Preprint

ON DETERMINANTS INVOLVING SECOND-ORDER RECURRENT SEQUENCES

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Abstract. Let A and B be complex numbers, and let $(w_n)_{n\ge0}$ be a sequence of complex numbers with $w_{n+1}=Aw_n-Bw_{n-1}$ for all $n=1,2,3,\ldots$. When $w_0=0$ and $w_1=1$, the sequence $(w_n)_{n\ge0}$ is just the Lucas sequence $(u_n(A,B))_{n\ge0}$. In this paper, we evaluate the determinants

$$\det[w_{|j-k|}]_{1\le j,k\le n}\ \ \text{and}\ \ \det[w_{|j-k+1|}]_{1\le j,k\le n}.$$

In particular, we have

$$\det[u_{|j-k|}(A,B)]_{1\le j,k\le n}=(-1)^{n-1}u_{n-1}(2A,(B+1)^2).$$

When $B=-1$ and $2\mid n$, we also determine the characteristic polynomial of the matrix $[w_{j+k}]_{0\le j,k\le n-1}$.

1. Introduction

In 1934 R. Robinson proposed the evaluation of the determinant $\det[|j-k|]_{1\le j,k\le n}$ as a problem in Amer. Math. Monthly, later its solutions appeared in [2]. Namely, we have

$$\det[|j-k|]_{1\le j,k\le n}=(-1)^{n-1}(n-1)2^{n-2}. \tag{1.1}$$

Let $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$, and let T be any (undirected) tree with n vertices $v_1,\ldots,v_n$. For $j,k=1,\ldots,n$, let $d(v_j,v_k)$ denote the distance between the vertices v_j and v_k. In 1971 R.L. Graham and H.O. Pollak [1] established the following celebrated formula:

$$\det[d(v_j,v_k)]_{1\le j,k\le n}=(-1)^{n-1}(n-1)2^{n-2}. \tag{1.2}$$

This is a further extension of (1.1) as a path with n vertices is a tree. Based on the idea in [4], in 2007 W. Yan and Y.-N. Yeh [5, Corollary 2.3] obtained the following q-analogue of (1.2) for $n>1$:

$$\det[[d(v_j,v_k)]_q]_{1\le j,k\le n}=(-1)^{n-1}(n-1)(1+q)^{n-2}, \tag{1.3}$$

where $[m]_q$ with $m\in\mathbb{N}$ denotes the q-analogue of m given by

$$[m]_q:=\sum_{0\le k<m}q^k=\begin{cases}(1-q^m)/(1-q) & \text{if } q\ne1,\\ m & \text{if } q=1.\end{cases}$$

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(Throughout this paper, we consider 0^0 as 1.) Another result of Yan and Yeh [5, Corollary 2.2] states that

$$\det[q^{d(v_j,v_k)}]_{1\le j,k\le n} = (1-q^2)^{n-1}, \tag{1.4}$$

Let R be a commutative ring with identity. The Lucas sequences $(u_n(x,y))_{n\ge0}$ and $(v_n(x,y))_{n\ge0}$ over R are defined as follows:

$$u_0(x,y)=0,\ u_1(x,y)=1,\ u_{n+1}(x,y)=xu_n(x,y)-yu_{n-1}(x,y) \text{ for } n\in\mathbb{Z}^+;$$

$$v_0(x,y)=2,\ v_1(x,y)=x,\ u_{n+1}(x,y)=xu_n(x,y)-yu_{n-1}(x,y) \text{ for } n\in\mathbb{Z}^+.$$

It is well known that

$$u_n(x,y)=\frac{x^n-y^n}{x-y}=\sum_{0\le k<n}x^ky^{n-1-k} \quad\text{and}\quad v_n(x,y)=x^n+y^n$$

for all $n\in\mathbb{N}$. Note that for any $n\in\mathbb{N}$ we have $u_n(q+1,q)=[n]_q$, in particular $u_n(2,1)=n$.

Let $A,B\in\mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers. Let $(w_n)_{n\ge0}$ be a sequence of complex numbers satisfying the recurrence

$$w_{n+1}=Aw_n-B_{n-1}\ (n=1,2,3,\ldots). \tag{1.5}$$

When $w_0=0$ and $w_1=1$, we have $w_n=u_n(A,B)$ for all $n\in\mathbb{N}$. When $w_0=2$ and $w_1=A$, we have $w_n=v_n(A,B)$ for any $n\in\mathbb{N}$. In this paper, we evaluate

$$\det[w_{|j-k|}]_{1\le j,k\le n} \quad\text{and}\quad \det[w_{|j-k+1|}]_{1\le j,k\le n},$$

which extends (1.1) in a new way.

Theorem 1.1. *Let A and B be elements of a commutative ring R with identity, and let $(w_n)_{n\ge0}$ be a sequence of elements of R satisfying the recurrence (1.5). For any $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$, we have*

$$\det[w_{|j-k|}]_{1\le j,k\le n}=w_0u_n(A',B')+((Bw_0)^2-(Aw_0-w_1)^2)u_{n-1}(A',B'), \tag{1.6}$$

where

$$A'=(A^2-B^2+1)w_0-2Aw_1 \text{ and } B'=((Aw_0-(B+1)w_1)^2.$$

Taking $w_0=0$ and $w_1=1$ in Theorem 1.1 and noting that

$$u_n(xz,yz^2)=u_n(x,y)z^{n-1}\quad(n=1,2,3,\ldots), \tag{1.7}$$

we immediately obtain the following corollary.

Corollary 1.1. *Let A and B be elements of a commutative ring R with identity. Then, for any positive integer n, we have*

$$\det[u_{|j-k|}(A,B)]_{1\le j,k\le n}=(-1)^{n-1}u_{n-1}(2A,(B+1)^2). \tag{1.8}$$

Let $n>1$ be an integer. In view of (1.7),

$$u_{n-1}(2A,4)=2^{n-2}u_{n-1}(A,1).$$

So, (1.8) with $B=1$ gives the identity

$$\det[u_{|j-k|}(A,1)]_{1\le j,k\le n}=(-1)^{n-1}2^{n-2}u_{n-1}(A,1). \tag{1.9}$$

In the case $A=2$, this turns out to be the classical formula (1.1). Note also that (1.8) with $B=-1$ yields the identity

$$\det[u_{|j-k|}(A,-1)]_{1\le j,k\le n}=(-1)^{n-1}(2A)^{n-2}. \tag{1.10}$$

The identity (1.8) with $A=1$ and $B=-2$ gives the formula

$$\det[u_{|j-k|}(1,-2)]_{1\le j,k\le n}=(-1)^{n-1}(n-1). \tag{1.11}$$

In the case $B=q$ and $A=q+1$, with the aid of (1.7), from the identity (1.8) we obtain the q-analogue of (1.1):

$$\det[[|j-k|]_q]_{1\le j,k\le n}=(-1)^{n-1}(n-1)(q+1)^{n-2}. \tag{1.12}$$

One may wonder whether the identity (1.8) can be extended to trees. The answer is negative. Let's consider a tree T with vertices v_1,v_2,v_3,v_4 and edges v_1v_2,v_2v_3,v_2v_4. For any $A,B\in\mathbb{C}$, we clearly have

$$\det\left[u_{d(v_j,v_k)}(A,B)\right]_{1\le j,k\le 4}=\begin{vmatrix}0&1&A&A\\1&0&1&1\\A&1&0&A\\A&1&A&0\end{vmatrix}=-3A^2$$

which is independent of B, while the right-hand side of (1.8) indeed depends on B.

It is easy to see that

$$u_n(1,1)=(-1)^{n-1}\left(\frac{n}{3}\right)\quad\text{for all } n\in\mathbb{N},$$

where $(-)$ denotes the Legendre symbol. In view of this and (1.7), the identity (1.8) with $B=\pm2A-1$ yields the following corollary.

Corollary 1.2. *Let A be any element of a commutative ring with identity. For any integer $n\ge2$ we have*

$$\begin{aligned}\det[u_{|j-k|}(A,2A-1)]_{1lsj,k\le n}&=\det[u_{|j-k|}(A,-2A-1)]_{1lsj,k\le n}\\&=\left(\frac{1-n}{3}\right)(2A)^{n-2}.\end{aligned} \tag{1.13}$$

Taking $w_0=2$ and $w_1=A$ in Theorem 1.1 and making use of (1.7), we get the following corollary.

Corollary 1.3. *Let A and B be elements of a commutative ring R with identity. For any integer $n>1$, we have*

$$\begin{aligned}\det[v_{|j-k|}(A,B)]_{1\le j,k\le n}=&\,2(1-B)^{n-1}u_n(2(1+B),A^2)\\&+(4B^2-A^2)(1-B)^{n-2}u_{n-1}(2(1+B),A^2).\end{aligned} \tag{1.14}$$

By Corollary 1.3, for any $A \in \mathbb{C}$, we have

$$\det[v_{|j-k|}(A,1)]_{1\le j,k\le n}=0 \quad \text{for all } n=3,4,5,\ldots. \tag{1.15}$$

In the case $B=-1$, Corollary 1.3 yields the following result.

Corollary 1.4. *For any $A\in\mathbb{C}$ and $n\in\{2,3,4,\ldots\}$, we have*

$$\det[v_{|j-k|}(A,-1)]_{1\le j,k\le n}=(-1)^{\lfloor (n-1)/2\rfloor}(2A)^{n-2}\times\begin{cases}4A & \text{if } 2\nmid n,\\ 4-A^2 & \text{if } 2\mid n.\end{cases} \tag{1.16}$$

Applying Corollary 1.3 with $A=\pm 2B$ and using the identity (1.7), we obtain the following corollary.

Corollary 1.5. *Let R be a commutative ring with identity. For any $B\in R$ and $n\in\mathbb{Z}^+$, we have*

$$\begin{aligned}\det[v_{|j-k|}(2B,B)]_{1\le j,k\le n}&=\det[v_{|j-k|}(-2B,B)]_{1\le j,k\le n}\\&=2^n(1-B)^{n-1}u_n(1+B,B^2).\end{aligned} \tag{1.17}$$

Recall that the Fibonacci numbers are those $F_n=u_n(1,-1)$ with $n\in\mathbb{N}$. For any $n\in\mathbb{N}$, we clearly have

$$F_{2n+2}=F_{2n}+(F_{2n}+F_{2n-1})=2F_{2n}+(F_{2n}-F_{2n-2})=3F_{2n}-F_{2n-2}.$$

Thus $F_{2n}=u_n(3,1)$ for all $n\in\mathbb{N}$.

Corollary 1.6. *For any integer $n\ge 2$, we have*

$$\det[v_{|j-k|}(2,2)]_{1\le j,k\le n}=(-2)^nF_{2n-4}. \tag{1.18}$$

Proof. In view of Corollary 1.3 with $A=B=2$ and the identity (1.7),

$$\begin{aligned}\det[v_{|j-k|}(2,2)]_{1\le j,k\le n}&=(-1)^{n-1}2(u_n(6,4)-6u_{n-1}(6,4))\\&=(-1)^{n-1}2(-4u_{n-2}(6,4))\\&=-8(-1)^{n-1}2^{n-3}u_{n-2}(3,1)=(-2)^nu_{n-2}(3,1).\end{aligned}$$

This implies (1.18) since $u_m(3,1)=F_{2m}$ for all $m\in\mathbb{N}$. □

Corollary 1.7. *Let R be a commutative ring with identity, and let $A,B\in R$ and $\varepsilon\in\{\pm1\}$. Suppose that*

$$w_{-1}=\varepsilon,\ w_0=1,\ \text{and } w_{n+1}=Aw_n-Bw_{n-1} \text{ for } n\in\mathbb{N}.$$

Then, for any $n\in\mathbb{Z}^+$, we have

$$\det[w_{|j-k|}]_{1\le j,k\le n}=u_n(1-(A-\varepsilon B)^2,B^2(1+B-\varepsilon A)^2). \tag{1.19}$$

Proof. Note that $w_1=Aw_0-Bw_{-1}=A-\varepsilon B$ and $(Bw_0)^2=(Aw_0-w_1)^2$. Applying Theorem 1.1, we immediately get the desired identity (1.19). □

Corollary 1.8. (i) *For any integer $n\ge 2$, we have*

$$\det[q^{|j-k|}+t]_{1\le j,k\le n}=(1-q)^{n-1}(1+q)^{n-2}((n(1-q)+2q)t+q+1). \tag{1.20}$$

(ii) *For any positive integer n, we have*

$$\det[q^{|j-k|}-q^j-q^k+1]_{1\le j,k\le n}=(1-q^2)^n+n(1+q)^{n-1}(1-q)^{n+1}. \tag{1.21}$$

Proof. (i) Let $w_n = q^n + t$ for $n \in \mathbb{N}$. Then $w_0 = t + 1$, $w_1 = q + t$, and

$$w_{n+1} = (q+1)w_n - qw_{n-1} \text{ for all } n = 1, 2, 3, \ldots.$$

Thus, applying Theorem 1.1 we get the desired identity (1.20).

(ii) By [3, Lemma 2.1],

$$\det[q^{|j-k} + t]_{1\le j,k\le n+1} = \det[q^{|j-k|}]_{1\le j,k\le n+1} + t\det(M), \tag{1.22}$$

where $M = [m_{j,k}]_{2\le j,k\le n+1}$ with $m_{j,k} = q^{|j-k|} - q^{|j-1|} - q^{|1-k|} + q^{|1-1|}$. In view of Theorem 1.1(i) and the identity (1.22),

$$\begin{aligned}\det(M) &= (1-q)^n(1+q)^{n-1}((n+1)(1-q)+2q)\\ &= (1-q)^n(1+q)^{n-1}(n(1-q)+1+q)\\ &= n(1-q)^{n+1}(1+q)^{n-1} + (1-q)^n(1+q)^n.\end{aligned}$$

Note that $M = [m_{j+1,k+1}]_{1\le j,k\le n}$ and

$$m_{j+1,k+1} = q^{|j-k|} - q^j - q^k + 1 \quad \text{for all } j, k = 1, \ldots, n.$$

So we have the identity (1.21). This ends our proof. □

In contrast with Theorem 1.1, we also have the following (relatively easier) result.

Theorem 1.2. *Let A and B be elements of a commutative ring R with identity, and let $(w_n)_{n\ge0}$ be a sequence of elements of R satisfying the recurrence (1.5). For any integer $n > 1$, we have*

$$\det[w_{|j-k+1|}]_{1\le j,k\le n} = (w_1^2 - Aw_0w_1 + Bw_0^2)((B+1)w_1 - Aw_0)^{n-2}. \tag{1.23}$$

Clearly, Theorem 1.2 has the following consequence.

Corollary 1.9. *Let A and B be elements of a commutative ring R with identity. For any integer $n > 1$, we have*

$$\det[u_{|j-k+1|}(A,B)]_{1\le j,k\le n} = (B+1)^{n-2} \tag{1.24}$$

and

$$\det[v_{|j-k+1|}(A,B)]_{1\le j,k\le n} = (4B - A^2)(A(B-1))^{n-2}. \tag{1.25}$$

Now, we present our third theorem.

Theorem 1.3. *For any positive integer n, we have*

$$\begin{aligned}\det[q^{|j-k|} + x\delta_{jk}]_{1\le j,k\le n} = {}&(x+1)u_n(1-q^2+(1+q^2)x, q^2x^2)\\ &- q^2x^2u_{n-1}(1-q^2+(1+q^2)x, q^2x^2),\end{aligned} \tag{1.26}$$

where the Kronecker symbol δ_{jk} is 1 or 0 according as $j = k$ or not.

Taking $x = \pm1$ in (1.26) and recalling the identity (1.7), we obtain the following corollary.

Corollary 1.10. *Let n be any positive integer. Then*

$$\det[q^{|j-k|}+\delta_{jk}]_{1\le j,k\le n}=u_{n+1}(2,q^2) \tag{1.27}$$

and

$$\det[q^{|j-k|}-\delta_{jk}]_{1\le j,k\le n}=(-1)^{n-1}q^n u_{n-1}(2q,1). \tag{1.28}$$

Applying (1.27) with $q=2$, and noting that

$$u_{n+1}(2,4)=2^n u_{n+1}(1,1)=2^n(-1)^n\left(\frac{n+1}{3}\right)$$

with the aid of (1.7), we get from Corollary 1.10 the following consequence.

Corollary 1.11. *For any positive integer n, we have the identity*

$$\det[2^{|j-k|}+\delta_{jk}]_{1\le j,k\le n}=(-2)^n\left(\frac{n+1}{3}\right). \tag{1.29}$$

Now we state our last theorem.

Theorem 1.4. *Let $A\in\mathbb{C}$ with $A(A^2+4)\ne 0$. And let $(w_i)_{i\ge 0}$ be a sequence of complex numbers with $w_{i+1}=Aw_i+w_{i-1}$ for all $i=1,2,3,\ldots$. For any positive even integer n,* $\det[x\delta_{jk}-w_{j+k}]_{0\le j,k\le n-1}$ *(the characteristic polynomial of the matrix $W=[w_{j+k}]_{0\le j,k\le n-1}$) equals*

$$\begin{aligned}&x^n-(w_1v_{n-1}(A,-1)+w_0v_{n-2}(A,-1))\frac{u_n(A,-1)}{A}x^{n-1}\\&\quad+(w_0^2+Aw_0w_1-w_1^2)\frac{u_n(A,-1)^2}{A^2}x^{n-2}.\end{aligned} \tag{1.30}$$

Taking $w_0=0$ and $w_1=1$ in Theorem 1.4, we get the following corollary.

Corollary 1.12. *Let $A\in\mathbb{C}$ with $A(A^2+4)\ne 0$. For any positive even integer n, we have*

$$\begin{aligned}&A^2\det[x\delta_{jk}-u_{j+k}(A,-1)]_{0\le j,k\le n-1}\\&=A^2x^n-Au_n(A,-1)v_{n-1}(A,-1)x^{n-1}-u_n(A,-1)^2x^{n-2}.\end{aligned} \tag{1.31}$$

Applying Theorem 1.4 with $w_i=u_{i+2}(A,-1)$ for all $i\in\mathbb{N}$, and noting that

$$\begin{aligned}v_{n+1}(A,-1)&=Av_n(A,-1)+v_{n-1}(A,-1)\\&=A(Av_{n-1}(A,1)+v_{n-2}(A,-1))+v_{n-1}(A,-1)\\&=(A^2+1)v_{n-1}(A,-1)+Av_{n-2}(A,-1)\end{aligned}$$

for all $n=2,3,\ldots$, we obtain the following corollary.

Corollary 1.13. *Let $A\in\mathbb{C}$ with $A(A^2+4)\ne 0$. For any positive even integer n, we have*

$$\begin{aligned}&A^2\det[x\delta_{jk}-u_{j+k}(A,-1)]_{1\le j,k\le n}\\&=A^2x^n-Au_n(A,-1)v_{n+1}(A,-1)x^{n-1}-u_n(A,-1)^2x^{n-2}.\end{aligned} \tag{1.32}$$

Taking $w_0 = 2$ and $w_1 = A$ in Theorem 1.4, and noting that

$$Av_m(A,-1) + 2v_{m-1}(A,-1) = (A^2+4)u_m(A,-1) \text{ for all } m = 1,2,3,\dots$$

(which can be easily proved by induction), we obtain the following result.

Corollary 1.14. *Let $A \in \mathbb{C}$ with $A(A^2+4) \neq 0$. For any positive even integer n, we have*

$$\begin{aligned} &A^2 \det[x\delta_{jk} - v_{j+k}(A,-1)]_{0\le j,k\le n-1} \\ =\ &A^2x^n - A(A^2+4)u_n(A,-1)u_{n-1}(A,-1)x^{n-1} + (A^2+4)u_n(A,-1)^2x^{n-2}. \end{aligned} \tag{1.33}$$

Applying Theorem 1.4 with $w_i = v_{i+2}(A,-1)$ for all $i \in \mathbb{N}$, and noting that

$$A(A^2+3)v_{n-1}(A,-1) + (A^2+2)v_{n-2}(A,-1) = (A^2+4)u_{n+1}(A,-1)$$

for all $n = 2,3,\dots$ (which can be easily proved by induction), we get the following corollary.

Corollary 1.15. *Let $A \in \mathbb{C}$ with $A(A^2+4) \neq 0$. For any positive even integer n, we have*

$$\begin{aligned} &A^2 \det[x\delta_{jk} - v_{j+k}(A,-1)]_{1\le j,k\le n} \\ =\ &A^2x^n - A(A^2+4)u_n(A,-1)u_{n+1}(A,-1)x^{n-1} + (A^2+4)u_n(A,-1)^2x^{n-2}. \end{aligned} \tag{1.34}$$

The Lucas numbers are those $L_n = v_n(1,-1)$ $(n \in \mathbb{N})$. Taking $x = -1$ in Corollaries 1.13 and 1.15, we obtain the following consequence.

Corollary 1.16. *Let n be a positive even number. For any $A \in \mathbb{C}$ with $A(A^2+4) \neq 0$, we have*

$$A^2 \det[u_{j+k}(A,-1)+\delta_{jk}]_{1\le j,k\le n} = (A-1)(A+u_n(A,-1)^2)+Au_{n+1}(A,-1)^2 \tag{1.35}$$

and

$$A^2 \det[v_{j+k}(A,-1) + \delta_{jk}] = v_{n+1}(A,-1)^2. \tag{1.36}$$

In particular,

$$\det[F_{j+k} + \delta_{jk}]_{1\le j,k\le n} = F_{n+1}^2 \tag{1.37}$$

and

$$\det[L_{j+k} + \delta_{jk}]_{1\le j,k\le n} = L_{n+1}^2. \tag{1.38}$$

Similarly, taking $x = -1$ and $A = 1$ in Corollaries 1.12 and 1.14, we find that for any positive even integer n we have

$$\det[F_{j+k} + \delta_{jk}]_{0\le j,k\le n-1} = F_{n-1}^2 \tag{1.39}$$

and

$$\det[L_{j+k} + \delta_{jk}]_{0\le j,k\le n-1} = L_nL_{n+1} - 1 = L_{2n+1}. \tag{1.40}$$

We are going to prove Theorems 1.1-1.2 and Theorems 1.3-1.4 in Sections 2 and 3, respectively. We will propose some conjectures in Section 4.

2. Proofs of Theorems 1.1-1.2

Proof of Theorem 1.1. Let M_n denote the matrix $[w_{|j-k|}]_{1\le j,k\le n}$. Clearly,

$$\det(M_1)=w_0=w_0u_1(A',B')+((Bw_0)^2-(w_1-Aw_0)^2)u_0(A',B')$$

and

$$\begin{aligned}\det(M_2)=\begin{vmatrix}w_0 & w_1\\ w_1 & w_0\end{vmatrix}&=w_0^2-w_1^2\\ &=w_0u_2(A',B')+((Bw_0)^2-(w_1-Aw_0)^2)u_1(A',B').\end{aligned}$$

So (1.6) holds for $n=1,2$.

Now suppose $n\ge 3$, and assume that

$$\det(M_k)=w_0u_k(A',B')+((Bw_0)^2-(w_1-Aw_0)^2)u_{k-1}(A',B') \tag{2.1}$$

for each $k=1,\ldots,n-1$. Observe that

$$\begin{aligned}&Bw_{|(n-2)-k|}-Aw_{|(n-1)-k|}+w_{|n-k|}\\ =&\begin{cases}Bw_{n-2-k}-Aw_{n-1-k}+w_{n-k}=0 & \text{if } 1\le k<n-1,\\ Bw_1-Aw_0+w_1=(B+1)w_1-Aw_0 & \text{if } k=n-1,\\ Bw_2-Aw_1+w_0=B(Aw_1-Bw_0)-Aw_1+w_0 & \text{if } k=n.\end{cases}\end{aligned}$$

Thus, adding the $(n-2)$-th row times B and the $(n-1)$-th row times $-A$ to the last row of M_n, we find that $\det(M_n)=\det(M_n')$, where

$$M_n':=\begin{bmatrix}w_0 & w_1 & w_2 & \cdots & w_{n-3} & w_{n-2} & w_{n-1}\\ w_1 & w_0 & w_1 & \cdots & w_{n-4} & w_{n-3} & w_{n-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ w_{n-4} & w_{n-5} & w_{n-6} & \cdots & w_1 & w_2 & w_3\\ w_{n-3} & w_{n-4} & w_{n-5} & \cdots & w_0 & w_1 & w_2\\ w_{n-2} & w_{n-3} & w_{n-4} & \cdots & w_1 & w_0 & w_1\\ 0 & 0 & 0 & \cdots & 0 & C & (B-1)D\end{bmatrix}$$

with $C=(B+1)w_1-Aw_0$ and $D=Aw_1-(B+1)w_0$. Adding the $(n-2)$-th column times B and the $(n-1)$-th column times $-A$ to the last column of M_n', we see that $\det(M_n')=\det(M_n'')$, where

$$M_n'':=\begin{bmatrix}w_0 & w_1 & w_2 & \cdots & w_{n-3} & w_{n-2} & 0\\ w_1 & w_0 & w_1 & \cdots & w_{n-4} & w_{n-3} & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ w_{n-4} & w_{n-5} & w_{n-6} & \cdots & w_1 & w_2 & 0\\ w_{n-3} & w_{n-4} & w_{n-5} & \cdots & w_0 & w_1 & 0\\ w_{n-2} & w_{n-3} & w_{n-4} & \cdots & w_1 & w_0 & C\\ 0 & 0 & 0 & \cdots & 0 & C & (B-1)D-AC\end{bmatrix}.$$

Expanding $\det(M_n'')$ via its last row, we get

$$\det(M_n'') = ((B-1)D - AC)\begin{vmatrix} w_0 & w_1 & w_2 & \cdots & w_{n-3} & w_{n-2} \\ w_1 & w_0 & w_1 & \cdots & w_{n-4} & w_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{n-4} & w_{n-5} & w_{n-6} & \cdots & w_1 & w_2 \\ w_{n-3} & w_{n-4} & w_{n-5} & \cdots & w_0 & w_1 \\ w_{n-2} & w_{n-3} & w_{n-4} & \cdots & w_1 & w_0 \end{vmatrix}$$

$$- C\begin{vmatrix} w_0 & w_1 & w_2 & \cdots & w_{n-3} & 0 \\ w_1 & w_0 & w_1 & \cdots & w_{n-4} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{n-4} & w_{n-5} & w_{n-6} & \cdots & w_1 & 0 \\ w_{n-3} & w_{n-4} & w_{n-5} & \cdots & w_0 & 0 \\ w_{n-2} & w_{n-3} & w_{n-4} & \cdots & w_1 & C \end{vmatrix}.$$

Therefore

$$\det(M_n) = \det(M_n') = \det(M_n'') = ((B-1)D-AC)\det(M_{n-1}) - C^2\det(M_{n-2}).$$

Note that $C^2 = B'$ and

$$(B-1)D - AC = (B-1)(Aw_1 - (B+1)w_0) - A((B+1)w_1 - Aw_0) = A'.$$

Thus, with the aid of (2.1) for $k = n-1, n-2$, we have

$$\begin{aligned} \det(M_n) &= A'\det(M_{n-1}) - B'\det(M_{n-2}) \\ &= A'(w_0u_{n-1}(A',B') + ((Bw_0)^2 - (w_1 - Aw_0)^2)u_{n-2}(A',B')) \\ &\quad - B'(w_0u_{n-2}(A',B') + ((Bw_0)^2 - (w_1 - Aw_0)^2)u_{n-3}(A',B')) \\ &= w_0u_n(A',B') + ((Bw_0)^2 - (w_1 - Aw_0)^2)u_{n-1}(A',B'). \end{aligned}$$

In view of the above, by induction the identity (1.6) holds for any $n \in \mathbb{Z}^+$. $\square$

Proof of Theorem 1.2. Let $W_n = \det[w_{|j-k+1|}]_{1\le j,k\le n}$. Clearly,

$$W_2 = \begin{vmatrix} w_1 & w_0 \\ w_2 & w_1 \end{vmatrix} = w_1^2 - w_0(Aw_1 - Bw_0) = w_1^2 - Aw_0w_1 + Bw_0^2.$$

Now, assume $n \ge 3$. Observe that

$$\begin{aligned} & Bw_{|(n-2)-k+1|} - Aw_{|(n-1)-k+1|} + w_{|n-k+1|} \\ &= \begin{cases} Bw_{n-k-1} - Aw_{n-k} + w_{n-k+1} = 0 & \text{if } 1 \le k < n, \\ Bw_1 - Aw_0 + w_1 = (B+1)w_1 - Aw_0 & \text{if } k = n. \end{cases} \end{aligned}$$

Thus, adding the $(n-2)$-th row times B and the $(n-1)$-th row times $-A$ to the last row of the determinant W_n, we find that the last row turns to be

$$\underbrace{0\ \cdots\ 0}_{n-1}\ (B+1)w_1 - Aw_0.$$

Therefore

$$W_n = ((B+1)w_1 - Aw_0)W_{n-1}.$$

In view of the above, by induction we have

$$W_n = (w_1^2 - Aw_0w_1 + Bw_0^2)((B+1)w_1 - Aw_0)^{n-2}$$

for all $n = 2, 3, \ldots$. This ends our proof. $\Box$

3. Proofs of Theorems 1.3-1.4

Proof of Theorem 1.3. Let $a_{jk} = q^{|j-k|} + x\delta_{jk}$ for all $j, k = 1, \ldots, n$, and let Q_n denote the matrix $[a_{jk}]_{1\le j,k\le n}$. Clearly, $\det(Q_1) = q^0 + x = x + 1$ and

$$\det(Q_2) = \begin{vmatrix} q^0 + x & q \\ q & q^0 + x \end{vmatrix} = (x+1)^2 - q^2 = (x+1)(1-q^2+(1+q^2)x) - q^2x^2.$$

Thus, (1.26) holds for $n \in \{1, 2\}$.

Now, we let $n \ge 3$ and assume the equality

$$\begin{aligned} \det[q^{|j-k|} + x\delta_{jk}]_{1\le j,k\le m} = (x+1)&u_m(1-q^2+(1+q^2)x, q^2x^2) \\ &- q^2x^2u_{m-1}(1-q^2+(1+q^2)x, q^2x^2) \end{aligned} \tag{3.1}$$

for any positive integer $m < n$.

Observe that

$$\begin{aligned} -qa_{n-1,k} + a_{nk} &= -q(q^{|n-1-k|} + x\delta_{n-1,k}) + q^{|n-k|} + x\delta_{nk} \\ &= \begin{cases} 0 & \text{if} 1 \le k \le n-2, \\ -q(1+x) + q = -qx & \text{if } k = n-1, \\ -q^2 + 1 + x & \text{if } k = n. \end{cases} \end{aligned}$$

Thus, via adding the $(n-1)$-th row times $-q$ to the n-th row of Q_n, we see that $\det(Q_n) = \det(Q_n')$, where

$$Q_n' := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-3,1} & a_{n-3,2} & a_{n-3,3} & \cdots & a_{n-3,n-2} & a_{n-3,n-1} & a_{n-3,n} \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & -qx & x+1-q^2 \end{bmatrix}.$$

Via adding the $(n-1)$-th column times $-q$ to the n-th column of Q_n', we get that $\det(Q_n')=\det(Q_n'')$, where

$$Q_n'':=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2} & a_{1,n-1} & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-2} & a_{2,n-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-3,1} & a_{n-3,2} & a_{n-3,3} & \cdots & a_{n-3,n-2} & a_{n-3,n-1} & 0 \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} & a_{n-2,n-1} & 0 \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & -qx \\ 0 & 0 & 0 & \cdots & 0 & -qx & f(q,x) \end{bmatrix},$$

where

$$f(q,x):=1-q^2+(1+q^2)x.$$

Expanding $\det(Q_n'')$ via its last row, we see that

$$\begin{aligned}\det(Q_n'')=&\ f(q,x)\det(Q_{n-1})\\ &-(-qx)\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2} & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-2} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-3,1} & a_{n-3,2} & a_{n-3,3} & \cdots & a_{n-3,n-2} & 0 \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} & 0 \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-2} & -qx \end{vmatrix}.\end{aligned}$$

Therefore,

$$\det(Q_n)=\det(Q_n')=\det(Q_n'')=f(q,x)\det(Q_{n-1})-(-qx)^2\det(Q_{n-2}),$$

Combining this with (3.1) for $m=n-1,n-2$, we find that

$$\begin{aligned}\det(Q_n)=&\ f(q,x)\left((x+1)u_{n-1}(f(q,x),q^2x^2)-q^2x^2u_{n-2}(f(q,x),q^2x^2)\right)\\ &-q^2x^2\left((x+1)u_{n-2}(f(q,x),q^2x^2)-q^2x^2u_{n-3}(f(q,x),q^2x^2)\right)\\ =&\ (x+1)u_n(f(x,q),q^2x^2)-q^2x^2u_{n-1}(f(q,x),q^2x^2).\end{aligned}$$

In view of the above, we have proved the desired result by induction n. □

Lemma 3.1. *Let $A,B\in\mathbb{C}$, and let*

$$w_0,w_1\in\mathbb{C},\ \text{and}\ w_{i+1}=Aw_i-Bw_{i-1}\ \text{for all}\ i=1,2,3,\ldots.$$

Then, for any $j\in\mathbb{N}$ and $k\in\mathbb{Z}^+$, we have

$$w_{j+k}=w_{j+1}u_k(A,B)-Bw_ju_{k-1}(A,B). \tag{3.2}$$

Proof. This can be easily proved by induction on k. □

Proof of Theorem 1.4. Let α and β be the two distinct roots of the quadratic equation $x^2-Ax+B=0$ with $B=-1$. Then $\alpha+\beta=A$ and $\alpha\beta=B=-1$. It is well known that there are $a,b\in\mathbb{C}$ such that $w_m=a\alpha^m+b\beta^m$ for all $m\in\mathbb{Z}$. As $a+b=w_0$ and $a\alpha+b\beta=w_1$, we find that

$$a=\frac{w_1-\beta w_0}{\alpha-\beta}\quad\text{and}\quad b=\frac{\alpha w_0-w_1}{\alpha-\beta}. \tag{3.3}$$

It follows that

$$ab = \frac{(\alpha+\beta)w_0w_1 - Bw_0^2 - w_1^2}{(\alpha-\beta)^2} = \frac{w_0^2 + Aw_0w_1 - w_1^2}{(\alpha-\beta)^2}. \tag{3.4}$$

Observe that $\alpha \neq \pm 1$ since $\alpha\beta = B = -1$ and $\alpha + \beta = A \neq 0$. For any $j \in \mathbb{N}$, we clearly have

$$\begin{aligned}\sum_{k=0}^{n-1} w_{j+k}\alpha^k &= \sum_{k=0}^{n-1}(a\alpha^{j+k} + b\beta^{j+k})\alpha^k \\ &= a\alpha^j \sum_{k=0}^{n-1}\alpha^{2k} + b\beta^j \sum_{k=0}^{n-1} B^k = a\alpha^j \frac{\alpha^{2n}-1}{\alpha^2-1} + b\beta^j \frac{B^n-1}{B-1}.\end{aligned}$$

Note that $B^n = 1$ as $B = -1$ and $2 \mid n$. Thus

$$\sum_{k=0}^{n-1} w_{j+k}\alpha^k = a\lambda^j \frac{\alpha^{2n} - (\alpha\beta)^n}{\alpha^2 + \alpha\beta} = \alpha^j \times \frac{a}{A}\alpha^{n-1}(\alpha^n - \beta^n).$$

So, the matrix $W = [w_{j+k}]_{0 \le j,k \le n-1}$ has an eigenvalue

$$\lambda_0 = \frac{a}{A}\alpha^{n-1}(\alpha^n - \beta^n)$$

with the eigenvector $(1, \alpha, \alpha^2, \ldots, \alpha^{n-1})^T$. Similarly,

$$\lambda_1 = \frac{b}{A}\beta^{n-1}(\beta^n - \alpha^n) = -\frac{b}{A}\beta^{n-1}(\alpha^n - \beta^n)$$

is an eigenvalue of W with the eigenvector $(1, \beta, \beta^2, \ldots, \beta^{n-1})^T$. If $c, d \in \mathbb{C}$ are not all zero, and $c\alpha^k + d\beta^k = 0$ for all $k = 0, \ldots, n-1$, then $c = -d \neq 0$ and $\alpha = \beta$. As $\alpha \neq \beta$, the two vectors $(1, \alpha, \alpha^2, \ldots, \alpha^{n-1})^T$ and $(1, \beta, \beta^2, \ldots, \beta^{n-1})^T$ are linearly independent over $\mathbb{C}$.

For each $2 \le m \le n-1$, since

$$w_n(-Bu_{m-1}) + w_{j+1}u_m + w_{j+m} \times (-1) = 0$$

by Lemma 3.1, the vector $V_m = (v_{m0}, v_{m1}, \ldots, v_{m,n-1})^T$ is an eigenvector associated to the eigenvalue $\lambda_m = 0$ of the matrix W, where

$$v_{m0} = -Bu_{m-1} = u_{m-1},\ v_{m1} = u_m,\ v_{mm} = -1,$$

and $v_{mk} = 0$ for all $2 \le k \le n-1$ with $k \neq m$. If $c_2, \ldots, c_{n-1} \in \mathbb{C}$ and $\sum_{m=2}^{n-1} c_m v_{mk} = 0$ for all $k = 0, 1, \ldots, n-1$, then for each $2 \le k \le n-1$ we have

$$0 = \sum_{m=2}^{n-1} c_m v_{mk} = c_k v_{kk} = c_k.$$

So, the vectors $V_2, \ldots, V_{n-1}$ are linearly independent over $\mathbb{C}$.

In view of the above, $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are all the n eigenvalues of the matrix $W = [w_{j+k}]_{0\le j,k\le n-1}$. So we have

$$\det[x\delta_{jk} - w_{j+k}]_{0\le j,k\le n-1} = \prod_{m=0}^{n-1}(x-\lambda_m) = x^{n-2}(x-\lambda_0)(x-\lambda_1)$$
$$= x^{n-2}(x^2 - (\lambda_0+\lambda_1)x + \lambda_0\lambda_1).$$

In view of (3.3),

$$\lambda_0+\lambda_1 = \frac{u_n}{A}\left((w_1-\beta w_0)\alpha^{n-1} - (\alpha w_0 - w_1)\beta^{n-1}\right) = \frac{u_n}{A}\left(w_1 v_{n-1} - Bw_0 v_{n-2}\right).$$

Also,

$$\lambda_0\lambda_1 = -\frac{abB^{n-1}}{A^2}(\alpha^n-\beta^n)^2 = -\frac{B^{n-1}u_n^2}{A^2}(w_0^2 + Aw_0w_1 - w_1^2)$$

by (3.4).

4. Some Conjectures

Conjecture 4.1. *For any positive integer n, we have*

$$\det[2^{|j-k|} - 1 + \delta_{j,k}]_{1\le j,k\le n} = 2^n + (-1)^n 2^{n-1}\left(2\binom{n}{3} + n\binom{n+1}{3}\right). \tag{4.1}$$

Remark 4.1. For any positive integer n, [6, Theorem 1.4] implies that

$$\det[2^{j+k} - 1 + \delta_{jk}]_{1\le j,k\le n} = 4(2^n-1)^2 - (n-1)\frac{4^{n+1}-1}{3}.$$

Conjecture 4.2. *For any positive odd integer n, we have*

$$\det[F_{j+k} + \delta_{jk}]_{0\le j,k\le n-1} = F_{n+1}^2 + 1 \tag{4.2}$$
$$\det[F_{j+k} + \delta_{jk}]_{1\le j,k\le n} = F_{n+1}^2 + 1 = F_nF_{n+2}, \tag{4.3}$$
$$\det[L_{j+k} + \delta_{jk}]_{0\le j,k\le n-1} = L_nL_{n+1} = L_{2n+1} - 1. \tag{4.4}$$

Remark 4.2. For any positive integer n, H. Wang and Z.-W. Sun [6, Theorem 1.1(ii)] proved that

$$\det[F_{|j-k|} + \delta_{jk}]_{1\le j,k\le n} = \begin{cases} 1 & \text{if } n \equiv 0, \pm 1 \pmod{6}, \\ 0 & \text{otherwise.} \end{cases}$$

Based on our computation via `Mathematica`, we also propose the following two conjectures.

Conjecture 4.3. *For any $A \in \mathbb{C}$ and $n \in \mathbb{Z}^+$, we have*

$$\det[v_{j+k}(A,1) + \delta_{jk}]_{1\le j,k\le n} = u_{n+1}(A,1)^2 - n^2. \tag{4.5}$$

Conjecture 4.4. *Let n be any positive odd integer. For any $A \in \mathbb{C}$ we have*

$$A^2\det[v_{j+k}(A,-1) + \delta_{jk}]_{1\le j,k\le n} = v_{n+1}(A,-1)^2 - A^2 - 4. \tag{4.6}$$

In particular,

$$\det[L_{j+k} + \delta_{jk}]_{1\le j,k\le n} = L_{n+1}^2 - 5. \tag{4.7}$$

Remark 4.3. For any $A, B \in \mathbb{C}$ and $n \in \mathbb{N}$, it is well known that

$$v_n(A,B)^2 - (A^2 - 4B)u_n(A,B)^2 = 4B^n$$

which can be easily proved.

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