)$ and $(l, h_1 + k_1 e)$. This implies that position $(l+2, h_1 + k_1 e)$ is empty, or position $(1, h_1 + k_1 e - e)$ is empty. Moreover, we have in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ that position $(1, h_2 - k_2 e - e)$ and $(r, h_1 + k_1 e)$ is empty and there are a bead at position $(r, h_2 - k_2 e)$. We have gathered all conditions required by Corollary 4.1.5.

(3) If $l=1$, we consider the value of k_1. If $k_1 \neq 0$, then position (l, h_1+k_1e) is after position (i, h_1). This forces positions (r, h_1+k_1e) and $(1, h_1+k_1e-e)$ being between positions $(i+2, h_1)$ and $(1, h_1+k_1e)$, and therefore they are empty. In addition, position (r, h_2-k_2e) has a bead. This implies that $h_2-k_2e-e \neq h_1+k_1-e$. Here is a summary of useful information in abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$.

- there is a bead at (r, h_2-k_2e) and position at $(1, h_2-k_2e-e)$ is empty;
- there is a bead at $(1, h_1+k_1e)$ and position at $(2, h_1+k_1e)$ is empty.
- Position at $(1, h_1+k_1e-e)$ is empty and $h_1+k_1e-e \neq h_2-k_2-e$.

which are requirements of Lemma 4.1.6.

If $k_1=0$, we consider positions $(i+1, h_2-k_2e)$ and (i, h_2-k_2e). They are between positions $(1, h_2-k_2e-e)$ and $(i-1, h_2)$ and consequently occupied by beads. By combining this with the following facts

- positions (i, h_1) and $(i+1, h_1)$ are empty
- there is a beads at position (r, h_2-ke) and position $(1, h_2-ke-e)$ is empty;
- there is a bead at position $(1, h_1)$ and position $(2, h_1)$ is empty;

we obtain all conditions of Lemma 4.1.9.

Subcase 3. $l=r$ and $j<r$.

It is dual to Subcase 2.

Subcase 4. $l<r$ and $j<r$. We consider the relationship between j and l. If $|l-j|>1$, then the following facts are just required by Lemma 4.1.3.

- There is a bead at position (l, h_1+k_1e) and position $(l+1, h_1+k_1e)$ is empty.
- There is a bead at position (j, h_2-k_2e) and position $(j+1, h_2-k_2e)$ is empty.

On the other hand, if $|l-j| \leq 1$, we need to handle three possibilities, $l=j$ or $l+1=j$ or $l-1=j$.

(1) $l=j$. We need to discuss the situation. If $h_1+k_1e \neq h_2-k_2e$, then in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,

- there is a bead at position (l, h_1+k_1e) and position $(l+1, h_1+k_1e)$ is empty,
- there is a bead at position (l, h_2-k_2e) and position $(l+1, h_2-k_2e)$ is empty.

Then the correctness of the result is ensured by Lemma 4.1.1.

If $h_1+k_1e=h_2-k_2e$, then positions positions (i, h_1) and $(i+1, h_1)$ are empty and there are beads at positions $(i, h_1+(k_1+k_2)e)$ and $(i+1, (k_1+k_2e)e)$, which are just conditions of Corollary 4.1.12

(2) $l+1=j$. Note that position $(j-1, h_2-k_2e)$ is between positions $(j+1, h_2-k_2e)$ and $(i-1, h_2)$. This implies that there is a bead at position $(j-1, h_2-k_2e)$. Then in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,

- there is a bead at position (l, h_1+k_1e) and position $(l+1, h_1+k_1e)$ is empty.
- there are beads at positions (l, h_2-k_2e) and $(l+1, h_2-k_2e)$ and position $(l+2, h_2-k_2e)$ is empty.

The all conditions of Corollary 4.1.5 are satisfied.

(3) $l-1=j$. We need to discuss according to the values of k_1 and k_2.

If $k_2 \geq 2$, then positions $(j+1, h_2-k_2e+e)$ and $(j+2, h_2-k_2e+e)$ are between positions $(j+1, h_2-k_2e)$ and $(i-1, h_2)$, and therefore they are occupied by beads. This forces $h_2-k_2e+e \neq h_1+k_1e$ because position $(j+2, h_1+k_1e)$ is empty. Now in abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$, we have

- position $(j, h_2 - k_2e)$ has a bead and position $(j+1, h_2 - k_2e)$ is empty;
- position $(j+1, h_1 + k_1e)$ has a bead and position $(j+2, h_1 + k_1e)$ is empty;
- there is a bead at $(j+1, h_2 - k_2e + e)$, $h_2 - k_2e + e \neq h_1 + k_1e$,

which are all conditions required by Lemma 4.1.6 and we obtain the result. Similarly, we can prove the result if $k_1 \geq 2$. So we only need to consider $k_1 < 2$ and $k_2 < 2$. It is necessary to point out here that we can prove the result by the way above if $k_2 = 1$ and $i \leq j+1$ or if $k_1 = 1$ and $l \leq i+1$.

Let us divide the proof into the following.

(i) $k_1 = k_2 = 1$. By the analysis above, we only need to work under the assumptions $i > j+1$ and $l > i+1$. Obviously, this is contradict with $l - 1 = j$.

(ii) $k_1 = 0$ and $k_2 = 0$. Note that position $(l, h_1 + k_1e)$ is after position (i, h_1). Then $k_1 = 0$ implies $l < i$. Moreover, position $(j+1, h_2 - k_2e)$ is before position $(i+1, h_2)$. A simple corollary of $k_2 = 0$ is $i+1 < j+1$, or $i < j$. As a result, we get $l < j$, which is contradict to $l = j+1$.

(iii) $k_1 = 0$ and $k_2 = 1$. We only need to give a proof under condition $j+1 < i$. Then in $L_{\mathbf{s}}(\boldsymbol{\lambda})$,

- positions (i, h_1) and $(i+1, h_1)$ are empty and there are beads at positions (i, h_2) and $(i+1, h_2)$;
- position $(j+1, h_1)$ has a bead and position $(j+2, h_1)$ is empty.
- position $(j, h_2 - e)$ has a bead and position $(j+1, h_2 - e)$ is empty.

All conditions of Lemma 4.1.9 are satisfied.

(iv) $k_1 = 1$ and $k_2 = 0$. It is a dual case of (ii).

Finally, if $e = \infty$, then the proof is a copy of case $j \neq r$, $l \neq r$, in which both k_1 and $=k_2$ are replaced by 0.

Case 6. *There are beads at positions* $(i+1, h_1)$ *and* $(i+1, h_2)$.

Because positions $(i+1, h_1)$ and $(i+1, h_2)$ have beads and $[(i, h_1), *]$, $[(i, h_2), *] \in \mathcal{F}$, the operation set $\mathcal{F}$ has to contain $[(i+1, h_1), *]$ and $[(i+1, h_2), *]$. If one of positions $(i+2, h_1)$ and $(i+2, h_2)$ is empty, then we arrive at Case 1 or Case 2. Otherwise, repeat the above analysis process. Clearly, the process will end after finite times. since $\mathcal{F}$ is finite and this completes the proof.

Case 7. *Positions* (i, h_1) *and* (i, h_2) *are empty.*

Note that positions (i, h_1) and (i, h_2) are empty and $[(i, h_1), *]$, $[(i, h_2), *] \in \mathcal{F}$. This forces $[(i-1, h_1), *]$, $[(i-1, h_2), *] \in \mathcal{F}$. If one of positions $(i-1, h_1)$ and $(i-1, h_2)$ is not empty, then we are in the circumstances of Case 1 or Case 3. Otherwise, we can repeat the above analysis process. Note that $\mathcal{F}$ is finite. This implies that we only need to do the analysis process finite times.

4.3. **Proof of Type II.** In this subsection, we will prove the result os this section as long as in block $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$, there is a pair $(\boldsymbol{\lambda}, \mathbf{s})$ such that the operation set $\mathcal{F}$ from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to its core is of Type II. Assume $m_i \geq 2$ and the operations happen in column h.

We divide the proof into the following **4 cases**.

Case 1. *Position* (i, h) *is empty and there is a bead at position* $(i+1, h)$.

By Lemma 3.4.12, we can transform the case $i = r$ into the case $i < r$. So we only need to handle the case $i < r$. Let us first give the proof under the assumption $e \neq \infty$. In this case, it is clear that there exist at least two empty positions before position $(i+1, h)$. Let position $(j_1, h - k_1e)$ and $(j_2, h - k_2e)$ be the first one and the second one, respectively, where $k_1 \leq k_2 \in \mathbb{N}$. This implies that all positions between $(j_1, h - k_1e)$ and $(i+1, h)$ and those between $(j_2, h - k_2e)$ and $(j_1, h - k_1e)$ are occupied by beads. If $k_1 = 0$ then $i+1 < j_1 \leq r$ and if $k_2 = 0$ then $i+1 < j_1 < j_2 \leq r$. On the other hand, there exist at least two positions occupied by beads after position $(i+1, h)$. Let positions $(j_3, h+k_3e)$ and $(j_4, h+k_4e)$ be the first one and the second one, respectively, where $k_3 \leq k_4 \in \mathbb{N}$. Clearly, all positions between (i, h) and $(j_3, h+k_3e)$ and those between $(j_3, h+k_3e)$ and $(j_4, h+k_4e)$ are empty. Moreover, if $k_3 = 0$ then $1 \leq j_3 < i$ and if $k_4 = 0$ then $1 \leq j_4 < j_3 < i$. We now analyze the possible values of k_2 and k_4. If $k_2 > 1$, then by Lemma 3.4.14 $[(r, h), *]$, $[(r, h-e), *] \in \mathcal{F}$. It is contradict with the conditions of this type and therefore, $k_2 \leq 1$. Similarly, we have $k_4 \leq 1$. Furthermore, if $k_4 = k_2 = 1$, then $[(r, h), *]$, $[(r, h+e), *] \in \mathcal{F}$. It is also a contradiction. That is, we have proved $k_2 + k_4 \leq 1$. In addition, if $k_2 + k_4 = 1$, by conditions of this type, we have $j_2 \leq j_4$.

Now let us divide the proof into five subcases according to the values of k_1 and k_2 and the relationship between j_2 and j_4.

Subcase 1. $k_2 = k_4 = 0$. Under this condition, it is easy to check that

- positions (j_1, h) and (j_2, h) are empty;
- there are beads at positions (j_4, h) and (j_3, h);
- $1 \leq j_4 < j_3 < j_1 < j_2 \leq r$.

By Lemma 4.1.13, the result follows.

Subcase 2. $k_2 = 1$, $k_4 = 0$ *and* $j_2 = j_4$. The given conditions implies that

- positions (i, h) and $(j_4, h-e)$ are empty;
- there are beads at positions (j_3, h) and (j_4, h);
- $1 \leq j_4 < j_3 < i$.

Clearly, this is just the case Lemma 4.1.15 handled.

Subcase 3. $k_2 = 1$, $k_4 = 0$ *and* $j_2 < j_4$. Move the runners $1, \cdots, j_2$ of $L_{\mathbf{s}}(\boldsymbol{\lambda})$ e positions to the right and put them on the top in the original order. Denote by $L_{\mathbf{s}'}(\overline{\boldsymbol{\lambda}})$ the new abacus obtained. Then by Lemma 3.4.12 (5), $L_{\mathbf{s}'}(\overline{\boldsymbol{\lambda}}) \in \mathcal{H}_{\beta}^{\Lambda}$, $\mathbf{s}' \in \overline{A}_e^r$. By analyzing the shape of $L_{\mathbf{s}}(\boldsymbol{\lambda})$ and using the relationship between $L_s(\boldsymbol{\lambda})$ and $L_{\mathbf{s}'}(\overline{\boldsymbol{\lambda}})$, we get that in $L_{\mathbf{s}'}(\overline{\boldsymbol{\lambda}})$,

- positions $(i - j_2, h)$ and (r, h) are empty;
- there are beads at positions $(j_3 - j_2, h)$ and $(j_4 - j_2, h)$;
- $1 \leq j_4 - j_2 < j_3 - j_2 < i - j_2 < r$,

which are conditions required by Lemma 4.1.13.

Subcase 4. $k_2 = 0$, $k_4 = 1$ *and* $j_2 < j_4$. Move the runners $j_4, \cdots, r$ of $L_{\mathbf{s}}(\boldsymbol{\lambda})$ e positions to the left and put them at the bottom in the original order. Then the proof left is similar to Subcase 3.

Subcase 5. $k_2 = 0$, $k_4 = 1$ *and* $j_2 = j_4$. By observing abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$, we have

- positions (j_1, h) and (j_2, h) are empty;
- there are beads at positions $(i+1, h)$ and $(j_2, h+e)$;
- $i+1 < j_1 < j_2 \leq r$.

Then Lemma 4.1.14 provides us a pair of incomparable abaci.

Finally, if $e < \infty$, then the proof is the same as Subcase 1.

The following three cases can be handled similarly as Case 1. We omit the details here.

Case 2. *Position (i, h) is occupied by a bead and position $(i+1, h)$ is empty.*

Case 3. *Both position (i, h) and $(i+1, h)$ are occupied by beads.*

Case 4. *Both positions (i, h) and $(i+1, h)$ are empty.*

5. Proof of Main Theorem: Part II

Given a pair $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $\mathbf{s} \in \overline{\mathcal{A}}_e^r$ and $r > 2$, let $\mathcal{M} = (1, 1, \cdots, 1)$ the block move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to its core. We prove in this section that $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ has infinite representation type. Clearly, the condition on $\mathcal{M}$ in this section forces $e < \infty$. Moreover, $\mathbf{s}$ has to be one of the following two types.

Type I: There exists $1 \leq j < r$ such that $s_j \neq s_{j+1}$.

Type II: $s_j = s_{j+1}$ for all $1 \leq j < r$.

5.1. **Proof of Type I.** Let us first prove two lemmas under the assumptions of this type.

Lemma 5.1.1. *Let $L_{\mathbf{s}}(\boldsymbol{\lambda})$ be $(e, \mathbf{s})$-complete, where $\mathbf{s} \in \mathcal{A}_e^r$. Then there exists $h \in \mathbb{Z}$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, position (r, h) has a bead and position $(1, h)$ is empty.*

Proof. Let us use reduction to absurdity. Assume that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, if position (r, h) has a bead, then so is position $(1, h)$. Note that $(\boldsymbol{\lambda}, \mathbf{s})$ is complete. This implies that for arbitrary $h \in \mathbb{Z}$, if position $(1, h)$ has a bead, then all positions (x, h) have beads for $1 \leq x \leq r$, and if position $(1, h)$ is empty, then all positions (x, h) are empty for all $1 \leq x \leq r$. That is, all runners in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ are the same. By Lemma 3.4.7, this forces $s_1 = s_2 = \cdots = s_r$. It contradicts with the assumption of Type I and this completes the proof. □

For the other lemma, we continue to use the method of disproof.

Lemma 5.1.2. *There exists $k \in \mathbb{Z}$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, position $(1, k)$ has a bead and position $(r, k+e)$ is empty.*

Proof. Suppose for arbitrary $l \in \mathbb{Z}$, the bead at position $(1, l)$ implies a bead being at position $(r, l+e)$. Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the abacus obtained by in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ deleting $L_{s_1}(\boldsymbol{\lambda}^{(1)})$ and putting $L_{s_1+e}(\boldsymbol{\lambda}^{(1)})$ on the top. It follows from Lemma 3.1.9 that $u_{r-1} = s_r$ and $u_r = s_1 + e$. Then in $L_{\mathbf{u}}(\boldsymbol{\mu})$, a bead at position (r, l) forces there being a bead at position $(r-1, l)$. Let h be an integer such that all positions (i, x) are occupied by beads, where $1 \leq i \leq r$ and $x \leq h$. Then in $L_{\mathbf{u}}(\boldsymbol{\mu})$, positions $(r-1, y)$ with $y > h$ having beads are more than that in runner r. Combining this fact with Lemma 3.1.9, we get $u_{r-1} \geq u_r$, or $s_r \geq s_1 + e$. This contradict with $\mathbf{s} \in \mathcal{A}_e^r$. We have completed the proof. □

We are ready to give the proof of Type I. The strategy is to construct incomparable abaci.

Proof of Type I. Let $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ be the core of $L_{\mathbf{s}}(\boldsymbol{\lambda})$. Then a direct corollary of Lemma 3.4.7 is $\mathbf{s}^* = \mathbf{s}$. Note that $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$ is $(e, \mathbf{s})$-complete. We have from Lemma 3.1.12 (2) that there exists $h \in \mathbb{Z}$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$, position (r, h) has a bead and position $(1, h)$ is empty. Take the biggest one and denote it still by h. According to the definition of $(e, \mathbf{s})$-complete, we can find $1 < j \leq r$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$, positions (x, h) have beads for $j \leq x \leq r$ and positions (y, h) are empty for $1 \leq y < j$. On the other hand, Lemma 5.1.2 implies the existence of $k \in \mathbb{Z}$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$, position $(1,k)$ has a bead and position $(r,k+e)$ is empty. Write the biggest one still by k for simplicity. It is clear $h \neq k+e$ by assumptions above.

We consider two cases.

Case 1. $h < k+e$.

In $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$, move the beads at positions $(1,k)$ and (j,h) to positions $(r,k+e)$ and $(1,h)$, respectively, and then move successively the beads at positions (x,h) to positions $(x-1,h)$ for all $j < x \leq r$ from $j+1$ to r. Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the new abacus obtained. Furthermore, suppose that in $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$, $\bullet_1^2$ is at position $(2,m)$. Move in $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$ the bead at position $(2,m)$ to position $(2,m+e)$ and denote by $L_{\mathbf{v}}(\boldsymbol{\nu})$ the abacus obtained. By Lemma 3.4.13 , both move vectors from $L_{\mathbf{u}}(\boldsymbol{\mu})$ and $L_{\mathbf{v}}(\boldsymbol{\nu})$ to $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$ are equal to $\mathcal{M} = \{1,1,\cdots,1\}$. We deduce from Lemma 3.4.9 that $\mathbf{u} = \mathbf{v} = \mathbf{s}$ and $(\boldsymbol{\mu},\mathbf{s}),(\boldsymbol{\nu},\mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. To prove $L_{\mathbf{s}}(\boldsymbol{\mu}) \parallel L_{\mathbf{s}}(\boldsymbol{\nu})$, we can take $(\kappa_1,\iota_1) = (1,h)$, $(\kappa_2,\iota_2) = (2,h)$.

Case 2. $k+e < h$. We divide the proof into two subcases.

Subcase 1. $j > 2$. This implies that in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ position $(2,h)$ is empty. Suppose that in $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$, the bead $\bullet_1^1$ is at position $(1,m)$. Move this bead to $(1,m+e)$ and denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the new abacus obtained. On the other hand, move in $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$ the bead at position $(1,k)$ to position $(r,k+e)$ and then move the bead at position $(2,k)$ to position $(1,k)$. Move the bead at position (j,h) to position $(2,h)$, and next move successively the beads at positions (x,h) to positions $(x-1,h)$ for $j < x \leq r$ from $j+1$ to r. Denote the final abacus obtained by $L_{\mathbf{v}}(\boldsymbol{\nu})$. In the light of Lemma 3.4.13, both the move vectors from $L_{\mathbf{u}}(\boldsymbol{\mu})$ and $L_{\mathbf{v}}(\boldsymbol{\nu})$ to $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$ are equal to $\mathcal{M}$. According to Lemma 3.4.9, $\mathbf{u} = \mathbf{v} = \mathbf{s}$ and $(\boldsymbol{\mu},\mathbf{s}),(\boldsymbol{\nu},\mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. It is a routine task to check $L_{\mathbf{u}}(\boldsymbol{\mu}) \parallel L_{\mathbf{v}}(\boldsymbol{\nu})$ by taking $(\kappa_1,\iota_1) = (1,m+e)$ and $(\kappa_2,\iota_2) = (r,h)$.

Subcase 2. $j = 2$. This implies that in $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$ position $(2,h)$ has a bead. Assume bead $\bullet_1^2$ in $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$ is at position $(2,m)$. Let us construct two incomparable abaci by $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$. Firstly, denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the abacus obtained by moving bead at position $(2,m)$ to position $(2,m+e)$. Secondly, move in $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$ the bead at position $(1,k)$ to position $(r,k+e)$, and move successively beads at positions (x,h) to $(x-1,h)$ for $2 \leq x \leq r$ from 2 to r. Denote by $L_{\mathbf{v}}(\boldsymbol{\nu})$ the new abacus. Then $\mathbf{u} = \mathbf{v} = \mathbf{s}$ and $(\boldsymbol{\mu},\mathbf{s}),(\boldsymbol{\nu},\mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. By taking $(\kappa_1,\iota_1) = (2,m+e)$ and $(\kappa_2,\iota_2) = (r,k+e)$, we get $L_{\mathbf{u}}(\boldsymbol{\mu}) \parallel L_{\mathbf{v}}(\boldsymbol{\nu})$.

5.2. **Proof of Type II.** Let us first depict the frame of abaci in a block with block move vector being $(1,1,\cdots,1)$. For this aim, we need the following lemma.

Lemma 5.2.1. *If $L_{\mathbf{s}}(\boldsymbol{\lambda})$ is $(e,\mathbf{s})$-complete, then all runners in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ are the same.*

Proof. Let $h \in \mathbb{Z}$ such that in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ all positions (x,y) are occupied by beads, where $1 \leq x \leq r$ and $y \leq h$. Assume the bead at position $(1,h)$ is $\bullet_j^1$. Note that $s_1 = s_r$. So by Lemma 3.1.9, the bead at position (r,h) is $\bullet_j^r$. On the other hand, We can derive from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ being complete that for all $l \in \mathbb{Z}$,

- if position (r,l) is empty, then all positions (x,l) in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ are empty;
- if there is a bead at position $(1,l)$, then all positions (x,l) in $L_{\mathbf{s}}(\boldsymbol{\lambda})$ have beads.

As a result, for $1 \leq k < j$, $\bullet_k^1$ and $\bullet_k^r$ are in the same column. This forces all runners are the same. $\square$

In the following two lemmas, we always assume $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ with $\mathbf{s} \in \mathcal{A}_e^r$ and $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ its core. The pair $(\boldsymbol{\lambda}, \mathbf{s})$ satisfies assumptions of this subsection. Based on Lemma 5.2.1, we can describe all the abaci in $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$.

Lemma 5.2.2. *The following are equivalent.*

(1) $(\boldsymbol{\mu}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$.
(2) In $L_{\mathbf{s}}(\boldsymbol{\mu})$ *there is an empty position* (j, l) *such that position* $(j, l+e)$ *has a bead and* $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ *can be obtained from* $L_{\mathbf{s}}(\boldsymbol{\mu})$ *by moving the bead at position* $(j, l+e)$ *to position* (j, l).
(3) $\boldsymbol{\lambda}^*$ *can be obtained from* $\boldsymbol{\mu}$ *by deleting a rim e-hook in* $\boldsymbol{\mu}^{(j)}$.

Proof. (2) $\Leftrightarrow$ (3) is a clear corollary of Lemma 3.4.16 and "(2) $\Rightarrow$ (1)" is a direct corollary of Lemma 3.4.9 and 3.4.16. Thus we only need to prove"(1) $\Rightarrow$ (2)".

By Lemma 3.4.7, we have $\mathbf{s}^* = \mathbf{s}$. Since $(\boldsymbol{\mu}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$, we deduce from Lemma 3.4.10 that the core of $L_{\mathbf{s}}(\boldsymbol{\mu})$ is $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$. Obviously, $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$ is $(e, \mathbf{s})$-complete, and then all its runners are the same in the light of Lemma 5.2.1. This forces the final operation from $L_{\mathbf{s}}(\boldsymbol{\mu})$ to $L_{\mathbf{s}}(\boldsymbol{\lambda}^*)$ is $[(r, l+e)*]$ for some $l \in \mathbb{Z}$. Suppose the final operation is done in abacus $L_{\mathbf{v}}(\boldsymbol{\nu})$. Then in $L_{\mathbf{v}}(\boldsymbol{\nu})$, we have

- position $(1, l)$ is empty and all the other positions in column l have beads;
- position $(r, l+e)$ has a bead and all the other positions in column $l+e$ are empty;
- for each $h \neq l, l+e$, all positions in column h are either occupied by beads or empty.

Consequently, operations from $L_{\mathbf{s}}(\boldsymbol{\mu})$ to $L_{\mathbf{v}}(\boldsymbol{\nu})$ must happen in columns l and $l+e$. Furthermore, in $L_{\mathbf{s}}(\boldsymbol{\mu})$ column l, there is only one empty position (i, l) and in column $l+e$ there is only one position $(j, l+e)$ occupied by a bead. Note that the block move vector is $(1, 1, \cdots, 1)$. This forces $i = j$. $\square$

The following corollary is easy and we omit its proof.

Corollary 5.2.3. *All multipartitions in* $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ *is a totally ordered set with respect to the dominance order. Moreover, assume one can obtain* $\boldsymbol{\lambda}^*$ *from* $\boldsymbol{\mu}$ *and* $\boldsymbol{\nu}$ *by deleting a rim e-hook in* $\boldsymbol{\mu}^{(i)}$ *and* $\boldsymbol{\nu}^{(j)}$*, respectively. If* $i < j$*, then* $\boldsymbol{\mu} \rhd \boldsymbol{\nu}$.

Next we consider the simple modules of a block satisfying assumptions of this subsection.

Lemma 5.2.4. *If pair* $(\boldsymbol{\lambda}, \mathbf{s})$ *is a Kleshchev one, then* $\boldsymbol{\lambda}^*$ *can be obtained from* $\boldsymbol{\lambda}$ *by deleting a rim e-hook in* $\boldsymbol{\mu}^{(r)}$.

Proof. By Lemma 5.2.2, $\boldsymbol{\lambda}^*$ can be obtained from $\boldsymbol{\lambda}$ by deleting a rim e-hook in $\boldsymbol{\mu}^j$, where $1 \leq j \leq r$. If $j < r$, let σ be transposition (j, r) and write $\boldsymbol{\lambda}^\sigma = \boldsymbol{\nu}$. Since $\boldsymbol{\lambda}^*$ can be obtained from $\boldsymbol{\nu}$ by deleting a rim e hook in $\boldsymbol{\nu}^r$, we have from Lemma 5.2.2 that $(\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. It is clear that if $\mathfrak{t}$ is a standard $(\boldsymbol{\lambda}, \mathbf{s})$-tableau, then $\mathfrak{t}_\sigma$ is a standard $(\boldsymbol{\nu}, \mathbf{s})$ tableau. Note that $s_1 = s_2 = \cdots = s_r$. This implies that $\mathrm{res}_{\boldsymbol{\lambda}, \mathbf{s}}(\mathfrak{t}) = \mathrm{res}_{\boldsymbol{\nu}, \mathbf{s}}(\mathfrak{t}_\sigma)$. By Corollary 5.2.3, $\boldsymbol{\lambda} \rhd \boldsymbol{\nu}$. Then we can deduce from Lemma 3.1.2 that pair $(\boldsymbol{\lambda}, \mathbf{s})$ is not Kleshchev. This completes the proof. $\square$

We divide the proof into two cases according to whether or not $e = 2$.

Case 1. $e \neq 2$

It is a classical result (see [46, Exercise 5.10]) that there are exactly e partitions $\lambda_{(1)} \rhd \lambda_{(2)} \rhd \cdots \lambda_{(e)}$ of weight 1 with e-core $\boldsymbol{\lambda}^{(r-1)*}$, and because $e \neq 2$, the numbers of standard $\lambda_{(e)}$-tableaux and standard $\boldsymbol{\lambda}_{(e-1)}$-tableaux are not same. Let

$$\boldsymbol{\mu} = (\boldsymbol{\lambda}^{*(1)}, \cdots, \boldsymbol{\lambda}^{*(r-2)}, \lambda_{(e)}, \boldsymbol{\lambda}^{*(r)})$$

and

$$\boldsymbol{\nu} = (\boldsymbol{\lambda}^{*(1)}, \cdots, \boldsymbol{\lambda}^{*(r-2)}, \lambda_{(e-1)}, \boldsymbol{\lambda}^{*(r)}).$$

Clearly, the numbers of standard $\boldsymbol{\mu}$-tableaux and standard $\boldsymbol{\nu}$-tableaux are not same. We have from Lemma 5.2.2 that $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$. Moreover, it follows from Lemma 5.2.4 that neither $\boldsymbol{\mu}$ nor $\boldsymbol{\nu}$ is a Kleshchev multipartition, and from Corollary 5.2.3 that both $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are not maximal. We deduce from Corollary 2.3.9 that block $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ has infinite representation type.

Case 2. $e = 2$.

Given a pair $(\boldsymbol{\lambda}, \mathbf{s})$ satisfying the assumptions of this subsection with $(\boldsymbol{\lambda}^*, \mathbf{s}^*)$ its core, we deduce from Lemma 5.2.2 that

(1) $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ can be obtained from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ by moving certain bead at position (j, h) to empty position $(j, h-2)$;
(2) $\mathbf{s}^* = \mathbf{s}$;
(3) all runners in $L_{\mathbf{s}^*}(\boldsymbol{\lambda})$ are same and $\boldsymbol{\lambda}^{*(1)} = \boldsymbol{\lambda}^{*(2)} = \cdots = \boldsymbol{\lambda}^{*(r)}$ are 2-cores.

It is easy to check that $\boldsymbol{\lambda}^{*(i)}$ is of the form $(m, m-1, \cdots, 1)$ with $m \geq 0$. The 2-core is $\varnothing$ when $m = 0$. Assume the bead $\bullet^1_1$ in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ is at position $(1, x)$. Move this bead to position $(1, x+2)$ and denote by $L_{\mathbf{s}}(\tilde{\boldsymbol{\lambda}}(m))$ the new abacus obtained. It follows from Lemma 5.2.2 that $(\tilde{\boldsymbol{\lambda}}(m), \mathbf{s})$ and $(\boldsymbol{\lambda}, \mathbf{s})$ belong to the same block. By Lemma 3.1.5, we can assume that $s_1 = s_2 = \cdots = s_r = 0$. Then $\boldsymbol{\Lambda}_{\tilde{\boldsymbol{\lambda}}(m), \mathbf{s}} = r\boldsymbol{\Lambda}_0$ and

$$\boldsymbol{\beta}_{\tilde{\boldsymbol{\lambda}}(m), \mathbf{s}} = \begin{cases} r[k^2\boldsymbol{\alpha}_0 + k(k+1)\boldsymbol{\alpha}_1] + \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1, & \text{if } m = 2k; \\ r[(k+1)^2\boldsymbol{\alpha}_0 + k(k+1)\boldsymbol{\alpha}_1] + \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1, & \text{if } m = 2k+1. \end{cases} \tag{5.2.1}$$

We claim that

$$r\boldsymbol{\Lambda}_0 - \boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1 = \begin{cases} (\sigma_0\sigma_1)^k(r\boldsymbol{\Lambda}_0 - \boldsymbol{\beta}_{\tilde{\boldsymbol{\lambda}}(m), \mathbf{s}}), & \text{if } m = 2k; \\ (\sigma_0\sigma_1)^k\sigma_0(r\boldsymbol{\Lambda}_0 - \boldsymbol{\beta}_{\tilde{\boldsymbol{\lambda}}(m), \mathbf{s}}), & \text{if } m = 2k+1. \end{cases}$$

We use induction on m. Clearly, the claim holds when $m = 0$. Assume the claim holds for $m-1$. We now prove the claim holds for m. Let us consider two cases.

If $m = 2k$, then

$$\begin{aligned}
& \sigma_1(r\boldsymbol{\Lambda}_0 - \boldsymbol{\beta}_{\tilde{\boldsymbol{\lambda}}(m), \mathbf{s}}) \\
= \; & \sigma_1(r\boldsymbol{\Lambda}_0 - r[k^2\boldsymbol{\alpha}_0 + k(k+1)\boldsymbol{\alpha}_1] - \boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1) \text{ (Substituting (5.2.1))} \\
= \; & r\boldsymbol{\Lambda}_0 - r[k^2\boldsymbol{\alpha}_0 + k(k+1)\boldsymbol{\alpha}_1] - \boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1 \\
& -(\boldsymbol{\alpha}_1, r\boldsymbol{\Lambda}_0 - r[k^2\boldsymbol{\alpha}_0 + k(k+1)\boldsymbol{\alpha}_1] - \boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1)\boldsymbol{\alpha}_1 \text{ (Definition of } \sigma_1) \\
= \; & r\boldsymbol{\Lambda}_0 - r[k^2\boldsymbol{\alpha}_0 + k(k-1)\boldsymbol{\alpha}_1] - \boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1 \text{ (Definition of bilinear form (,))} \\
= \; & r\boldsymbol{\Lambda}_0 - \boldsymbol{\beta}_{\tilde{\boldsymbol{\lambda}}(m-1), \mathbf{s}}. \text{ (Substituting (5.2.1))}
\end{aligned}$$

That is, $(\sigma_0\sigma_1)^k(r\boldsymbol{\Lambda}_0 - \boldsymbol{\beta}_{\tilde{\boldsymbol{\lambda}}(m), \mathbf{s}}) = (r_0r_1)^{k-1}r_0(r\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\tilde{\boldsymbol{\lambda}}(m-1), \mathbf{s}})$. By the induction hypothesis, the claim follows in this case.

If $m = 2k+1$, then

$$\begin{aligned}
&\sigma_0(r\boldsymbol{\Lambda}_0 - \boldsymbol{\beta}_{\tilde{\boldsymbol{\lambda}}(m),\mathbf{s}})\\
= \;&\sigma_0(r\boldsymbol{\Lambda}_0 - r[(k+1)^2\boldsymbol{\alpha}_0 + k(k+1)\boldsymbol{\alpha}_1] - \boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1) \text{ (Substituting (5.2.1))}\\
= \;&r\boldsymbol{\Lambda}_0 - r[(k+1)^2\boldsymbol{\alpha}_0 + k(k+1)\boldsymbol{\alpha}_1] - \boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1\\
&-(\boldsymbol{\alpha}_0, r\boldsymbol{\Lambda}_0 - r[(k+1)^2\boldsymbol{\alpha}_0 + k(k+1)\boldsymbol{\alpha}_1] - \boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1)\boldsymbol{\alpha}_0 \text{ (Definition of } \sigma_0)\\
= \;&r\boldsymbol{\Lambda}_0 - r[k^2\boldsymbol{\alpha}_0 + k(k+1)\boldsymbol{\alpha}_1] - \boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1 \text{ (Definition of bilinear form (,))}\\
= \;&r\Lambda_0 - \beta_{\tilde{\boldsymbol{\lambda}}(m-1),\mathbf{s}}. \text{ (Substituting (5.2.1))}
\end{aligned}$$

This implies that $(\sigma_0\sigma_1)^k\sigma_0(r\boldsymbol{\Lambda}_0 - \boldsymbol{\beta}_{\tilde{\boldsymbol{\lambda}}(m),\mathbf{s}}) = (\sigma_0\sigma_1)^k(r\boldsymbol{\Lambda} - \boldsymbol{\beta}_{\tilde{\boldsymbol{\lambda}}(m-1),\mathbf{s}})$. By the induction hypothesis, the claim holds in this case and we have proven the claim. According to a classical result proved in [18] by Chuang and Rouquier, we only need to consider block $\mathcal{H}^{r\boldsymbol{\Lambda}_0}_{\boldsymbol{\alpha}_0+\boldsymbol{\alpha}_1}$. We have from Lemma 3.1.4 that this block has a basis $\{e_{\boldsymbol{\alpha}_0+\boldsymbol{\alpha}_1}y_1^iy_2^j \mid 0 \le i < r, 0 \le j < 2\}$, or the block is isomorphic to $K[y_1, y_2]/\langle y_1^r, y_2^2\rangle$. Clearly, the block has infinite representation type.

6. Proof of Main Theorem: Part III

Let $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ be a block with block move vector $\mathcal{M} = (m_1, \cdots, m_r)$, where $r > 2$, satisfying

(1) $w = \sum_i m_i \ge 2$;
(2) $m_i \le 1$ for $1 \le i \le r$;
(3) $\prod_i m_i = 0$.

Define an oriented quiver Γ_r associated with $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ as follows. The vertex set is $I = \mathbb{Z}/r\mathbb{Z} = \{\bar{1}, \bar{2}, \cdots, \bar{r}\}$ and directed edges are $\bar{i} \longrightarrow \overline{i+1}$ for all $m_i = 1$.

If $e = \infty$, then the meaning of $0 \le j \le e-1$ is $j \in \mathbb{Z}$.

6.1. **Infinite representation type cases.** To simplify the proof, we can assume according to Lemma 3.4.12 in this subsection that $m_r = 0$. A direct benefit is that we can write elements $\bar{i}$ in $\mathbb{Z}/r\mathbb{Z}$ as i and compare them as natural numbers without any confusion. Let $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ with $\mathbf{s} \in \overline{\mathcal{A}}^r_e$ and Γ_r the associated oriented quiver. Denote by $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the core of $L_{\mathbf{s}}(\boldsymbol{\lambda})$. We handle four cases in this subsection, which are all of infinite representation type.

Case 1. *There exist at least two connected components (not isolated dots) in Γ_r.*

Since there exist at least two connected components (not isolated dots) in Γ_r, there exist $1 \le i_1 < i_2 + 1 < i_3 \le i_4 < r$ such that the path from i_1 to i_2+1 and path from i_3 to i_4+1 are two connected components. To prove block $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ has infinite representation type in this case, we construct two incomparable abaci from $L_{\mathbf{s}}(\boldsymbol{\lambda})$.

By Lemma 3.4.7, we have $s^*_{i_1} = s_{i_1} - 1$, $s^*_{i_2+1} = s_{i_2+1} + 1$, $s^*_{i_3} = s_{i_3} - 1$ and $s^*_{i_4+1} = s_{i_4+1} + 1$. Note that $\mathbf{s} \in \overline{\mathcal{A}}^r_e$. This implies $s_{i_1} \le s_{i_2+1}$ and $s_{i_3} \le s_{i_4+1}$. Therefore, $s^*_{i_1} + 2 \le s^*_{i_2+1}$, $s^*_{i_2} + 2 \le s^*_{i_4+1}$. Then we can deduce from Lemma 3.1.12 (4) that there exist $h_1, h_2, h_3, h_4 \in \mathbb{Z}$ with $h_1 < h_2$ and $h_3 < h_4$ such that in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$,

- positions (i_1,h_1) and (i_1,h_2) are empty and positions (i_2+1,h_1) and (i_2+1,h_2) have beads;
- positions (i_3,h_3) and (i_3,h_4) are empty and positions (i_4+1,h_3) and (i_4+1,h_4) have beads.

Define $L_{\tilde{\mathbf{s}}}(\tilde{\boldsymbol{\lambda}})$ to be the abacus such that the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to it is $\mathcal{M}=(m_1,\cdots,m_r)$, where

$$m_j=\begin{cases}1, & \text{if } i_1\le j\le i_2 \,\text{or}\, i_3\le j\le i_4;\\ 0, & \text{otherwise.}\end{cases}$$

It is clear that $L_{\tilde{\mathbf{s}}}(\tilde{\boldsymbol{\lambda}})$ is uniquely determined and runners $i_1,\cdots,i_2,i_2+1$ and $i_3,\cdots,i_4,i_4+1$ in $L_{\tilde{\mathbf{s}}}(\tilde{\boldsymbol{\lambda}})$ and $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ are the same, respectively. Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the abacus obtained by moving in $L_{\tilde{\mathbf{s}}}(\tilde{\boldsymbol{\lambda}})$ the bead at positions (i_2+1,h_2) and (i_4+1,h_3) to positions (i_1,h_2) and (i_3,h_3) respectively. Moreover, move in $L_{\tilde{\mathbf{s}}}(\tilde{\boldsymbol{\lambda}})$ the beads at positions (i_2+1,h_1) and (i_4+1,h_4) to positions (i_1,h_1) and (i_3,h_4), respectively. Denote by $L_{\mathbf{v}}(\boldsymbol{\nu})$ the new abacus obtained. By Lemma 3.4.13, both the move vectors from abaci $L_{\mathbf{u}}(\boldsymbol{\mu})$ and $L_{\mathbf{v}}(\boldsymbol{\nu})$ to $L_{\tilde{\mathbf{s}}}(\tilde{\boldsymbol{\lambda}})$ are equal to $\mathcal{M}$. It follows from Lemma 3.4.9 that $\mathbf{u}=\mathbf{v}=\mathbf{s}$ and $(\boldsymbol{\mu},\mathbf{s})$, $(\boldsymbol{\nu},\mathbf{s})\in\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$. To prove $L_{\mathbf{s}}(\boldsymbol{\mu})\parallel L_{\mathbf{s}}(\boldsymbol{\nu})$, we only need to take $(\kappa_1,\iota_1)=(i_1,h_2)$, $(\kappa_2,\iota_2)=(i_4+1,h_3)$ and this complete the proof of this case.

In the following three cases, we assume in Γ_r, there is only one non isolated dot connected component, which is a path from i_1 to i_2+1 with length not less than 2. It is easy to check that the move vector from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ is $\mathcal{M}=(m_1,\cdots,m_r)$, where

$$m_j=\begin{cases}1, & \textit{if } i_1\le j\le i_2;\\ 0, & \textit{otherwise.}\end{cases}$$

.

Case 2. *There exists $i_1<i_3<i_2$ such that $s_{i_3}\neq s_{i_3+1}$.*

We have from Lemma 3.4.7 that $s^*_{i_1}=s_{i_1}-1$, $s^*_{i_2+1}=s_{i_2+1}+1$ and $s^*_j=s_j$ for all $i_1<j\le i_2$. As a result, $s^*_{i_3}=s_{i_3}$. Since $\mathbf{s}\in\overline{\mathcal{A}}^r_e$ and $s_{i_3}\neq s_{i_3+1}$, we have $s_{i_3}<s_{i_3+1}$. Then we come to a conclusion $s^*_{i_3}=s_{i_3}<s_{i_3+1}\le s^*_{i_3+1}$, or $s^*_{i_3}+1\le s^*_{i_3+1}$. Furthermore, combining $s^*_{i_1+1}=s_{i_1+1}$, $s^*_{i_1}+1=s_{i_1}$ and $s_{i_1}\le s_{i_1+1}$ gives $s^*_{i_1}+1\le s^*_{i_1+1}$. Similarly, $s^*_{i_2}+1\le s^*_{i_2+1}$. Note that $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ is complete. Then by Lemma 3.1.12 (4), the fact $s^*_{i_1}+1\le s^*_{i_1+1}$ implies that there exists $l_1\in\mathbb{Z}$ such that in $L(\boldsymbol{\lambda}^*,\mathbf{s}^*)$, position (i_1,l_1) is empty and positions $(i_1+1,l_1),\cdots,(i_2+1,l_1)$ have beads. Similarly, there exists $l_2\in\mathbb{Z}$ such that positions $(i_1,l_2),\cdots,(i_3,l_2)$ are empty and positions $(i_3+1,l_2),\cdots,(i_2+1,l_2)$ have beads, and there exists $l_3\in\mathbb{Z}$ such that position $(i_1,l_3),\cdots,(i_2,l_3)$ are empty and position (i_2+1,l_3) has bead. The configuration of $L(\boldsymbol{\lambda}^*,\mathbf{s}^*)$ described above implies that l_1, l_2 and l_3 are different from each other. Define $\{h_1,h_2,h_3\}$ to be equal to $\{l_1,l_2,l_3\}$ as a set satisfying $h_1<h_2<h_3$, and define $\{j_1,j_2,j_3\}$ to be equal to $\{i_1+1,i_3+1,i_2+1\}$ as a set such that in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$,

- positions $(i_1,h_1),\cdots,(j_1-1,h_1)$ are empty and positions $(j_1,h_1),\cdots,(i_2+1,h_1)$ have beads;
- positions $(i_1,h_2),\cdots,(j_2-1,h_2)$ are empty and positions $(j_2,h_2),\cdots,(i_2+1,h_2)$ have beads;

- positions $(i_1,h_3),\cdots,(j_3-1,h_3)$ are empty and positions $(j_3,h_3),\cdots,(i_2+1,h_3)$ have beads.

We consider two possibilities.

(1) $j_1>j_3$. Move in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the beads at positions (j_3,h_3) and (i_2+1,h_1) to positions (i_1,h_3) and (j_3,h_1), respectively. Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the new abacus. Another abacus $L_{\mathbf{v}}(\boldsymbol{\nu})$ is obtained by moving in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the beads at position (i_2+1,h_2) to position (i_1,h_3). By Lemma 3.4.13, both the move vectors from $L_{\mathbf{u}}(\boldsymbol{\mu})$ and $L_{\mathbf{v}}(\boldsymbol{\nu})$ to $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ are equal to $\mathcal{M}$. We reach a conclusion by Lemma 3.4.9 that $\mathbf{u}=\mathbf{v}=\mathbf{s}$ and $(\boldsymbol{\mu},\mathbf{s}),(\boldsymbol{\nu},\mathbf{s})\in\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$. By taking $(\kappa_1,\iota_1)=(i_1,h_3)$, $(\kappa_2,\iota_2)=(i_2+1,h_1)$, we get $L_{\mathbf{s}}(\boldsymbol{\mu})\parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

(2) $j_1<j_3$. Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the abacus obtained by moving in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the bead at position (i_2+1,h_2) to position (i_1,h_2). On the other hand, move in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the beads at positions (j_1,h_1) and (i_2+1,h_3) to positions (i_1,h_1) and (j_1,h_3), respectively, and denote the new abacus by $L_{\mathbf{v}}(\boldsymbol{\nu})$. For the same reason as Subcase 1, we have $\mathbf{u}=\mathbf{v}=\mathbf{s}$ and $(\boldsymbol{\mu},\mathbf{s}),(\boldsymbol{\nu},\mathbf{s})\in\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$. To reach $L_{\mathbf{s}}(\boldsymbol{\mu})\parallel L_{\mathbf{s}}(\boldsymbol{\nu})$, we can choose $(\kappa_1,\iota_1)=(i_1,h_2)$, $(\kappa_2,\iota_2)=(i_2+1,h_2)$.

Case 3. $s_{i_1}\neq s_{i_1+1}$.

By analyzing similarly as in Case 2, we get $s^*_{i_1}+2\leq s^*_{i_1+1}$ and $s^*_{i_2}+1\leq s^*_{i_2+1}$. According to Lemma 3.1.12 (4), there exist integers $h_1<h_2$ such that in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$, positions (i_1,h_1) and (i_1,h_2) are empty and at positions h_1 and h_2 all runners $i_1+1,\cdots,i_2+1$ have beads. For the same reason ,there exists $l_1\in\mathbb{Z}$ such that in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ positions $(i_1,l_1),\cdots,(i_2,l_1)$ are empty and position (i_2+1,l_1) has a bead. Let us consider three possibilities. We will only illustrate incomparable abaci in each situation and omit the details.

(1) $h_1<h_2<l_1$. Move in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the bead at position (i_2+1,h_2) to position (i_1,h_2). The new abacus is $L_{\mathbf{s}}(\boldsymbol{\mu})$. Move in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the bead at positions (i_1+1,h_1) and (i_2+1,l_1) to positions (i_1,h_1) and (i_1+1,l_1), respectively. The abacus obtained is $L_{\mathbf{s}}(\boldsymbol{\nu})$. By taking $(\kappa_1,\iota_1)=(i_1,h_2)$, $(\kappa_2,\iota_2)=(i_2+1,h_2)$, we arrive at $L_{\mathbf{s}}(\boldsymbol{\mu})\parallel L_{\mathbf{s}}(\boldsymbol{\nu})$.

(2) $l_1<h_1<h_2$. The proof is similar to (1).

(3) $h_1<l_1<h_2$.

By moving in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the bead at position (i_2+1,h_1) to position (i_1,h_1), we get abacus $L_{\mathbf{s}}(\boldsymbol{\mu})$. Abacus $L_{\mathbf{s}}(\boldsymbol{\nu})$ is obtained by moving in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the beads at positions (i_2+1,l_1) and (i_1+1,h_2) to positions (i_1+1,l_1) and (i_1,h_2), respectively. It is a routine task to check $L_{\mathbf{s}}(\boldsymbol{\mu})\parallel L_{\mathbf{s}}(\boldsymbol{\nu})$ by taking $(\kappa_1,\iota_1)=(i_1+1,h_2)$, $(\kappa_2,\iota_2)=(i_2+1,h_1)$.

Case 4. $s_{i_2}\neq s_{i_2+1}$.

The proof is similar to Case 3.

6.2. **Finite representation type case.** Let $(\boldsymbol{\lambda},\mathbf{s})\in\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ with $\mathbf{s}\in\mathcal{A}^r_e$. and Γ_r the associated oriented quiver. Denote by $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the core of $L_{\mathbf{s}}(\boldsymbol{\lambda})$. Suppose that there is only one non isolated dot connected component in Γ_r, which is a path from i to $i+w-1$. According to results of previous subsection, if $m_r=1$, then the block has infinite representation type. Then based on the previous subsection there is only one case to consider, that is, $s_i=s_{i+1}=\cdots=s_{i+w}$ and $m_r=0$. Fix the meaning of i and w throughout this subsection. Our aim now is to prove the blocks have finite representation type. Let us first study the configuration of $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$.

Lemma 6.2.1. *There exist $h, h' \in \mathbb{Z}$ such that in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$,*

1. *positions $(1,h), \cdots, (i,h)$ are empty and and positions $(i+1,h), \cdots, (r,h)$ have beads;*
2. *positions $(1,h'), \cdots, (i+w-1,h')$ are empty and and positions $(i+w,h')$, $\cdots$, (r,h') have beads;*
3. *runners $i, i+1, \cdots, i+w$ are the same except for columns h and h'.*
4. *for each $l \in \mathbb{Z}$ with $l \neq h, h'$, the number of beads in column l is either more than $r-i$, or less than $r-i-w+1$.*

Proof. It follows from Lemma 3.4.7 that $s_i^*+1 = s_i$, $s_{i+w}^* - 1 = s_{i+w}$ and $s_j^* = s_j$ for $i < j < i+w$. Note that $s_i = s_{i+1} = \cdots = s_{i+w}$ we have

$$s_i^* + 1 = s_{i+1}^* = \cdots = s_{i+w-1}^* = s_{i+w}^* - 1.$$

Then (1) and (2) follow from Lemma 3.1.12 (4).

(3) Let k be an integer such that all positions (x,y) in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ are occupied by beads, where $1 \leq x \leq r$ and $y \leq k$. In the light of Lemma 3.1.9, we have

$$\mathfrak{n}_{i+1}^k = \cdots = \mathfrak{n}_{i+w-1}^k, \tag{6.2.1}$$

$$\mathfrak{n}_{i+1}^k = \mathfrak{n}_i^k + 1 \tag{6.2.2}$$

and

$$\mathfrak{n}_{i+w}^k = \mathfrak{n}_{i+w-1}^k + 1. \tag{6.2.3}$$

Let us use reduction to absurdity. Suppose that there exists $l \neq h, h'$ such that in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$, position (j,l) is empty and position $(j+1,l)$ has a bead, where $i \leq j \leq i+w-1$. If $j = i$, then $\mathfrak{n}_{i+1}^k = \mathfrak{n}_i^k + 2$. It contradicts to equality (6.2.2). If $i < j < i+w-1$, then $\mathfrak{n}_j^k - \mathfrak{n}_{j+1}^k \geq 1$. It contradicts to equality (6.2.1). If $j = i+w-1$, then $\mathfrak{n}_{i+w}^k - \mathfrak{n}_{i+w-1}^k \geq 2$. It contradicts to equality (6.2.3) and the proof of (3) is completed.

(4) Let $l \in \mathbb{Z}$ with $l \neq h, h'$. If position (i,l) has a bead, then it follows from $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ being complete that all positions (j,l) have beads for $i \leq j \leq r$, that is, column l has at least $r-i+1$ beads.

On the other hand, if position (i,l) is empty, then by (3), all positions (j,l) are empty for $1 \leq j \leq i+w$. This implies that column l has at most $r-i-w$ beads. $\square$

From now on, we fix the meaning of h and h'.

Lemma 6.2.2. *Under the assumptions of this subsection, we have*

1. *each elementary operation from abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ is of the form $[(x,h),*]$ or $[(x,h'),*]$;*
2. *the abaci $L_{\mathbf{s}}(\boldsymbol{\lambda})$ and $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ are the same except for positions $\{(x,y) \mid i \leq x \leq i+w,\ y = h, h'\}$;*
3. *in $L_{\mathbf{s}}(\boldsymbol{\lambda})$, the number of beads in column h is $r-i$ and that in column h' is $r-i-w+1$.*

Proof. (1) We use disproof. Assume that $[(i+j,y),*] \in \mathcal{F}$ with $0 \leq j < w$ and $y \neq h, h'$. This implies that at least one of the positions (i,y), $(i+1,y)$, $\cdots$, $(i+j,y)$ is empty and at least one of the positions $(i+j+1,y)$, $\cdots$, $(i+w,y$ have a bead. This contradicts to Lemma 6.2.1 (3). Therefore, y is equal to h or h'.

(2) is a corollary of (1) and the assumption on block move vector.

(3) Note that $m_r = 0$ forces all elementary operations being of the first kind. Therefore, the number of beads in each column is not changed. Then the result follows from Lemma 6.2.1 (1). $\square$

Lemma 6.2.3. *In $L_{\mathbf{s}}(\boldsymbol{\lambda})$, only one of the positions (i,h), $(i+1,h)$, $\cdots$, $(i+w,h)$ is empty and only one of positions (i,h'), $(i+1,h')$, $\cdots$, $(i+w,h')$ has a bead.*

Proof. According to the assumptions of this subsection, all w elementary operations from $L_{\mathbf{s}}(\boldsymbol{\lambda})$ to $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ are of the first kind, and happens in columns h and h' in the light of Lemma 6.2.2. Note that $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$, we have from Lemma 6.2.1 (1) that only one of the positions (i,h), $(i+1,h)$, $\cdots$, $(i+w,h)$ is empty and only one of positions (i,h'), $(i+1,h')$, $\cdots$, $(i+w,h')$ has a bead. Clearly, the number of beads in each column does not change under the first kind elementary operations. Then the lemma follows. $\square$

Now all pairs $(\boldsymbol{\mu},\mathbf{s})$ in block $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ can be determined completely.

Lemma 6.2.4. *Pair $(\boldsymbol{\mu},\mathbf{s})$ is in $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ if and only if*

(1) $L_{\mathbf{s}}(\boldsymbol{\mu})$ and $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ are the same except for positions $\{(x,y) \mid i \leq x \leq i+w,\ y = h,h'\}$;
(2) there exists $i \leq j \leq i+w$ such that in $L_{\mathbf{u}}(\boldsymbol{\mu})$, position (j,h) is empty and position (j,h') has a bead.

Proof. "$\Rightarrow$" By Lemma 6.2.2 (2), we only need to prove (2). Let l be an integer such that all positions (x,y) in $L_{\mathbf{s}}(\boldsymbol{\mu})$ are occupied by beads, where $1 \leq x \leq r$ and $y \leq l$. By Lemma 3.1.9, condition $s_i = s_{i+1} = \cdots = s_{i+w}$ implies that $\mathfrak{n}^l_i = \mathfrak{n}^l_{i+1} = \cdots = \mathfrak{n}^l_{i+w}$. Then (2) follows from Lemma 6.2.2 and 6.2.3.

"$\Leftarrow$" By moving in $L_{\mathbf{u}}(\boldsymbol{\mu})$ the beads at positions (i,h) and (j,h') to positions (j,h) and $(i+w,h')$, respectively, we get $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$. By Lemma 3.4.13, the move vector from $L_{\mathbf{u}}(\boldsymbol{\mu})$ to $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ is just the block move vector. Then we have from Lemma 3.4.9 that $(\boldsymbol{\mu},\mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$. $\square$

We can give some more details on the shape of abaci of pairs in $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$. Since $m_r = 0$, the following lemma is clear.

Lemma 6.2.5. *Let $(\boldsymbol{\mu},\mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ and $e \neq \infty$. If in $L_{\mathbf{s}}(\boldsymbol{\mu})$, position (j,l) has a bead, then all positions $(x, l-ke)$ have beads for $k \in \mathbb{N}^+$. In particular, $h \neq h' \pmod{e}$.*

According to the definition of the action of affine Weyl group W, it is easy to check that in each W-orbit of blocks, there exists $\boldsymbol{\Lambda} - \hat{\boldsymbol{\beta}}$ such that $(\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \hat{\boldsymbol{\beta}}) \geq 0$ for all $0 \leq j \leq e-1$. Take a pair $(\hat{\boldsymbol{\lambda}},\mathbf{s})$ in block $\mathcal{H}^{\boldsymbol{\Lambda}}_{\hat{\boldsymbol{\beta}}}$. Because block move vector and the multicharge are invariant under the action of W, block $\mathcal{H}^{\boldsymbol{\Lambda}}_{\hat{\boldsymbol{\beta}}}$ satisfies the assumptions of this subsection. Then abaci $L_{\mathbf{s}}(\hat{\boldsymbol{\lambda}})$ has an interesting property, which is described as follows.

Lemma 6.2.6. *In $L_{\mathbf{s}}(\hat{\boldsymbol{\lambda}})$, the number of beads in column $l-1$ is not less than that in column l for arbitrary $l \in \mathbb{Z}$.*

Proof. We first prove the lemma under the assumption $e \neq \infty$. We consider three cases according to c_l, which is the number of beads in column l of $L_{\mathbf{s}}(\hat{\boldsymbol{\lambda}})$.

Case 1. $0 < c_l < r$.

Subcase 1. $0 < c_{l-1}$. Let us use disproof. Suppose that $c_{l-1} < c_l$. We have from Lemma 6.2.5 that $c_{l-ke} = r = c_{l-1-ke}$ and $c_{l-1+ke} = 0 = c_{l+ke}$ for each $k \in \mathbb{N}^+$. This implies $\mathfrak{m}_l^{l-1} < 0$. On the other hand, let $l = j \pmod e$ with $0 \le j \le e-1$. It follows from Lemma 3.2.6 that

$$\boldsymbol{\Lambda} - \hat{\boldsymbol{\beta}} - (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \hat{\boldsymbol{\beta}})\boldsymbol{\alpha}_j = \sigma_j(\boldsymbol{\Lambda} - \hat{\boldsymbol{\beta}}) = \boldsymbol{\Lambda} - \boldsymbol{\beta}_{\sigma(\hat{\boldsymbol{\lambda}}),\mathbf{s}} = \boldsymbol{\Lambda} - \hat{\boldsymbol{\beta}} - \mathfrak{m}_l^{l-1}\boldsymbol{\alpha}_j,$$

which implies that $\mathfrak{m}_l^{l-1} = (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \hat{\boldsymbol{\beta}}) \ge 0$. We reach a contradiction.

Subcase 2. $c_{l-1} = 0$. Suppose $c_{l-1-ae} = r$, where $a \in \mathbb{N}^+$ such that $c_y < r$ for all $y \in \mathbb{Z}$ with $y > l-1-ae$ and $y \equiv l-1 \pmod e$. By Lemma 6.2.5 $c_{l-1-ae-ke} = r = c_{l-ke}$ and $c_{l-1+ke} = 0 = c_{l+ke}$ for all $k \in \mathbb{N}^+$. Consequently, $\mathfrak{m}_l^{l-1} < 0$. We can also get a contradiction similar to Subcase 1.

Case 2. $c_l = r$.

Clearly, we only need to prove $c_{l-1} = r$. Suppose $c_{l+ae} = 0$, where $a \in \mathbb{N}^+$ such that $c_y > 0$ for all $y \in \mathbb{Z}$ with $y < l + ae$ and $y \equiv l \pmod e$. If $0 < c_{l-1} < r$, then by Lemma 6.2.5, $c_{l-ke} = r = c_{l-1-ke}$and $c_{l+ae+ke} = 0 = c_{l-1+ke}$ for $k \in \mathbb{N}^+$. Therefore, $\mathfrak{m}_l^{l-1} < 0$, which leads to a contradiction.

If $c_{l-1} = 0$, one can analyze similarly. We omit the details here.

Case 3. $c_l = 0$.

Nothing need to prove in this case.

Finally, if $e = \infty$, then for $j \in \mathbb{Z}$ by Lemma 3.2.6, the number of beads in column $j-1$ is m more than that in column j, where $m = (\boldsymbol{\alpha}_j, \boldsymbol{\Lambda} - \boldsymbol{\beta})$. Since $m \ge 0$, the result follows. □

Based on the preparations above, we can prove the result of this subsection now. By the result given by Chuang and Rouquier in [18], we only need to consider block $L_{\mathbf{s}}(\hat{\boldsymbol{\lambda}})$. Let us first summarize characteristics on the shape of abaci of pairs $(\hat{\boldsymbol{\lambda}}, \mathbf{s})$ in $\mathcal{H}^{\boldsymbol{\Lambda}}_{\hat{\boldsymbol{\beta}}}$.

By Lemma 6.2.1 (4), 6.2.2 (3) and Lemma 6.2.6, we have

(1) $c_l > r - i$ if $l < h$, $c_l < c_{r-i-w+1}$ if $l > h'$ and $c_l \ge c_{l+1}$ for all $l \in \mathbb{Z}$;
(2) $h' = h + 1$.

According to Lemma 6.2.4, assume in $L_{\mathbf{s}}(\hat{\boldsymbol{\lambda}})$, position $(i+j, h)$ is empty and position $(i+j, h')$ has a bead, where $0 \le j \le w$. Then by combing the characteristics (1), (2) above with Lemma 6.2.6, we get that in $L_{\mathbf{s}}(\hat{\boldsymbol{\lambda}})$,

(3) if position (x, y) has a bead, where $x \neq h'$, then so is position $(x, y-1)$;
(4) position $(i+j, h-1)$ has a bead.

In light of Lemma 3.1.8, characteristic (3) and (4) force $|\hat{\boldsymbol{\lambda}}| = 1$. As a result, all multipartitions in $\mathcal{H}^{\boldsymbol{\Lambda}}_{\hat{\boldsymbol{\beta}}}$ are of the form

$$(\varnothing_1, \cdots, \varnothing_{i-1}, (1)_i, \varnothing_{i+1}, \cdots \varnothing_{i+w}, \varnothing_{i+w+1}, \cdots, \varnothing_r)$$
$$(\varnothing_1, \cdots, \varnothing_{i-1}, \varnothing_i, (1)_{i+1}, \cdots \varnothing_{i+w}, \varnothing_{i+w+1}, \cdots, \varnothing_r)$$
$$\cdots$$
$$(\varnothing_1, \cdots, \varnothing_{i-1}, \varnothing_i, \varnothing_{i+1}, \cdots (1)_{i+w}, \varnothing_{i+w+1}, \cdots, \varnothing_r)$$

From Lemma 3.1.4, we deduce that the block has a basis $\{1, x, \cdots, x^w\}$. That is, $\mathcal{H}^{\boldsymbol{\Lambda}}_{\hat{\boldsymbol{\beta}}}$ is isomorphic to $k[x]/(x^{w+1})$, which has finite representation type. Consequently, all blocks that are in the same W-orbit with $\mathcal{H}^{\boldsymbol{\Lambda}}_{\hat{\boldsymbol{\beta}}}$ are of finite representation type. By [7, Theorem 6.8], this forces these blocks being Morita equivalent with a Brauer tree algebra, whose Brauer tree is T_1 in Example 2.2.2 with $m(1) = w$. That is, these blocks are Morita equivalent to $k[x]/(x^{w+1})$.

Remark 6.2.7. *The result of this subsection implies that in cyclotomic case, there exist blocks of finite representation type are Brauer tree algebras whose Brauer trees have exceptional vertex. Clearly, this can only appear when $r \geq 3$.*

6.3. **Remarks on finite type.** In this subsection, we give two additional results on blocks of finite representation type in this section. One is about multipartition and the other is an equivalence characterization of conditions in Subsection 2. In our opinion, they are interesting, although they are not necessary in the proof of the Main Theorem.

It is known that all multipartitions in $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ form a totally ordered set with respect to dominance order $\trianglelefteq$. In fact, we can write them out in order. For this goal, we first introduce a new notation. Let $(\boldsymbol{\mu}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. If in abacus $L_{\mathbf{s}}(\boldsymbol{\mu})$, position $(i+j, h)$ is empty and position $(i+j, h')$ has a bead, where $0 \leq j \leq w$, then we rewrite multipartition $\boldsymbol{\mu}$ by $\boldsymbol{\lambda}[j]$.

Lemma 6.3.1. *For $0 \leq j < w$, we have*

(1) if $h < h'$, then $\boldsymbol{\lambda}[j] \triangleright \boldsymbol{\lambda}[j+1]$;
(2) if $h > h'$, then $\boldsymbol{\lambda}[j] \triangleleft \boldsymbol{\lambda}[j+1]$.

Proof. (1) Let the bead at position $(i+j, h')$ in $L_{\mathbf{s}}(\boldsymbol{\lambda}[j])$ be $\bullet_a^{i+j}$ and let the bead at position $(i+j, h)$ in $L_{\mathbf{s}}(\boldsymbol{\lambda}[j+1])$ be $\bullet_b^{i+j}$. By observing the shape of the two abaci, it is easy to know

(i) $a \leq b$;
(ii) $\mathfrak{n}_{i+j}^{h-1}(L_{\mathbf{s}}(\boldsymbol{\lambda}[j])) = \mathfrak{n}_{i+j}^{h-1}(L_{\mathbf{s}}(\boldsymbol{\lambda}[j+1]))$;
(iii) the number of empty positions on the left side of $h'+1$-th position in runner $i+j$ of $L_{\mathbf{s}}(\boldsymbol{\lambda}[j])$ is equal to that of $L_{\mathbf{s}}(\boldsymbol{\lambda}[j+1])$.

Combining this observation with Lemma 3.1.8 (2), we get that for arbitrary x with $1 \leq x < a$ or $x > b$,

$$\boldsymbol{\lambda}[j]_x^{(i+j)} = \boldsymbol{\lambda}[j+1]_x^{(i+j)}. \tag{6.3.1}$$

For $a \leq x \leq b$, the number empty positions in $L_{\mathbf{s}}(\boldsymbol{\lambda}[j])$ on the left side of position $\bullet_x^{i+j}$ is one more than that in $L_{\mathbf{s}}(\boldsymbol{\lambda}[j+1])$. Note that the empty position is just $(i+j, h)$. As a result of Lemma 3.1.8 (2), if $a \leq x \leq b$, then

$$\boldsymbol{\lambda}[j]_x^{(j)} > \boldsymbol{\lambda}[j+1]_x^{(j)}. \tag{6.3.2}$$

A direct corollary of (6.3.1) and (6.3.2) is

$$|\boldsymbol{\lambda}[j]^{(i+j)}| > |\boldsymbol{\lambda}[j+1]^{(i+j)}|. \tag{6.3.3}$$

For each $1 \leq c \leq r$ with $c \neq i+j$, $i+j+1$, runner c in $L_{\mathbf{s}}(\boldsymbol{\lambda}[j])$ is the same as that in $L_{\mathbf{s}}(\boldsymbol{\lambda}[j+1])$. This fact leads to

$$\boldsymbol{\lambda}[j]^{(c)} = \boldsymbol{\lambda}[j+1]^{(c)}, \tag{6.3.4}$$

and consequently,

$$\sum_{y=1}^{i+j-1} |\boldsymbol{\lambda}[j]^{(y)}| = \sum_{y=1}^{i+j-1} |\boldsymbol{\lambda}[j+1]^{(y)}|. \tag{6.3.5}$$

Combining (6.3.3) with (6.3.5) leads to

$$\sum_{y=1}^{i+j} |\boldsymbol{\lambda}\,[j]^{(y)}| > \sum_{y=1}^{i+j} |\boldsymbol{\lambda}\,[j+1]^{(y)}|. \tag{6.3.6}$$

Note that all multipartitions in $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ is a totally order with respect to the dominance order. Then (6.3.6) implies $\boldsymbol{\lambda}\,[j] \rhd \boldsymbol{\lambda}\,[j+1]$.

(2) is proved similarly. $\square$

In order to describe the second result, we need to define a new vector for a given block.

Definition 6.3.2. *Let $(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}^{\Lambda}_{\boldsymbol{\beta}}$ and $\mathcal{F}$ be the operation set from $L_s(\boldsymbol{\lambda})$ to its core. Define $\mathcal{W}_{\boldsymbol{\lambda}} = (w_{0\boldsymbol{\lambda}}, w_{1\boldsymbol{\lambda}}, \cdots, w_{e-1\boldsymbol{\lambda}})$, where*

$$w_i = \sharp\{[(x,h),*] \in \mathcal{F} \mid 1 \le x \le r,\ h \equiv i \,(\operatorname{mod} e)\}.$$

Then $\mathcal{W}(\mathcal{H}^{\Lambda}_{\boldsymbol{\beta}}) = (w_0, w_1, \cdots, w_{e-1}) = \sum_{(\boldsymbol{\lambda},\mathbf{s}) \in \mathcal{H}^{\Lambda}_{\boldsymbol{\beta}}} \mathcal{W}_{\boldsymbol{\lambda}}$ is called the subabacus move vector of block $\mathcal{H}^{\Lambda}_{\boldsymbol{\beta}}$.

Lemma 6.3.3. *Let $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ be a block satisfying the assumptions of Subsection 2. Then the number of non-zero components in $\mathcal{W}$ is 2.*

Proof. Keep notation as in Subsection 2. Let $(\boldsymbol{\mu}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$. Then by Lemma 6.2.2 all the elementary operations from $L_{\mathbf{s}}(\boldsymbol{\mu})$ to $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ happen in columns h and h'. Moreover, we from Lemma 6.2.5 that columns h and h' are in different subabaci of $L_{\mathbf{s}}(\boldsymbol{\mu})$. As a result, the number of non-zero components in $\mathcal{W}$ is not more than two.

On the other hand, we know that $[(i,h),*]$, $[(i+1,h'),*] \in \mathcal{F}$, where $\mathcal{F}$ is the operation set from $L_{\mathbf{s}}(\boldsymbol{\lambda}\,[1])$ to $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$, that is, the number of non-zero components in $\mathcal{W}$ is two. $\square$

Lemma 6.3.4. *Let $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$ be a block satisfying assumptions of this section and assume in Γ_r, there is only one non isolated dot connected component, which is a path from i_1 to i_2+1 ($1 \le i_1 < i_2 < r$) with length not less than 2. If there exists $i_1 \le i_3 \le i_2$ such that $s_{i_3} \ne s_{i_3+1}$, then the number of non zero components in $\mathcal{W}$ is not less than 3.*

Proof. We consider three cases.

Case 1. $i_1 < i_3 < i_2$.

Let l_1, l_2, l_3 be integers as Case 2 of Subsection 1. By Lemma 3.1.12 (3), it is easy to check that columns l_1, l_2 and l_3 are in different subabaci. Move in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$, which is the core of $\mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$, the beads at positions (i_1+1, l_1), (i_3+1, l_2) and (i_2+1, l_3) to positions (i_1, l_1), (i_1+1, l_2) and (i_3+1, l_3), respectively. Denote by $L_{\mathbf{u}}(\boldsymbol{\mu})$ the new abacus obtained. It follows from Lemma 3.4.13 that the move vector from $L_{\mathbf{u}}(\boldsymbol{\mu})$ to $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ is equal to $\mathcal{M}$. In light of Lemma 3.4.9, we have $\mathbf{u} = \mathbf{s}$ and $(\boldsymbol{\mu}, \mathbf{s}) \in \mathcal{H}^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta}}$. This implies that in $\mathcal{W} = (w_0, \cdots, w_{e-1})$, components w_{j_1}, w_{j_2} and w_{j_3} are nonzero, where $0 \le j_1, j_2, j_3 < e$ and $j_1 \equiv l_1 \,(\operatorname{mod} e)$, $j_2 \equiv l_2 \,(\operatorname{mod} e)$, $j_3 \equiv l_3 \,(\operatorname{mod} e)$.

Case 2. $i_3 = i_1$.

Let l_1 be an integer as Case 3 of Subsection 1. By Lemma 3.1.12(3), it is easy to check that columns h_1, h_2 and l_1 are in different subabaci.

Move in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the bead at positions (i_1+1, h_1) and (i_2+1, l_1) to positions (i_1, h_1) and (i_1+1, l_1). Denote the new abacus by $L_{\mathbf{u}}(\boldsymbol{\mu})$. Move in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the beads at positions (i_1+1, h_2) and (i_2+1, l_1) to positions (i_1, h_2) and (i_1+1, l_1). Denote the new abacus by $L_{\mathbf{v}}(\boldsymbol{\nu})$. It is routine to check that $\mathbf{u} = \mathbf{v} = \mathbf{s}$ and $(\boldsymbol{\mu}, \mathbf{s}), (\boldsymbol{\nu}, \mathbf{s}) \in \mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$. This implies that in $\mathcal{W} = (w_0, \cdots, w_{e-1})$, components w_{j_1}, w_{j_2} and w_{j_3} are nonzero, where $0 \le j_1, j_2, j_3 < e$ and $j_1 \equiv h_1 \pmod e$, $j_2 \equiv h_2 \pmod e$, $j_3 \equiv l_1 \pmod e$.

Case 3. $i_3 = i_2$.

The proof is similar to Case 2. □

Combining Lemmas 6.3.3 and 6.3.4, we get the following result.

Theorem 6.3.5. *Let $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ be a block of weight more than 1 and $\mathcal{M}$ its block move vector. Then $\mathcal{H}_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}$ has finite representation type if and only if*

(1) $m_r = 0$;
(2) Γ_r has only one non isolated dot connected component;
(3) the number of non zero components of $\mathcal{W}$ is 2.

7. Derived equivalence of blocks

In this section, we study the derived equivalence of blocks of a cyclotomic Hecke algebra by using the theory developed in this paper.

Recall that two blocks of a Hecke algebra of type A are derived equivalent if and only if their weights are equal and if and only if they are in the same orbit under the adjoint action of the affine Weyl group. However, this does not hold in cyclotomic case ($r > 2$). Firstly, we can easily find two blocks with the same weights, and one of them has infinite representation type, and the other is Morita equivalent to a truncated polynomial ring, which has finite representation type. This implies that equal in weight does not mean derived equivalence. So our discussion in this section will center on constructing examples of derived equivalent blocks being in different orbits under the adjoint action of the affine Weyl group.

It is an open problem to give a necessary and sufficient condition for two blocks of a cyclotomic Hecke algebra being derived equivalent. In the light of Ariki's result [7, Theorem 6.8], a finite representation type indecomposable self-injective cellular algebra is Morita equivalent to a Brauer tree algebra whose Brauer tree is a straight line with at most one exceptional vertex. The problem of derived equivalence of Brauer tree algebras is complete resolved in [48]. We will translate the results [7, Theorem 6.8] and [48, Theorem 4.2] into the language of weight and charge for blocks of a cyclotomic Hecke algebra. Obviously, two blocks being Morita equivalent to truncated polynomial rings are derived equivalent if and only if they have the same weights. Thus we only need to handle the weight one case.

Given two pairs $(\boldsymbol{\lambda}, \mathbf{s})$ and $(\boldsymbol{\mu}, \mathbf{u})$ with $\mathbf{s}, \mathbf{u} \in \overline{\mathcal{A}}_e^r$ and $(\boldsymbol{\lambda}^*, \mathbf{s}^*)$ and $(\boldsymbol{\mu}^*, \mathbf{u}^*)$ the corresponding cores, respectively, denote by $B_{\boldsymbol{\lambda},\mathbf{s}}$ and $B_{\boldsymbol{\mu},\mathbf{u}}$ the blocks containing $(\boldsymbol{\lambda}, \mathbf{s})$ and $(\boldsymbol{\mu}, \mathbf{u})$, respectively. Suppose that $w(B_{\boldsymbol{\lambda},\mathbf{s}}) = w(B_{\boldsymbol{\mu},\mathbf{u}}) = 1$. Then we have the following proposition.

Proposition 7.0.1. *Blocks $B_{\boldsymbol{\lambda},\mathbf{s}}$ and $B_{\boldsymbol{\mu},\mathbf{u}}$ are derived equivalent if and only if the number of non-zero components in $\mathcal{W}(B_{\boldsymbol{\lambda},\mathbf{s}})$ is equal to that of $\mathcal{W}(B_{\boldsymbol{\mu},\mathbf{u}})$.*

Proof. Let the block move vector of $B_{\boldsymbol{\lambda},\mathbf{s}}$ be $\mathcal{M} = (m_1, m_2, \cdots, m_r)$ with $m_j = 1$ and $m_i = 0$ for all $1 \leq i \leq r$, $i \neq j$. Assume that $s_{j+1} - s_j = a$ (s_{r+1} is defined to be $s_1 + e$ if $e < \infty$). We claim that the number of multipartitions in $B_{\boldsymbol{\lambda},\mathbf{s}}$ is $a+2$. In fact, if $j < r$, we have from Lemma 3.4.7 that $s^*_{j+1} = s_{j+1} + 1$ and $s^*_j = s_j - 1$ and thus $s^*_j + a + 2 = s^*_{j+1}$. It follows from Lemma 3.1.12 (4) that there exist $h_1, \cdots, h_{a+2} \in \mathbb{Z}$ such that in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$, positions $(j, h_1), \cdots, (j, h_{a+2})$ are empty and positions $(j+1, h_1), \cdots, (j+1, h_{a+2})$ have beads. Let h be an integer such that all positions (x, y) in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ are occupied by beads, where $1 \leq x \leq r$ and $y \leq h$. In the light of Lemma 3.1.9, $\mathfrak{n}^h_{j+1} - \mathfrak{n}^h_j = a + 2$. Note that $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ is complete. This implies that if position (j, k) is empty and position $(j+1, k)$ has a bead, then $k \in \{h_1, \cdots, h_{a+2}\}$. Because $w(B_{\boldsymbol{\lambda},\mathbf{s}}) = 1$, this forces each abacus of a pair in $B_{\boldsymbol{\lambda},\mathbf{s}}$ has to be obtained by sliding in $L_{\mathbf{s}^*}(\boldsymbol{\lambda}^*)$ the bead at position $(j+1, k)$ to position (j, k) with $k \in \{h_1, \cdots, h_{a+2}\}$. The case $j = r$ can be transformed into the case $j < r$ by using Lemma 3.4.12.

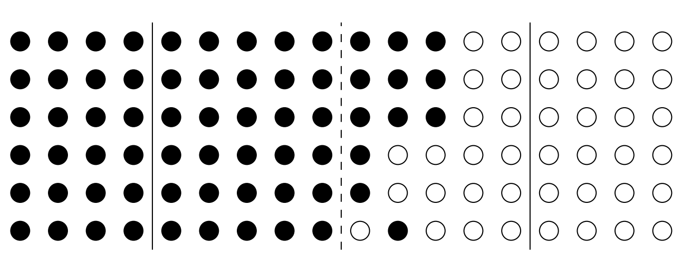
Combining [24, Theorem 4.12] with [7, Theorem 6.8] gives that $B_{\boldsymbol{\lambda},\mathbf{s}}$ is Morita equivalent to a Brauer tree algebra, whose Brauer tree is a straight line with $a+1$ edges and without exceptional vertex.

Moreover, by Lemma 3.1.12 (3) we have $e \nmid h_x - h_y$ for all $x, y \in \{1, 2, \cdots, a+2\}$. This implies that for arbitrary pair $(\boldsymbol{\lambda}, \mathbf{s}) \in B_{\boldsymbol{\lambda},\mathbf{s}}$, columns $h_1, \cdots, h_{a+2}$ are in different subabaci of $L_{\mathbf{s}}(\boldsymbol{\lambda})$, and consequently, the number of non-zero components of $\mathcal{W}(B_{\boldsymbol{\lambda},\mathbf{s}})$ is $a+2$.

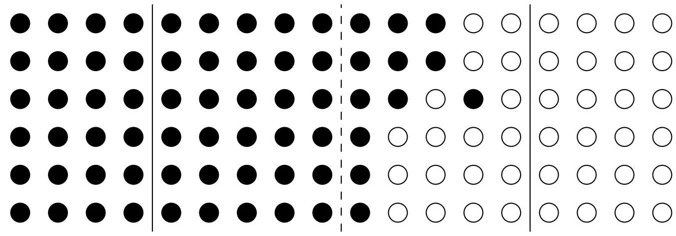
By the same reason, block $B_{\boldsymbol{\mu},\mathbf{u}}$ is also Morita equivalent to a Brauer tree algebra, whose Brauer tree is a straight line without exceptional vertex. According to [48, Theorem 4.2], derived equivalence classes of Brauer tree algebras are determined by the number of the edges and the multiplicity of the exceptional vertex. Then the proposition follows. □

Now let us give some examples. The first one is from the blocks Morita equivalent to truncated polynomial rings.

Example 7.0.2. *Let $e = 5$ and $\mathbf{s} = (1, 1, 1, 3, 3, 3)$. Take $\boldsymbol{\lambda} = ((1), \varnothing, \varnothing, \varnothing, \varnothing, \varnothing)$ and $\boldsymbol{\mu} = (\varnothing, \varnothing, \varnothing, (1), \varnothing, \varnothing)$. Then abacus $L_{\mathbf{s}}(\boldsymbol{\lambda})$ is*



and abacus $L_{\mathbf{s}}(\boldsymbol{\mu})$ is

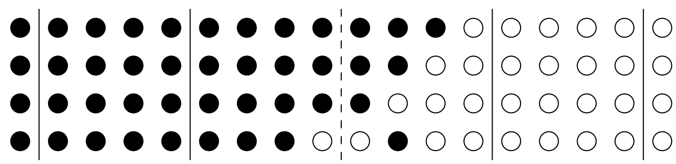


It is easy to check that $\mathcal{M}(B_{\boldsymbol{\lambda},\mathbf{s}}) = (1,1,0,0,0,0)$ *and* $\mathcal{M}(B_{\boldsymbol{\mu},\mathbf{s}}) = (0,0,0,1,1,0)$. *According to the result obtained in Subsection 6.2, both* $B_{\boldsymbol{\lambda},\mathbf{s}}$ *and* $B_{\boldsymbol{\mu},\mathbf{s}}$ *are isomorphic to* $K[x]/(x^3)$. *On the other hand, all abaci in block* $B_{\boldsymbol{\mu},\mathbf{s}}$ *can be determined completely by Subsection 6.2. In the accordance with the definition of the action of an affine Weyl group on abaci, non of the abaci in* $B_{\boldsymbol{\mu},\mathbf{s}}$ *is belong to the image of* $L_{\mathbf{s}}(\boldsymbol{\lambda})$. *This implies by Proposition 3.2.6 that blocks* $B_{\boldsymbol{\lambda},\mathbf{s}}$ *and* $B_{\boldsymbol{\mu},\mathbf{s}}$ *are in different orbits.*

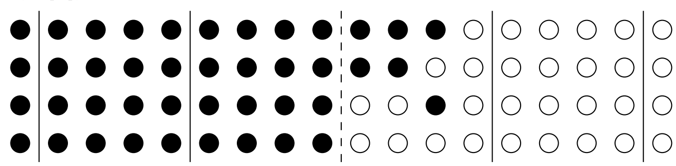
Another example is from the blocks of weight one.

Example 7.0.3. *For arbitrary* $r \geq 3$, *let* $e = r$ *and* $\mathbf{s} = (0,1,2,\cdots,r-1)$. *For* $1 \leq i \leq r$, *define* $\boldsymbol{\lambda}[i] = (\underbrace{\varnothing,\cdots,\varnothing}_{i-1},(2),\underbrace{\varnothing,\cdots,\varnothing}_{r-i})$. *Clearly,* $\mathcal{M}(B_{\boldsymbol{\lambda}[i],\mathbf{s}}) = (m_1,m_2,\cdots,m_r)$, *where* $m_i = 1$ *and* $m_j = 0$ *if* $j \neq i$. *It follows from Proposition 7.0.1 that all blocks* $B_{\boldsymbol{\lambda}[i],\mathbf{s}}$ *are derived equivalent, and that all abaci in* $B_{\boldsymbol{\lambda}[i],\mathbf{s}}$ *can be listed completely. By Proposition 3.2.6, all blocks* $B_{\boldsymbol{\lambda}[i],\mathbf{s}}$ *are in different orbits.*

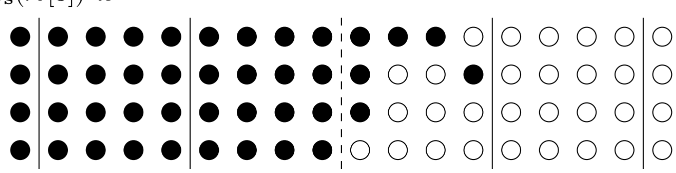
For example, let $r = 4$. *Then* $L_{\mathbf{s}}(\boldsymbol{\lambda}[1])$ *is*



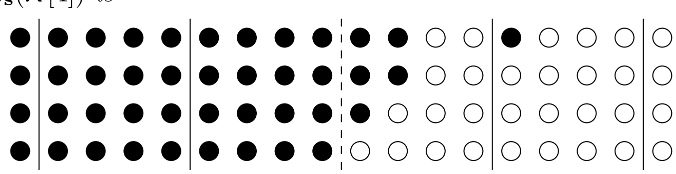
Abacus $L_{\mathbf{s}}(\boldsymbol{\lambda}[2])$ *is*



Abacus $L_{\mathbf{s}}(\boldsymbol{\lambda}[3])$ *is*



Abacus $L_{\mathbf{s}}(\boldsymbol{\lambda}[4])$ *is*



The last example is a derived equivalent class of infinite representation type blocks.

Example 7.0.4. *Let $k \geq 2$ be an integer. Take $r = e = 3k$ and $\mathbf{s} = (1, 1, 2, \cdots,)$. For $1 \leq i \leq k$, define $\boldsymbol{\lambda}[i] = (\underbrace{\varnothing, \cdots, \varnothing}_{3i-1}, (2), \underbrace{\varnothing, \cdots, \varnothing}_{r-3i})$. It is easy to check $B_{\boldsymbol{\lambda},\mathbf{s}}$ is of infinite representation type. Moreover, all blocks $B_{\boldsymbol{\lambda},\mathbf{s}}$, $1 \leq i \leq k$, are derived equivalent and in different orbits.*

8. Blocks of cyclotomic q-Schur algebra

We end our paper by a remark on representation type of blocks of a cyclotomic q-Schur algebra.

Combining Theorem A and Theorem B in [5] gives that a block of a Hecke algebra of type B has finite representation type if and only if its weight is not more than one. This implies that the representation type of a block of the cyclotomic q-Schur algebra associated to a type B Hecke algebra is finite if and only if its weight is not more than one. On the other hand, in [50, Corollary 3.20], Wada proved that any block of $\mathcal{S}_{n,r}(q, Q_1, Q_2, \cdots, Q_r)$ is Morita equivalent to a certain block of $\mathcal{S}_{n',2}(q, Q_i, Q_j)$ for some $i, j \in \{1, 2, \cdots, r\}$. Therefore, the representation type of each block of a cyclotomic q-Schur algebra can be determined in this sense. By using the Main Theorem, we can give a direct result without using the Morita equivalence mentioned above.

We point out that if a block B of $\mathcal{H}_n(q, Q)$ is Morita equivalent to a truncated polynomial ring, then the Auslander algebra (see [13] for definition) of B is just the corresponding block of $\mathcal{S}_{n,r}$. It is well-known that the Auslander algebra of $K[x]/(x^i)$ has finite representation type if and only if $0 < i < 4$. Let $\mathcal{B}$ be a block of $\mathcal{S}_{n,r}$ and B the corresponding block in $\mathcal{H}_n(q, Q)$. Then by Main Theorem, we get

(1) If $w(\mathcal{B}) > 2$, then $\mathcal{B}$ has infinite representation type.
(2) If $w(\mathcal{B}) < 2$, then $\mathcal{B}$ has finite representation type.
(3) If $w(\mathcal{B}) = 2$, then the representation type of $\mathcal{B}$ is the same as that of B.

According to the above result, $w(\mathcal{B}) = 1$ is only a sufficient condition for $\mathcal{B}$ being of finite representation type (non simple).

Acknowledgement The authors are grateful to Prof. Wei Hu for some helpful conversations. Most of this work was done when Li visited Institute of Algebra and Number Theory at University of Stuttgart from August 2021 to September 2022. He takes this opportunity to express his sincere thanks to the institute and Prof. S. Koenig for the hospitality during the visit.

References

[1] J. Alperin, Local representation theory Cambridge University Press. Cambridge, 1986.

[2] S. Ariki, *On the decomposition numbers of the Hecke algebra of $G(m,1,n)$*, J. Math. Kyoto Univ. **36** (1996) 789-808.

[3] S. Ariki, *On the classification of simple modules for cyclotomic Hecke algebras of type $G(r,1,n)$ and Kleshchev multi-partitions*, Osaka J. Math. **38** (2001), 827-837.

[4] S. Ariki, *Hecke algebras of classical type and their representation type*, Proc. Lond. Math. Soc. **91** (2005) 355-413, Corrigendum: **92** (2006) 342-344.

[5] S. Ariki, *Representation type for block algebras of Hecke algebras of classical type*, Adv. Math. **317** (2017), 823-845.

[6] S. Ariki, K. Iijima, E. Park, *Representation type of finite quiver Hecke algebras of type $A_l^{(1)}$ for arbitrary parameters*, Int. Math. Res. Not. IMRN **2015** (2015) 6070-6135.

[7] S. Ariki, R. Kase, K. Miyamoto, K. Wada, *Self-injective cellular algebras of polynomial growth representation type*, Algebras and Representation Theory **23** (2020) 833-871.

[8] S. Ariki and K. Koike, *A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations*, Adv. Math. **106** (1994), 216-243.

[9] S. Ariki, A. Mathas, *The representation type of Hecke algebras of type B*, Adv. Math. **181** (2004) 134-159, Corrigendum: **192** (2005) 228-230.

[10] S. Ariki, E. Park, *Representation type of finite quiver Hecke algebras of type $A_{2l}^{(2)}$*, J. Algebra **397** (2014) 457-488.

[11] S. Ariki, E. Park, *Representation type of finite quiver Hecke algebras of type $D_{l+1}^{(2)}$*, Trans. Amer. Math. Soc. **368** (2016) 3211-3242.

[12] S. Ariki, E. Park, *Representation type of finite quiver Hecke algebras of type $C_l^{(1)}$*, Osaka J. Math. **53** (2016) 463-489.

[13] M. Auslander, I. Reiten and S. O. Smalo, Representation Theory of Artin Algebras. Cambridge University Press, Cambridge, 1995.

[14] M. Broué and G. Malle, *Zyklotomische Heckealgebren*, Astérisque **212** (1993), 119-189.

[15] J. Brundan and A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math. **178** (2009) 451-484.

[16] J. Brundan, A. Kleshchev, and W. Wang, *Graded Specht modules*, J. reine angew. Math. **655** (2011) 61-87.

[17] I.V. Cherednik, *A new interpretation of Gelfand-Tzetlin bases*, Duke Math. J., **54** (1987) 563-577.

[18] J. Chuang and R. Rouquier, *Derived equivalences for symmetric groups and sl2-categorification*, Ann. of Math. (2) 167 (2008) 245-298.

[19] R. Dipper, G. James and A. Mathas, *Cyclotomic q-Schur algebras,* Math. Z. **229** (1999), 385-416.

[20] R. Dipper and A. Mathas, *Morita equivalences of Ariki-Koike algebras*, Math. Z. **240** (2002) 579-610.

[21] Yu. A. Drozd, Tame and wild matrix problems, representations and quadratic forms Institute of Mathematics, Academy of Sciences, Ukrainian SSR, Kiev 1979. A.M.S Transl. 128 (1986), 31-55.

[22] K. Erdmann, Blocks of tame representation type and related algebras, Lecture Notes in Mathematics, vol.1428, Springer-Verlag, 1990.

[23] K. Erdmann, D.K. Nakano, *Representation type of Hecke algebras of type A*, Trans. Amer. Math. Soc. **354** (2002) 275-285.

[24] M. Fayers, *Weights of multipartitions and representations of Ariki-Koike algebras*, Adv. Math. **206** (2008), 112-144.

[25] M. Fayers, *Weights of multipartitions and representations of Ariki-Koike algebras II: canonical bases*, J. Algebra **319** (2008), 2963-78.

[26] M. Geck, *Brauer trees of Hecke algebras*, Comm. Algebra **20** (1992) 2937-2973.

[27] M. Geck, *Hecke algebras of finite type are cellular*, Invent. Math. **169** (2007) 501-517.

[28] J. Graham and G. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996), 1-34.

[29] Jun Hu, *On a generalization of the Dipper–James–Murphy conjecture*, Journal of Combinatorial Theory, Series A **118** (2011) 78-93.

[30] J. Hu and A. Mathas, *Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type A*, Adv. Math. **225** (2010), 598-642.

[31] Jun Hu and Lei Shi, *Proof of the Center Conjectures for the cyclotomic Hecke and KLR algebras of type A*, arxiv: 2211.07069.

[32] N. Jacon *Kleshchev multipartitions and extended Young diagrams*, arxiv: 1706.07595

[33] N. Jacon and C. Lecouvey, *Cores of Ariki-Koike algebras*, arxiv: 1912.08461

[34] G. James, Some combinatorial results involving Young diagrams, Proc. Cambridge Philos. Soc. 83 (1978) 1-10.

[35] V. G. Kac, *Infinite dimensional Lie algebras*, CUP, Cambridge, third ed., 1994.

[36] S. Koenig and C. C. Xi, *On the structure of cellular algebras*, In: I.Reiten, S.Smalø and Ø. Solberg (Eds.): Algebras and Modules II. Canadian Mathematical Society Conference Proc. **24** (1998), 365-386.

[37] S. Koenig and C.C. Xi, *Cellular algebras: Inflations and Morita equivalences*, J. London Math. Soc. (2) **60** (1999), 700-722.

[38] S. Koenig and A. Zimmermann, Derived equivalences for group rings, Lecture Notes in Mathematics 1685 Springer 1998

[39] H. Krause, *Stable equivalence preserves representation type*, Comment. Math. Helv. **72** (1997) 266-284.

[40] S. Lyle, A. Mathas, *Blocks of cyclotomic Hecke algebras*, Adv. Math. **216** (2007) 854-878.

[41] Yanbo Li, *Centers of symmetric cellular algebras*, Bull. Aust. Math. Soc. **82** (2010), 511-522.

[42] Y. Li, *Radicals of symmetric cellular algebras*, Colloq. Math. **133** (2013), 67-83.

[43] Y. Li and D. Zhao, *On graded symmetric cellular algebras*, J. Aust. Math. Soc. **108** (2020), 349-362.

[44] S. Lyle and A. Mathas, *Blocks of cyclotomic Hecke algebras*, Adv. Math. **216** (2007), 854-878.

[45] G. Malle and A. Mathas, *Symmetric cyclotomic Hecke algebras*, J. Algebra **205** (1998), 275-293.

[46] A. Mathas, *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, University Lecture Series 15, Amer. Math. Soc. (1999)

[47] G. Murphy, *The representations of Hecke algebras of type A_n*, J. Algebra **173** (1995), 97-121.

[48] J. Rickard, *Derived categories and stable equivalence*, J. Pure. Appl. Alg. **61** (1989), 303-317.

[49] D. Uglov, Canonical bases of higher level q-deformed Fock spaces and Kazhdan-Lusztig polynomials, in Physial Combinatorics (ed. M. Kashiwara, T. Miwa), Progress in Math. 191, Birkhauser (2000)

[50] K. Wada, *The representation type of Ariki-Koike algebras and cyclotomic q-Schur algebras*, Adv. Math. **224** (2010) 539-560.

Li: School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Qinhuangdao, 066004, P.R. China

Email address: liyanbo707@163.com

Qi: Department of mathematics, School of Science, Northeastern University, Shenyang, 110819, P.R. China

Email address: 2974042081@qq.com