CATEGORIFICATIONS OF NON-INTEGER QUIVERS: TYPE $I_2(2n)$

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ABSTRACT. We use weighted unfoldings of quivers to provide a categorification of mutations of quivers of types $I_2(2n)$, thus extending the construction of categorifications of mutations of quivers to all finite types.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

This is the second paper in a series started in [7].

In [7], we constructed a categorification of mutations of non-integer quivers of finite types H_4, H_3 and $I_2(2n+1)$. The main tool in the construction is a *weighted (un)folding* of quivers of types E_8, D_6 and A_{2n}, the application of which follows the projection of root systems developed in [15, 22, 16]. The (un)foldings induce projections of dimension vectors of objects in module categories of integer quivers to the roots associated to folded quivers; they also induce semiring actions on categories associated to integer quivers. We also define the tropical seed patterns of folded quivers and show that our definition is consistent with both the folding and the categorical definition of g-vectors.

In this paper, we extend the results of [7] to the last remaining finite type of quivers, $I_2(2n)$, thus completing the construction of categorifications of mutations for quivers of all finite types. Unlike the settings of [7], there is more than one possible folding that could be considered for every quiver of type $I_2(2n)$. More precisely, a quiver of type $I_2(2n)$ admits an unfolding to a quiver of type A_{2n-1} and to a quiver of type D_{n+1}. Moreover, there are three exceptional foldings $E_6 \to I_2(12)$, $E_7 \to I_2(18)$, and $E_8 \to I_2(30)$. We consider all of these different possible foldings to construct categorifications via semiring actions on the categories associated to different integer quivers, and we see that aside from the semiring action itself, the theory between these different foldings is very much the same. The key upshot of this is that the module, bounded derived, and cluster categories associated to any bipartite Dynkin quiver has a non-crystallographic interpretation via a semiring action and the associated non-integer quiver. This includes the ability to associate non-integer g-vectors and mutation to the categories of any bipartite Dynkin quiver.

Another difference to the setting of [7] is that for the foldings considered in this paper, it is more natural to consider the root system of $I_2(2n)$ in such a way that it contains roots of two different lengths. We therefore use non-trivial rescaling throughout the paper. This approach to the root system generalises how one classically considers the crystallographic root systems $I_2(4) = B_2$ and $I_2(6) = G_2$ to higher n. It is entirely possible to categorify the root system of $I_2(2n)$ via an unfolding with a trivial rescaling (and thus with all roots of $I_2(2n)$ of the same length) and a projection map from the categories associated to the unfolding. However, when it comes to considering the semiring action on the cluster category of an unfolded Dynkin quiver, the g-vectors one obtains are precisely those that are associated to the non-trivial rescaling, and thus we consider this approach to be the most fitting.

We recall all necessary definitions and details of the construction from [7] in Section 2.

Our first main result is Theorem 3.5 (see also Corollaries 3.6 and 3.7), in which we prove that, given a weighted folding $F\colon Q^\Delta \to Q^{I_2(2n)}$, there is a weighting on the rows of the Auslander-Reiten quiver of $\operatorname{mod} KQ^\Delta$ and $\mathcal{D}^b(\operatorname{mod} KQ^\Delta)$ consistent with F, with the projection of the dimension vectors of objects, and with the projection of the corresponding root systems. Objects in the rows of weight 1 in the Auslander-Reiten quiver map precisely to the roots of $I_2(2n)$, thus generalising Gabriel's Theorem for quivers of type $I_2(2n)$. In fact, Theorem 3.5 says much more than this. The folding F has a strong relationship with the structure of the Auslander-Reiten quiver. For example, objects that reside in the same column of the Auslander-Reiten quiver project onto a multiple of the same root, and these multiples are determined by the weights of the rows. Moreover, Auslander-Reiten translation acts by rotation with respect to the projection.

The projection map induced by the folding gives rise to a semiring action on the module and bounded derived categories of the integer quivers, where the semiring (we denote it by R_+) is defined separately for each folding (Section 2.4) using Chebyshev polynomials. The main results here (Theorem 4.7 and Corollary 4.8) state that $\operatorname{mod} KQ^\Delta$ has an action of the corresponding semiring, and there exists a collection of indecomposable objects of $\operatorname{mod} KQ^\Delta$ that are in bijection with the positive roots of $I_2(2n)$, and these objects generate the whole category under the action of the semiring. These results naturally extend to the bounded derived category $\mathcal{D}^b(\operatorname{mod} KQ^\Delta)$.

The semiring action also applies to the cluster category of Q^Δ, and this allows us to extend the results of [2] to categorify mutations of folded quivers by considering distinguished objects with respect to the action, which we call R_+-*tilting objects* (Section 5). We prove (Theorem 5.2 and Corollary 5.3) that for a folding $F\colon Q^\Delta \to Q^{I_2(2n)}$, the following hold for the cluster category $\mathcal{C}_\Delta$ of Q^Δ: every basic R_+-tilting object $T \in \mathcal{C}_\Delta$ (injectively) corresponds to a basic tilting object $\widehat{T} \in \mathcal{C}_\Delta$; every basic R_+-tilting object has precisely 2 indecomposable direct summands; every almost complete R_+-tilting object has exactly two complements, and changing the complement corresponds to a single mutation; if T is a basic R_+-tilting object, and A_T is the cluster-tilted algebra corresponding to $\widehat{T}$, then there exists an R_+-action on $\operatorname{mod} A_T$.

Finally, we define the tropical seed patterns of folded quivers. We define c-vectors and C-matrices as in the integer case, and we show that c-vectors of $Q^{I_2(2n)}$ are roots of $I_2(2n)$ and are thus sign-coherent (Corollary 6.12). Going along the results of [19], this allows us to define G-matrices as the inverse of transposed C-matrices, and we define g-vectors as the column vectors of G-matrices. We then prove (see Section 6 and Theorem 6.15 for details) that these definitions are compatible with both rescaling and the projection of C-matrices and G-matrices with respect to the folding. We note that these results provide a categorical interpretation of g-vectors and G-matrices of $Q^{I_2(2n)}$ (Corollary 6.17).

The paper is organised as follows. In Section 2 we recall all essential definitions and details from [7] about mutations, unfoldings, and semirings. In Section 3, we describe the projections of module and bounded derived categories induced by the (un)foldings of quivers. Section 4 is devoted to the description of the semiring action on the module and bounded derived categories. In Section 5, we extend the semiring action to the cluster categories of the unfolded quivers, thus providing a categorification of mutations of the folded quivers. Section 6 is devoted to the construction of the tropical seed pattern, and to the compatibility of the projections and mutations. Finally, Appendix A provides two detailed worked examples (one for unfoldings of type A_{2n-1} and one for unfoldings of type D_{n+1}) that showcases the entire theory of the paper for the benefit of the reader.

2. Preliminaries

For the benefit of the reader, we will briefly recall some notation and definitions from [7] that we will use throughout the paper. Further details can be found in the aforementioned reference.

2.1. **General setup and notation.** Throughout the paper, K is an algebraically closed field and Q^Δ is a quiver of Dynkin type Δ. The vertex set of Q^Δ is denoted by Q_0^Δ and the arrow set is denoted by Q_1^Δ. We denote by KQ^Δ the path algebra of Q^Δ over the field K. All KQ^Δ-modules in this paper are right modules, and thus we read paths in the quiver from left to right. We denote by $\operatorname{mod} KQ^\Delta$ the category of finitely generated right KQ^Δ modules and by $\mathcal{D}^b(KQ^\Delta)$ the bounded derived category.

Each vertex $i \in Q_0^\Delta$ simultaneously corresponds to a simple, indecomposable projective, and indecomposable injective KQ^Δ-module, which we will denote by $S(i)$, $P(i)$ and $I(i)$ respectively. The shift functor of $\mathcal{D}^b(KQ^\Delta)$ shall be written as $\Sigma \colon \mathcal{D}^b(KQ^\Delta) \to \mathcal{D}^b(KQ^\Delta)$, and where appropriate, we will adopt the abuse of notation where for each object $M \in \operatorname{mod} KQ^\Delta$, the object $M \in \mathcal{D}^b(KQ^\Delta)$ is the corresponding object concentrated in degree 0. Additionally, we denote by $\tau_\mathcal{A}$ the Auslander-Reiten translate in the category $\mathcal{A}$, where $\mathcal{A}$ is either a module, bounded derived, or cluster category of the appropriate type. Whenever the context is clear, we will omit this subscript and simply write τ.

2.2. **Exchange matrices over a ring, R-quivers, and the $I_2(2n)$ root system.** Let R be a totally ordered ring throughout. By an *exchange matrix* over R, we mean a square skew-symmetrisable matrix with entries in R whose rows and columns are indexed by a set Q_0^B. The R-*quiver* associated to a *skew-symmetric* exchange matrix $B = (b_{ij})$ over R is the R-arrow-weighted quiver Q^B with vertex set Q_0^B, arrow set Q_1^B, and arrow weighting function $\xi \colon Q_1^B \to R_{>0}$ determined such that $b_{ij} > 0$ if and only if there exists an arrow $a \colon i \to j$ in Q_1^B with weight $\xi(a) = b_{ij}$.

This paper will be working with a particular family of R-quivers and exchange matrices over R. Namely, we will be working with exchange matrices of type $I_2(2n)$ over the ring $R = \mathbb{Z}[2\cos\frac{\pi}{2n}]$, whose corresponding R-quiver we will denote by

$$Q^{I_2(2n)} \colon \quad [0] \xrightarrow{\;2\cos\frac{\pi}{2n}\;} [1]$$
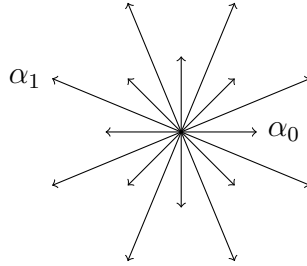
where the weight of the unique arrow is given by its label.

FIGURE 2.1. The standard root system of type $I_2(8)$ corresponding to the quiver $Q^{I_2(2n)}$, with simple positive roots labelled by α_0 and α_1.

The quiver $Q^{I_2(2n)}$ is associated to a *non-crystallographic root system* that is related to the Coxeter group $I_2(2n)$ in the sense of [4]. The non-crystallographic root system of type $I_2(2n)$ has $2n$ positive roots, two of which are simple. Being non-crystallographic, the root system can appear in various forms which depends on the relative lengths of the simple roots. This paper will work with two versions of the root system.

Version 1: The standard form. In this version, the positive roots may be partitioned into two equally sized classes, called *short* and *long* positive roots. In particular, one of the simple positive roots will be longer than the other. The short roots of $I_2(2n)$ will be of length λ and the long roots of $I_2(2n)$ will be of length $2\lambda \cos \frac{\pi}{2n}$. For the quiver $Q^{I_2(2n)}$ above, we will choose the convention that the short positive roots correspond to the points $\lambda \exp(\frac{k\pi i}{n})$ and the long positive roots correspond to the points $(2\lambda \cos \frac{\pi}{2n}) \exp(\frac{(2k+1)\pi i}{2n})$ for $0 \leq k \leq n-1$. For the opposite quiver $(Q^{I_2(2n)})^{\mathrm{op}}$, this convention is reversed. See Figure 2.1 for an example. Throughout the paper, we will denote the set of short roots (resp. short positive roots) of $I_2(2n)$ by Φ_{2n}^s (resp. $\Phi_{2n}^{s,+}$) and we will denote the set of long roots (resp. long positive roots) of $I_2(2n)$ by Φ_{2n}^l (resp. $\Phi_{2n}^{l,+}$).

We call this version of the root system of $I_2(2n)$ standard, as it agrees with the root systems $B_2 = I_2(4)$ and $G_2 = I_2(6)$, and is the one that is most compatible with the theory of semiring actions on module categories that we will develop.

Version 2: The rescaled form. In this version the positive roots all have the same length, which by convention, we choose to be 1. The positive roots thus correspond to the points $\lambda \exp(\frac{k\pi i}{2n})$ with $0 \leq k \leq 2n-1$. We call this the *rescaled root system of $I_2(2n)$*. We only use the rescaled root system in Section 6.

To make precise which version of the root system we are using when considering the R-quiver $Q^{I_2(2n)}$, we will define a *valuation* on the quiver.

Definition 2.1. Let S be a totally ordered ring. An *S-valuation* of an R-quiver is a positive S-vertex-weighting $\varsigma \colon Q_0^B \to S_{>0}$.

The standard form of the root system of type $I_2(2n)$ is associated to the R-valuation of $Q^{I_2(2n)}$ given by $\varsigma([0]) = \lambda$ and $\varsigma([1]) = 2\lambda \cos \frac{\pi}{2n}$. The rescaled version of $I_2(2n)$ is instead given by the R-valuation $\varsigma([0]) = \varsigma([1]) = \lambda$. In all but one case in this paper, $\lambda = 1$ (where $\lambda = 2$ in the other case).

Henceforth, whenever we refer to the root system of $I_2(2n)$ and its roots, we mean the standard form (version 1) and its elements. We will always refer to version 2 of $I_2(2n)$ as the *rescaled root system* and its elements as *rescaled roots*.

2.3. **Weighted (un)foldings.** Here we will recall the definition of a weighted (un)folding from [7]. The definition is a generalisation of the classical definition of unfolding due to Zelevinsky (see [8, 9] for details) and makes use of rescaling introduced by Reading in [21].

Definition 2.2. Let $B = (b_{ij})$ be an exchange matrix over $\mathbb{Z}$ and B' be an exchange matrix over R. Suppose there exists a disjoint collection of index sets $\{E_{[i]} : [i] \in Q_0^{B'}\}$ such that $\bigcup_{[i] \in Q_0^{B'}} E_{[i]} = Q_0^B$. Then B has the structure of a block matrix $(B_{[i][j]})$ with blocks indexed by $Q_0^{B'}$. In this case, we call a pair (B, B') of exchange matrices an *origami pair* if the following hold:

(O1) For each $[i], [j] \in Q_0^{B'}$, the sum of entries in each column of $B_{[i][j]}$ is $b'_{[i][j]}$.

(O2) If $b'_{[i][j]} > 0$ then the $B_{[i][j]}$ has all entries non-negative.

Suppose R is an integral domain and let $\mathrm{Frac}(R)$ be the field of fractions of R. An exchange matrix B over $\mathbb{Z}$ is said to be a *weighted unfolding* of an exchange matrix B' over R if the following hold.

(U1) There exist diagonal matrices $W = (w_i)$ and $P = (p_{[i]})$ with positive entries in $R \subseteq \mathrm{Frac}(R)$ such that $(WBW^{-1}, PB'P^{-1})$ is an origami pair with blocks indexed by the collection $\{E_{[i]} : [i] \in Q_0^{B'}\}$. Here, multiplication occurs over $\mathrm{Frac}(R)$.

(U2) For any sequence of iterated mutations $\mu_{[k_1]} \ldots \mu_{[k_l]}$ of B', the pair

$$(W\widehat{\mu}_{[k_1]} \ldots \widehat{\mu}_{[k_l]}(B)W^{-1}, P\mu_{[k_1]} \ldots \mu_{[k_l]}(B')P^{-1})$$

is origami, where each $\widehat{\mu}_{[j]}$ is the *composite mutation* of B given by

$$\widehat{\mu}_{[j]} = \prod_{i \in E_{[j]}} \mu_i.$$

The composite mutations of B in the above definition are well-defined because the mutations indexed by each block are pairwise commutative. We call the matrix W the *weight matrix* of the unfolding (whose entries w_i we call *weights*), and we call P the *rescaling matrix* of the unfolding. We also call the exchange matrix B' the *folded exchange matrix* of B, and we call B the *unfolded exchange matrix* of B'.

A weighted unfolding associated to an origami pair $(WBW^{-1}, PB'P^{-1})$ equips both the associated $\mathbb{Z}$-quiver Q^B and the R-quiver $Q^{B'}$ with an R-valuation. For the $\mathbb{Z}$-quiver, we denote the valuation with the function $\kappa \colon Q_0^B \to R_{>0}$ defined by $\kappa(i) = w_i$ for each $i \in Q_0^B$. For the R-quiver $Q^{B'}$, we denote the valuation with the function $\varsigma \colon Q_0^{B'} \to R_{>0}$ defined by $\varsigma([i]) = p_{[i]}$. To each weighted unfolding B of B', we define a *weighted folding* of quivers $F \colon Q^B \to Q^{B'}$. That is, F is a surjective morphism of quivers such that $F(i) = [j]$ whenever $i \in E_{[j]}$. Given such a folding F, we call Q^B the *unfolded quiver* and $Q^{B'}$ the *folded quiver*. In any given folding, the quivers Q^B and $Q^{B'}$ are implicitly assumed to have the aforementioned structure of R-valued quivers (with weight function κ for Q^B and ς for $Q^{B'}$).

This paper is concerned with three families of foldings onto R-quivers of type $I_2(2n)$. We call these families foldings of type A, D and E, respectively. The foldings of type A in particular are consistent with similar constructions in [22, 17].

2.3.1. *Foldings of type A.* The first family to consider is the folding from a bipartite quiver of type A_{2n-1}, which we denote by $Q^{A_{2n-1}}$. Specifically, we have $F^{A_{2n-1}} \colon Q^{A_{2n-1}} \to Q^{I_2(2n)}$, where $Q^{A_{2n-1}}$ is given by

$$0 \longrightarrow 1 \longleftarrow 2 \longrightarrow \cdots \longleftarrow 2n-4 \longrightarrow 2n-3 \longleftarrow 2n-2$$

with vertex weights such that $\kappa(i) = U_i(\cos\frac{\pi}{2n})$, where U_i is the i-th Chebyshev polynomial of the second kind. Furthermore, each vertex i is such that $F(i) = [0]$ if i is even and $F(i) = [1]$ if i is odd. The valuation of $Q^{I_2(2n)}$ is such that $\varsigma([0]) = 1$ and $\varsigma([1]) = 2\cos\frac{\pi}{2n}$. It is not difficult to verify that on the level of unfoldings, this class of unfoldings factors through an unfolding of an exchange matrix of Dynkin type C_n. The above quiver has

$\mathbb{Z}_2$-symmetry via the action that maps $i \in Q_0^{A_{2n-1}}$ to $2n - 2 - i \in Q_0^{A_{2n-1}}$. This action also respects the vertex-weights in the sense that $U_i(\cos \frac{\pi}{2n}) = U_{2n-2-i}(\cos \frac{\pi}{2n})$. In light of this group action, it is sometimes convenient to work with the alternative labelling

$$0^+ \longrightarrow 1^+ \longleftarrow \cdots \longrightarrow (n-2)^+ \longrightarrow n-1 \longrightarrow (n-2)^- \longrightarrow \cdots \longrightarrow 1^- \longleftarrow 0^- \, .$$

2.3.2. *Foldings of type D.* The second family of foldings we consider are from bipartite quivers of type D_{n+1}, which we denote by $Q^{D_{n+1}}$. Specifically, we have $F^{D_{n+1}} : Q^{D_{n+1}} \to Q^{I_2(2n)}$, where $Q^{D_{n+1}}$ is given by

$$0 \longrightarrow 1 \longleftarrow 2 \longrightarrow \cdots \longrightarrow n-2 \begin{array}{c} \nearrow (n-1)^+ \\ \\ \searrow (n-1)^- \end{array}$$

with vertex weights such that $\kappa(i) = 2U_i(\cos \frac{\pi}{2n})$ for $0 \leq i \leq n - 2$ and $\kappa((n-1)^\pm) = U_{n-1}(\cos \frac{\pi}{2n})$. Furthermore, each vertex i is such that $F(i) = [0]$ if i is even and $F(i) = [1]$ if i is odd (the same rule applies to the vertices $(n-1)^\pm$ with the integer $n - 1$). The valuation of $Q^{I_2(2n)}$ is such that $\varsigma([0]) = 2$ and $\varsigma([1]) = 4 \cos \frac{\pi}{2n}$. It is not difficult to verify that on the level of unfoldings, this class of unfoldings factors through an unfolding of an exchange matrix of Dynkin type B_n. This quiver also has $\mathbb{Z}_2$-symmetry via the action that fixes each $i \in Q_0^{D_{n+1}}$ and maps $(n+1)^\pm \in Q_0^{D_{n+1}}$ to $(n+1)^\mp \in Q_0^{D_{n+1}}$.

2.3.3. *Foldings of type E.* There are three exceptional foldings of type E. Namely we have foldings

$$F^{E_6} : Q^{E_6} \to Q^{I_2(12)},$$
$$F^{E_7} : Q^{E_7} \to Q^{I_2(18)},$$
$$F^{E_8} : Q^{E_8} \to Q^{I_2(30)},$$

where Q^{E_6}, Q^{E_7} and Q^{E_8} are bipartite R-valued quivers of type E_6, E_7 and E_8, respectively. Throughout, the valuation of $Q^{I_2(2n)}$ is such that $\varsigma([0]) = 1$ and $\varsigma([1]) = 2 \cos \frac{\pi}{2n}$.

The quiver Q^{E_6} is given by

$$\begin{array}{c} v_6 \\ \uparrow \\ 0^+ \longrightarrow 1^+ \longleftarrow 2 \longrightarrow 1^- \longleftarrow 0^- \end{array}$$

with weighting such that $\kappa(i^\pm) = U_i(\cos \frac{\pi}{12})$, $\kappa(2) = U_2(\cos \frac{\pi}{12})$ and $\kappa(v_6) = \sqrt{2}$. Moreover, we have $F^{E_6}(0^\pm) = F^{E_6}(2) = [0]$ and $F^{E_6}(1^\pm) = F^{E_6}(v_6) = [1]$. This unfolding can be shown to factor through an unfolding of an exchange matrix of Dynkin type F_4.

The quiver Q^{E_7} is given by

$$\begin{array}{c} \widetilde{1} \\ \downarrow \\ 0 \longrightarrow 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow \widetilde{2} \longrightarrow v_7 \end{array}$$

with weighting such that $\kappa(i) = U_i(\cos \frac{\pi}{18})$, $\kappa(\widetilde{i}) = U_i(\cos \frac{\pi}{9})$ and $\kappa(v_7) = 2 \cos \frac{5\pi}{18}$. It worth noting that $U_0(\cos \frac{\pi}{9}) = U_0(\cos \frac{\pi}{18})$ and $U_3(\cos \frac{\pi}{9}) = U_2(\cos \frac{\pi}{18})$. It is therefore

convenient to occasionally adopt the labelling $\widetilde{0} = 0$ and $\widetilde{3} = 2$. Moreover, F^{E_7} maps the source vertices to $[0]$ and sink vertices to $[1]$.

The quiver Q^{E_8} is given by

$$v_8$$
$$\uparrow$$
$$0 \longrightarrow 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow \phi_1 \longleftarrow \phi_0$$

with weighting such that $\kappa(i) = U_i(\cos\frac{\pi}{30})$, $\kappa(\phi_i) = \varphi U_i(\cos\frac{\pi}{30})$ and $\kappa(v_8) = \varphi^{-1}U_3(\cos\frac{\pi}{30})$, where $\varphi = 2\cos\frac{\pi}{5}$ is the golden ratio and $\varphi^{-1} = 2\cos\frac{2\pi}{5}$ is its inverse. Similar to above, F^{E_8} maps the source vertices to $[0]$ and sink vertices to $[1]$.

2.4. Chebyshev polynomials and associated (semi)rings.

As previously mentioned, we denote throughout the paper the i-th Chebyshev polynomial of the second kind by U_i. That is, U_i satisfies the identity

$$U_i(\cos\theta)\sin\theta = \sin((i+1)\theta).$$

In [7], we defined a family of rings related to Chebyshev polynomials of the second kind that would later be used to define a semiring action on the module, derived and cluster categories of unfolded quivers. We will do the same here, except the (semi)rings we consider in this paper will be slightly different. Throughout the paper, denote $\chi^{(2n)} = \mathbb{Z}[2\cos\frac{\pi}{2n}]$ for each $n \geq 2$. The following (semi)rings will be used in foldings of type A.

Definition 2.3. For each $n \geq 2$, define the families of commutative rings

$$\chi^{A_{2n-1}} = \mathbb{Z}[\psi_2, \psi_4, \ldots, \psi_{2n-2}] \qquad \subset \qquad \widehat{\chi}^{A_{2n-1}} = \mathbb{Z}[\psi_1, \psi_2, \psi_3, \ldots, \psi_{2n-2}],$$

subject to the following product rule for any $1 \leq j \leq i \leq 2n-2$:

$$\psi_i\psi_j = \psi_j\psi_i = \sum_{k=0}^{j} \psi_{i-j+2k},$$

where any element ψ_k with $k \notin \{1, \ldots, 2n-2\}$ resulting from the above product is such that $\psi_0 = 1$, $\psi_{2n-1} = 0$ and $\psi_{2n+k} = -\psi_{2n-2-k}$. After cancellation, the above product will always produce a sum of elements ψ_k with positive coefficients, and so it makes sense to define the corresponding semirings:

$$\chi_+^{A_{2n-1}} = \mathbb{Z}_{\geq 0}[\psi_2, \psi_4, \ldots, \psi_{2n-2}] \qquad \subset \qquad \widehat{\chi}_+^{A_{2n-1}} = \mathbb{Z}_{\geq 0}[\psi_1, \psi_2, \psi_3, \ldots, \psi_{2n-2}],$$

which are subsemirings of $\chi^{A_{2n-1}}$ and $\widehat{\chi}^{A_{2n-1}}$ respectively.

In the next family of (semi)rings, we use a signed notation on the elements. Here, elements ψ_i^+ and ψ_i^- are distinct, but in some statements we collectively refer to them as $\psi_i^\pm$. We also have statements dependent on elements which either have the same sign or opposite signs. For example, by $\psi_i^\pm\psi_j^\pm$, we mean either $\psi_i^+\psi_j^+$ or $\psi_i^-\psi_j^-$. On the other hand, by $\psi_i^\pm\psi_j^\mp$, we mean $\psi_i^+\psi_j^-$ or $\psi_i^-\psi_j^+$.

Definition 2.4. Define commutative (semi)rings of type D_4 by

$$\chi^{D_4} = \mathbb{Z}[g] \qquad \subset \qquad \widehat{\chi}^{D_4} = \mathbb{Z}[g, \psi_1],$$
$$\chi_+^{D_4} = \mathbb{Z}_{\geq 0}[g] \qquad \subset \qquad \widehat{\chi}_+^{D_4} = \mathbb{Z}_{\geq 0}[g, \psi_1],$$

satisfying the relations $g^3 = 1$, $g\psi_1 = \psi_1$ and $\psi_1^2 = 1 + g + g^2$. For $n > 3$, define further families of commutative (semi)rings of type D_{n+1} by

$$\chi^{D_{n+1}} = \mathbb{Z}\left[\psi_0^-, \psi_2^\pm, \ldots, \psi_{2\lfloor\frac{n-1}{2}\rfloor}^\pm\right] \qquad \subset \qquad \widehat{\chi}^{D_{n+1}} = \mathbb{Z}[\psi_0^-, \psi_1^\pm, \psi_2^\pm, \psi_3^\pm, \ldots, \psi_{n-1}^\pm],$$

$$\chi_+^{D_{n+1}} = \mathbb{Z}_{\geq 0}\left[\psi_0^-, \psi_2^\pm, \ldots, \psi_{2\left\lfloor \frac{n-1}{2}\right\rfloor}^\pm\right] \quad \subset \quad \widehat{\chi}_+^{D_{n+1}} = \mathbb{Z}_{\geq 0}[\psi_0^-, \psi_1^\pm, \psi_2^\pm, \psi_3^\pm, \ldots, \psi_{n-1}^\pm],$$

subject to the following product rules for any $1 \leq j \leq i \leq n-1$:

$$\psi_i^\pm \psi_j^\pm = \sum_{k=0}^j \omega_{i-j+2k}, \qquad \psi_0^- \psi_j^\pm = \psi_j^\mp, \qquad (\psi_0^-)^2 = \psi_0^+ = 1,$$

where each element ω_k resulting from the above product is such that

$$\omega_k = \begin{cases} \psi_k^+ & \text{if } k < n-1, \\ \psi_{n-1}^+ & \text{if } k = n-1 \text{ and } i+j-n+1 \equiv 0 \pmod 4, \\ \psi_{n-1}^- & \text{if } k = n-1 \text{ and } i+j-n+1 \equiv 2 \pmod 4, \\ \psi_{2n-2-k}^- & \text{if } k > n-1. \end{cases}$$

Definition 2.5. Define the following exceptional rings of type E.

$$\begin{array}{lll}
\chi^{E_6} = \mathbb{Z}[\psi_0^-, \psi_2]/(S_6) & \subseteq & \widehat{\chi}^{E_6} = \mathbb{Z}[\psi_0^-, \psi_1^+, \psi_1^-, \psi_2, \psi_{v_6}]/(\widehat{S}_6), \\
\chi^{E_7} = \mathbb{Z}[\psi_2, \widetilde{\psi}_2]/(S_7) & \subseteq & \widehat{\chi}^{E_7} = \mathbb{Z}[\psi_1, \widetilde{\psi}_1, \psi_2, \widetilde{\psi}_2, \psi_3, \psi_{v_7}]/(\widehat{S}_7), \\
\chi^{E_8} = \mathbb{Z}[\varphi, \psi_2]/(S_8) & \subseteq & \widehat{\chi}^{E_8} = \mathbb{Z}[\psi_1, \psi_2, \psi_{v_8}, \varphi]/(\widehat{S}_8),
\end{array}$$

where each $(\widehat{S}_i)$ is the ideal generated by a set of relations $\widehat{S}_i = S_i \cup S_i'$ with

$$\begin{aligned}
S_6 &= \{(\psi_0^-)^2 - 1, (\psi_2)^2 - 1 - 2\psi_2 - \psi_0^-, \psi_0^- \psi_2 - \psi_2\}, \\
S_6' &= \{\psi_0^- \psi_1^+ - \psi_1^-, (\psi_1^+)^2 - 1 - \psi_2, \psi_1^+ \psi_2 - \psi_1^+ - \psi_1^- - \psi_{v_6}, \psi_1^+ \psi_{v_6} - \psi_2\}, \\
S_7 &= \{\psi_2^2 - 1 - \psi_2\widetilde{\psi}_2, \widetilde{\psi}_2^2 - 1 - \widetilde{\psi}_2 - \psi_2\}, \\
S_7' &= \{\psi_1^2 - 1 - \psi_2, \widetilde{\psi}_1 - \psi_2\widetilde{\psi}_2 + \psi_2 + \widetilde{\psi}_2, \psi_1\psi_2 - \psi_3 - \psi_1, \psi_1\widetilde{\psi}_2 - \psi_3 - \psi_{v_7}\}, \\
S_8 &= \{\varphi^2 - \varphi - 1, \psi_2^2 - \varphi\psi_2 - \psi_2 - 1\}, \\
S_8' &= \{\psi_1^2 - 1 - \psi_2, \psi_1\psi_2 - \psi_1 - \varphi\psi_{v_8}, \psi_1\psi_{v_8} - \varphi\psi_2\}.
\end{aligned}$$

Since the product of any two elements with only positive coefficients in the rings χ^{E_6}, χ^{E_7} and χ^{E_8} will produce an element which again has only positive coefficients, it is natural to define semirings $\chi_+^{E_6}$, $\chi_+^{E_7}$ and $\chi_+^{E_8}$ by restricting the coefficient ring from $\mathbb{Z}$ to $\mathbb{Z}_{\geq 0}$.

Remark 2.6. Note that we could also define semirings $\widehat{\chi}_+^{E_6}$ and $\widehat{\chi}_+^{E_8}$ similarly, but we do not need this in our construction. Moreover, the ring $\widehat{\chi}^{E_7}$ is somewhat unusual in that the product of any two elements with only positive coefficients in this ring will not necessarily produce an element which has only positive coefficients. It is thus preferable to avoid working with such notions for rings of type E.

The relationship between the above (semi)rings and Chebyshev polynomials of the second kind is highlighted by the existence of ring homomorphisms

$$\begin{aligned}
\sigma^\Delta &\colon \chi^\Delta \to \chi^{(2n)} \\
\widehat{\sigma}^\Delta &\colon \widehat{\chi}^\Delta \to \chi^{(2n)} \\
\psi_i^\bullet &\mapsto U_i\left(\cos\frac{\pi}{2n}\right) & (\Delta \in \{A_{2n-1}, D_{n+1}, E_6, E_7, E_8\}), \\
g &\mapsto 1 & (\Delta = D_4), \\
\widetilde{\psi}_i &\mapsto U_i\left(\cos\frac{\pi}{9}\right) & (\Delta = E_7), \\
\varphi &\mapsto 2\cos\frac{\pi}{5} & (\Delta = E_8),
\end{aligned}$$

$$\psi_{v_i} \mapsto \kappa(v_i) \qquad\qquad (\Delta \in \{E_6, E_7, E_8\}),$$

where $n = 6, 9, 15$ for $\Delta = E_6, E_7, E_8$ respectively. This homomorphism along with the following will be used at various points in the paper.

Definition 2.7. Let $\Delta \in \{A_{2n-1}, D_{n+1}, E_6, E_7, E_8\}$ and let $\sigma^{(2n)} \colon \chi^{(2n)} \to \mathbb{R}$ be the canonical embedding. Then χ^Δ has a partial ordering given by

$$r \leq s \Leftrightarrow r = s \text{ or } \sigma^{(2n)}\sigma^\Delta(r) < \sigma^{(2n)}\sigma^\Delta(s).$$

This endows χ_+^Δ with a partial ordering in the natural way via the embedding $\sigma_+^\Delta \colon \chi_+^\Delta \to \chi^\Delta$.

3. The folding projections of the module and bounded derived categories

In [7], it is shown that a folding onto a quiver of type H_3, H_4 or $I_2(2n+1)$ induces a map that projects the dimension vectors of objects in the module and derived categories of the unfolded quiver to certain multiples of the roots of the folded quiver. Additionally, the folding induces a weighting on the rows of the Auslander-Reiten quiver of the module and derived categories associated to the unfolded quiver such that the dimension vectors of the objects in the rows with weight 1 are projected to the roots of the folded quiver — providing an analogue of Gabriel's Theorem (c.f. [11, 6]) for R-quivers. On the other hand, the dimension vectors of the objects in the rows with weight w are mapped to the w-multiple of the projected dimension vector of an object in a corresponding row with weight 1. We will show that the same is true for all possible foldings onto quivers of type $I_2(2n)$. Throughout this section, we denote by $\mathrm{Frac}(R)$ to be the field of fractions of an integral domain R. First let us make precise the notion of the weighting of the rows of the Auslander-Reiten quiver.

Definition 3.1. Throughout, let $F \colon Q^\Delta \to Q^{I_2(2n)}$ be a folding of quivers from Section 2.3 (so $\Delta \in \{A_{2n-1}, D_{n+1}, E_6, E_7, E_8\}$) and let $\mathcal{A}$ be either the module or bounded derived category of the path algebra KQ^Δ. For each $i \in Q^\Delta$, we denote by $\mathcal{P}_i^{\mathcal{A}}$ the row of the Auslander-Reiten quiver of $\mathcal{A}$ containing the module $P(i)$. That is,

$$\mathcal{P}_i^{\mathcal{A}} = \{[\tau^m P(i)] : m \in \mathbb{Z}\}$$

where $[M]$ denotes the iso-class of an object $M \in \mathcal{A}$. We define the *weight of the row* $\mathcal{P}_i^{\mathcal{A}}$ to be the value $\varepsilon_i = \frac{\kappa(i)}{\varsigma(F(i))} \in \mathrm{Frac}(\chi^{(2n)})$.

Remark 3.2. One may note that since KQ^Δ is representation-finite, we could equivalently define rows of the Auslander-Reiten quiver by sets

$$\mathcal{I}_i^{\mathcal{A}} = \{[\tau^m I(i)] : m \in \mathbb{Z}\}.$$

For the foldings with $\Delta \in \{D_{2m}, E_7, E_8\}$, we have $\mathcal{P}_i^{\mathcal{A}} = \mathcal{I}_i^{\mathcal{A}}$. For foldings with $\Delta = A_{2n-1}$, we note that $\mathcal{P}_i^{\mathcal{A}} = \mathcal{I}_{2n-2-i}^{\mathcal{A}}$, which mirrors the relation $U_i(\cos \frac{\pi}{2n}) = U_{2n-2-i}(\cos \frac{\pi}{2n})$. We also note in this case that $F(i) = F(2n-2-i)$. For the foldings with $\Delta = D_{n+1}$ with n even, we have $\mathcal{P}_i^{\mathcal{A}} = \mathcal{I}_i^{\mathcal{A}}$ for all $i \neq (n-1)^\pm$ and $\mathcal{P}_{(n-1)^\pm}^{\mathcal{A}} = \mathcal{I}_{(n-1)^\mp}^{\mathcal{A}}$. For foldings with $\Delta = E_6$, we note that $\mathcal{P}_{i^\pm}^{\mathcal{A}} = \mathcal{I}_{i^\mp}^{\mathcal{A}}$ for $i = 0, 1$, that $\mathcal{P}_i^{\mathcal{A}} = \mathcal{I}_i^{\mathcal{A}}$ for $i = 2, v_6$, and that $F(i^\pm) = F(i^\mp)$. It is then easy to see that in all cases, we could equivalently define the weight of a row containing both $P(i)$ and $I(j)$ as $\varepsilon_i = \varepsilon_j$.

Now we will recall the definition from [7] of the projection map that will be central to this paper, which we modify slightly to account for the non-trivial R-valuation of $Q^{I_2(2n)}$.

Definition 3.3. Given an arbitrary weighted folding $F\colon Q \to Q^{I_2(2n)}$, define a map $d_F\colon \mathbb{Z}^{|Q_0|} \to (\mathrm{Frac}(\chi^{(2n)}))^2$ by the weighted sum

$$(u_i)_{i\in Q_0} \mapsto \left(\sum_{F(i)=[0]} \frac{\kappa(i)}{\varsigma([0])}u_i, \ \sum_{F(i)=[1]} \frac{\kappa(i)}{\varsigma([1])}u_i \right).$$

For each object $M \in \mathrm{mod}\, KQ$, we define

$$\delta_F^{\mathrm{mod}\, KQ}(M) = d_F\, \mathbf{dim}(M).$$

In addition, for each object $X = \bigoplus_{j\in\mathbb{Z}} \Sigma^j M_j \in \mathcal{D}^b(\mathrm{mod}\, KQ)$, we define

$$\delta_F^{\mathcal{D}^b(\mathrm{mod}\, KQ)}(X) = \sum_{j\ \mathrm{even}} \delta_F^{\mathrm{mod}\, KQ}(M_j) - \sum_{j\ \mathrm{odd}} \delta_F^{\mathrm{mod}\, KQ}(M_j).$$

Given any object $X \in \mathcal{A}$, where $\mathcal{A}$ is either $\mathrm{mod}\, KQ$ or $\mathcal{D}^b(\mathrm{mod}\, KQ)$, we call the vector $\delta_F^{\mathcal{A}}(X)$ the *F-projected (dimension) vector of X*.

Remark 3.4. It is easy to see that $\delta_F^{\mathcal{A}}(X \oplus X') = \delta_F^{\mathcal{A}}(X) + \delta_F^{\mathcal{A}}(X')$ for any object $X, X' \in \mathcal{A}$. This fact will be useful in several proofs.

We will now state the main theorem of this section.

Theorem 3.5. *Let $F\colon Q^{\Delta} \to Q^{I_2(2n)}$ be a weighted folding and let $\mathcal{A}$ be either $\mathrm{mod}\, KQ^{\Delta}$ or $\mathcal{D}^b(\mathrm{mod}\, KQ^{\Delta})$. Then the following hold:*

(a) *For any $i \in Q_0^{\Delta}$ such that $F(i) = [0]$, we have*

$$\delta_F^{\mathcal{A}}(\tau^m I(i)) = \varepsilon_i \alpha_m,$$

where α_m is the short root corresponding to the point $\varsigma([0])\exp(\frac{m\pi\mathrm{i}}{n})$.

(b) *For any $i \in Q_0^{\Delta}$ such that $F(i) = [1]$, we have*

$$\delta_F^{\mathcal{A}}(\tau^m I(i)) = \varepsilon_i \beta_m,$$

where β_m is the long root corresponding to the point $\varsigma([1])\exp(\frac{(2m+1)\pi\mathrm{i}}{2n})$.

Essentially, the theorem above says that the map $\delta_F^{\mathcal{A}}$ maps an indecomposable object in the category $\mathcal{A}$ to a specific multiple of a root of the standard root system of $I_2(2n)$. In fact, it says much more than this: objects in the same row map to vectors of the same length/multiple and that the Auslander-Reiten translation of an object corresponds to a rotation of the corresponding vector. Before we begin the proof, we will highlight some immediate corollaries for the benefit of the reader. The first Corollary may be viewed as a generalisation of Gabriel's Theorem to quivers of type $I_2(2n)$.

Corollary 3.6. *The map $\delta_F^{\mathcal{A}}$ induces a bijection from each row of the Auslander-Reiten quiver of $\mathcal{A}$ to a subset of roots of $I_2(2n)$. In particular,*

$$\delta_F^{\mathcal{A}}(\mathcal{P}_i^{\mathcal{A}}) = \begin{cases} \Phi_{2n}^{s,+} & \text{if } \varepsilon_i = 1, \ F(i) = [0] \text{ and } \mathcal{A} = \mathrm{mod}\, KQ^{\Delta}, \\ \Phi_{2n}^{s} & \text{if } \varepsilon_i = 1, \ F(i) = [0] \text{ and } \mathcal{A} = \mathcal{D}^b(\mathrm{mod}\, KQ^{\Delta}), \\ \Phi_{2n}^{l,+} & \text{if } \varepsilon_i = 1, \ F(i) = [1] \text{ and } \mathcal{A} = \mathrm{mod}\, KQ^{\Delta}, \\ \Phi_{2n}^{l} & \text{if } \varepsilon_i = 1, \ F(i) = [1] \text{ and } \mathcal{A} = \mathcal{D}^b(\mathrm{mod}\, KQ^{\Delta}). \end{cases}$$

Corollary 3.7. *For any $i \in Q_0^{\Delta}$ and $m \in \mathbb{Z}$, the length of the vector $\delta_F^{\mathcal{A}}(\tau^m P(i))$ with respect to the standard basis of $\mathbb{R}^2$ is*

$$\ell(\delta_F^{\mathcal{A}}(\tau^m P(i))) = \kappa(i).$$

The proof of these results follows the same reasoning as that used for the foldings onto $I_2(2n + 1)$ featuring in [7]. The proof works along these lines: We first show that the the relative length of the F-projected vectors of any two indecomposable projective (resp. injective) modules that correspond to the same folded vertex is determined by the R-weights of the unfolded vertices. Next, we show that the Auslander-Reiten translate of an object corresponds to an anticlockwise rotation of its F-projected vector by an angle of $\frac{\pi}{n}$ about the origin. The proof is then finalised with the calculation of the F-projected vectors of two objects (one corresponding to a short root and the other a long root).

Lemma 3.8. *For any $i, j \in Q_0^\Delta$ such that $F(i) = F(j)$, we have*

$$\kappa(i)\delta_F^{\mathrm{mod}\,KQ^\Delta}(P(j)) = \kappa(j)\delta_F^{\mathrm{mod}\,KQ^\Delta}(P(i))$$

$$\kappa(i)\delta_F^{\mathrm{mod}\,KQ^\Delta}(I(j)) = \kappa(j)\delta_F^{\mathrm{mod}\,KQ^\Delta}(I(i)).$$

Proof. One can see from the multiplication rule for Chebyshev polynomials of the second kind that

$$\delta_F^{\mathrm{mod}\,KQ^\Delta}(P(i)) = \begin{cases} (\varepsilon_i, 0) & \text{if } P(i) \text{ is simple,} \\ (\varepsilon_i, \varepsilon_i) & \text{if } P(i) \text{ is non-simple.} \end{cases}$$

$$\delta_F^{\mathrm{mod}\,KQ^\Delta}(I(i)) = \begin{cases} (0, \varepsilon_i) & \text{if } I(i) \text{ is simple,} \\ (4\varepsilon_i \cos^2 \frac{\pi}{2n}, \varepsilon_i) & \text{if } I(i) \text{ is non-simple.} \end{cases}$$

Thus, the result follows from the commutativity of $\chi^{(2n)}$. $\qquad\square$

Remark 3.9. Recall that $\ell((1,0)) = \varsigma([0]) = \lambda$ and $\ell((0,1)) = \varsigma([1]) = 2\lambda \cos \frac{\pi}{2n}$, where $\lambda = 1$ if $\Delta \neq D_{n+1}$ and $\lambda = 2$ if $\Delta = D_{n+1}$. Thus by the law of cosines, the length of the vector $(1,1)$ is λ. From this, the proof of the above lemma, and the definition of the foldings, we can conclude that $\ell(\delta_F^{\mathcal{A}}(P(i))) = \kappa(i)$. Likewise, we have $\ell(\delta_F^{\mathcal{A}}(I(i))) = \kappa(i)$.

Lemma 3.10. *Let $M \in \mathcal{A}$. Then $\delta_F^{\mathcal{A}}(\tau M)$ is obtained from $\delta_F^{\mathcal{A}}(M)$ by an anticlockwise rotation of $\frac{\pi}{n}$ about the origin.*

Proof. Suppose $M \in \mathrm{mod}\,KQ^\Delta$ is a non-projective indecomposable. Let

$$0 \to P_1 \to P_0 \to M \to 0,$$
$$0 \to \tau M \to I_1 \to I_0 \to 0$$

be the minimal projective resolution of M and the corresponding injective resolution of τM given by applying the Nakayama functor ν. Since KQ^Δ is radical square zero, P_1 and I_0 are semisimple modules. Consequently, we have

$$\delta_F^{\mathcal{A}}(P_1) = (0, r'), \qquad\qquad \delta_F^{\mathcal{A}}(P_0) = (r, r),$$
$$\delta_F^{\mathcal{A}}(I_1) = (4s' \cos^2 \tfrac{\pi}{2n}, s'), \qquad\qquad \delta_F^{\mathcal{A}}(I_0) = (s, 0).$$

for some $r, r', s, s' \in \mathrm{Frac}(\chi^{(2n)})$. But since $\nu P(i) = I(i)$ for each $i \in Q_0^\Delta$, we in fact have $r = s$ and $r' = s'$. This, along with Remark 3.9, implies that

$$\ell(\delta_F^{\mathcal{A}}(P_0)) = \ell(\delta_F^{\mathcal{A}}(I_0)) \qquad \text{and} \qquad \ell(\delta_F^{\mathcal{A}}(P_1)) = \ell(\delta_F^{\mathcal{A}}(I_1)).$$

Now note that the vector $-\delta_F^{\mathcal{A}}(I_0)$ is obtained from $\delta_F^{\mathcal{A}}(P_0)$ by an anticlockwise rotation of $\frac{\pi}{n}$ about the origin. Likewise, $\delta_F^{\mathcal{A}}(I_1)$ is obtained from $-\delta_F^{\mathcal{A}}(P_1)$ by an anticlockwise rotation of $\frac{\pi}{n}$ about the origin. But such a rotation is a linear transformation, and we have

$$\delta_F^{\mathcal{A}}(M) = \delta_F^{\mathcal{A}}(P_0) - \delta_F^{\mathcal{A}}(P_1),$$
$$\delta_F^{\mathcal{A}}(\tau M) = \delta_F^{\mathcal{A}}(I_1) - \delta_F^{\mathcal{A}}(I_0)$$

by dimension counting the above projective/injective resolutions. Thus, $\delta_F^{\mathcal{A}}(\tau M)$ is an anticlockwise rotation of the vector $\delta_F^{\mathcal{A}}(M)$ by an angle of $\frac{\pi}{n}$ about the origin.

Now suppose instead that $M \in \mathcal{D}^b(\operatorname{mod} KQ^\Delta)$ corresponds to an indecomposable projective module concentrated in some degree k. Then $\tau M \cong \Sigma^{k+1} I$ for some indecomposable injective module I whose corresponding vertex has the same weight as with M. So $\delta_F^{\mathcal{A}}(\tau M) = -\delta_F^{\mathcal{A}}(\Sigma^k I)$, which we have already shown to be the appropriate rotated vector. By Remark 3.4, we therefore conclude that for any object $M \in \mathcal{A}$, the vector $\delta_F^{\mathcal{A}}(\tau M)$ is obtained from $\delta_F^{\mathcal{A}}(M)$ by an anticlockwise rotation of $\frac{\pi}{n}$ about the origin. $\quad\square$

We can now prove Theorem 3.5.

Proof of Theorem 3.5. (a) If $F(i) = [0]$ then $I(i)$ is simple. So $\delta_F^{\mathcal{A}}(I(i)) = (\varepsilon_i, 0)$. The result then follows from Lemma 3.10.

(b) If $F(i) = [1]$ then $I(i)$ is non-simple. As shown in the proof of Lemma 3.10, $\delta_F^{\mathcal{A}}(I(i))$ is given by an anticlockwise rotation of the vector $-\delta_F^{\mathcal{A}}(P(i)) = (0, -\varepsilon_i)$ by $\frac{\pi}{n}$ about the origin. Thus, $\delta_F^{\mathcal{A}}(I(i)) = \varepsilon_i \beta_0$, where β_0 is the long root corresponding to the point $\varsigma([1])\exp(\frac{\pi i}{2n})$. The result then follows from Lemma 3.10. $\quad\square$

Remark 3.11. The results of this section can be viewed as an 'unfolded categorification' of the root system of type $I_2(2n)$. Equivalently, this can be approached from the perspective of a 'folded categorification', as presented in [12]. In particular, Theorem 4.11 of [12] in the $I_2(2n)$ case can be obtained from our Theorem 3.5 with $\Delta = A_{2n-1}$ and $\mathcal{A} = \operatorname{mod} KQ^{A_{2n-1}}$.

4. The semiring actions on the module and bounded derived categories

Each folding $F \colon Q^\Delta \to Q^{I_2(2n)}$ and the non-crystallographic projection it induces gives rise to a semiring action of $R_+ = \chi_+^\Delta$ on $\operatorname{mod} KQ^\Delta$ and $\mathcal{D}^b(\operatorname{mod} KQ^\Delta)$ in the sense of [7, Definition 6.1], where χ_+^Δ is as defined in Section 2.4. As in [7], we call the appropriate category an R_+-*coefficient category* when it is equipped with a semiring action of R_+. Whilst the semiring actions are slightly different for each folding, the general principle underlying each action is the same. We will first define the semiring actions for $\mathcal{A} = \operatorname{mod} KQ^\Delta$, as the semiring action on $\mathcal{D}^b(\operatorname{mod} KQ^\Delta)$ is a straightforward extension of this action.

4.1. **The action on iso-classes of objects.** For each positive root α of $I_2(2n)$, define a set of iso-classes

$$\mathbf{M}_\alpha = \{[M] : M \in \mathcal{A} \text{ indecomposable and } r'\delta_F^{\mathcal{A}}(M) = r\alpha \text{ for some } r, r' \in \chi^{(2n)}\}.$$

Since Q^Δ is bipartite, it follows from Theorem 3.5 that the iso-classes in $\mathbf{M}_\alpha$ bijectively correspond to objects in the same column of the Auslander-Reiten quiver. That is, $\mathbf{M}_\alpha = \{[\tau^m I(i)] : F(i) = j\}$, where $j = [0]$ if $\alpha \in \Phi_{2n}^{s,+}$ and corresponds to the point $\varsigma([0])\exp(\frac{m\pi i}{2})$, and $j = [1]$ if $\alpha \in \Phi_{2n}^{l,+}$ and corresponds to the point $\varsigma([1])\exp(\frac{(m+1)\pi i}{2n})$. In both cases, define $M_{i,\alpha} = \tau^m I(i)$. Hence $[M_{i,\alpha}] \in \mathcal{I}_i^{\mathcal{A}}$ and $\varsigma(F(i))\delta_F^{\mathcal{A}}(M_{i,\alpha}) = \kappa(i)\alpha$. Consequently, we have

$$\mathbf{M}_\alpha = \begin{cases} \{[M_{i,\alpha}] : F(i) = [0]\} & \text{if } \alpha \in \Phi_{2n}^{s,+} \\ \{[M_{i,\alpha}] : F(i) = [1]\} & \text{if } \alpha \in \Phi_{2n}^{l,+}. \end{cases}$$

The semiring R_+ will act on $\operatorname{mod} KQ^\Delta$ in such a way that for any positive root α, for any $[M] \in \mathbf{M}_\alpha$ and for any $r \in R_+$, the object rM will be isomorphic to an object whose indecomposable direct summands belong to iso-classes in $\mathbf{M}_\alpha$. We will describe this precisely for each folding.
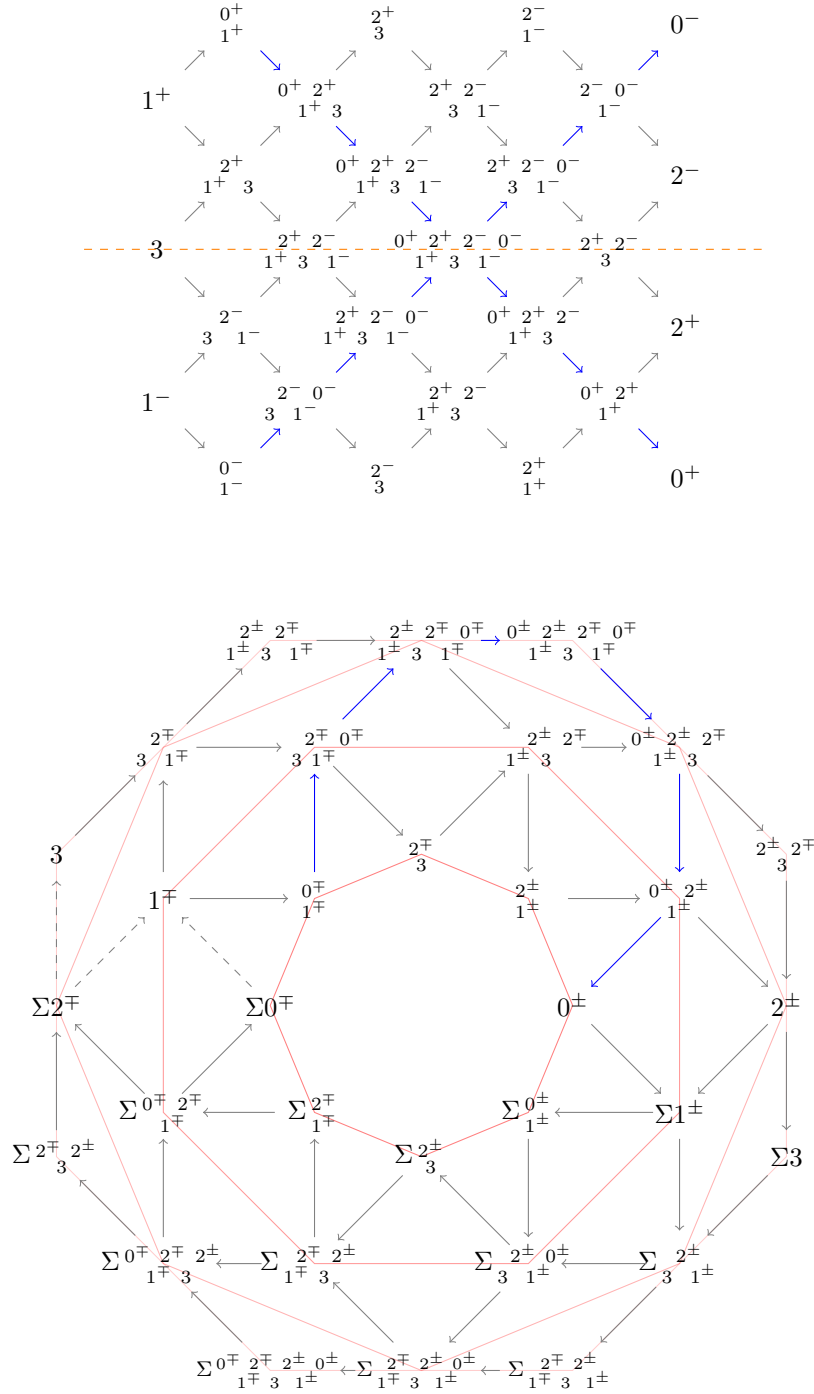
FIGURE 3.1. The folding $F^{A_7}: Q^{A_7} \to Q^{I_2(8)}$. Top: The Auslander-Reiten quiver of $\bmod KQ^{A_7}$. The category has $\mathbb{Z}_2$-symmetry determined by reflection in the dashed line. Bottom: The non-crystallographic projection of the Auslander-Reiten quiver of $\mathcal{A} = \mathcal{D}^b(\bmod KQ^{A_7})$ under the map δ_F^A, with irreducible morphisms superimposed. Objects in the same $\mathbb{Z}_2$-orbit map to the same point. Objects concentrated in odd (resp. even) degree map to the points labelled with (resp. without) Σ. One of the sectors of morphisms is dashed to indicate that they are not morphisms between objects of degrees k and $k - 1$, but rather between objects of degrees k and $k + 1$. Rays in the Auslander-Reiten quiver map to octagonal arcs in the projection (eg. the blue arrows).

4.1.1. *Isomorphism conditions of type A_{2n-1}*. We will adopt the unsigned labelling of the vertices for $Q^{A_{2n-1}}$ (the first quiver in Section 2.3). For any positive root α, we have a function $\omega_\alpha \colon \mathbf{M}_\alpha \to \widehat{\chi}_+^{A_{2n-1}}$ defined by $\omega_\alpha([M_{i,\alpha}]) = \psi_i$ for each $[M_{i,\alpha}] \in \mathbf{M}_\alpha$. Now recall the product rule for the semiring $\widehat{\chi}_+^{A_{2n-1}}$, which gives

$$\psi_j \omega_\alpha([M_{i,\alpha}]) = \sum_{k \in V_{ji}} \psi_k$$

for each $\psi_j \in \chi_+^{A_{2n-1}}$ and for some index set $V_{ji} \subset \{0, 1, \ldots, 2n-2\}$. Since the index j is even, note that the set V_{ji} consists of indices that are either all odd if i is odd, or all even if i is even. In particular, this implies that there exists $[M_{k,\alpha}] \in \mathbf{M}_\alpha$ for each $k \in V_{ji}$. We will thus define the semiring action of $\chi_+^{A_{2n-1}}$ on $\operatorname{mod} KQ^{A_{2n-1}}$ such that the following isomorphism holds for each positive root α, each $[M_{i,\alpha}] \in \mathbf{M}_\alpha$ and each $\psi_j \in \chi_+^{A_{2n-1}}$.

$$\psi_j M_{i,\alpha} \cong \bigoplus_{k \in V_{ji}} M_{k,\alpha}. \tag{A1}$$

An example demonstrating the action of $\chi_+^{A_{2n-1}}$ on $\operatorname{mod} KQ^{A_{2n-1}}$ is given in Section A.1.

4.1.2. *Isomorphism conditions of type D_4*. The semiring action of $\chi_+^{D_4}$ on $\operatorname{mod} KQ^{D_4}$ is a minor extension of the classical $\mathbb{Z}_3$-group action on $\operatorname{mod} KQ^{D_4}$. That is we define for any positive root α

$$gM_{0,\alpha} \cong M_{2^+,\alpha}, \qquad gM_{2^+,\alpha} \cong M_{2^-,\alpha}, \qquad gM_{2^-,\alpha} \cong M_{0,\alpha},$$
$$gM_{1,\alpha} \cong M_{1,\alpha}.$$

The action of $\chi_+^{D_4}$ is only different from the action of $\mathbb{Z}_3$ in that we have $(r + r')M \cong rM \oplus r'M$ for any $r, r' \in \chi_+^{D_4}$ and $M \in \operatorname{mod} KQ^{D_4}$.

4.1.3. *Isomorphism conditions of type D_{n+1}*. We assume $n \geq 4$, since $D_3 = A_3$ and D_4 is covered separately. For any positive root α, we define a map $\omega_\alpha \colon \mathbf{M}_\alpha \to \widehat{\chi}_+^{D_{n+1}}$ by

$$\omega_\alpha([M_{i,\alpha}]) = \begin{cases} \psi_i^+ + \psi_i^- & \text{if } 0 \leq i < n-1 \\ \psi_{n-1}^\pm & \text{if } i = (n-1)^\pm. \end{cases}$$

For each $[M_{i,\alpha}] \in \mathbf{M}_\alpha$ such that $i \neq (n-1)^\pm$, we have

$$\psi_j^\pm \omega_\alpha([M_{i,\alpha}]) = \sum_{k=0}^{n-1} q_{ijk}(\psi_k^+ + \psi_k^-)$$

$$= \sum_{k=0}^{n-2} q_{ijk}\omega_\alpha([M_{k,\alpha}]) + q_{ij(n-1)}(\omega_\alpha([M_{(n-1)^+,\alpha}]) + \omega_\alpha([M_{(n-1)^-,\alpha}]))$$

where each $q_{ijk} \in \{0, 1, 2\}$. In particular, $q_{ijk} = 0$ for all even k if i is odd and $q_{ijk} = 0$ for all odd k if i is even. We will thus define the semiring action of $\chi_+^{D_{n+1}}$ on $\operatorname{mod} KQ^{D_{n+1}}$ to be such that the following isomorphism holds for each positive root α, each $[M_{i,\alpha}] \in \mathbf{M}_\alpha$ with $i \neq (n-1)^\pm$ and each $\psi_j^\pm \in \chi_+^{D_{n+1}}$.

$$\psi_j^\pm M_{i,\alpha} \cong \bigoplus_{k=0}^{n-2} \bigoplus_{l=1}^{q_{ijk}} M_{k,\alpha} \oplus \bigoplus_{l=1}^{q_{ij(n-1)}} (M_{(n-1)^+,\alpha} \oplus M_{(n-1)^-,\alpha}). \tag{D1}$$

This is a slight abuse of notation, as $M_{k,\alpha}$ might not exist if $q_{ijk} = 0$. However, $M_{k,\alpha}$ is not a direct summand of the above if $q_{ijk} = 0$, so this is fine.

On the other hand, if $i = (n-1)^\pm$ then

$$\psi_j^\pm \omega_\alpha([M_{(n-1)^\pm,\alpha}]) = \begin{cases} \sum_{k \in V_{ji}} (\psi_k^+ + \psi_k^-) + \psi_{n-1}^+ & \text{if } \frac{j}{2} \text{ even,} \\ \sum_{k \in V_{ji}} (\psi_k^+ + \psi_k^-) + \psi_{n-1}^- & \text{if } \frac{j}{2} \text{ odd,} \end{cases}$$

$$\psi_j^\mp \omega_\alpha([M_{(n-1)^\pm,\alpha}]) = \begin{cases} \sum_{k \in V_{ji}} (\psi_k^+ + \psi_k^-) + \psi_{n-1}^- & \text{if } \frac{j}{2} \text{ even,} \\ \sum_{k \in V_{ji}} (\psi_k^+ + \psi_k^-) + \psi_{n-1}^+ & \text{if } \frac{j}{2} \text{ odd.} \end{cases}$$

for some index set $V_{ji} \subset \{0, \ldots, n-1\}$ whose elements are odd if $n-1$ is odd, or even if $n-1$ is even. Thus, we will define the action of $\chi_+^{D_{n+1}}$ on $\operatorname{mod} KQ^{D_{n+1}}$ to be such that

$$\psi_j^\pm M_{i,\alpha} \cong \begin{cases} \bigoplus_{k \in V_{ji}} M_{k,\alpha} \oplus M_{(n-1)^+,\alpha} & \text{if } i = (n-1)^\pm \text{ and } \frac{j}{2} \text{ even} \\ & \quad \text{or if } i = (n-1)^\mp \text{ and } \frac{j}{2} \text{ odd,} \\ \bigoplus_{k \in V_{ji}} M_{k,\alpha} \oplus M_{(n-1)^-,\alpha} & \text{if } i = (n-1)^\mp \text{ and } \frac{j}{2} \text{ even} \\ & \quad \text{or if } i = (n-1)^\pm \text{ and } \frac{j}{2} \text{ odd.} \end{cases} \tag{D2}$$

An example demonstrating the action of $\chi_+^{D_{n+1}}$ on $\operatorname{mod} KQ^{D_{n+1}}$ is given in Section A.2.

4.1.4. *Isomorphism conditions of type E_6.* Let α be a positive root of $I_2(12)$ and let $\mathbf{M}_\alpha$ be as before. Then R_+ will act on $\operatorname{mod} KQ^{E_6}$ such that the following isomorphisms hold.

$$\psi_0^- M_{i,\alpha} \cong \begin{cases} M_{j^\mp,\alpha} & \text{if } i = j^\pm \text{ with } j \in \{0,1\}, \\ M_{i,\alpha} & \text{otherwise,} \end{cases} \tag{E6.1}$$

$$\psi_2 M_{i,\alpha} \cong \begin{cases} M_{2,\alpha} & \text{if } i = 0^\pm, \\ M_{1^+,\alpha} \oplus M_{v_6,\alpha} \oplus M_{1^-,\alpha} & \text{if } i = 1^\pm, \\ M_{0^+,\alpha} \oplus M_{2,\alpha} \oplus M_{2,\alpha} \oplus M_{0^-,\alpha} & \text{if } i = 2, \\ M_{1^+,\alpha} \oplus M_{1^-,\alpha} & \text{if } i = v_6, \end{cases} \tag{E6.2}$$

4.1.5. *Isomorphism conditions of type E_7.* The isomorphism conditions for the folding F^{E_7} are such that for any positive root α, we have the following.

$$\psi_2 M_{i,\alpha} \cong \begin{cases} M_{2,\alpha} & \text{if } i = 0, \\ M_{1,\alpha} \oplus M_{3,\alpha} & \text{if } i = 1, \\ M_{0,\alpha} \oplus M_{\widetilde{1},\alpha} \oplus M_{\widetilde{2},\alpha} \oplus M_{2,\alpha} & \text{if } i = 2, \\ M_{1,\alpha} \oplus M_{3,\alpha} \oplus M_{3,\alpha} \oplus M_{v_7,\alpha} & \text{if } i = 3, \\ M_{\widetilde{2},\alpha} \oplus M_{2,\alpha} & \text{if } i = \widetilde{1}, \\ M_{\widetilde{1},\alpha} \oplus M_{\widetilde{2},\alpha} \oplus M_{2,\alpha} & \text{if } i = \widetilde{2}, \\ M_{3,\alpha} & \text{if } i = v_7, \end{cases} \tag{E7.1}$$

$$\widetilde{\psi}_2 M_{i,\alpha} \cong \begin{cases} M_{\widetilde{2},\alpha} & \text{if } i = 0, \\ M_{3,\alpha} \oplus M_{v_7,\alpha} & \text{if } i = 1, \\ M_{\widetilde{1},\alpha} \oplus M_{\widetilde{2},\alpha} \oplus M_{2,\alpha} & \text{if } i = 2, \\ M_{1,\alpha} \oplus M_{3,\alpha} \oplus M_{3,\alpha} & \text{if } i = 3, \\ M_{\widetilde{1},\alpha} \oplus M_{2,\alpha} & \text{if } i = \widetilde{1}, \\ M_{0,\alpha} \oplus M_{\widetilde{2},\alpha} \oplus M_{2,\alpha} & \text{if } i = \widetilde{2}, \\ M_{1,\alpha} \oplus M_{v_7,\alpha} & \text{if } i = v_7, \end{cases} \tag{E7.2}$$

It is easy to verify that the above is compatible with the relations $\widetilde{\psi}_2^2 = 1 + \widetilde{\psi}_2 + \psi_2$ and $\psi_2^2 = 1 + \psi_2 \widetilde{\psi}_2$.

4.1.6. *Isomorphism conditions of type* E_8. The isomorphism conditions for this folding are such that for any positive root α, we have the following.

$$\psi_2 M_{i,\alpha} \cong \begin{cases} M_{2,\alpha} & \text{if } i = 0, \\ M_{1,\alpha} \oplus M_{3,\alpha} & \text{if } i = 1, \\ M_{0,\alpha} \oplus M_{2,\alpha} \oplus M_{4,\alpha} & \text{if } i = 2, \\ M_{1,\alpha} \oplus M_{3,\alpha} \oplus M_{v_8,\alpha} \oplus M_{\phi_1,\alpha} & \text{if } i = 3, \\ M_{2,\alpha} \oplus M_{4,\alpha} \oplus M_{4,\alpha} \oplus M_{\phi_0,\alpha} & \text{if } i = 4, \\ M_{3,\alpha} \oplus M_{\phi_1,\alpha} & \text{if } i = v_8, \\ M_{3,\alpha} \oplus M_{v_8,\alpha} \oplus M_{\phi_1,\alpha} & \text{if } i = \phi_1, \\ M_{4,\alpha} & \text{if } i = \phi_0, \end{cases} \tag{E8.1}$$

$$\varphi M_{i,\alpha} \cong \begin{cases} M_{\phi_0,\alpha} & \text{if } i = 0, \\ M_{\phi_1,\alpha} & \text{if } i = 1, \\ M_{4,\alpha} & \text{if } i = 2, \\ M_{3,\alpha} \oplus M_{v_8,\alpha} & \text{if } i = 3, \\ M_{2,\alpha} \oplus M_{4,\alpha} & \text{if } i = 4, \\ M_{3,\alpha} & \text{if } i = v_8, \\ M_{1,\alpha} \oplus M_{\phi_1,\alpha} & \text{if } i = \phi_1, \\ M_{0,\alpha} \oplus M_{\phi_0,\alpha} & \text{if } i = \phi_0, \end{cases} \tag{E8.2}$$

4.2. **The action on morphisms.** The action of R_+ on morphisms in $\mathrm{mod}\, KQ^\Delta$ is best described by adopting a particular basis for the modules. Under this basis, the semiring can be seen to act diagonally on morphisms. This requires us to establish additional notation for an explicit explanation. As in [7, Section 6], let $\{\eta_i : i \in Q_0^\Delta\}$ be a complete set of primitive orthogonal idempotents of KQ^Δ and let N be a KQ^Δ-module. Let $N_{i,j} \colon N_i \to N_j$ be the restriction of the linear action of the unique arrow $a_{i,j} \colon i \to j \in Q_1^\Delta$ on N to the vector subspace $N_i = N\eta_i$, composed with the canonical surjection onto $N_j = N\eta_j$. The data given by all vector subspaces N_i and all linear maps $N_{i,j}$ entirely determine the structure of N — this is equivalently the module N expressed as a K-representation of Q^Δ, which in this setting, is more convenient to work with. For any $L \in \mathcal{M}_F$ and morphism $f \in \mathrm{Hom}_{\mathcal{M}_F}(N, L)$, we then recall by Schur's lemma that f can be written diagonally as $(f_i)_{i \in Q_0^\Delta}$ with $f_i = f|_{N_i} \colon N_i \to L_i$.

To define the appropriate basis of $\psi_j N$, first let $[M_{i,\alpha}] \in \mathbf{M}_\alpha$ for some positive root α of $I_2(2n)$ and write

$$\psi_j M_{i,\alpha} \cong \bigoplus_{k \in Q_0^\Delta} \bigoplus_{l=1}^{q_{ijk}} M_{k,\alpha}$$

for some non-negative integers q_{ijk} determined by the relevant isomorphism conditions. Then we define the vector subspace $(\psi_j N)_i$ of $\psi_j N$ to be the vector space

$$(\psi_j N)_i = \bigoplus_{k \in Q_0^\Delta} \bigoplus_{l=1}^{q_{ijk}} N_k^{(l)},$$

where each $N_k^{(l)}$ is a copy of the vector subspace $N_k \subseteq N$. For each arrow $a\colon i \to i'$ in Q^Δ, the linear map $(\psi_j N)_{i,i'}\colon (\psi_j N)_i \to (\psi_j N)_{i'}$ is defined blockwise by linear maps

$$N_k^{(l)} \to N_{k'}^{(l')}\colon v \mapsto \zeta_{a,k,k'}^{(l,l')} N_{k,k'}(v)$$

for some constants $\zeta_{a,k,k'}^{(l,l')} \in K$ determined such that the isomorphism conditions hold. Under such a basis, one can define the action of ψ_j on a morphism $f = (f_i)_{i \in Q_0^\Delta}\colon N \to L$ by the morphism $\psi_j f\colon \psi_j N \to \psi_j L$ given by

$$(\psi_j f)_i = \bigoplus_{k \in Q_0^\Delta} \bigoplus_{l=1}^{q_{ijk}} f_l.$$

Finally, we define

$$(r + r')f\colon (r + r')N \to (r + r')L = rf \oplus r'f\colon rN \oplus r'N \to rL \oplus r'L$$

for any $r, r' \in R_+$ such that $r + r'$ is irreducible, and we define $\psi_0 = 1 \in R_+$ to act by the identity functor.

Remark 4.1. For the case where $\Delta = A_{2n-1}$, the semiring action on the category $\operatorname{mod} KQ^\Delta$ can also be described via the equivalence given in Theorem 3.1 (and Section 6.3) of [12].

4.3. **Basic properties and actions on the bounded derived category.** Let $F\colon Q^\Delta \to Q^{I_2(2n)}$ be a weighted folding. Since the action of R_+ on $\operatorname{mod} KQ^\Delta$ has been defined such that certain isomorphism conditions are met and that the actions on morphisms are diagonalisable with respect to a given basis, it is easy to see that $\mathcal{M}_\Delta = \operatorname{mod} KQ^\Delta$ has the structure of an R_+-coefficient category. In particular, the action of any $r \in R_+$ is exact, faithful, and induces an injective map $\operatorname{Ext}^1_{\mathcal{M}_\Delta}(M, N) \to \operatorname{Ext}^1_{\mathcal{M}_\Delta}(rM, rN)$. As is the case in [7], it is straightforward to extend this action to the bounded derived category $\mathcal{D}_\Delta^b = \mathcal{D}^b(\mathcal{M}_\Delta)$. One simply notes the faithful action of R_+ on $\operatorname{Ext}^1_\mathcal{M}$-spaces and defines $r\Sigma M = \Sigma rM$ for each object M. One can then see that $\mathcal{D}_\Delta^b$ is also an R_+-coefficient category.

Throughout, we let $\mathcal{A}$ be either of the R_+-coefficient categories $\mathcal{M}_\Delta$ or $\mathcal{D}_\Delta^b$. We have the following basic properties.

Remark 4.2. Let $M \in \mathcal{A}$ be indecomposable and $r \in R_+$. Write $rM \cong M_1 \oplus \ldots \oplus M_m$, where each M_i is indecomposable.

(a) $M_1, \ldots, M_m$ reside in the same column of the Auslander-Reiten quiver. In particular, r commutes with the Auslander-Reiten translate τ.
(b) The Auslander-Reiten quiver starting (resp. ending) in M is mapped under r to the direct sum of each Auslander-Reiten sequence starting (resp. ending) in M_i.

In [7], the notion of basic and minimal R_+-generators was defined. We will briefly recall those here.

Definition 4.3. Suppose $\mathcal{A}$ is an abelian R_+-coefficient category and let R be the ring obtained from R_+ by applying the Grothendieck group construction on the additive commutative monoid structure of R_+. We say that an object $N \in \mathcal{A}$ is R_+-*generated by* $M \in \mathcal{A}$ if for some $r, r' \in R_+$, there exists a split exact sequence

$$0 \to r'M \to rM \to N' \to 0$$

with $N' \cong N$. Similarly for a triangulated R_+-coefficient category $\mathcal{A}$, we say an object $N \in \mathcal{A}$ is R_+-*generated by* $M \in \mathcal{A}$ if for some $r, r' \in R_+$, there exists a split triangle

$$r'M \to rM \to N' \to \Sigma r'M$$

with monic first morphism such that $N' \cong N$. In both the abelian and triangulated settings, we call the pair (r, r') the R_+-*generating pair of N by M* and call the value $r - r' \in R$ the R-*index of N with respect to M*. We denote the class of all objects R_+-generated by M in a category $\mathcal{A}$ by $\mathcal{G}_\mathcal{A}(M)$.

Given a set Γ of objects of $\mathcal{A}$, we denote by $\mathcal{A}(\Gamma)$ the full subcategory of $\mathcal{A}$ whose objects are R_+-generated by the objects in Γ. The objects of $\mathcal{A}(\Gamma)$ may be endowed with a partial ordering by defining $N_1 \leq N_2$ if and only if $N_1 \cong N_2$ or there exists $M \in \Gamma$ such that N_1 is R_+-generated by M with the pair (r_1, r_1') and N_2 is R_+-generated by M with the pair (r_2, r_2') and $r_1 + r_2' < r_2 + r_1'$.

We say Γ is a set of R_+-*generators for $\mathcal{A}$* if $\mathcal{A}(\Gamma) \simeq \mathcal{A}$. We say Γ is *basic* if the elements of Γ are pairwise non-isomorphic indecomposable objects. We say a basic set of R_+-generators Γ is *minimal* if for any other basic set of R_+-generators Γ', there exists an injective map of sets $\theta \colon \Gamma \to \Gamma'$ and inclusions $\iota \colon \Gamma \to \Gamma \cup \Gamma'$ and $\iota' \colon \Gamma' \to \Gamma \cup \Gamma'$ such that $\iota(M) \leq \iota'\theta(M)$ for each $M \in \Gamma$. We say Γ is τ-*closed* if for any (non-projective if $\mathcal{A}$ is abelian) $M \in \Gamma$, we have $\tau M \in \Gamma$.

For the foldings outlined in this paper, there is a close relationship between the projection map $\delta_F^\mathcal{A}$ and R_+-generated objects. The proof of the following remarks is identical to [7, Section 6].

Remark 4.4. Recall that for each folding F^Δ we have $R_+ = \chi_+^\Delta$. Let $\sigma_+^\Delta \colon \chi_+^\Delta \to \chi^\Delta$ be the canonical embedding and let $\sigma^\Delta \colon \chi^\Delta \to \chi^{(2n)}$ be the homomorphism from Section 2.4. Let $\ell(v)$ denote the length of the vector $v \in \mathbb{R}^2$ with respect to the standard basis of $\mathbb{R}^2$

(a) For any $M \in \mathcal{A}$ and any $r \in \chi_+^\Delta$, we have $\ell(\delta_F^\mathcal{A}(rM)) = \sigma^\Delta \sigma_+^\Delta(r)\ell(\delta_F^\mathcal{A}(M))$ and $\delta_F^\mathcal{A}(rM)$ is collinear to $\delta_F^\mathcal{A}(M)$.

(b) Suppose $\mathcal{A}$ has a set of R_+-generators and that $N_1, N_2 \in \mathcal{A}$ are indecomposable. Then N_1 and N_2 have a common χ_+^Δ-generator if and only if N_1 and N_2 are concentrated in the same degree and $\delta_F^\mathcal{A}(N_1)$ and $\delta_F^\mathcal{A}(N_2)$ are collinear. Furthermore, $N_1 < N_2$ if and only if N_1 and N_2 are concentrated in the same degree, $\delta_F^\mathcal{A}(N_1)$ and $\delta_F^\mathcal{A}(N_2)$ are collinear, and $\ell(\delta_F^\mathcal{A}(N_1)) < \ell(\delta_F^\mathcal{A}(N_2))$.

One of the main results of [7] is that the R_+-coefficient categories that arise from foldings onto quivers of type H_4, H_3 and $I_2(2n + 1)$ all have basic, minimal, τ-closed sets of R_+-generators that are unique up to isomorphic elements. The same is not always true for R_+-coefficient categories that arise from foldings onto quivers of type $I_2(2n)$ (except for the foldings F^{E_7} and F^{E_8}). One always has sets of basic, τ-closed R_+-generators, but the condition that is lost is the minimality condition. The problem in this setting is, in effect, the category is too big. One has non-isomorphic indecomposable objects that map to the same (multiple of a) root, which leads to some objects being incomparable under the partial ordering. This happens precisely when there is a $\mathbb{Z}_2$-action on the category and when an object is not stable under this action. So to fix this, we need a weaker notion of minimality that takes into account the group action.

Lemma 4.5. *Suppose there exists a group $G \subset R_+$ and a set Γ of R_+-generators for $\mathcal{A}$. Then for each $g \in G$ and $M \in \Gamma$, the set*

$$\Gamma^{gM} = (\Gamma \setminus \{M\}) \cup \{gM\}$$

is a set of R_+-generators for $\mathcal{A}$

Proof. Since G is a group embedded in the semiring R_+, it follows that M is R_+-generated by gM by the pair $(g^{-1}, 0)$. This recovers the original set Γ. Hence, $\mathcal{A}(\Gamma^{gM}) \simeq \mathcal{A}(\Gamma) \simeq \mathcal{A}$, as required. $\square$

The above lemma implies that we have an equivalence relation $\sim_G$ on sets of R_+-generators whenever there is a group embedded in R_+. That is, we say that $\Gamma \sim_G \Gamma'$ if and only if for each $M \in \Gamma$, there exists $g_M \in G$ such that $\Gamma' = \{g_M M : M \in \Gamma\}$.

Definition 4.6. Suppose there exists a group $G \subset R_+$. We say that a basic set of R_+-generators Γ is *G-minimal* if for any basic set of R_+-generators $\Gamma' \not\sim_G \Gamma$, there exists an injective map of sets $\theta \colon \Gamma \to \Gamma'$ and inclusions $\iota \colon \Gamma \to \Gamma \cup \Gamma'$ and $\iota' \colon \Gamma' \to \Gamma \cup \Gamma'$ such that $\iota(M) \leq \iota'\theta(M)$ for each $M \in \Gamma$.

Theorem 4.7. *Let $F^\Delta \colon Q^\Delta \to Q^{I_2(2n)}$ be the weighted folding of type Δ. Then there exists a basic, τ-closed set of R_+-generators of $\mathcal{A}$, where $R_+ = \chi_+^\Delta$. In particular, we have the following.*

(a) *If $\Delta = A_{2n-1}$, then there exist four distinct (up to isomorphic elements) such sets of $\chi_+^{A_{2n-1}}$-generators that are $\mathbb{Z}_2$-minimal. Namely,*

$$\Gamma_{i,j} = \{M \in \mathcal{A} : [M] \in \mathcal{P}_i^{\mathcal{A}} \cup \mathcal{P}_j^{\mathcal{A}}\}$$

with $i \in \{0, 2n-2\}$ and $j \in \{1, 2n-3\}$. These are pairwise equivalent under $\sim_{\mathbb{Z}_2}$.

(b) *If $\Delta = D_4$, then there exist three distinct (up to isomorphic elements) such sets of $\chi_+^{D_4}$-generators that are $\mathbb{Z}_3$-minimal. Namely,*

$$\Gamma_i = \{M \in \mathcal{A} : [M] \in \mathcal{P}_i^{\mathcal{A}} \cup \mathcal{P}_1^{\mathcal{A}}\},$$

with $i \neq 1$. In addition, $\Gamma_i \sim_{\mathbb{Z}_3} \Gamma_j$ for any distinct $i, j \neq 1$.

(c) *If $\Delta = D_{n+1}$, then there exist two distinct (up to isomorphic elements) such sets of $\chi_+^{D_{n+1}}$-generators that are $\mathbb{Z}_2$-minimal. Namely,*

$$\Gamma^\pm = \{M \in \mathcal{A} : [M] \in \mathcal{P}_i^{\mathcal{A}} \cup \mathcal{P}_{(n-1)^\pm}^{\mathcal{A}}\},$$

where $i = 0$ if n is even and $i = 1$ if n is odd. In addition, $\Gamma^+ \sim_{\mathbb{Z}_2} \Gamma^-$.

(d) *If $\Delta = E_6$, then there exist four distinct (up to isomorphic elements) such sets of $\chi_+^{E_6}$-generators that are $\mathbb{Z}_2$-minimal. Namely,*

$$\Gamma^{(\pm,\pm)} = \{M \in \mathcal{A} : [M] \in \mathcal{P}_{0^\pm}^{\mathcal{A}} \cup \mathcal{P}_{1^\pm}^{\mathcal{A}}\},$$

$$\Gamma^{(\pm,\mp)} = \{M \in \mathcal{A} : [M] \in \mathcal{P}_{0^\pm}^{\mathcal{A}} \cup \mathcal{P}_{1^\mp}^{\mathcal{A}}\}.$$

These are pairwise equivalent under $\sim_{\mathbb{Z}_2}$.

(e) *If $\Delta = E_7$, then there exists a unique minimal such set of $\chi_+^{E_7}$-generators (up to isomorphic elements). Namely,*

$$\Gamma = \{M \in \mathcal{A} : [M] \in \mathcal{P}_0^{\mathcal{A}} \cup \mathcal{P}_{v_7}^{\mathcal{A}}\}.$$

(f) *If $\Delta = E_8$, then there exists a unique minimal such set of $\chi_+^{E_8}$-generators (up to isomorphic elements). Namely,*

$$\Gamma = \{M \in \mathcal{A} : [M] \in \mathcal{P}_0^{\mathcal{A}} \cup \mathcal{P}_1^{\mathcal{A}}\}.$$

Proof. (a) We will first show that $\Gamma_{0,1}$ is a set of R_+-generators. We have

$$\mathbf{M}_\alpha = \{[M_{0,\alpha}], [M_{2,\alpha}], \ldots, [M_{2n-2,\alpha}]\} \qquad (\alpha \in \Phi_{2n}^{s,+}),$$

$$\mathbf{M}_\beta = \{[M_{1,\beta}], [M_{3,\beta}], \ldots, [M_{2n-3,\beta}]\} \qquad (\beta \in \Phi_{2n}^{l,+}).$$

It is easy to see from (A1) that $\psi_{2i} M_{0,\alpha} \cong M_{2i,\alpha}$. Thus the iso-classes of $\mathbf{M}_\alpha$ are R_+-generated by $M_{0,\alpha}$. To see that the iso-classes of $\mathbf{M}_\beta$ are R_+-generated by $M_{1,\beta}$, first define $r_1 = \psi_2$ and $r_1' = 1$. It then follows from (A1) that $M_{2i+1,\beta}$ is R_+-generated by (r_{2i+1}, r_{2i+1}'), where $r_{2i+1} = \psi_{2i} + r_{2i-1}'$ and $r_{2i+1}' = r_{2i-1}$. Thus by Theorem 3.5 and Remark 4.2, $\Gamma_{0,1}$ is a set of R_+-generators for $\mathcal{A}$, which is clearly also basic and τ-closed.

It remains to show that $\Gamma_{0,1}$ is $\mathbb{Z}_2$-minimal. By (A1), we have $\mathbb{Z}_2 \cong \{1, \psi_{2n-2}\} \subset R_+$ as $\psi_{2n-2} M_{i,\gamma} \cong M_{2n-1-i,\gamma}$ for any i and positive root γ. From this and Lemma 4.5, we can see that $\Gamma_{0,1} \sim_{\mathbb{Z}_2} \Gamma_{2n-2,1} \sim_{\mathbb{Z}_2} \Gamma_{0,2n-3} \sim_{\mathbb{Z}_2} \Gamma_{2n-2,2n-3}$. Moreover, $\ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{0,\alpha})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{2i,\alpha}))$ for all $i \neq n-1$ and $\ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{1,\beta})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{2i+1,\beta}))$ for all $i \neq n-2$. So each $\Gamma_{i,j}$ is $\mathbb{Z}_2$-minimal by Remark 4.4, as required.

(b) This is a consequence of the sole generating element $g \in \chi_+^{D_4}$ being the $\mathbb{Z}_3$-group action on the category, along with the fact that $\ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{0,\alpha})) = \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{2^+,\alpha})) = \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{2^-,\alpha}))$ for each short positive root α.

(c) We will first show that in both the odd and even cases, the sets $\Gamma^\pm$ are both basic, τ-closed sets of R^+-generators that are equivalent under the group action $\mathbb{Z}_2 \cong \{1, \psi_0^-\} \subset R_+$. Suppose n is even. In this case we have

$$\mathbf{M}_\alpha = \{[M_{0,\alpha}], [M_{2,\alpha}], \ldots, [M_{n-2,\alpha}]\} \qquad (\alpha \in \Phi_{2n}^{s,+}),$$

$$\mathbf{M}_\beta = \{[M_{1,\beta}], [M_{3,\beta}], \ldots, [M_{(n-1)^+,\beta}], [M_{(n-1)^-,\beta}]\} \qquad (\beta \in \Phi_{2n}^{l,+}).$$

It is straightforward to see from (D1) that $M_{0,\alpha}$ R_+-generates the iso-classes in $\mathbf{M}_\alpha$ – the argument is identical to that used in type A. We can also see that $M_{(n-1)^\pm,\beta}$ both R_+-generate the iso-classes in $\mathbf{M}_\beta$. This follows from (D2), where we can see that each $M_{n-1-2i,\beta}$ is R_+-generated from $M_{(n-1)^\pm,\beta}$ by the pair $(\psi_{2i}^+, \psi_{2(i-1)}^-)$. Thus $\Gamma^\pm$ are both sets of R_+-generators of $\mathcal{A}$.

Now suppose n is odd. In this case we have

$$\mathbf{M}_\alpha = \{[M_{0,\alpha}], [M_{2,\alpha}], \ldots, [M_{(n-1)^+,\alpha}], [M_{(n-1)^-,\alpha}]\} \qquad (\alpha \in \Phi_{2n}^{s,+}),$$

$$\mathbf{M}_\beta = \{[M_{1,\beta}], [M_{3,\beta}], \ldots, [M_{n-2,\beta}]\} \qquad (\beta \in \Phi_{2n}^{l,+}).$$

By (D2), $M_{(n-1)^\pm,\alpha}$ R_+-generates the iso-classes in $\mathbf{M}_\alpha$ in the same way as $M_{(n-1)^\pm,\beta}$ does for $\mathbf{M}_\beta$ in the even case. By (D1), $M_{1,\beta}$ R_+-generates the iso-classes in $\mathbf{M}_\beta$ in a similar way to how $M_{1,\beta}$ does for $\mathbf{M}_\beta$ in type A — the pairs are the same except with $\pm$ superscripts. Thus, $\Gamma^\pm$ are again both sets of R_+-generators of $\mathcal{A}$.

It now remains to show that these sets of R_+-generators satisfy the required properties. That they are basic and τ-closed follows trivially by construction. The equivalence under $\sim_{\mathbb{Z}_2}$ is also obvious, as $\psi_0^- M_{i,\gamma} = M_{i,\gamma}$ for any γ and any $i \neq (n-1)^\pm$, and $\psi_0^- M_{(n-1)^\pm,\gamma} = M_{(n-1)^\mp,\gamma}$. Thus, it remains to show that these sets are $\mathbb{Z}_2$-minimal. Here, it is important to note the following: if $[M_{(n-1)^\pm,\gamma}] \in \mathbf{M}_\gamma$ for some γ, then every $M_{i,\gamma}$ (with $i \neq (n-1)^\pm$) does not R_+-generate $M_{(n-1)^\pm,\gamma}$. This follows from (D1), where we can see that for any object N that is R_+-generated by $M_{i,\gamma}$, $M_{(n-1)^+,\gamma}$ is isomorphic to direct summand of N if and only if $M_{(n-1)^-,\gamma}$ is isomorphic to direct summand of N. That is, it may be possible to R_+-generate $M_{(n-1)^+,\gamma} \oplus M_{(n-1)^-,\gamma}$ from $M_{i,\gamma}$, but not one of these summands individually. So even though we may have $M_{i,\gamma} < M_{(n-1)^\pm,\gamma}$, this is irrelevant, as any set that does not contain one of $M_{(n-1)^\pm,\gamma}$ is not a set of R_+-generators. As for the set of iso-classes such that $[M_{(n-1)^\pm,\gamma}] \notin \mathbf{M}_\gamma$, the $\mathbb{Z}_2$-minimality of $M_{0,\gamma}$ or $M_{1,\gamma}$ is obvious from Remark 4.4. Hence, $\Gamma^\pm$ are both $\mathbb{Z}_2$-minimal.

(d) Let α be a short positive root and β be a long positive root of $I_2(12)$. Then we can see $\Gamma^{(i,j)}$ (with $i,j \in \{\pm, \mp\}$) is a set of R_+-generators by the following table of R_+-generating pairs (where columns are R_+-generated by rows with pairs in the given cell).

	$M_{0^\pm,\alpha}$	$M_{0^\mp,\alpha}$	$M_{2,\alpha}$	$M_{1^\pm,\beta}$	$M_{1^\mp,\beta}$	$M_{v_6,\beta}$
$M_{0^\pm,\alpha}$	$(1,0)$	$(\psi_0^-, 0)$	$(\psi_2, 0)$			
$M_{1^\pm,\beta}$				$(1,0)$	$(\psi_0^-, 0)$	$(\psi_2, 1+\psi_0^-)$

By Theorem 3.5 and Remark 4.2, this is basic and τ-closed, as required. The group is $\mathbb{Z}_2 \cong \{1, \psi_0^-\} \subset R_+$, and thus it is clear from the table that these sets of R_+-generators are pairwise equivalent under $\sim_{\mathbb{Z}_2}$. To see that these are $\mathbb{Z}_2$-minimal, we note that

$$\ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{0^+,\alpha})) = \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{0^-,\alpha})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{2,\alpha})),$$

$$\ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{v_6,\beta})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{1^+,\beta})) = \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{1^-,\beta})),$$

but also that $M_{v_6,\beta}$ does not R_+-generate either $M_{1^\pm,\beta}$ (the best we can do is R_+-generate $M_{1^+,\beta} \oplus M_{1^-,\beta}$, which is insufficient). Hence, each $\Gamma^{(i,j)}$ is $\mathbb{Z}_2$-minimal by Remark 4.4, as required.

(e) Let α be a short positive root and β be a long positive root of $I_2(18)$. Then we can see Γ is a set of R_+-generators by the following table of R_+-generating pairs.

	$M_{0,\alpha}$	$M_{\widetilde{1},\alpha}$	$M_{\widetilde{2},\alpha}$	$M_{2,\alpha}$	$M_{v_7,\beta}$	$M_{1,\beta}$	$M_{3,\beta}$
$M_{0,\alpha}$	$(1,0)$	$(\psi_2\widetilde{\psi_2}, \psi_2 + \widetilde{\psi_2})$	$(\widetilde{\psi_2}, 0)$	$(\psi_2, 0)$			
$M_{v_7,\beta}$					$(1,0)$	$(\widetilde{\psi_2}, 1)$	$(\psi_2, 0)$

By Theorem 3.5 and Remark 4.2, Γ is basic and τ-closed, as required. It also follows from Remark 4.4 that Γ is minimal, since

$$\ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{0,\alpha})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{\widetilde{1},\alpha})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{\widetilde{2},\alpha})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{2,\alpha})),$$

$$\ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{v_7,\beta})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{1,\beta})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{3,\beta})),$$

as required.

(f) Let α be a short positive root and β be a long positive root of $I_2(30)$. Then we can see Γ is a set of R_+-generators by the following table of R_+-generating pairs.

	$M_{0,\alpha}$	$M_{2,\alpha}$	$M_{4,\alpha}$	$M_{\phi_0,\alpha}$	$M_{1,\beta}$	$M_{3,\beta}$	$M_{\phi_1,\beta}$	$M_{v_8,\beta}$
$M_{0,\alpha}$	$(1,0)$	$(\psi_2,0)$	$(\varphi\psi_2,0)$	$(\varphi,0)$				
$M_{1,\beta}$					$(1,0)$	$(\psi_2,1)$	$(\varphi,0)$	$(\varphi\psi_2+1, \psi_2+\varphi)$

By Theorem 3.5 and Remark 4.2, Γ is basic and τ-closed, as required. It also follows from Remark 4.4 that Γ is minimal, since

$$\ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{0,\alpha})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{\phi_0,\alpha})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{2,\alpha})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{4,\alpha})),$$

$$\ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{1,\beta})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{v_8,\beta})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{\phi_1,\beta})) < \ell(\delta_{F^\Delta}^{\mathcal{A}}(M_{3,\beta})),$$

as required. $\square$

Corollary 4.8. *Let F^Δ be a folding onto $I_2(2n)$ and let Γ be a basic, τ-closed set of minimal (or G-minimal) set of R_+-generators of $\mathcal{M}_\Delta$. Then the elements of Γ are in bijective correspondence with the positive roots of $I_2(2n)$.*

Proof. We have by construction (and Theorem 3.5) that the sets $\mathbf{M}_\alpha$ (with $\alpha \in \Phi_{2n}^+$) partition the iso-classes of indecomposable objects of $\mathcal{M}_\Delta$. By Theorem 4.7, $\Gamma = \{N_\alpha : \alpha \in \Phi_{2n}^+\}$, where each $[N_\alpha] \in \mathbf{M}_\alpha$. Thus, the elements of Γ are in bijective correspondence with the roots in Φ_{2n}^+, as required. $\square$

Similar to the situation in [7], a useful consequence of the above Theorem is the following.

Remark 4.9. Let $R_+ \subset R$, where R is the ring obtained by performing the Grothendieck group construction on the additive commutative monoid structure of R_+. Then the Grothendieck group $K_0(\mathcal{M}_\Delta)$ of the category $\mathcal{M}_\Delta$ has the structure of a free R-module of rank $|Q_0^{I_2(2n)}| = 2$. For each iso-class $[M] \in K_0(\mathcal{M}_\Delta)$, mulitplication by $r - r' \in R$

(where $r, r' \in R_+$) is given by $(r - r')[M] = [rM] - [r'M]$. A basis of $K_0(\mathcal{M}_\Delta)$ is given by simple objects that belong to a set of basic R_+-generators of $\mathcal{M}_\Delta$.

5. CLUSTER-R_+-TILTING THEORY FOR FOLDINGS ONTO $I_2(2n)$

Cluster algebras and mutations of Dynkin $\mathbb{Z}$-quivers have previously been categorified via the cluster category (constructed in [2], see [1] for further details). The semiring action on the category $\mathcal{D}_\Delta^b$ naturally extends to the cluster category $\mathcal{C}_\Delta = \mathcal{D}_\Delta^b / \langle \tau_{\mathcal{D}_\Delta^b} \Sigma^{-1} \rangle$ of Q^Δ. In particular, we have a canonical triangulated quotient functor $G \colon \mathcal{D}_\Delta^b \to \mathcal{C}_\Delta$ (due to [13]) and we define the action of R_+ on $\mathcal{C}_\Delta$ by $rG = Gr$ for each $r \in R_+$. Given any set of R_+-generators $\Gamma_{\mathcal{D}_\Delta^b}$ of $\mathcal{D}_\Delta^b$, we obtain a set of R_+-generators

$$\Gamma_{\mathcal{C}_\Delta} = \{G(M) : M \in \Gamma_{\mathcal{D}_\Delta^b}\}$$

of $\mathcal{C}_\Delta$. Moreover, $\Gamma_{\mathcal{C}_\Delta}$ is basic if $\Gamma_{\mathcal{D}_\Delta^b}$ is basic and $\Gamma_{\mathcal{C}_\Delta}$ is G-minimal if $\Gamma_{\mathcal{D}_\Delta^b}$ is G-minimal. For worked examples of the theory that follows, see Section A.

Definition 5.1. Let Γ be a set of R_+-generators of $\mathcal{C}_\Delta$. We say an object $T \in \mathcal{C}_\Delta$ is *(cluster)-R_+-tilting* with respect to Γ if the following hold.

(T1) $T \cong \bigoplus_{i=1}^{|T|} T_i$ with each $T_i \in \Gamma$.
(T2) T is R_+-rigid: $\operatorname{Ext}^1_{\mathcal{C}_\Delta}(T', T'') = 0$ for any $T', T'' \in \mathcal{G}_T$.
(T3) T is maximal: If there exists $X \in \Gamma$ such that $T \oplus X$ is R_+-rigid, then X is isomorphic to a direct summand of T.

In addition, we say an R_+-tilting object $T \in \mathcal{C}_\Delta$ is *basic* if the direct summands of T are pairwise non-isomorphic and T is with respect to a basic, G-minimal set of R_+-generators, where G is some (possibly trivial) group embedded in R_+.

Henceforth, we will assume that Γ is a basic, τ-closed, G-minimal set of R_+-generators of $\mathcal{C}_\Delta$ that corresponds to a basic, τ-closed, G-minimal set of R_+-generators Γ' of $\mathcal{D}_\Delta^b$ under the quotient functor $G \colon \mathcal{D}_\Delta^b \to \mathcal{C}_\Delta$.

Theorem 5.2. *Let $T \cong X_1 \oplus \ldots \oplus X_m \in \mathcal{C}_\Delta$ with each $X_i \in \Gamma$. For each i, denote by $\mathbf{I}_{X_i}$ the set of all iso-classes of indecomposable objects in $\mathcal{C}_\Delta$ that are R_+-generated by X_i. Then the following are equivalent.*

(a) T is basic R_+-tilting with respect to Γ.
(b) $T \cong X_1 \oplus X_2$, where $X_1, X_2 \in \Gamma$ reside in (distinct) adjacent columns of the Auslander-Reiten quiver of $\mathcal{C}_\Delta$.
(c) The object

$$\widehat{T} \cong \bigoplus_{i=1}^{m} \bigoplus_{[Y_i] \in \mathbf{I}_{X_i}} Y_i \in \mathcal{C}_\Delta$$

is basic tilting.

Proof. (c) $\Rightarrow$ (b): Since $\widehat{T}$ is basic, the direct summands of $\widehat{T}$ are pairwise non-isomorphic. By construction, this is true only if we have $X_i \not\cong X_j$ for all $i \neq j$. By Remark 4.2, the iso-classes in $\mathbf{I}_{X_i}$ are precisely the iso-classes of indecomposable objects that reside in the same column as X_i in the Auslander-Reiten quiver of $\mathcal{C}_\Delta$. Thus we conclude that $\widehat{T}$ is basic only if each X_i resides in a distinct column of the Auslander-Reiten quiver of Q_0^Δ. To see that $m = 2$, we note that if $m \neq 2$, then $|\widehat{T}| \neq |Q_0^\Delta|$, and thus cannot be tilting by the results of [2]. Thus, $T \cong X_1 \oplus X_2$.

It remains to show that X_1 and X_2 reside in adjacent columns. Suppose for a contrapositive argument that this is not the case. Then at least one of X_1 and X_2 is represented by an object $M_1 \in \mathcal{D}_\Delta^b$ concentrated in degree 0 — for convenience, we will say (without

loss of generality) that this is a representative of X_1. Then by assumption, X_2 is represented either by another object M_2 concentrated in degree 0, or by an object $\Sigma P(i) \in \mathcal{D}_\Delta^b$ for some $i \in Q_0^\Delta$.

In the latter case, if $P(i)$ is simple (or equivalently, i is a sink vertex), then $M_1 \not\cong S(j)$ for any source vertex $j \in Q_0^\Delta$ (otherwise, X_1 and X_2 are adjacent). Thus we must have $S(k) \subseteq M_1$ for some sink vertex $k \in Q_0^\Delta$ and hence $\mathrm{Ext}^1_{\mathcal{D}_\Delta^b}(\Sigma S(k), M_1) = \mathrm{Hom}_{\mathcal{D}_\Delta^b}(\Sigma S(k), \Sigma M_1) \neq 0$. But each $\Sigma S(k)$ (k a source vertex) is a representative of some object $Y_2 \in \mathcal{C}_\Delta$ in the same column of the Auslander-Reiten quiver as X_2 (that is, $[Y_2] \in \mathbf{I}_{X_2}$). So $\mathrm{Ext}^1_{\mathcal{D}_\Delta^b}(\widehat{T}, \widehat{T}) \neq 0$, which implies $\mathrm{Ext}^1_{\mathcal{C}_\Delta}(\widehat{T}, \widehat{T}) \neq 0$, and hence $\widehat{T}$ is not tilting, as required. On the other hand, if $P(i)$ is not simple (or equivalently, i is a source vertex), then $M_1 \not\cong S(j)$ for any sink vertex $j \in Q_0^\Delta$ (otherwise, X_1 and X_2 are again adjacent). In this subcase, we trivially have $\mathrm{Ext}^1_{\mathcal{D}_\Delta^b}(\Sigma P, M_1) = \mathrm{Hom}_{\mathcal{D}_\Delta^b}(\Sigma P, \Sigma M_1) \neq 0$ for some indecomposable object P in the same column as $P(i)$, as this occurs as a component of a projective cover of M_1 in the category $\mathcal{M}_\Delta$. Thus we again have $\mathrm{Ext}^1_{\mathcal{C}_\Delta}(\widehat{T}, \widehat{T}) \neq 0$, which implies $\widehat{T}$ is not tilting.

Now we consider the former case where we have representatives M_1 and M_2 both concentrated in degree 0. Suppose without loss of generality that M_2 resides in a column strictly to the right of M_1 (that is, in the direction of the irreducible morphisms in the Auslander-Reiten quiver). Then M_1 does not correspond to a injective object in $\mathcal{M}$ and M_2 does not correspond to a projective object in $\mathcal{M}$ (otherwise we are forced to have X_1 and X_2 in adjacent columns). Thus by the Auslander-Reiten formula,

$$\dim_K \mathrm{Ext}^1_{\mathcal{D}_\Delta^b}(M_2', M_1') = \dim_K \mathrm{Hom}_{\mathcal{D}_\Delta^b}(M_2', \Sigma M_1') = \dim_K \mathrm{Hom}^1_{\mathcal{M}_\Delta}(M_1', \tau M_2'),$$

where each M_i' is some object in the same column of the Auslander-Reiten quiver as M_i. This quantity must be non-zero for some choice of M_1' and M_2', since $\tau M_2'$ is not injective and thus has a simple projective subobject P such that the inclusion of P into $\tau M_2'$ factors through M_1'. This implies that we must have $\mathrm{Ext}^1_{\mathcal{C}_\Delta}(\widehat{T}, \widehat{T}) \neq 0$ in this case, which implies $\widehat{T}$ is not tilting, as required. This was the final case to consider, and hence we conclude that X_1 and X_2 must reside in (distinct) adjacent columns of the Aulsander-Reiten quiver.

(b) $\Rightarrow$ (a): Let $Y_1 \in \mathcal{G}_{X_1}$ and $Y_2 \in \mathcal{G}_{X_2}$ be indecomposable and let $M_1, M_2 \in \mathcal{D}_\Delta^b$ be representatives of Y_1 and Y_2 respectively. Note that by Remark 4.2 that Y_1 resides in the same column as X_1 and Y_2 resides in the same column as X_2 in the Auslander-Reiten quiver of $\mathcal{C}_\Delta$.

If M_1 and M_2 are concentrated in the same degree, then they also reside in adjacent columns of the Auslander-Reiten quiver of $\mathcal{D}_\Delta^b$, and clearly we have

$$\mathrm{Ext}^1_{\mathcal{D}_\Delta^b}(M_1, M_2) = \mathrm{Ext}^1_{\mathcal{D}_\Delta^b}(M_2, M_1) = 0$$

So assume Y_1 and Y_2 are concentrated in the degrees k and $k+1$ (where we assume without loss of generality that Y_1 is concentrated in degree k), then clearly we have $\mathrm{Ext}^1_{\mathcal{D}_\Delta^b}(Y_1, Y_2) = \mathrm{Hom}_{\mathcal{D}_\Delta^b}(Y_1, \Sigma Y_2) = 0$, since $\mathcal{D}_\Delta^b$ is hereditary. To compute $\mathrm{Ext}^1_{\mathcal{D}_\Delta^b}(Y_2, Y_1) = \mathrm{Hom}_{\mathcal{D}_\Delta^b}(Y_2, \Sigma Y_1)$, we note that either Y_2 and ΣY_1 are concentrated in the same degree. By construction, there are two possibilities: either Y_2 and ΣY_1 are non-isomorphic simple objects or ΣY_1 necessarily resides in a column to the left of Y_2 in the Auslander-Reiten quiver of $\mathcal{D}_\Delta^b$. In both cases, we trivially have $\mathrm{Ext}^1_{\mathcal{D}_\Delta^b}(Y_2, Y_1) = 0$.

Now assume Y_1 and Y_2 are concentrated in the degrees k and $k+l$ with $l \geq 2$ respectively. Again we have $\mathrm{Ext}^1_{\mathcal{D}_\Delta^b}(Y_1, Y_2) = 0$ from the fact that $\mathcal{D}_\Delta^b$ is hereditary and that

$\text{Ext}^1_{\mathcal{D}^b_\Delta}(Y_2, Y_1) = 0$ since ΣY_1 resides in a column to the left of Y_2 in the Auslander-Reiten quiver of $\mathcal{D}^b_\Delta$.

Since Γ is a set of R_+-generators, and every object R_+-generated by X_i resides in the same column as X_i, and we have made an argument based on arbitrary indecomposables in each column, we have $\text{Ext}_{\mathcal{C}_\Delta}(X'_1, X'_2) = \text{Ext}_{\mathcal{C}_\Delta}(X'_2, X'_1) = 0$ for any $X'_1 \in \mathcal{G}_{X_1}$ and $X'_2 \in \mathcal{G}_{X_2}$. By similar arguments to the above, one also has $\text{Ext}_{\mathcal{C}_\Delta}(X'_1, X'_1) = \text{Ext}_{\mathcal{C}_\Delta}(X'_2, X'_2) = 0$ for any $X'_1 \in \mathcal{G}_{X_1}$ and $X'_2 \in \mathcal{G}_{X_2}$. Thus, T must be R_+-rigid. It is also basic, since X_1 and X_2 are non-isomorphic by assumption. That T is maximal follows from the fact that there exists an element $r \in R_+$ such that every indecomposable object residing in the columns of both X_1 and X_2 belong as subobjects of rT. Namely,

$$r = \begin{cases} \sum_{j=0}^{n-1} \psi_{2j} & \text{if } \Delta = A_{2n-1}, \\ \sum_{j=0}^{2\lfloor \frac{n-1}{2} \rfloor} \psi_{2j}^+ + \psi_{2j}^- & \text{if } \Delta = D_{n+1}, \\ 1 + \psi_0^- + \psi_2 & \text{if } \Delta = E_6, \\ 1 + \psi_2 + \widetilde{\psi}_2 + \psi_2\widetilde{\psi}_2 & \text{if } \Delta = E_7, \\ 1 + \psi_2 + \varphi + \varphi\psi_2 & \text{if } \Delta = E_8. \end{cases}$$

In particular, $|rT| = |Q_0^\Delta|$, where $|rT|$ denotes the number of distinct (up to isomorphism) indecomposable direct summands of rT. Thus rT is a basic cluster tilting object. But then if X_3 is distinct from X_1 and X_2 (which since Γ is G-minimal, implies it resides in a distinct column), we must then have direct summands $Z_1, Z_2 \subseteq r(T \oplus X_3)$ such that $\text{Ext}_{\mathcal{C}_\Delta}(Z_1, Z_2) \neq 0$. Hence, T must be maximal and hence basic R_+-tilting with respect to Γ, as required.

(a) $\Rightarrow$ (c): This is immediate from the last part of the proof of (b) $\Rightarrow$ (a). $\square$

Recall from [7] that we call an object $T \in \mathcal{C}_\Delta$ *almost complete basic R_+-tilting (with respect to Γ)* if T is R_+-rigid and there exists $X \in \Gamma$ such that $T \oplus X$ is basic R_+-tilting. We call such an object X a *complement* to T. The following is a trivial consequence of the above theorem.

Corollary 5.3. *Let $T \in \mathcal{C}_\Delta$.*

(a) *If T is basic R_+-tilting, then T has precisely $|Q_0^{I_2(2n)}| = 2$ direct summands.*
(b) *If T is almost complete basic R_+-tilting, then T is indecomposable and has precisely two complements.*

Also recall from [7] that given a basic R_+-tilting object $T \in \mathcal{C}_\Delta$, we call the algebra $\text{End}_{\mathcal{C}_\Delta}(\widehat{T})^{\text{op}}$ the *cluster-R_+-tilted algebra*, where $\widehat{T}$ is the cluster-tilting object given in Theorem 5.2(c).

Remark 5.4. For any basic R_+-tilting objects $T_1 = X \oplus X_1, T_2 = X \oplus X_2 \in \mathcal{C}_\Delta$, the respective cluster-R_+-tilted algebras A_1 and A_2 are Morita equivalent to either KQ^Δ or $(KQ^\Delta)^{\text{op}}$. In particular, $A_1 \simeq KQ^\Delta$ if and only if $A_2 \simeq (KQ^\Delta)^{\text{op}}$, and thus changing complement corresponds to mutation of the folded quiver $Q^{I_2(2n)}$. Furthermore, the associated module, bounded derived and cluster categories of the cluster-R_+-tilted algebras have the structure of R_+-coefficient categories in the natural way.

6. THE c- AND g-VECTORS OF EXCHANGE MATRICES OF TYPE $I_2(2n)$

The c- and g-vectors of an integer exchange matrix were introduced in [10]. In [7], we defined c- and g-vectors for quivers of type H_4, H_3 and $I_2(2n+1)$. We will do the same now for quivers of type $I_2(2n)$ using similar methods.

6.1. **Representations of $\widehat{\chi}^\Delta$.** The first step in defining c- and g-vectors for $I_2(2n)$ is to realise the exchange matrix of the unfolded quiver Q^Δ as a block matrix with blocks given (in some way) by representations of an associated ring. The situation is slightly more complicated in the $I_2(2n)$ case, as on the one hand we have a subring $\chi^\Delta \subset \widehat{\chi}^\Delta$ whose role is to act on the larger ring, and on the other hand, the exchange matrix of $I_2(2n)$ has a rescaling due to the simple roots being of different size.

It is helpful to bypass the problems caused by the embedding of rings and different root sizes by instead considering the doubled quiver $\bar{\bar{Q}}^\Delta = Q^\Delta \sqcup (Q^\Delta)^{\mathrm{op}}$ with vertex weights given as before. That is, Q^Δ has the same R-valuation as given in the folding F^Δ, and $(Q^\Delta)^{\mathrm{op}}$ is a copy of this quiver with the same R-valuation, but with the orientation of the arrows reversed.

Part of the idea here is that the exchange matrix of $\bar{\bar{Q}}^\Delta$ has nicer properties (which we will later demonstrate), however it is also the most canonical setting when considering unfoldings of the rescaled/skew-symmetric form of $I_2(2n)$ (where roots are considered to be of the same length). That is, the doubled-quiver $\bar{\bar{Q}}^\Delta$ allows us to consider a folding onto the *'rescaled'* quiver $\vec{Q}^{I_2(2n)}$, which is the same as $Q^{I_2(2n)}$, except we now have an R-valuation of $\varsigma([0]) = \varsigma([1]) = \lambda$, where we define

$$\lambda = \begin{cases} 1 & \text{if } \Delta \neq D_{n+1}, \\ 2 & \text{if } \Delta = D_{n+1}. \end{cases}$$

We thus have a corresponding weighted (double-)folding $\bar{\bar{F}}^\Delta \colon \bar{\bar{Q}}^\Delta \to \vec{Q}^{I_2(2n)}$ such that $\bar{\bar{F}}^\Delta$ maps source vertices in $\bar{\bar{Q}}^\Delta$ to the source vertex $[0] \in Q_0^{I_2(2n)}$, and likewise maps sink vertices to the sink vertex $[1] \in Q_0^{I_2(2n)}$.

We will recall the definition of the regular representation of a ring R from [7].

Definition 6.1. Suppose R has a finite $\mathbb{Z}$-basis $\mathcal{B}$. Define a ring homomorphism

$$\rho \colon R \to \mathbb{Z}^{|\mathcal{B}| \times |\mathcal{B}|} \colon r \mapsto \rho(r),$$

where the j-th column of $\rho(r)$ (with $j \in \mathcal{B}$) is the vector $(a_{ij})_{i \in \mathcal{B}}$ given by $jr = \sum_{i \in \mathcal{B}} a_{ij} i$. We call the map ρ the *regular representation of R with respect to $\mathcal{B}$*.

For $\Delta \in \{A_{2n-1}, D_4, E_6, E_7, E_8\}$, we will later see that the regular representation of $\widehat{\chi}^\Delta$ is sufficient to describe the blocks of the exchange matrix of $\bar{\bar{Q}}^\Delta$. This motivates the following definition.

Definition 6.2. Let $\Delta \in \{A_{2n-1}, D_4, E_6, E_7, E_8\}$ and consider the $\mathbb{Z}$-bases

$$\begin{aligned}
\mathcal{B}^{A_{2n-1}} &= \{\psi_0, \psi_1, \ldots, \psi_{2n-2}\}, \\
\mathcal{B}^{D_4} &= \{\psi_0, \psi_1, g, g^2\}, \\
\mathcal{B}^{E_6} &= \{\psi_0^+, \psi_0^-, \psi_1^+, \psi_1^-, \psi_2, \psi_{v_6}\}, \\
\mathcal{B}^{E_7} &= \{\psi_0, \psi_1, \widetilde{\psi_1}, \psi_2, \widetilde{\psi_2}, \psi_3, \psi_{v_7}\}, \\
\mathcal{B}^{E_8} &= \{\psi_0, \varphi, \psi_1, \varphi\psi_1, \psi_2, \varphi\psi_2, \psi_{v_8}, \varphi\psi_{v_8}\},
\end{aligned}$$

where both ψ_0 and ψ_0^+ denote $1 \in \widehat{\chi}^\Delta$. We call the regular representation ρ^Δ of $\widehat{\chi}^\Delta$ with respect to $\mathcal{B}^\Delta$ the *representation of $\widehat{\chi}^\Delta$ corresponding to $\bar{\bar{F}}^\Delta$*.

The situation for type D_{n+1} with $n > 3$ is slightly more complicated. In this case, the ring $\widehat{\chi}^{D_{n+1}}$ has a positive $\mathbb{Z}$-basis

$$\mathcal{B}^{D_{n+1}} = \{\psi_0^\pm, \psi_1^\pm, \ldots, \psi_{n-1}^\pm\}$$

that is useful to us, but is too big to describe the blocks of the exchange matrix of $\bar{\bar{Q}}^{D_{n+1}}$. Instead, we have the following.

Definition 6.3. Let $n > 3$ and consider the ideal (and proper submodule) of $\widehat{\chi}^{D_{n+1}}$

$$\langle \psi_0^+ + \psi_0^-, \psi_1^+ + \psi_1^-, \ldots, \psi_{n-2}^+ + \psi_{n-2}^-, \psi_{n-1}^+, \psi_{n-1}^- \rangle \subset \widehat{\chi}^{D_{n+1}},$$

whose integral basis $\widehat{\mathcal{B}}^{D_{n+1}}$ is given by its listed set of generators. From this, define a ring homomorphism

$$\rho^{D_{n+1}} \colon \widehat{\chi}^{D_{n+1}} \to \mathbb{Z}^{n+1 \times n+1} \colon r \mapsto \rho^{D_{n+1}}(r),$$

where the j-th column of $\rho^{D_{n+1}}(r)$ ($j \in \widehat{\mathcal{B}}^{D_{n+1}}$) is the vector $(a_{ij})_{i \in \widehat{\mathcal{B}}^{D_{n+1}}}$ given by $jr = \sum_{i \in \widehat{\mathcal{B}}^{D_{n+1}}} a_{ij} i$. We call the map $\rho^{D_{n+1}}$ the *representation of $\widehat{\chi}^{D_{n+1}}$ corresponding to* $\bar{\bar{F}}^{D_{n+1}}$.

The bases for the above representations have been deliberately chosen such that one has a bijective map

$$
\begin{aligned}
\vartheta^\Delta \colon \mathcal{B}^\Delta &\to Q_0^\Delta & (\Delta \in \{A_{2n-1}, D_4, E_6, E_7, E_8\}) \\
\widehat{\mathcal{B}}^\Delta &\to Q_0^\Delta & (\Delta = D_{n+1}, n > 3) \\
\psi_i &\mapsto i & (\Delta \in \{A_{2n-1}, E_6, E_7, E_8\}) \\
g &\mapsto 2^+ & (\Delta = D_4) \\
g^2 &\mapsto 2^- & (\Delta = D_4) \\
\psi_i^+ + \psi_i^- &\mapsto i & (\Delta = D_{n+1}, n > 3) \\
\psi_i^\pm &\mapsto i^\pm & (\Delta \in \{D_{n+1}, E_6\}) \\
\varphi \psi_i &\mapsto \phi_i & (\Delta = E_8, i \in \{0, 1\}) \\
\varphi \psi_{v_8} &\mapsto 3 & (\Delta = E_8) \\
\varphi \psi_2 &\mapsto 4 & (\Delta = E_8)
\end{aligned}
$$

which satisfies $\kappa \vartheta^\Delta(e) = \widehat{\sigma}^\Delta(e)$ for each e.

Remark 6.4. Consider the element $s \in \widehat{\chi}^\Delta$ with

$$
s = \begin{cases} \psi_1 & \Delta \in \{A_{2n-1}, D_4, E_7, E_8\}, \\ \psi_1^+ & \Delta \in \{D_{n+1}, E_6\}. \end{cases}
$$

Then we have the following that result from the structure of $\widehat{\chi}^\Delta$.

(a) For each $e \in \mathcal{B}^\Delta$, the product $se = \sum_{e' \in \mathcal{B}^\Delta} a_{e'} e'$ is such that each $a_e \in \{0, 1\}$.

(b) There exists an arrow $\vartheta^\Delta(e) \to \vartheta^\Delta(e')$ or an arrow $\vartheta^\Delta(e') \to \vartheta^\Delta(e)$ in Q^Δ if and only if e' is a summand of the product se.

(c) Let $\ll$ be a total ordering of the set $\bar{\bar{Q}}_0^\Delta$ satisfying $v_0 \ll v_1$ for any vertices $v_0, v_1 \in \bar{\bar{Q}}_0^\Delta$ such that $\bar{\bar{F}}^\Delta(v_0) = [0]$ and $\bar{\bar{F}}^\Delta(v_1) = [1]$. Then the exchange matrix of $\bar{\bar{Q}}^\Delta$ with rows and columns ordered with respect to $\ll$ is given by the block matrix

$$\bar{\bar{B}}^\Delta = \begin{pmatrix} 0 & \rho^\Delta(s) \\ -\rho^\Delta(s) & 0 \end{pmatrix}.$$

(d) Let $<$ be a total ordering of the set Q_0^Δ. Then the exchange matrix of Q^Δ is

$$B^\Delta = \Lambda^\Delta \bar{\bar{B}}^\Delta \Theta^\Delta,$$

where Θ^Δ is the matrix corresponding to the canonical inclusion $Q_0^\Delta \to \bar{\bar{Q}}_0^\Delta$ and Λ^Δ is the matrix corresponding to the canonical surjection $\bar{\bar{Q}}_0^\Delta \to Q_0^\Delta$ of $\mathbb{Z}$-bases/index sets with respect to the order $<$.

(e) Since $\bar{\bar{Q}}_\Delta = Q^\Delta \sqcup (Q^\Delta)^{\mathrm{op}}$, the exchange matrix $B^\Delta \oplus (B^\Delta)^T$ is indexed by the set $\bar{\bar{Q}}_0^\Delta$, where $\oplus$ is the direct sum of linear maps and $(B^\Delta)^T$ is the matrix transpose of B^Δ. In particular,

$$\bar{\bar{B}}^\Delta = V(B^\Delta \oplus (B^\Delta)^T)V^{-1}$$

where V is the orthogonal permutation matrix that orders the $\mathbb{Z}$-basis/index set for the rows of $B^\Delta \oplus (B^\Delta)^T$ with respect to $\ll$ from (c). It therefore follows that, the operation $\Lambda^\Delta(-')\Theta^\Delta$ is left inverse to $V(- \oplus -^T)V^{-1}$, where $-$ is a placeholder for a $|Q_0^\Delta| \times |Q_0^\Delta|$ integer matrix with respect to the ordering $<$, and $-'$ is a placeholder for a $|\bar{\bar{Q}}_0^\Delta| \times |\bar{\bar{Q}}_0^\Delta|$ integer matrix with respect to the ordering $\ll$.

Remark 6.5. Let $e, e' \in \mathcal{B}^\Delta$.
  (a) If $\Delta \neq E_7$, then ee' is a sum of positive multiples of elements of $\mathcal{B}^\Delta$. Thus the entries of the matrix $\rho^\Delta(e)$ are all positive.
  (b) If $\Delta = E_7$ and $e \notin \{\widetilde{\psi}_1, \psi_{v_7}\}$, then ee' is also sum of positive multiples of elements of $\mathcal{B}^{E_7}$ and hence the entries of the matrix $\rho^{E_7}(b)$ are all positive. On the other hand, we have

$$\rho^{E_7}(\widetilde{\psi}_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \qquad \text{and} \qquad \rho^{E_7}(\psi_{v_7}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the ordering $\psi_0 < \widetilde{\psi}_1 < \widetilde{\psi}_2 < \psi_2 < \psi_1 < \psi_3 < \psi_{v_7}$.

6.2. C-matrices and c-vectors.
We shall recall the following general definitions for C-matrices and c-vectors of R-quivers from [7].

Definition 6.6. Let J be an index set and let $\mathbb{T}_J$ be the $|J|$-regular tree, where any pair of distinct edges incident to a common vertex in $\mathbb{T}_J$ are respectively labelled by distinct indices in J. Choose a distinguished vertex $t_0 \in \mathbb{T}_J$ and consider an exchange matrix B over a ring R with rows and columns indexed by J. We define the *extended exchange matrix of B over R* to be the $2|J| \times |J|$ matrix

$$\widetilde{B}_{t_0} = \begin{pmatrix} B_{t_0} \\ C_{t_0} \end{pmatrix},$$

where $B_{t_0} = B$ and C_{t_0} is the identity matrix. For any edge $t \overset{k}{\underline{\quad}} t'$ in $\mathbb{T}_J$, one then defines matrices

$$\widetilde{B}_t = \begin{pmatrix} B_t \\ C_t \end{pmatrix} \qquad \text{and} \qquad \widetilde{B}_{t'} = \begin{pmatrix} B_{t'} \\ C_{t'} \end{pmatrix}$$

such that $\widetilde{B}_{t'}$ is obtained from $\widetilde{B}_t$ by mutation at $k \in J$. We thus define the mutation of C_t at $k \in J$ as $\mu_k(C_t) = C_{t'}$.

With this in mind, we define a *tropical y-seed pattern* by assigning to each vertex $t \in \mathbb{T}_J$ a *tropical y-seed* $\{B_t; \mathbf{c}_{t,j} : j \in J\}$, where each $\mathbf{c}_{t,j}$ is the j-th column of the matrix C_t. We call the tropical y-seed at the vertex t_0 the *initial tropical y-seed*. We call the matrices C_t (where $t \in \mathbb{T}_J$) *C-matrices*, and we call the vectors $\mathbf{c}_{t,j}$ *c-vectors*. The c-vectors $\mathbf{c}_{t_0,j}$ in particular are called the *initial c-vectors*.

The folded quivers $Q^{I_2(2n)}$ arise from skew-symmetrisable exchange matrices $B^{I_2(2n)} = (b_{[i][j]})_{[i][j] \in Q_0^{I_2(2n)}}$ that have been rescaled to a skew-symmetric exchange matrix

$$\vec{B}^{I_2(2n)} = PB^{I_2(2n)}P^{-1} = (\vec{b}_{[i][j]})_{[i][j] \in Q_0^{I_2(2n)}}.$$

The relationship between these exchange matrices, their respective C-matrices and mutation can be summarised by the following due to [21].

Lemma 6.7. *Consider the initial extended exchange matrices*

$$\begin{pmatrix} B_{t_0}^{I_2(2n)} \\ C_{t_0}^{I_2(2n)} \end{pmatrix} \qquad and \qquad \begin{pmatrix} \vec{B}_{t_0}^{I_2(2n)} \\ \vec{C}_{t_0}^{I_2(2n)} \end{pmatrix}$$

of $B^{I_2(2n)}$ and $\vec{B}^{I_2(2n)}$, respectively. Mutation of $B^{I_2(2n)}$ and its C-matrices commutes with rescaling. In particular, for any vertex $t \in \mathbb{T}_{Q_0^{I_2(2n)}}$ we have

(a) $\vec{B}_t^{I_2(2n)} = PB_t^{I_2(2n)}P^{-1}$,
(b) $\vec{C}_t^{I_2(2n)} = PC_t^{I_2(2n)}P^{-1}$.

Proof. (a) The effect of rescaling by the diagonal matrix $P = (p_{[i]})_{[i] \in Q_0^{I_2(2n)}}$ is such that

$$\vec{b}_{[i][j]} = \frac{p_{[i]}}{p_{[j]}} b_{[i][j]}.$$

Since the rescaling matrix has positive entries, it is clear from the above and the mutation formula for exchange matrices that rescaling commutes with mutation. That is,

$$\mu_{[k]}(\vec{B}^{I_2(2n)}) = P\mu_{[k]}(B^{I_2(2n)})P^{-1}$$

for each $[k] \in Q_0^{I_2(2n)}$, and thus $\vec{B}_t^{I_2(2n)} = PB_t^{I_2(2n)}P^{-1}$ as required.

(b) Since initial C-matrices are identity matrices, the extension of the initial rescaled exchange matrix of type $Q^{I_2(2n)}$ is such that

$$\begin{pmatrix} \vec{B}_{t_0}^{I_2(2n)} \\ \vec{C}_{t_0}^{I_2(2n)} \end{pmatrix} = \begin{pmatrix} PB_{t_0}^{I_2(2n)}P^{-1} \\ \vec{C}_{t_0}^{I_2(2n)} \end{pmatrix} = \begin{pmatrix} PB_{t_0}^{I_2(2n)}P^{-1} \\ PC_{t_0}^{I_2(2n)}P^{-1} \end{pmatrix}.$$

Also recall the mutation formula for C-matrices corresponding to an edge $t \overset{[k]}{\rule{1.2em}{0.4pt}} t'$:

$$c_{[i][j],t'} = \begin{cases} -c_{[i][k],t} & \text{if } [j] = [k], \\ c_{[i][j],t} + \text{sgn}(c_{[i][k],t})[c_{[i][k],t}b_{[k][j],t}]_+ & \text{otherwise,} \end{cases}$$

where for any value a, $\text{sgn}(a)$ denotes the sign of a and $[a]_+ = \max(0, a)$. From this, it is clear that mutation of C-matrices also commutes with rescaling. Thus,

$$\vec{C}_{t'}^{I_2(2n)} = PC_{t'}^{I_2(2n)}P^{-1},$$

for each vertex $t' \in \mathbb{T}_{Q_0^{I_2(2n)}}$. □

Since we have weighted foldings $F^\Delta \colon Q^\Delta \to Q^{I_2(2n)}$ and $\bar{\bar{F}}^\Delta \colon \bar{\bar{Q}}^\Delta \to Q^{I_2(2n)}$, each vertex $t \in \mathbb{T}_{Q_0^{I_2(2n)}}$ is associated to vertices $\hat{t} \in \mathbb{T}_{Q_0^\Delta}$ and $\bar{\bar{t}} \in \mathbb{T}_{\bar{\bar{Q}}_0^\Delta}$ obtained via the unfolding procedure. That is, if t is obtained from t_0 by traversing edges $k_1, \ldots, k_m$ in $\mathbb{T}_{Q_0^{I_2(2n)}}$, then $\hat{t}$ is obtained from the initial vertex $\hat{t}_0 \in \mathbb{T}_{Q_0^\Delta}$ by traversing sequences of edges $\hat{k}_1, \ldots, \hat{k}_m$ in $\mathbb{T}_{Q_0^\Delta}$, where $\hat{k}_l$ corresponds to the composite mutation $\hat{\mu}_{[k_l]} = \prod_{F^\Delta(i)=[k_l]} \mu_i$. Similarly, $\bar{\bar{t}}$ is obtained from the initial vertex $\bar{\bar{t}}_0 \in \mathbb{T}_{\bar{\bar{Q}}_0^\Delta}$ by traversing sequences of edges $\bar{\bar{k}}_1, \ldots, \bar{\bar{k}}_m$ in $\mathbb{T}_{\bar{\bar{Q}}_0^\Delta}$, where $\bar{\bar{k}}_l$ corresponds to the composite mutation $\bar{\bar{\mu}}_{[k_l]} = \prod_{\bar{F}^\Delta(i)=[k_l]} \mu_i$. The only vertices of $\mathbb{T}_{Q_0^\Delta}$ and $\mathbb{T}_{\bar{\bar{Q}}_0^\Delta}$ that we will consider in this paper are those associated to the vertices $t \in \mathbb{T}_{Q_0^{I_2(2n)}}$ by unfolding. Thus for the purposes of readability, we will abuse notation and write $\hat{t}$ and $\bar{\bar{t}}$ as t. Likewise, if the context is clear, we will write $\hat{\mu}_{[k]}$ and $\bar{\bar{\mu}}_{[k]}$ as $\mu_{[k]}$.

The relationship between the C-matrices of the double-unfolded quiver $\bar{\bar{Q}}^\Delta$ and the C-matrices of the single-unfolded quiver Q^Δ is similar to what happens with rescaling.

Lemma 6.8. *Consider the initial extended exchange matrices*

$$\begin{pmatrix} B^\Delta_{t_0} \\ C^\Delta_{t_0} \end{pmatrix} \qquad and \qquad \begin{pmatrix} \bar{\bar{B}}^\Delta_{t_0} \\ \bar{\bar{C}}^\Delta_{t_0} \end{pmatrix}$$

of B^Δ and $\bar{\bar{B}}^\Delta$, respectively. Composite mutation (with respect to unfolding) of $\bar{\bar{B}}^\Delta$ and its C-matrices commutes with the operation $\Lambda^\Delta(-)\Theta^\Delta$. Similarly, composite mutation of B^Δ and its C-matrices commutes with the operation $V(- \oplus -^T)V^{-1}$. In particular, for any vertex $t \in \mathbb{T}_{Q_0^{I_2(2n)}}$ we have

(a) $B^\Delta_t = \Lambda^\Delta \bar{\bar{B}}^\Delta_t \Theta^\Delta$ *and* $\bar{\bar{B}}^\Delta_t = V(B^\Delta_t \oplus (B^\Delta_t)^T)V^{-1}$,
(b) $C^\Delta_t = \Lambda^\Delta \bar{\bar{C}}^\Delta_t \Theta^\Delta$ *and* $\bar{\bar{C}}^\Delta_t = V(C^\Delta_t \oplus (C^\Delta_t)^T)V^{-1}$.

Proof. Similar to the previous lemma, we have

$$\begin{pmatrix} B^\Delta_{t_0} \\ C^\Delta_{t_0} \end{pmatrix} = \begin{pmatrix} \Lambda^\Delta \bar{\bar{B}}^\Delta_{t_0} \Theta^\Delta \\ \Lambda^\Delta \bar{\bar{C}}^\Delta_{t_0} \Theta^\Delta \end{pmatrix}.$$

Since $\bar{\bar{Q}}^\Delta$ is a disjoint union of Q^Δ and $(Q^\Delta)^{\mathrm{op}}$, the result follows from the unfolding procedure and mutation formulae. The commutativity of $V(- \oplus -^T)V^{-1}$ with mutation follows by the same argument. $\qquad\square$

In what follows, we will utilise the block matrix structure of unfolded exchange matrices and C-matrices. In particular, for the double-folding $\bar{\bar{F}}^\Delta$, we will denote by $\bar{\bar{B}}^\Delta_{[i][j],t}$ and $\bar{\bar{C}}^\Delta_{[i][j],t}$ the $([i],[j])$-th block of $\bar{\bar{B}}^\Delta_t$ and $\bar{\bar{C}}^\Delta_t$, respectively. Here, we note that $\bar{\bar{B}}^\Delta_{[i][j],t}$ corresponds to the $([i],[j])$-th entry of B_t under the unfolding procedure.

Proposition 6.9. *Let $t \in \mathbb{T}_{Q_0^{I_2(2n)}}$ and let $\Delta \in \{A_{2n-1}, D_{n+1}, E_6, E_7, E_8\}$.*

(a) *For any $[i],[j] \in Q_0^{I_2(2n)}$, there exists $r_{[i][j],t} \in \widehat{\chi}^\Delta$ such that*

$$\bar{\bar{C}}^\Delta_{[i][j],t} = \rho^\Delta(r_{[i][j],t}).$$

   In particular, $r_{[i][j],t} = \sum_{e \in \mathcal{B}^\Delta} a_e e$, where each $a_e \in \mathbb{Z}$ is such that $\mathrm{sgn}(a_e) = \mathrm{sgn}(a_{e'})$ for any $e, e' \in \mathcal{B}^\Delta$.

(b) *$\bar{\bar{C}}^\Delta_t$ is block-sign-coherent. That is for any $[i],[j] \in Q_0^{I_2(2n)}$, the entries of $\bar{\bar{C}}^\Delta_{[i][j],t}$ are either all positive or all negative.*

(c) *Let $t \overset{[k]}{\rule{1cm}{0.4pt}} t'$ be an edge of $\mathbb{T}_{Q_0^{I_2(2n)}}$. Then the matrix $\bar{\bar{C}}^\Delta_{t'} = \mu_{[k]}(\bar{\bar{C}}^\Delta_t)$ is such that*

$$\bar{\bar{C}}^\Delta_{[i][j],t'} = \begin{cases} -\bar{\bar{C}}^\Delta_{[i][k],t} & \text{if } [j] = [k], \\ \bar{\bar{C}}^\Delta_{[i][j],t} + \mathrm{sgn}(\bar{\bar{C}}^\Delta_{[i][k],t})[\bar{\bar{C}}^\Delta_{[i][k],t}\bar{\bar{B}}^\Delta_{[k][j],t}]_+ & \text{otherwise,} \end{cases}$$

   where $\mathrm{sgn}(\bar{\bar{C}}^\Delta_{[i][k],t})$ is the sign of the $([i],[k])$-th block of $\bar{\bar{C}}^\Delta_t$, and for any matrix $A = (a_{ij})$, we define $[A]_+$ as the matrix whose entries are given by $[a_{ij}]_+ = \max(0, a_{ij})$.

(d) *The blocks of $\bar{\bar{C}}^\Delta_t$ commute. That is,*

$$\bar{\bar{C}}^\Delta_{[i][j],t}\bar{\bar{C}}^\Delta_{[k][l],t} = \bar{\bar{C}}^\Delta_{[k][l],t}\bar{\bar{C}}^\Delta_{[i][j],t}$$

   for any $[i],[j],[k],[l] \in Q_0^{I_2(2n)}$.

Proof. The proof of (a)-(d) is completed by induction on composite mutation. The initial C-matrix of $\bar{\bar{Q}}^\Delta$ is

$$\bar{\bar{C}}_{t_0}^\Delta \begin{pmatrix} \rho^\Delta(1) & \rho^\Delta(0) \\ \rho^\Delta(0) & \rho^\Delta(1) \end{pmatrix},$$

which clearly satisfies (a), (b) and (d). For the induction argument, we note that the blocks of any C-matrix $\bar{\bar{C}}_t^\Delta$ that satisfies (a) are just representations of the commutative ring χ^Δ. Thus, (a) clearly implies (d). Moreover, any C-matrix $\bar{\bar{C}}_t^\Delta$ that satisfies both (a) and (b) also satisfies (c) — the proof of this is identical to the proof of [7, Proposition 8.7(b)]. Thus, the induction argument is completed if we can show that given $\bar{\bar{C}}_t^\Delta$ satisfying (a)-(d), any matrix $\bar{\bar{C}}_{t'}^\Delta$ that arises from applying (c) satisfies both (a) and (b). The proof is very similar to that used in [7, Proposition 8.7], except in this paper, we cannot take for granted that (a) implies (b) — see Remark 6.5(b).

So suppose $\bar{\bar{C}}_t^\Delta$ satisfies (a)-(d) and let $\bar{\bar{C}}_{t'}^\Delta = \mu_{[k]}(\bar{\bar{C}}_t^\Delta)$. Clearly we have

$$\bar{\bar{C}}_{[i][k],t'}^\Delta = -\rho^\Delta(r_{[i][k],t}) = \rho^\Delta(-r_{[i][k],t}).$$

This shows that the particular blocks $\bar{\bar{C}}_{[i][k],t'}^\Delta$ are expressible as in (a). Moreover, the block-sign-coherence of $\bar{\bar{C}}_t^\Delta$ then implies that each block $\bar{\bar{C}}_{[i][k],t'}^\Delta$ is sign-coherent, so (b) is satisfied for these blocks too.

For the blocks with $[j] \neq [k]$, we have

$$\bar{\bar{C}}_{[i][j],t'}^\Delta = \rho^\Delta(r_{[i][j],t}) \pm [\rho^\Delta(r_{[i][k],t})\rho^\Delta(\pm s)]_+$$
$$= \rho^\Delta(r_{[i][j],t} \pm [\pm s r_{[i][k],t}]_+),$$

where s is as in Remark 6.4. To see that $r_{[i][j],t'} = r_{[i][j],t} \pm [\pm s r_{[i][k],t}]_+$ is as in (a), we note that since $\bar{\bar{C}}_{[i][j],t'}^\Delta$ is the C-matrix of a direct product of Dynkin quivers, its columns are sign-coherent (c.f. [10, 5, 18]). In particular, if $r_{[i][j],t'} = \sum_{e\in\mathcal{B}^\Delta} a_e e$ with $\mathrm{sgn}(a_e) \neq \mathrm{sgn}(a_{e'})$ for some $e, e' \in \mathcal{B}^\Delta$, then the column of $\bar{\bar{C}}_{[i][j],t'}^\Delta = \rho^\Delta(r_{[i][j],t'})$ that is indexed by $1 \in \widehat{\chi}^\Delta$ is not sign-coherent — a contradiction. So we must have $\mathrm{sgn}(a_e) = \mathrm{sgn}(a_{e'})$ for all $e, e' \in \mathcal{B}^\Delta$ instead.

If $\Delta \neq E_7$, then Remark 6.5 automatically implies that $\bar{\bar{C}}_{[i][j],t'}^\Delta$ satisfies (b). Otherwise if $\Delta = E_7$, we note that (b) fails only if $a_{\widetilde{\psi}_1}, a_{\psi_{v_7}} \neq 0$. But by Remark 6.5, the columns of $\rho^{E_7}(\widetilde{\psi}_1)$ and $\rho^{E_7}(\psi_{v_7})$ are not sign-coherent. In particular, since $\bar{\bar{C}}_{[i][j],t'}^\Delta$ must be column-sign-coherent, this (and Remark 6.5) implies that there exist some $e_1, \ldots, e_m \in \mathcal{B}^{E_7} \setminus \{\widetilde{\psi}_1, \psi_{v_7}\}$ such that

$$\rho^{E_7}\left(a_{\widetilde{\psi}_1}\widetilde{\psi}_1 + a_{\psi_{v_7}}\psi_{v_7} + \sum_{i=1}^m a_{e_i}e_i\right)$$

is column-sign-coherent (where each a_e is as determined from (a) of the proposition). But since (a) is satisfied, all entries of such a matrix must have the same sign. So $\bar{\bar{C}}_{[i][j],t'}^\Delta$ satisfies (b), as required.

We have thus shown that any C-matrix produced by composite mutation from some other C-matrix satisfying (a)-(d) also satisfies (a)-(d). Since the initial C-matrix satisfies (a)-(d), the induction argument is complete. $\square$

The connection between C-matrices of type Δ and type $I_2(2n)$ is formalised by the following.

Definition 6.10. Let $R = \chi^{(2n)}$ and $\mathrm{Frac}(R)$ be its field of fractions. Given a weighted folding $F\colon Q \to Q'$, where Q' is a quiver of type $I_2(2n)$, let d_F be the map from Definition 3.3. Suppose there exist $i_0, i_1 \in Q_0$ such that $\varsigma([0]) = \kappa(i_0)$ and $\varsigma([1]) = \kappa(i_1)$. Then define the *matrix F-projection map* with respect to (i_0, i_1) to be the map

$$\mathbf{d}_{F,i_0,i_1}\colon \mathbb{Z}^{|Q_0| \times |Q_0|} \to (\mathrm{Frac}(R))^{2 \times 2}$$

$$(\mathbf{u}_k)_{k \in Q_0} \mapsto \begin{pmatrix} d_F(\mathbf{u}_{i_0}) & d_F(\mathbf{u}_{i_1}) \end{pmatrix}$$

where each $\mathbf{u}_k$ is a column vector. Similarly, define the *matrix transpose F-projection map* with respect to (i_0, i_1) to be the map

$$\mathbf{d}_{F,i_0,i_1}^T\colon \mathbb{Z}^{|Q_0| \times |Q_0|} \to (\mathrm{Frac}(R))^{2 \times 2}$$

$$(\mathbf{u}_k)_{k \in Q_0} \mapsto \begin{pmatrix} d_F(\mathbf{u}_{i_0}) \\ d_F(\mathbf{u}_{i_1}) \end{pmatrix}$$

where each $\mathbf{u}_k$ is a row vector.

We will henceforth adopt the following notation. For each vertex $i \in Q_0^\Delta \subset \bar{\bar{Q}}^\Delta$, we will label the corresponding vertex of the opposite quiver by $i' \in Q_0^{\Delta,\mathrm{op}} \subset \bar{\bar{Q}}^\Delta$. The matrix projection maps of primary importance are then the maps

(i) $\mathbf{d}_{\bar{F}^\Delta,0,0'}$ and $\mathbf{d}_{F^\Delta,0,1}$ if $\Delta \neq E_6$,

(ii) $\mathbf{d}_{\bar{F}^\Delta,0^+,(0^+)'}$ and $\mathbf{d}_{F^\Delta,0^+,1^+}$ if $\Delta = E_6$.

For readability purposes, we will thus write without any ambiguity $\mathbf{d}_{\bar{F}^\Delta}$ and $\mathbf{d}_{F^\Delta}$ to represent these maps respectively.

Remark 6.11. It is not difficult to verify that since the vertices of $\vec{Q}^{I_2(2n)}$ have the same R-valuation, we have $\mathbf{d}_{\bar{F}^\Delta} = \mathbf{d}_{\bar{F}^\Delta}^T$. However, the same is not true for $\mathbf{d}_{F^\Delta}$ and $\mathbf{d}_{F^\Delta}^T$.

Our next set of results follow naturally from the previous proposition. In particular, we show an analogue of [7, Corollaries 8.8, 8.10].

Corollary 6.12. Let $t \in \mathbb{T}_{Q_0^{I_2(2n)}}$. Then we have the following.

(a) $\mathbf{d}_{\bar{F}^\Delta}(\bar{\bar{C}}_t^\Delta) = \vec{C}_t^{I_2(2n)}$.

(b) $\mathbf{d}_{F^\Delta}(C_t^\Delta) = C_t^{I_2(2n)}$

(c) The c-vectors of $\vec{B}^{I_2(2n)}$ and $B^{I_2(2n)}$ are sign-coherent.

(d) The c-vectors of $\vec{B}^{I_2(2n)}$ are rescaled roots of $I_2(2n)$ and the c-vectors of $B^{I_2(2n)}$ are roots of $I_2(2n)$.

Proof. For clarity, we will denote the $\chi^{(2n)}$-valuation of $\vec{Q}^{I_2(2n)}$ and $Q^{I_2(2n)}$ by the functions $\vec{\varsigma}$ and ς respectively. In particular, we note that $\varsigma([0]) = \vec{\varsigma}([0]) = \lambda$ and $\varsigma([1]) = 2\vec{\varsigma}([1])\cos(\frac{\pi}{2n}) = 2\lambda\cos(\frac{\pi}{2n})$, where $\lambda = 2$ if $\Delta = D_{n+1}$ with $n > 3$ and $\lambda = 1$ otherwise.

(a) By Proposition 6.9(a), we have

$$\bar{\bar{C}}_t^\Delta = \begin{pmatrix} \bar{\bar{C}}_{[0][0],t}^\Delta & \bar{\bar{C}}_{[0][1],t}^\Delta \\ \bar{\bar{C}}_{[1][0],t}^\Delta & \bar{\bar{C}}_{[1][1],t}^\Delta \end{pmatrix} = \begin{pmatrix} \rho^\Delta(r_{[0][0],t}) & \rho^\Delta(r_{[0][1],t}) \\ \rho^\Delta(r_{[1][0],t}) & \rho^\Delta(r_{[1][1],t}) \end{pmatrix},$$

where $r_{[i][j],t} = \sum_{e \in \mathcal{B}^\Delta} a_{e,[i][j]} e$ with each $a_{e,[i][j]} \in \mathbb{Z}$. It then follows from the definitions that we have

$$\mathbf{d}_{\bar{F}^\Delta}(\bar{\bar{C}}_t^\Delta) = \frac{1}{\lambda} \sum_{e \in \mathcal{B}^\Delta} \begin{pmatrix} a_{e,[0][0]}\kappa\vartheta^\Delta(se) & a_{e,[0][1]}\kappa\vartheta^\Delta(se) \\ a_{e,[1][0]}\kappa\vartheta^\Delta(se) & a_{e,[1][1]}\kappa\vartheta^\Delta(se) \end{pmatrix}$$

$$= \frac{1}{\lambda} \begin{pmatrix} \hat{\sigma}^\Delta(\lambda r_{[0][0],t}) & \hat{\sigma}^\Delta(\lambda r_{[0][1],t}) \\ \hat{\sigma}^\Delta(\lambda r_{[1][0],t}) & \hat{\sigma}^\Delta(\lambda r_{[1][1],t}) \end{pmatrix}$$

$$= \begin{pmatrix} \widehat{\sigma}^\Delta(r_{[0][0],t}) & \widehat{\sigma}^\Delta(r_{[0][1],t}) \\ \widehat{\sigma}^\Delta(r_{[1][0],t}) & \widehat{\sigma}^\Delta(r_{[1][1],t}) \end{pmatrix},$$

where $s = \psi_0^+ + \psi_0^-$ if $\Delta = D_{n+1}$ with $n > 3$ and $s = 1$ otherwise. In particular, for the initial C-matrix, we have

$$\mathbf{d}_{\bar{F}^\Delta}(\bar{\bar{C}}_{t_0}^\Delta) = \begin{pmatrix} \widehat{\sigma}^\Delta(1) & \widehat{\sigma}^\Delta(0) \\ \widehat{\sigma}^\Delta(0) & \widehat{\sigma}^\Delta(1) \end{pmatrix} = \vec{C}_{t_0}^{I_2(2n)}$$

The proof of statement $\mathbf{d}_{\bar{F}^\Delta}(\bar{\bar{C}}_t^\Delta) = \vec{C}_t^{I_2(2n)}$ for each $t \in \mathbb{T}_{Q_0^{I_2(2n)}}$ is then given by an induction argument on composite mutation, which is identical to the proof of [7, Corollary 8.8].

(b) We will first show that

$$\mathbf{d}_{F^\Delta}(\Lambda^\Delta \bar{\bar{C}}_t^\Delta \Theta^\Delta) = P^{-1}\mathbf{d}_{\bar{F}^\Delta}(\bar{\bar{C}}_t^\Delta)P, \tag{$*$}$$

where the operations $\Lambda^\Delta(-)\Theta^\Delta$ and $P^{-1}(-)P$ are as in Remark 6.4(d) and Lemma 6.7. We begin by noting that

$$\Lambda^\Delta \bar{\bar{C}}_t^\Delta \Theta^\Delta = \begin{pmatrix} \Lambda_{[0]}^\Delta \rho^\Delta(r_{[0][0],t})\Theta_{[0]}^\Delta & \Lambda_{[0]}^\Delta \rho^\Delta(r_{[0][1],t})\Theta_{[1]}^\Delta \\ \Lambda_{[1]}^\Delta \rho^\Delta(r_{[1][0],t})\Theta_{[0]}^\Delta & \Lambda_{[1]}^\Delta \rho^\Delta(r_{[1][1],t})\Theta_{[1]}^\Delta \end{pmatrix},$$

where $\Theta_{[i]}^\Delta$ is the matrix corresponding to the canonical inclusion $\{j \in Q_0^\Delta : F^\Delta(j) = [i]\} \to \bar{\bar{Q}}_0^\Delta$ and $\Lambda_{[i]}^\Delta$ is the matrix corresponding to the canonical surjection $\bar{\bar{Q}}_0^\Delta \to \{j \in Q_0^\Delta : F^\Delta(j) = [i]\}$ of $\mathbb{Z}$-bases/index sets. That is, $\Lambda_{[i]}^\Delta \rho^\Delta(r_{[i][j],t})\Theta_{[j]}^\Delta$ is obtained from $\rho^\Delta(r_{[i][j],t})$ by removing the rows indexed by $\{k \in \bar{\bar{Q}}_0^\Delta : F^\Delta(k) \neq [i]\}$ and removing the columns indexed by $\{k \in \bar{\bar{Q}}_0^\Delta : F^\Delta(k) \neq [j]\}$. Thus,

$$\mathbf{d}_{F^\Delta}(\Lambda^\Delta \bar{\bar{C}}_t^\Delta \Theta^\Delta) = \begin{pmatrix} \frac{1}{\varsigma([0])}\widehat{\sigma}^\Delta(s_{[0]}r_{[0][0],t}) & \frac{1}{\varsigma([0])}\widehat{\sigma}^\Delta(s_{[1]}r_{[0][1],t}) \\ \frac{1}{\varsigma([1])}\widehat{\sigma}^\Delta(s_{[0]}r_{[1][0],t}) & \frac{1}{\varsigma([1])}\widehat{\sigma}^\Delta(s_{[1]}r_{[1][1],t}) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\widehat{\sigma}^\Delta(s_{[0]})}{\varsigma([0])}\widehat{\sigma}^\Delta(r_{[0][0],t}) & \frac{\widehat{\sigma}^\Delta(s_{[1]})}{\varsigma([0])}\widehat{\sigma}^\Delta(r_{[0][1],t}) \\ \frac{\widehat{\sigma}^\Delta(s_{[0]})}{\varsigma([1])}\widehat{\sigma}^\Delta(r_{[1][0],t}) & \frac{\widehat{\sigma}^\Delta(s_{[1]})}{\varsigma([1])}\widehat{\sigma}^\Delta(r_{[1][1],t}) \end{pmatrix}$$

$$= \begin{pmatrix} \varsigma([0]) & 0 \\ 0 & \varsigma([1]) \end{pmatrix}^{-1} \begin{pmatrix} \widehat{\sigma}^\Delta(r_{[0][0],t}) & \widehat{\sigma}^\Delta(r_{[0][1],t}) \\ \widehat{\sigma}^\Delta(r_{[1][0],t}) & \widehat{\sigma}^\Delta(r_{[1][1],t}) \end{pmatrix} \begin{pmatrix} \widehat{\sigma}^\Delta(s_{[0]}) & 0 \\ 0 & \widehat{\sigma}^\Delta(s_{[1]}) \end{pmatrix}$$

where

$$s_{[0]} = \begin{cases} \psi_0^+ + \psi_0^- & \text{if } \Delta = D_{n+1} \text{ with } n > 3, \\ 1 & \text{otherwise,} \end{cases}$$

$$s_{[1]} = \begin{cases} \psi_1^+ + \psi_1^- & \text{if } \Delta = D_{n+1} \text{ with } n > 3, \\ \psi_1^+ & \text{if } \Delta = E_6, \\ \psi_1 & \text{otherwise.} \end{cases}$$

But then $\widehat{\sigma}^\Delta(s_{[i]}) = \varsigma([i])$ for both $i \in \{0, 1\}$, and hence the rightmost matrix in the above product is precisely the rescaling matrix P. Thus, the required commutativity relation $(*)$ holds. The result then follows from (a) alongside Lemma 6.7(b) and Lemma 6.8(b) and the fact that the operations $\Lambda^\Delta(-)\Theta^\Delta$ and $P^{-1}(-)P$ are invertible.

(c) This follows from (a), (b), and Proposition 6.9(a), as all morphisms involved respect positivity/negativity.

(d) We note that the module category of $K\bar{\bar{Q}}^\Delta$ is equivalent to the category mod $KQ^\Delta \times$ mod $KQ^{\Delta,\mathrm{op}}$. The category mod $KQ^{\Delta,\mathrm{op}}$ also satisfies an analogue of Theorem 3.5, where

the rows of the Auslander-Reiten quiver corresponding to sink vertices now correspond to short roots and vice versa. Since the c-vectors of $\bar{\bar{Q}}^\Delta$ are ± 1-multiples of dimension vectors of indecomposable $K\bar{\bar{Q}}^\Delta$-modules, (a), (b) and the definition of $\mathbf{d}_{\bar{F}^\Delta}$ imply that for a C-matrix

$$\vec{C}_t^{I_2(2n)} = \begin{pmatrix} \vec{c}_{[0][0]} & \vec{c}_{[0][1]} \\ \vec{c}_{[1][0]} & \vec{c}_{[1][1]} \end{pmatrix} = \begin{pmatrix} \vec{\mathbf{c}}_{[0]} & \vec{\mathbf{c}}_{[1]} \end{pmatrix}$$

we have $\vec{\mathbf{c}}_{[0]} = \delta_{F^\Delta}^{\mathrm{mod}\, KQ^\Delta}(M)$ and $\vec{\mathbf{c}}_{[1]} = \delta_{F^{\Delta,\mathrm{op}}}^{\mathrm{mod}\, KQ^{\Delta,\mathrm{op}}}(M')$ for some $M \in \mathrm{mod}\, KQ^\Delta$ and $M' \in \mathrm{mod}\, KQ^{\Delta,\mathrm{op}}$ that reside in rows of weight 1, where

$$F^{\Delta,\mathrm{op}} \colon Q^{\Delta,\mathrm{op}} \to Q^{I_2(2n)}$$

is the folding of the opposite quiver defined in the natural way by mapping sources vertices to sources and sinks to sinks. It is then clear from Theorem 3.5 that the columns of the C-matrix $C_t^{I_2(2n)}$ given by the inverse rescaling of $\vec{C}_t^{I_2(2n)}$ are roots of $I_2(2n)$, as described in Section 2.2. □

6.3. **G-matrices and g-vectors.** Since c-vectors of exchange matrices of type $I_2(2n)$ are sign-coherent, we are permitted (by the results of [19] on the tropical duality of c- and g-vectors) to use the following definition of G-matrices and g-vectors.

Definition 6.13. Let J be an index set and let B be a $|J| \times |J|$ exchange matrix over R whose c-vectors are sign-coherent. For any $t \in \mathbb{T}_J$, we define for each C-matrix C_t the corresponding G-*matrix*

$$G_t = (C_t^T)^{-1}.$$

We call the collection $\{G_t : t \in \mathbb{T}_J\}$ the G-*matrices* of B and call each column of each G_t a g-vector of B. We define the mutation of a G-matrix G_t at index k to be the G-matrix $\mu_k(G_t) = G_{t'}$, where $t \overset{k}{\rule{1.5em}{0.4pt}} t'$ is an edge in $\mathbb{T}_J$.

Remark 6.14. The results of [19] show that the mutations of G-matrices defined above are given by the explicit formula provided in [10, (6.12)–(6.13)].

The next theorem shows that all of the definitions of this section are compatible with each other.

Theorem 6.15. *Let* $t \overset{[k]}{\rule{1.5em}{0.4pt}} t' \in \mathbb{T}_{Q_0^{I_2(2n)}}$. *Then the diagram of Figure 6.1 commutes.*

Proof. We begin by considering the front and back cubes of Figure 6.1:

Firstly, the left faces commute as a consequence of Lemma 6.8 and Lemma 6.7. Secondly, the commutativity of the top/bottom squares of both cubes are a trivial exercise in linear algebra — for the front cube it is perhaps easier to see with the inverse map

$V(-\oplus -^T)V^{-1}$. Thirdly, the front and back faces of both cubes follow by definition, and this is compatible with the classical notion of G-matrix mutation due to the fact that all c-vectors in this paper are sign-coherent (c.f. [19, 10]). Finally, the fact that $(-^T)^{-1}$ is (self-)invertible and all other squares of both cubes commute imply that the rightmost squares of both cubes commute. Thus the front and back cubes commute, as required.

We now turn our attention to the central cube of Figure 6.1. Namely, the following.
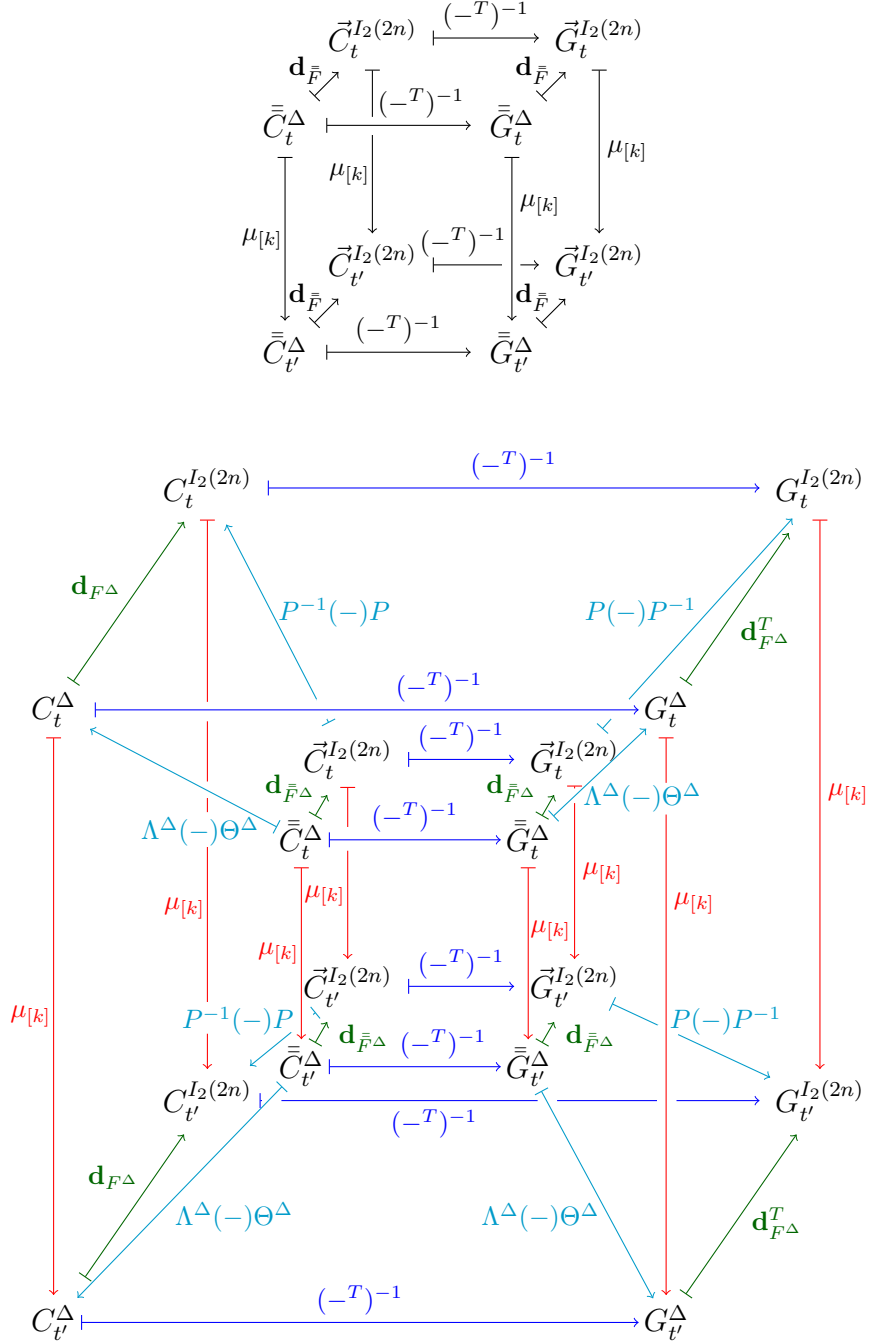
FIGURE 6.1. The relationship between the different notions of C- and G-matrices for foldings onto $I_2(2n)$ can be described by the above commutative diagram, which forms a tesseract. Arrows have been given different colours for clarity, where two arrows (and their corresponding operations) pointing in the same 'dimension' have the same colour. Notation is derived from Remark 6.4(e), Section 6.2 and Definition 6.10.

The left square commutes by Corollary 6.12. The front/back squares again follow by definition. If in addition to this, the top and bottom squares commute, then the right square commutes due to the fact that $(-^T)^{-1}$ is self-inverse. Thus this is the next step of the proof.

Note that for an arbitrary exchange matrix over a ring R (and indexed over a set J) whose C-matrices are sign-coherent, it follows that for each $t \in \mathbb{T}_J$, the determinant of C_t is $|C_t| = \pm 1$. This follows from the fact that C_{t_0} is the identity matrix, and that C-matrix mutation involves changing the sign of a column and/or (when sign-coherence is satisfied) adding a multiple of one column to another — an operation that preserves determinant up to sign. Now write

$$\vec{C}_t^{I_2(2n)} = \begin{pmatrix} \vec{c}_{[0][0]} & \vec{c}_{[0][1]} \\ \vec{c}_{[1][0]} & \vec{c}_{[1][1]} \end{pmatrix} \quad \text{and} \quad \bar{\bar{C}}_t^\Delta = \begin{pmatrix} \bar{\bar{C}}_{[0][0]} & \bar{\bar{C}}_{[0][1]} \\ \bar{\bar{C}}_{[1][0]} & \bar{\bar{C}}_{[1][1]} \end{pmatrix} = \begin{pmatrix} \rho^\Delta(r_{[0][0]}) & \rho^\Delta(r_{[0][1]}) \\ \rho^\Delta(r_{[1][0]}) & \rho^\Delta(r_{[1][1]}) \end{pmatrix}.$$

Since the blocks of $\bar{\bar{C}}_t^\Delta$ commute and the determinant of all C-matrices is ± 1, a simple computation shows that we have

$$\vec{G}_t^{I_2(2n)} = \pm \begin{pmatrix} \vec{c}_{[1][1]} & -\vec{c}_{[1][0]} \\ -\vec{c}_{[0][1]} & \vec{c}_{[0][0]} \end{pmatrix} \quad \text{and} \quad \bar{\bar{G}}_t^\Delta = \pm \begin{pmatrix} \rho^\Delta(r_{[1][1]}) & \rho^\Delta(-r_{[1][0]}) \\ \rho^\Delta(-r_{[0][1]}) & \rho^\Delta(r_{[0][0]}). \end{pmatrix}$$

From this, we can see that $\mathbf{d}_{\bar{F}^\Delta}(\bar{\bar{G}}_t^\Delta) = \vec{G}_t^\Delta$, as required. Thus, the top and bottom squares commute, and hence, the entire central cube commutes.

Next comes the left and right cubes of Figure 6.1:

The commutativity of the left cube is a consequence of Corollary 6.12 and the relation $(*)$ in its proof. For the right cube, we have already proven that the left, front and back faces commute. The proof of the commutativity of the top and bottom faces of the right cube follows a similar argument to the proof of Corollary 6.12(b). Namely, we note that

$$\mathbf{d}_{F^\Delta}^T(\Lambda^\Delta \bar{\bar{G}}_t^\Delta \Theta^\Delta) = \pm \begin{pmatrix} \frac{1}{\varsigma([0])} \widehat{\sigma}^\Delta(s_{[0]} r_{[1][1]}) & -\frac{1}{\varsigma([1])} \widehat{\sigma}^\Delta(s_{[0]} r_{[1][0]}) \\ -\frac{1}{\varsigma([0])} \widehat{\sigma}^\Delta(s_{[1]} r_{[0][1]}) & \frac{1}{\varsigma([1])} \widehat{\sigma}^\Delta(s_{[1]} r_{[0][0]}) \end{pmatrix}$$

$$= \pm \begin{pmatrix} \frac{\widehat{\sigma}^\Delta(s_{[0]})}{\varsigma([0])} \widehat{\sigma}^\Delta(r_{[1][1]}) & \frac{-\widehat{\sigma}^\Delta(s_{[0]})}{\varsigma([1])} \widehat{\sigma}^\Delta(r_{[1][0]}) \\ \frac{-\widehat{\sigma}^\Delta(s_{[1]})}{\varsigma([0])} \widehat{\sigma}^\Delta(r_{[0][1]}) & \frac{\widehat{\sigma}^\Delta(s_{[1]})}{\varsigma([1])} \widehat{\sigma}^\Delta(r_{[0][0]}) \end{pmatrix}$$

$$= \pm \begin{pmatrix} \widehat{\sigma}^\Delta(s_{[0]}) & 0 \\ 0 & \widehat{\sigma}^\Delta(s_{[1]}) \end{pmatrix} \begin{pmatrix} \vec{c}_{[1][1]} & -\vec{c}_{[1][0]} \\ -\vec{c}_{[0][1]} & \vec{c}_{[0][0]} \end{pmatrix} \begin{pmatrix} \varsigma([0]) & 0 \\ 0 & \varsigma([1]) \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \widehat{\sigma}^\Delta(s_{[0]}) & 0 \\ 0 & \widehat{\sigma}^\Delta(s_{[1]}) \end{pmatrix} \mathbf{d}_{\bar{F}^\Delta}(\bar{\bar{G}}_t^\Delta) P^{-1}$$

where each $s_{[i]}$ is as in the proof of Corollary 6.12(b). But then $\widehat{\sigma}^\Delta(s_{[i]}) = \varsigma([i])$ for both $i \in \{0, 1\}$, and hence the leftmost matrix in the above product is precisely the rescaling matrix P. Thus, the top and bottom squares commute. Since $\Lambda^\Delta(-)\Theta^\Delta$ and $P(-)P^{-1}$ are invertible, the right face of the right cube is also commutative, as required. Thus, we have just shown that both the left and right cubes commute.

The last thing to check are the top and bottom cubes. But for both of these remaining cubes, we have proved the commutativity of all but the topmost and bottommost faces respectively. Since all maps have an inverse, these squares are also commutative. Hence the entire tesseract of Figure 6.1 is commutative. $\qquad\square$

Using Theorem 6.15 and the results of [3, 20], one can obtain g-vectors of $I_2(2n)$ directly from the semiring action on the cluster category $\mathcal{C}_\Delta$ of Q^Δ. Let $\mathcal{P} \subset \mathcal{C}_\Delta$ be the full subcategory whose objects are the classes that correspond to the complexes of projective objects of $\mathrm{mod}\,KQ^\Delta$ concentrated in degree 0. Then for any object $X \in \mathcal{C}_\Delta$, we know from [14] that there exists a triangle

$$P' \to P \to X \to \Sigma P'$$

with $P, P' \in \mathcal{P}$. In our setting, we are particularly interested in objects of $\mathcal{C}_\Delta$ that reside in rows of the Auslander-Reiten quiver with weight 1. That is, rows that contain an object corresponding to an indecomposable projective $P(i)$ with $\kappa(i) = \varsigma(F^\Delta(i))$. Given a choice of indecomposable objects $P_0 \cong P(i_0)$ and $P_1 \cong P(i_1)$ in $\mathcal{P}$ with $F^\Delta(i_j) = [j]$ and $\kappa(i_j) = \varsigma([j])$, and given an indecomposable object $X \in \mathcal{C}_\Delta$ in a row of the Auslander-Reiten quiver with weight 1, we have a triangle

$$r_0' P_0 \oplus r_1' P_1 \to r_0 P_0 \oplus r_1 P_1 \to X \to \Sigma(r_0' P_0 \oplus r_1' P_1)$$

for some $r_0, r_0', r_1, r_1' \in \chi_+^\Delta$.

Definition 6.16. Let $X \in \mathcal{C}_\Delta$ be an indecomposable object in a row of the Auslander-Reiten quiver with weight 1. We call the vector

$$g^X = (\sigma^\Delta(r_1 - r_1'), \sigma^\Delta(r_0 - r_0'))$$

the *folded g-vector* of X with respect to $P_0, P_1 \in \mathcal{P}$, where P_0, P_1, r_0, r_0', r_1 and r_1' are as above.

An immediate consequence of Theorem 6.15 and known results on the categorification of g-vectors (c.f. [3, 20]) is the following.

Corollary 6.17. *Let $\mathbf{X}$ be the collection of all indecomposable objects of $\mathcal{C}_\Delta$ that reside in rows of the Auslander-Reiten quiver with weight 1.*

(a) *The folded g-vectors of the objects in $\mathbf{X}$ are precisely the g-vectors of $Q^{I_2(2n)}$.*

(b) *Basic χ_+^Δ-tilting objects (with respect to a basic, G-minimal set of χ_+^Δ-generators Γ) correspond to G-matrices of $Q^{I_2(2n)}$. In particular, let $T = Y_0 \oplus Y_1$ be basic χ_+^Δ-tilting with respect to Γ such that Y_i resides in a row that corresponds to $[i] \in Q_0^{I_2(2n)}$. Then we have the following.*

  (i) *If $X_0, X_1 \in \mathbf{X}$ are respectively χ_+^Δ-generated by Y_0 and Y_1, then*

$$G^{X_1, X_0} = (g^{X_1}, g^{X_0})$$

    is a G-matrix of $Q^{I_2(2n)}$.

  (ii) *Let $T^{(1)} = Y_0 \oplus Y_1'$ be given by changing complement and let $X_1' \in \mathbf{X}$ be an object χ_+^Δ-generated by Y_1'. Then the G-matrix*

$$G^{X_1', X_0} = (g^{X_1'}, g^{X_0})$$

    is obtained by mutating G^{X_1, X_0} with respect to the index $[0] \in Q^{I_2(2n)}$.

(iii) *Let $T^{(0)} = Y_0' \oplus Y_1$ be given by changing complement and let $X_0' \in \mathbf{X}$ be an object χ_+^Δ-generated by Y_0'. Then the G-matrix*

$$G^{X_1, X_0'} = (g^{X_1}, g^{X_0'})$$

is obtained by mutating G^{X_1, X_0} with respect to the index $[1] \in Q^{I_2(2n)}$.

APPENDIX A. WORKED EXAMPLES

Many of the constructions in this paper are combinatorial and seemingly technical in nature. However, it is not too difficult to construct examples of how the theory works in practice. We will show two examples: One example highlights the theory for foldings of type A and the other will highlight the theory for foldings of type D. The three exceptional foldings of type E work along a similar principle to the other two examples.

A.1. **The folding F^{A_7}.** Consider the example given by $F^{A_7} \colon Q^{A_7} \to Q^{I_2(8)}$. We already have an example of the projection maps $\delta_{F^{A_7}}^{\mathcal{A}}$ with $\mathcal{A} = \operatorname{mod} KQ^{A_7}$ or $\mathcal{A} = \mathcal{D}^b(\operatorname{mod} KQ^{A_7})$ with Figure 3. From the figure, it is straightforward to see the implications of Theorem 3.5 and Corollaries 3.6 and 3.7. In particular, we have $\delta_{F^{A_7}}^{\mathcal{A}}(0^\pm) = (1,0)$ and $\delta_{F^{A_7}}^{\mathcal{A}}(1^\pm) = (0,1)$, where the vectors are given with respect to the standard root system of $I_2(2n)$. We will thus focus on the impact of Sections 4-6 on this example.

The category $\mathcal{A}$ has an action of $\chi_+^{A_7} = \mathbb{Z}_{\geq 0}[\psi_2, \psi_4, \psi_6]$. Here, the element ψ_6 represents the $\mathbb{Z}_2$-symmetry on the quiver, and thus, $\psi_4 = \psi_2 \psi_6$. The isomorphism condition (A1) in Section 4.1 shows that the Auslander-Reiten sequence/triangle

$$\begin{smallmatrix} 2^+ \\ 1^+ \end{smallmatrix} \to \begin{smallmatrix} 0^+ & 2^+ \\ & 1^+ \end{smallmatrix} \to 0^+ \to$$

maps to sequences/triangles that are isomorphic to the following Auslander-Reiten sequences/triangles under the following actions.

$$\psi_2 \colon \quad \begin{smallmatrix} 0^+ & 2^+ & 2^- \\ & 1^+ & 3 \end{smallmatrix} \to \begin{smallmatrix} 0^+ & 2^+ \\ & 1^+ \end{smallmatrix} \oplus \begin{smallmatrix} 2^+ & 2^- \\ & 3 \end{smallmatrix} \to 2^+ \to$$

$$\psi_4 \colon \quad \begin{smallmatrix} 0^- & 2^- & 2^+ \\ & 1^- & 3 \end{smallmatrix} \to \begin{smallmatrix} 0^- & 2^- \\ & 1^- \end{smallmatrix} \oplus \begin{smallmatrix} 2^- & 2^+ \\ & 3 \end{smallmatrix} \to 2^- \to$$

$$\psi_6 \colon \quad \begin{smallmatrix} 2^- \\ 1^- \end{smallmatrix} \to \begin{smallmatrix} 0^- & 2^- \\ & 1^- \end{smallmatrix} \to 0^- \to$$

The middle term in the first sequence reflects the relation $\psi_2 \psi_1 = \psi_1 + \psi_3 \in \widehat{\chi}_+^{A_7}$ and the middle term in the second sequence reflects the relation $\psi_4 \psi_1 = \psi_3 + \psi_5 \in \widehat{\chi}_+^{A_7}$. One also notes from this that the original Auslander-Reiten sequence ends in an object whose F^{A_7}-projected dimension vector is of length 1, and that under the action of ψ_i, this maps to an Auslander-Reiten sequence ending in an object whose F^{A_7}-projected dimension vector is of length $\widehat{\sigma}^{A_7}(\psi_i)$. Similarly, the Auslander-Reiten sequence/triangle

$$\begin{smallmatrix} 2^+ & 2^- \\ 1^+ & 3 \end{smallmatrix} \to \begin{smallmatrix} 2^+ \\ 1^+ \end{smallmatrix} \oplus \begin{smallmatrix} 0^+ & 2^+ & 2^- \\ & 1^+ & 3 \end{smallmatrix} \to \begin{smallmatrix} 0^+ & 2^+ \\ & 1^+ \end{smallmatrix} \to$$

maps to sequences/triangles that are isomorphic to the following direct sums of Auslander-Reiten sequences/triangles under the following actions.

$$\psi_2 \colon \; \begin{smallmatrix} 2^+ & 2^- \\ 1^+ & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 0^+ & 2^+ & 2^- & 0^- \\ & 1^+ & 3 & 1^- \end{smallmatrix} \to \left(\begin{smallmatrix} 2^+ \\ 1^+ \end{smallmatrix} \oplus \begin{smallmatrix} 0^+ & 2^+ & 2^- \\ & 1^+ & 3 \end{smallmatrix} \right) \oplus \left(\begin{smallmatrix} 0^+ & 2^+ & 2^- \\ & 1^+ & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 0^- & 2^- & 2^+ \\ & 1^- & 3 \end{smallmatrix} \right) \to \begin{smallmatrix} 0^+ & 2^+ \\ & 1^+ \end{smallmatrix} \oplus \begin{smallmatrix} 2^+ & 2^- \\ & 3 \end{smallmatrix}$$

$$\psi_4 \colon \; \begin{smallmatrix} 2^- & 2^+ \\ 1^- & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 0^- & 2^- & 2^+ & 0^+ \\ & 1^- & 3 & 1^+ \end{smallmatrix} \to \left(\begin{smallmatrix} 2^- \\ 1^- \end{smallmatrix} \oplus \begin{smallmatrix} 0^- & 2^- & 2^+ \\ & 1^- & 3 \end{smallmatrix} \right) \oplus \left(\begin{smallmatrix} 0^- & 2^- & 2^+ \\ & 1^- & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 0^+ & 2^+ & 2^- \\ & 1^+ & 3 \end{smallmatrix} \right) \to \begin{smallmatrix} 0^- & 2^- \\ & 1^- \end{smallmatrix} \oplus \begin{smallmatrix} 2^- & 2^+ \\ & 3 \end{smallmatrix}$$

$$\psi_6 \colon \; \begin{smallmatrix} 2^- & 2^+ \\ 1^- & 3 \end{smallmatrix} \to \begin{smallmatrix} 2^- \\ 1^- \end{smallmatrix} \oplus \begin{smallmatrix} 0^- & 2^+ & 2^+ \\ & 1^- & 3 \end{smallmatrix} \to \begin{smallmatrix} 0^- & 2^- \\ & 1^- \end{smallmatrix} \to$$

This highlights multiple relations at play simultaneously. For example, we have the relations $\psi_2(1 + \psi_2) = 1 + 2\psi_2 + \psi_4$ and $\psi_4(1 + \psi_2) = \psi_2 + 2\psi_4 + \psi_6$ reflecting the action on the objects of the middle terms of the first two sequences. Moreover, we have the relations $\psi_2 \psi_1 = \psi_1 + \psi_3$ and $\psi_4 \psi_1 = \psi_3 + \psi_5$ reflecting the action on the starting/ending terms of the first two sequences.

The action on the Auslander-Reiten translate of the above sequences is given in the natural way by the fact that the action of $r \in \chi_+^{A_7}$ commutes with τ. A key upshot of this is that every Auslander-Reiten sequence in $\mathcal{A}$ can be labelled by an element $\psi_i \in \widehat{\chi}_+^{A_7}$ determined by the starting or ending term (which in turn, can be determined by the row in the Auslander-Reiten quiver that the starting/ending object resides in). If $\psi_j \psi_i = \sum_{k \in V_{ji}} \psi_k$ for some index set V_{ji}, then ψ_j maps an Auslander-Reiten sequence labelled by ψ_i to a sequence that is isomorphic to a direct sum of Auslander-Reiten sequences labelled by the summands ψ_k (whose ending terms all reside in the same column).

To showcase the theory of $\chi_+^{A_7}$-generators on categories, we will focus on $\operatorname{mod} KQ^{A_7}$. Consider the sets

$$\Gamma_{0\pm} = \left\{ 0^\pm, \begin{smallmatrix} 2^\pm \\ 1^\pm \end{smallmatrix}, \begin{smallmatrix} 2^\mp \\ 3 \end{smallmatrix}, \begin{smallmatrix} 0^\mp \\ 1^\mp \end{smallmatrix} \right\} \quad \text{and} \quad \Gamma_{1\pm} = \left\{ \begin{smallmatrix} 0^\pm \; 2^\pm \\ 1^\pm \end{smallmatrix}, \begin{smallmatrix} 2^\pm \; 2^\mp \\ 1^\pm \; 3 \end{smallmatrix}, \begin{smallmatrix} 2^\mp \; 0^\mp \\ 3 \; 1^\mp \end{smallmatrix}, 1^\mp \right\}$$

By Theorem 4.7(a), any set $\Gamma_{0^s,1^{s'}} = \Gamma_{0^s} \cup \Gamma_{1^{s'}}$ is a basic, τ-closed, $\mathbb{Z}_2$-minimal set of $\chi_+^{A_7}$-generators of $\operatorname{mod} KQ^{A_7}$, where $s, s' \in \{+, -\}$. Note that the elements of $\Gamma_{0^s,1^{s'}}$ bijectively correspond to the columns of the Auslander-Reiten quiver, which in turn, bijectively correspond to the positive roots of $I_2(8)$ (this is Corollary 4.8). The object $\begin{smallmatrix} 2^+ \; 2^- \\ 3 \end{smallmatrix}$, for example, is $\chi_+^{A_7}$-generated by the object $\begin{smallmatrix} 0^+ \; 2^+ \\ 1^+ \end{smallmatrix}$ with χ^{A_7}-index $\psi_2 - 1$, or equivalently, with the $\chi_+^{A_7}$-generating pair $(\psi_2, 1)$ (this is Definition 4.3). That is, we have an isomorphism of split exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \begin{smallmatrix} 0^+ \; 2^+ \\ 1^+ \end{smallmatrix} & \overset{f}{\longrightarrow} & \psi_2 \left(\begin{smallmatrix} 0^+ \; 2^+ \\ 1^+ \end{smallmatrix} \right) & \longrightarrow & \operatorname{Coker} f & \longrightarrow & 0 \\
& & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \\
0 & \longrightarrow & \begin{smallmatrix} 0^+ \; 2^+ \\ 1^+ \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 0^+ \; 2^+ \\ 1^+ \end{smallmatrix} \oplus \begin{smallmatrix} 2^+ \; 2^- \\ 3 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 2^+ \; 2^- \\ 3 \end{smallmatrix} & \longrightarrow & 0
\end{array}$$

By Remark 4.9, the Grothendieck group $K_0(\operatorname{mod} KQ^{A_7})$ has the structure of a χ^{A_7}-module of rank 2, whose χ^{A_7}-basis can (for example) be given by $\{[0^+], [1^+]\}$.

By Section 5, the action of $\chi_+^{A_7}$ naturally extends to the cluster category $\mathcal{C}_{A_7}$. We will choose our basic, $\mathbb{Z}_2$ set of $\chi_+^{A_7}$-generators of $\mathcal{C}_{A_7}$ to be

$$\Gamma = \left\{ \Sigma \begin{smallmatrix} 0^- \\ 1^- \end{smallmatrix}, 0^+, \begin{smallmatrix} 2^+ \\ 1^+ \end{smallmatrix}, \begin{smallmatrix} 2^- \\ 3 \end{smallmatrix}, \begin{smallmatrix} 0^- \\ 1^- \end{smallmatrix} \right\} \cup \left\{ \Sigma 1^-, \begin{smallmatrix} 0^+ \; 2^+ \\ 1^+ \end{smallmatrix}, \begin{smallmatrix} 2^+ \; 2^- \\ 1^+ \; 3 \end{smallmatrix}, \begin{smallmatrix} 2^- \; 0^- \\ 3 \; 1^- \end{smallmatrix}, 1^- \right\}$$

With respect to Γ, the object $\begin{smallmatrix} 0^- \\ 1^- \end{smallmatrix}$ is almost complete $\chi_+^{A_7}$-tilting with complements $\begin{smallmatrix} 2^- \; 0^- \\ 3 \; 1^- \end{smallmatrix}$ and 1^-. Note that these complements reside in adjacent columns of the Auslander-Reiten quiver to $\begin{smallmatrix} 0^- \\ 1^- \end{smallmatrix}$, as per Theorem 5.2. Thus for example, the object $T = \begin{smallmatrix} 0^- \\ 1^- \end{smallmatrix} \oplus 1^-$ is basic $\chi_+^{A_7}$-tilting with respect to Γ, and the act of changing complement corresponds to mutating the folded quiver $I_2(8)$ at a vertex. Note also that by Theorem 5.2, T corresponds to a cluster-tilting object

$$\widehat{T} = \left(\begin{smallmatrix} 0^- \\ 1^- \end{smallmatrix} \oplus \begin{smallmatrix} 2^- \\ 1^- \; 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2^+ \\ 1^+ \; 3 \end{smallmatrix} \oplus \begin{smallmatrix} 0^+ \\ 1^+ \end{smallmatrix} \right) \oplus \left(1^- \oplus 3 \oplus 1^+ \right)$$

and mutation/changing complement in T corresponds to composite mutation/iteratively changing complement for each object in a bracket in $\widehat{T}$.

Now we shall compute the folded g-vectors in $\mathcal{C}_{A_7}$, as per Definition 6.16. The set Γ consists of all objects in rows of weight 1 (up to the group action of ψ_6), so we will focus on computing the folded g-vectors of these objects — it is not difficult to check that the folded g-vectors g^X and $g^{\psi_6 X}$ are in fact the same. We will compute the g-vectors with respect to the projectives $\begin{smallmatrix} 0^- \\ 1^- \end{smallmatrix}$ and 1^-. We thus have the following triangles and corresponding folded g-vectors, where for readability purposes, we have denoted the zero object by $\mathbf{0}$ and the identity element $1 \in \chi_+^{A_7}$ by ψ_0 (or omitted it completely if the action

on an object is trivial).

$$0 \to 1^- \to 1^- \to 0 \qquad \rightsquigarrow \qquad g^{1^-} = (1,0),$$

$$0 \to {}_{1^-}^{0^-} \to {}_{1^-}^{0^-} \to 0 \qquad \rightsquigarrow \qquad g^{{}_{1^-}^{0^-}} = (0,1),$$

$$1^- \to (\psi_0 + \psi_2)\,{}_{1^-}^{0^-} \to {}_{3}^{2^-}{}_{1^-}^{0^-} \to \Sigma 1^- \qquad \rightsquigarrow \qquad g^{{}_{3}^{2^-}{}_{1^-}^{0^-}} = (-1, 2+\sqrt{2}),$$

$$1^- \to \psi_2\,{}_{1^-}^{0^-} \to {}_{3}^{2^-} \to \Sigma 1^- \qquad \rightsquigarrow \qquad g^{{}_{3}^{2^-}} = (-1, 1+\sqrt{2}),$$

$$\psi_2 1^- \to (\psi_2 + \psi_4)\,{}_{1^-}^{0^-} \to {}_{1^+}^{2^+}{}_{3}^{2^-} \to \Sigma \psi_2 1^- \qquad \rightsquigarrow \qquad g^{{}_{1^+}^{2^+}{}_{3}^{2^-}} = (-1-\sqrt{2}, 2+2\sqrt{2}),$$

$$\psi_2 1^- \to 1^- \oplus \psi_4\,{}_{1^-}^{0^-} \to {}_{1^+}^{2^+} \to \Sigma \psi_2 1^- \qquad \rightsquigarrow \qquad g^{{}_{1^+}^{2^+}} = (-\sqrt{2}, 1+\sqrt{2}),$$

$$\psi_4 1^- \to (\psi_4 + \psi_6)\,{}_{1^-}^{0^-} \to {}_{1^+}^{0^+}{}^{2^+} \to \Sigma \psi_4 1^- \qquad \rightsquigarrow \qquad g^{{}_{1^+}^{0^+}{}^{2^+}} = (-1-\sqrt{2}, 2+\sqrt{2}),$$

$$\psi_6 1^- \to \psi_6\,{}_{1^-}^{0^-} \to 0^+ \to \Sigma \psi_6 1^- \qquad \rightsquigarrow \qquad g^{0^+} = (-1, 1),$$

$$1^- \to 0 \to \Sigma 1^- \to \Sigma 1^- \qquad \rightsquigarrow \qquad g^{\Sigma 1^-} = (-1, 0),$$

$$\,{}_{1^-}^{0^-} \to 0 \to \Sigma\,{}_{1^-}^{0^-} \to \Sigma\,{}_{1^-}^{0^-} \qquad \rightsquigarrow \qquad g^{\Sigma\,{}_{1^-}^{0^-}} = (0, -1).$$

In the above, we have used the fact that $\sigma^{A_7}(\psi_2) = \sigma^{A_7}(\psi_4) = U_2(\cos\frac{\pi}{8}) = 1 + \sqrt{2}$ and $\sigma^{A_7}(\psi_0) = \sigma^{A_7}(\psi_6) = 1$. As per Corollary 6.17, these are g-vectors of the standard root system of type $I_2(8)$, and for each basic $\chi_+^{A_7}$-tilting object $T = X_0 \oplus X_1$, we obtain a G-matrix of $I_2(8)$ by $G^{X_1, X_0} = (g^{X_1}, g^{X_0})$. For example, the $\chi_+^{A_7}$-tilting object ${}_{1^-}^{0^-} \oplus 1^-$ corresponds to the initial G-matrix. Using Theorem 6.15, we can mutate the initial G-matrix $G^{1^-, {}_{1^-}^{0^-}}$ at index $[0]$ to obtain

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xleftarrow{\;(-T)^{-1}\;} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =\!\!=\!\!= G^{1^-, {}_{1^-}^{0^-}}$$

$$\Big\downarrow{\mu_{[0]}} \qquad\qquad \Big\downarrow{\mu_{[0]}} \qquad\qquad \Big\downarrow{\mu_{[0]}}$$

$$\begin{pmatrix} -1 & 2+\sqrt{2} \\ 0 & 1 \end{pmatrix} \xmapsto{\;(-T)^{-1}\;} \begin{pmatrix} -1 & 0 \\ 2+\sqrt{2} & 1 \end{pmatrix} =\!\!=\!\!= G^{{}_{3}^{2^-}{}_{1^-}^{0^-}, {}_{1^-}^{0^-}}$$

which corresponds to changing complement, as required.

A.2. **The folding** F^{D_5}. Inevitably, combinatorics in the setting of foldings of type D is more complicated than in the setting of type A. Since F^{D_5} is also a folding onto $Q^{I_2(8)}$, we will focus on the aspects of the theory that differ from the type A_7 setting. To further simplify things, most of this example will concern the category $\mathcal{A} = \operatorname{mod} KQ^{D_5}$, as the theory for $\mathcal{D}^b(\operatorname{mod} KQ^{D_5})$ is a straightforward extension of this. The non-crystallographic projection of $\mathcal{A}$ is illustrated in Figure A.1. We remind the reader that whilst the positive simple roots are of length 2 and $4\cos\frac{\pi}{8}$ respectively (under the standard basis of $\mathbb{R}^2$), the positive simple roots are written as $(1,0)$ and $(0,1)$ with respect to the standard root system of $I_2(8)$.

The category $\mathcal{A}$ has an action of $\chi_+^{D_5} = \mathbb{Z}_{\geq 0}[\psi_0^-, \psi_2^+, \psi_2^-]$, where we will also write $1 = \psi_0^+ \in \chi_+^{D_5}$ for clarity. Here, the element ψ_0^- represents the $\mathbb{Z}_2$-symmetry on the quiver. Similar to the theory for type A, we can label the Auslander-Reiten sequences by elements of $\widehat{\chi}_+^{D_5}$, and this label is determined by the row that the starting/ending term resides in. In particular, we can assign an Auslander-Reiten sequence a label of $\psi_i^+ + \psi_i^-$ (for $i \in \{0,1,2\}$) if the starting/ending term resides in the row $\mathcal{I}_i^{\mathcal{A}}$, and a
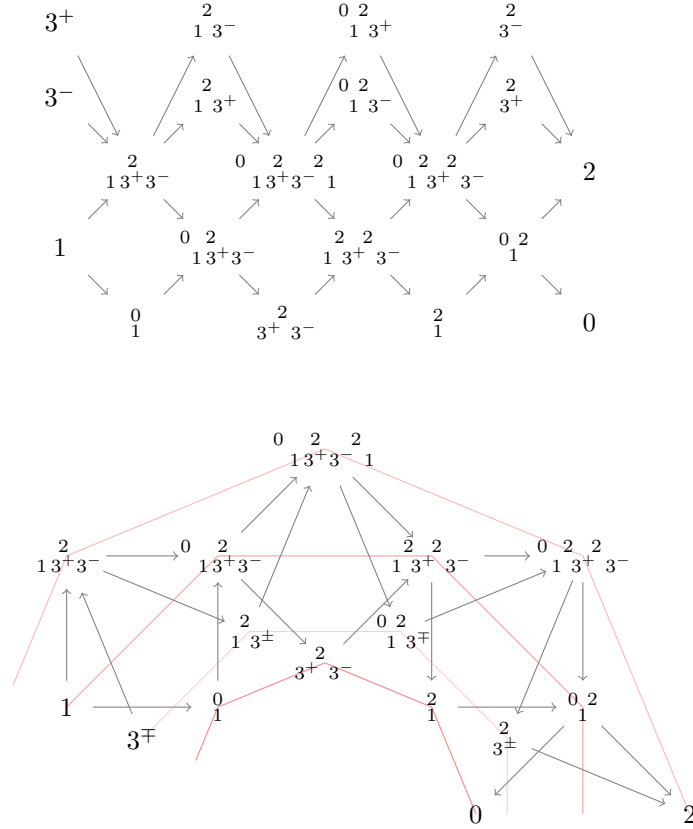
FIGURE A.1. The folding $F^{D_5}\colon Q^{D_5} \to Q^{I_2(8)}$. Top: The Auslander-Reiten quiver of $\bmod KQ^{D_5}$. Bottom: The non-crystallographic projection of the Auslander-Reiten quiver under the map $\delta_{F^{D_5}}^{\bmod KQ^{D_5}}$, with irreducible morphisms superimposed. The radius of each half-octagon (from the centre outwards) is 2, $U_3(\cos\frac{\pi}{8})$, $2U_1(\cos\frac{\pi}{8})$ and $2U_2(\cos\frac{\pi}{8})$.

label of $\psi_3^\pm$ if the starting/ending term resides in the row $\mathcal{I}_{3\pm}^{\mathcal{A}}$. The action of an element $r \in \chi_+^{D_5}$ on an Auslander-Reiten sequence labelled by $s \in \chi_+^{D_5}$ is then such that the sequence is mapped to an exact sequence isomorphic to a direct sum of Auslander-Reiten sequences whose labels are determined from the product rs. That is, we can write $rs = \sum_{i=0}^2 a_i(\psi_i^+ + \psi_i^-) + a_{3+}\psi_3^+ + a_{3-}\psi_3^-$, and thus the resulting sequence is isomorphic to a direct sum of a_i Auslander-Reiten sequences labelled by $\psi_i^+ + \psi_i^-$ and $a_{3\pm}$ Auslander-Reiten sequences labelled by $\psi_3^\pm$ (all in the same column). Likewise, we can also label individual indecomposable objects by elements of $\widehat{\chi}_+^{D_5}$, which is determined by the row the object resides in (as before). The action of $\chi_+^{D_5}$ on an object will also always respect this labelling (column-wise).

To demonstrate this, consider the Auslander-Reiten sequences

$$\mathbf{0} \to \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \to \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}2 \to 0 \to \mathbf{0} \tag{S0}$$

$$\mathbf{0} \to \begin{smallmatrix} 2 & 2 \\ 1 & 3^+ & 3^- \end{smallmatrix} \to \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 2 & 2 \\ 1 & 3^+ & 3^- \end{smallmatrix} \to \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}2 \to \mathbf{0} \tag{S1}$$

$$\mathbf{0} \to \begin{smallmatrix} 0 & 2 & 2 \\ 1 & 3^+ & 3^- \end{smallmatrix} \to \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}2 \oplus \begin{smallmatrix} 2 \\ 3^+ \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3^- \end{smallmatrix} \to 2 \to \mathbf{0} \tag{S2}$$

$$\mathbf{0} \to \begin{smallmatrix} 0 & 2 \\ 1 & 3^- \end{smallmatrix} \to \begin{smallmatrix} 0 & 2 & 2 \\ 1 & 3^+ & 3^- \end{smallmatrix} \to \begin{smallmatrix} 2 \\ 3^+ \end{smallmatrix} \to \mathbf{0} \tag{S3p}$$

$$\mathbf{0} \to \begin{smallmatrix} 0 & 2 \\ 1 & 3^+ \end{smallmatrix} \to \begin{smallmatrix} 0 & 2 & 2 \\ 1 & 3^+ & 3^- \end{smallmatrix} \to \begin{smallmatrix} 2 \\ 3^- \end{smallmatrix} \to \mathbf{0} \tag{S3m}$$

where for clarity, we have denoted the zero object by $\mathbf{0}$ to distinguish it from the simple module 0. The above Auslander-Reiten sequences (and the starting and ending terms)

may be labelled by $\psi_0^+ + \psi_0^-$, $\psi_1^+ + \psi_1^-$, $\psi_2^+ + \psi_2^-$, ψ_3^+ and ψ_3^-, respectively. The objects in the middle terms of each sequence therefore have labels $\psi_1^+ + \psi_1^-$, $\psi_0^+ + \psi_0^- + \psi_2 + \psi_2^-$, $\psi_1^+ + \psi_1^- + \psi_3^+ + \psi_3^-$, $\psi_2^+ + \psi_2^-$ and $\psi_2^+ + \psi_2^-$, respectively.

The action of $\psi_2^\pm$ on the Auslander-Reiten sequence (S0) is a map to (a sequence isomorphic to) the Auslander-Reiten sequence (S2). This reflects the product $\psi_2^\pm(\psi_0^+ + \psi_0^-) = \psi_2^+ + \psi_2^-$ for the sequence itself, and simultaneously, the product $\psi_2^\pm(\psi_1^+ + \psi_1^-) = \psi_1^+ + \psi_1^- + \psi_3^+ + \psi_3^-$ for the middle term. The action of $\psi_2^\pm$ on (S2), or equivalently the action of $(\psi_2^\pm)^2 = \psi_0^+ + \psi_2^+ + \psi_2^-$ on (S0), is such that we obtain a sequence isomorphic to (S0) $\oplus$ (S2) $\oplus$ (S2). Both of these examples showcase isomorphism condition (D1).

Another example is the action of ψ_0^-, which representing $\mathbb{Z}_2$-symmetry, maps (S3p) to (S3m) and vice versa. On the other hand the action of ψ_0^- on any other sequence in the list is trivial (up to isomorphism).

Finally, one needs to be careful of complicated sign changes with some of the products in $\widehat{\chi}_+^{D_5}$. The effect of this is highlighted in isomorphism condition (D2). For example, $\psi_2^+\psi_3^+ = \psi_1^+ + \psi_1^- + \psi_3^-$. Thus, the action of ψ_2^+ on (S3p) is such that we obtain (S1) $\oplus$ (S3m).

Now we define the following three sets.

$$\Gamma_0 = \{0, \, {}^2_{\,1}, \, {}_{3^+}2_{\,3^-}, \, {}^0_{\,1}\} \qquad \text{and} \qquad \Gamma_\pm = \{ \, {}_{\,3^\mp}2 \, , \, {}^0_{\,1}{}^2_{\,3^\pm} , \, {}_{\,1}2_{\,3^\mp}, 3^\pm\}$$

Theorem 4.7(c) then states that the sets $\Gamma_0 \cup \Gamma_\pm$ are pairwise equivalent, basic, τ-closed, $\mathbb{Z}_2$-minimal sets of $\chi_+^{D_5}$-generators of $\mathcal{A}$. Extending this to the cluster category $\mathcal{C}_{D_5}$, we obtain similar sets of basic, $\mathbb{Z}_2$-minimal $\chi_+^{D_5}$-generators

$$\Gamma^\pm = \{\Sigma^0_1, 0, \, {}^2_{\,1}, \, {}_{3^+}2_{\,3^-}, \, {}^0_{\,1}\} \cup \{\Sigma 3^\pm, \, {}_{\,3^\mp}2 \, , \, {}^0_{\,1}{}^2_{\,3^\pm} , \, {}_{\,1}2_{\,3^\mp}, 3^\pm\}.$$

An example of a basic $\chi_+^{D_5}$-tilting object with respect to Γ^+ is therefore ${}^2_{\,1} \oplus {}^0_{\,1}{}^2_{\,3^+}$ and the corresponding basic $\chi_+^{D_5}$-tilting object with respect to Γ^- is ${}^2_{\,1} \oplus {}^0_{\,1}{}^2_{\,3^-}$. The theory does not depend on which set we choose, as long as we are consistent with our choice.

Similar to the situation in type A, we can compute the g-vectors of $I_2(8)$ from the category $\mathcal{C}_{D_5}$. However this time, our choice of objects in the resolution do not always belong to a set of $\chi_+^{D_5}$-generators of $\mathcal{C}_{D_5}$. For example, the row containing $3^\pm$ does not have the correct weight, so we must instead choose the object 1. We therefore have the following triangles and corresponding folded g-vectors.

$$\mathbf{0} \to 1 \to 1 \to \mathbf{0} \qquad \rightsquigarrow \qquad g^1 = (1,0),$$

$$\mathbf{0} \to {}^0_{\,1} \to {}^0_{\,1} \to \mathbf{0} \qquad \rightsquigarrow \qquad g^{{}^0_{\,1}} = (0,1),$$

$$1 \to (\psi_0^+ + \psi_2^+){}^0_{\,1} \to {}^0_{\,1}{}^2_{\,3^+3^-} \to \Sigma 1 \qquad \rightsquigarrow \qquad g^{{}^0_{\,1}{}^2_{\,3^+3^-}} = (-1, 2+\sqrt{2}),$$

$$1 \to \psi_2^+{}^0_{\,1} \to {}_{3^+}2_{\,3^-} \to \Sigma 1 \qquad \rightsquigarrow \qquad g^{{}_{3^+}2_{\,3^-}} = (-1, 1+\sqrt{2}),$$

$$\psi_2 1 \to (\psi_2^+ + \psi_2^-){}^0_{\,1} \to {}_{\,1}2_{\,3^+}2_{\,3^-} \to \Sigma\psi_2^+ 1 \qquad \rightsquigarrow \qquad g^{{}_{\,1}2_{\,3^+}2_{\,3^-}} = (-1-\sqrt{2}, 2+2\sqrt{2}),$$

$$\psi_2^+ 1 \to 1 \oplus \psi_2^+{}^0_{\,1} \to {}^2_{\,1} \to \Sigma\psi_2^+ 1 \qquad \rightsquigarrow \qquad g^{{}^2_{\,1}} = (-\sqrt{2}, 1+\sqrt{2}),$$

$$\psi_2^+ 1 \to (\psi_0^+ + \psi_2^+){}^0_{\,1} \to {}^0_{\,1}{}^2 \to \Sigma\psi_2^+ 1 \qquad \rightsquigarrow \qquad g^{{}^0_{\,1}{}^2} = (-1-\sqrt{2}, 2+\sqrt{2}),$$

$$1 \to {}^0_{\,1} \to 0 \to \Sigma 1 \qquad \rightsquigarrow \qquad g^0 = (-1, 1),$$

$$1 \to \mathbf{0} \to \Sigma 1 \to \Sigma 1 \qquad \rightsquigarrow \qquad g^{\Sigma 1} = (-1, 0),$$

$${}^0_{\,1} \to \mathbf{0} \to \Sigma{}^0_{\,1} \to \Sigma{}^0_{\,1} \qquad \rightsquigarrow \qquad g^{\Sigma{}^0_{\,1}} = (0, -1).$$

As we can see, these are identical to the folded g-vectors computed before.

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