ON THE LAPLACIAN MATCHING ROOT INTEGRAL VARIATION

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Abstract. In this paper, we devote to investigating the circumstances under which the addition of an edge to a graph will cause the Laplacian matching roots to change only by integer quantities. We prove that the Laplacian matching root integral variation in one place is impossible and the Laplacian matching root integral variation in two places is also impossible under some constraints.

1. Introduction

There are several polynomials associated with a graph, including the characteristic polynomial, the chromatic polynomial, the matching polynomial , and the Tutte polynomial. One of the most fundamental topics in graph polynomial theory is to investigate the properties of roots of the polynomials. In this paper, we devote to studying that how the roots of the Laplacian matching polynomial of a graph change by integer quantities while adding an edge.

Throughout this paper, all graphs are assumed to be finite, undirected, and without loops or multiple edges. Let G be a graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G) = \{e_1, \dots, e_m\}$. For a vertex v of G, we denote by $N(v)$ the set of all vertices of G adjacent to v. The degree of v is defined as $|N(v)|$, and is denoted by $d(v)$. The maximum degree of the vertices of G is denoted by $\Delta(G)$. For a missing edge e in G, $G + e$ is the graph obtained from G by adding e as a new edge. For a subset W of $V(G)$, we shall use $G[W]$ to denote the induced graph of G induced by W. For a subset M of $E(G)$, we shall use $V(M)$ to denote the set of vertices of G each of which is an endpoint of one of the edges in M. If no two distinct edges in M share a common endpoint, then M is called a matching of G. The set of matchings of G is denoted by $\mathcal{M}(G)$. An i-matching is a matching of size i, we denote by $\phi_i(G)$ the number of i-matching of G, with the convention that $\phi_0(G) = 1$. The *matching polynomial* of G is

$$\mathscr{M}(G, x) = \sum_{M\in\mathcal{M}(G)} (-1)^{|M|} x^{|V(G)\setminus V(M)|} = \sum_{i\geq 0} (-1)^i \phi_i(G) x^{n-2i},$$

which was formally defined by Heilmann and Lieb [4] in studying statistical physics.

The matching polynomial is an absorbing mathematical object and is closely related to other topics in spectral graph theory. For instance, the well known Heilmann-Lieb root bound theorem [4] states that for a graph G with maximum degree $\Delta(G) \geq 2$, the roots of $\mathscr{M}(G, x)$ lie in the interval $(-2\sqrt{\Delta(G) - 1}, 2\sqrt{\Delta(G) - 1})$. Godsil and Gutman [3] shown that the average of

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adjacency characteristic polynomials of all signed graphs with underlying graph G is exactly the matching polynomial of G. Based on above two facts, Marcus, Spielman, and Srivastava [6] established that there are infinitely many bipartite Ramanujan graphs.

By observing above relation between the adjacency characteristic polynomial and the matching polynomial, it is natural to ask: What is the average of Laplacian characteristic polynomials of all signed graphs with underlying graph G? In 2020, Mohammadian [7] called it *the Laplacian matching polynomial* of G, denoted by $\mathscr{LM}(G,x)$, and proved that it has the following expression

$$\mathscr{LM}(G,x) = \sum_{M\in\mathcal{M}(G)} (-1)^{|M|} \prod_{v\in V(G)\setminus V(M)} (x-d(v)). \tag{1.1}$$

Independently, Zhang and Chen [12] also studied this polynomial and called it *the average Laplacian polynomial* of G.

The roots of $\mathscr{LM}(G,x)$, denoted by $\lambda_1(G),\dots,\lambda_n(G)$, are called the Laplacian matching roots of G. In [7], Mohammadian proved that all Laplacian matching roots of G are real and nonnegative. So, the Laplacian matching roots can be arranged as $\lambda_1(G)\ge\cdots\ge\lambda_n(G)\ge 0$. By (1.1), it is easy to observe that the sum of all Laplacian matching roots of G equals to $\sum_{v\in V(G)} d(v)$. This observation implies that for any $e\notin E(G)$,

$$\sum_{i=1}^{n}\lambda_i(G+e)-\sum_{i=1}^{n}\lambda_i(G)=2. \tag{1.2}$$

Recently, Wan, Wang, and Mohammdian [9] proved the following Theorem.

Theorem 1.1. *Let G be a graph of order n. Then the Laplacian matching roots of $G+e$ interlace those of G, that is,*

$$\lambda_1(G+e)\ge\lambda_1(G)\ge\lambda_2(G+e)\ge\lambda_2(G)\ge\cdots\ge\lambda_n(G+e)\ge\lambda_n(G). \tag{1.3}$$

Combining (1.2) and (1.3), we see that by adding an edge, none of the Laplacian matching roots can decrease, and that the sum of those roots will increase by 2. To investigate the integrality of Laplacian matching roots, in this paper, we only consider the circumstances under which the addition of an edge to a graph will cause the Laplacian matching roots to change only by integer quantities. Evidently, there are just two possible cases that can happen as follows:

(A) one root of $\mathscr{LM}(G,x)$ increasing by 2 and other $n-1$ roots of $\mathscr{LM}(G,x)$ remain unchanged;

(B) two roots of $\mathscr{LM}(G,x)$ increasing by 1 and other $n-2$ roots of $\mathscr{LM}(G,x)$ remain unchanged.

We refer to (A) and (B) by saying the *Laplacian matching root integral variation* (abbreviated by LMRIV) occurs to G in one place by adding an edge and LMRIV occurs to G in two places, respectively. The analogous discussion for the Laplacian eigenvalues of G was considered in serveral literatures. For more details, one can refer to [1], [5], [8] and [10].

In this paper, we will prove that LMRIV in one place is impossible and LMRIV in two places is also impossible if $\frac{g(G)}{c(G)}>\frac{7}{6}$, where $g(G)$ is the girth of G and $c(G)$ is the dimension of cycle space of G.

2. Preliminaries

In this section, we collect some concepts and known results about the matching polynomial and the Laplacian matching polynomial for later use. Let G be a graph of order n. If $v\in V(G)$, then $G-v$ is the graph obtained from G by deleting v together with all edges incident to v. The matching polynomial satisfies the following basic identity, which is called the expansion formula of $\mathscr{M}(G,x)$ at vertex v.

Proposition 2.1. [2] *Let G be a graph. For any $v \in V(G)$,*

$$\mathscr{M}(G,x) = x\mathscr{M}(G-v,x) - \sum_{u \in N(v)} \mathscr{M}(G-v-u,x).$$

The subdivision of G, denoted by $S(G)$, is the graph derived from G by replacing every edge $e = \{a,b\}$ of G with two edges $\{a, v_e\}$ and $\{b, v_e\}$ along with the new vertex v_e corresponding to the edge e. The following theorem establishs an important connection between the matching polynomial and the Laplacian matching polynomial, which is a useful tool to deal with the Laplacian matching roots of a graph.

Theorem 2.2. [9, 11, 12] *Let G be a graph. Then,*

$$\mathscr{M}(S(G),x) = x^{|E(G)|-|V(G)|}\mathscr{L}\mathscr{M}(G,x^2).$$

Another useful result due to Chen and Zhang [12] provides a combinatorial interpretation for the coefficients of the Laplacian matching polynomial by means of the weight of a TU-subgraph. A *TU-subgraph* of G is a subgraph whose components are trees or unicyclic graphs. Suppose that a TU-subgraph H of G consists of s unicyclic graphs and trees $T_1, \ldots, T_t$. Then the *weight* of H is defined as

$$\omega(H) = 2^s \prod_{i=1}^{t} |T_i|,$$

where $|T_i|$ is the order of T_i.

Theorem 2.3. [12] *Let G be a graph of order n, and $\mathscr{L}\mathscr{M}(G,x) = \sum_{i=0}^{n}(-1)^i b_i x^{n-i}$ be the Laplacian matching polynomial of G. Then,*

$$b_i = \omega(\mathscr{H}_i) = \sum_{H \in \mathscr{H}_i} \omega(H)$$

for $i = 1, 2, \ldots, n$, where $\mathscr{H}_i$ denotes the set of all the TU-subgraphs of G with i edges.

Recall that the Laplacian matching roots of G can be arranged as

$$\lambda_1(G) \geq \ldots \geq \lambda_n(G) \geq 0.$$

The following three theorems on the Laplacian matching roots for a connected graph, appeared in [9, 12], will be used frequently in the later section.

Theorem 2.4. [9, 12] *Let G be a connected graph. Then, $\lambda_n(G) = 0$ if and only if G is a tree.*

Theorem 2.5. [9] *Let G be a connected graph. Then,*

$$\lambda_1(G) \geq \Delta(G) + 1,$$

with the equality holds if and only if G is a star.

Theorem 2.6. [9] *Let G be a connected graph and $e \notin E(G)$. Then, $\lambda_1(G+e)$ has the multiplicity 1 and is strictly greater than $\lambda_1(G)$.*

3. Main results

The purpose of this section is to investigate the LMRIV. To begin with, we will prove that LMRIV in one place is impossible. Before proceeding, we introduce some needed notations. In what follows, we always suppose that G is a connected graph with $V(G) = \{v_1, \dots, v_n\}$ and $E(G) = \{e_1, \dots, e_m\}$, and always use $R(G) = \{\lambda_1, \dots, \lambda_n\}$ to denote the multiset of the Laplacian matching roots of G, where $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

Theorem 3.1. *The LMRIV will not occur in one place.*

Proof. Assume that LMRIV occurs to G in one place by adding a new edge $e = v_iv_j$. By Theorem 2.6, the largest Laplacian matching root must be changed by 2, so we can write $R(G+e) = \{\lambda_1 + 2, \lambda_2, \dots, \lambda_n\}$. It follows from Theorem 2.2 that

$$\mathscr{M}(S(G), x) = x^{m-n}\mathscr{L}\mathscr{M}(G, x^2) = x^{m-n}\prod_{\ell=1}^{n}(x^2 - \lambda_\ell)$$

and

$$\begin{aligned}\mathscr{M}(S(G+e), x) &= x^{m-n+1}\mathscr{L}\mathscr{M}(G+e, x^2)\\ &= x^{m-n+1}(x^2 - \lambda_1 - 2)\prod_{\ell=2}^{n}(x^2 - \lambda_\ell)\\ &= x\mathscr{M}(S(G), x) - 2x^{m-n+1}\prod_{\ell=2}^{n}(x^2 - \lambda_\ell).\end{aligned} \tag{3.1}$$

One the other hand, by Propostion 2.1, we have

$$\mathscr{M}(S(G+e), x) = x\mathscr{M}(S(G), x) - \mathscr{M}(S(G) - v_i, x) - \mathscr{M}(S(G) - v_j, x). \tag{3.2}$$

Combining (3.1) and (3.2), one can deduce that

$$\mathscr{M}(S(G) - v_i, x) + \mathscr{M}(S(G) - v_j, x) = 2x^{m-n+1}\prod_{\ell=2}^{n}(x^2 - \lambda_\ell). \tag{3.3}$$

By comparing the coefficient of x^{m+n-3} on two sides of (3.3), we observe that

$$4m - d(v_i) - d(v_j) = 2\sum_{\ell=2}^{n}\lambda_\ell = 2(2m - \lambda_1),$$

which implies that $d(v_i) + d(v_j) = 2\lambda_1$. This contradicts Theorem 2.5, completing the proof. $\square$

Now, we devote to considering the case in which the LMRIV occurs to G in two places. By Theorem 2.6, the largest matching root must be changed. In what follows, we always denote another changed root by λ_k.

Theorem 3.2. *If the LMRIV occurs to G in two places by adding a new edge $e = v_iv_j$ and the changed roots of G are λ_1 and λ_k, then*

$$\lambda_1 + \lambda_k = d(v_i) + d(v_j) + 1,$$
$$\lambda_1\lambda_k = d(v_i)d(v_j).$$

Proof. Write $R(G+e) = \{\lambda_1 + 1, \lambda_2, \dots, \lambda_{k-1}, \lambda_k + 1, \lambda_{k+1}, \dots, \lambda_n\}$. By Theorem 2.2, we have

$$\mathscr{M}(S(G), x) = x^{m-n}\mathscr{L}\mathscr{M}(G, x^2) = x^{m-n}\prod_{\ell=1}^{n}(x^2 - \lambda_\ell) \tag{3.4}$$

and

$$\begin{aligned}
\mathscr{M}(S(G+e),x) =& x^{m-n+1}\mathscr{L}\mathscr{M}(G+e,x^2)\\
=& x^{m-n+1}(x^2-\lambda_1-1)(x^2-\lambda_k-1)\prod_{\ell\neq 1,k}(x^2-\lambda_\ell)\\
=& x\mathscr{M}(S(G),x)-x^{m-n+1}(x^2-\lambda_1)\prod_{\ell\neq 1,k}(x^2-\lambda_\ell)\\
&-x^{m-n+1}(x^2-\lambda_k)\prod_{\ell\neq 1,k}(x^2-\lambda_\ell)+x^{m-n+1}\prod_{\ell\neq 1,k}(x^2-\lambda_\ell).
\end{aligned}\tag{3.5}$$

In addition, it follows from Proposition 2.1 that

$$\mathscr{M}(S(G+e),x)=x\mathscr{M}(S(G),x)-\mathscr{M}(S(G)-v_i,x)-\mathscr{M}(S(G)-v_j,x).\tag{3.6}$$

Combining the (3.5) and (3.6), one can deduce that

$$\begin{aligned}
&\mathscr{M}(S(G)-v_i,x)+\mathscr{M}(S(G)-v_j,x)\\
&=x^{m-n+1}(x^2-\lambda_1)\prod_{\ell\neq 1,k}(x^2-\lambda_\ell)+x^{m-n+1}(x^2-\lambda_k)\prod_{\ell\neq 1,k}(x^2-\lambda_\ell)\\
&-x^{m-n+1}\prod_{\ell\neq 1,k}(x^2-\lambda_\ell)\\
&=(2x^2-\lambda_1-\lambda_k-1)x^{m-n+1}\prod_{\ell\neq 1,k}(x^2-\lambda_\ell).
\end{aligned}\tag{3.7}$$

Note that both $S(G)-v_i$ and $S(G)-v_j$ contain $m+n-1$ vertices, and contain $2m-d(v_i)$ and $2m-d(v_j)$ edges, respectively. Further, by comparing the coefficients of x^{m+n-3} on two sides of (3.7), we observe that

$$4m-d(v_i)-d(v_j)=\lambda_1+\lambda_k+1+2\sum_{\ell\neq 1,k}\lambda_\ell=4m-\lambda_1-\lambda_k+1,$$

which implies that

$$\lambda_1+\lambda_k=d(v_i)+d(v_j)+1.$$

Now, we are going to prove the second statement. Recall that $\phi_2(G)$ deonte the number of the 2-matchings in G. For any $u\in V(G)$, it is clear that

$$\phi_2(S(G))-\phi_2(S(G)-u)=(2m-d(u)-1)d(u).\tag{3.8}$$

By comparing the coefficients of x^{m+n-4} on two sides of (3.4), we observe that

$$\phi_2(S(G))=\sum_{1\leq s<t\leq n}\lambda_s\lambda_t.\tag{3.9}$$

For $v_i,v_j\in V(G)$, combining (3.8) and (3.9), we have

$$\phi_2(S(G)-v_i)=\sum_{1\leq s<t\leq n}\lambda_s\lambda_t-(2m-d(v_i)-1)d(v_i)$$

and

$$\phi_2(S(G)-v_j)=\sum_{1\leq s<t\leq n}\lambda_s\lambda_t-(2m-d(v_j)-1)d(v_j).$$

Hence, writing $\phi(v_i,v_j)=\phi_2(S(G)-v_i)+\phi_2(S(G)-v_j)$, we have

$$\phi(v_i,v_j)=2\sum_{1\leq s<t\leq n}\lambda_s\lambda_t-(2m-d(v_i)-1)d(v_i)-(2m-d(v_j)-1)d(v_j).\tag{3.10}$$

By comparing the coefficients of x^{m+n-5} on two sides of (3.7), we have

$$\phi(v_i, v_j) = 2\sum_{s,t\neq 1,k} \lambda_s\lambda_t + (\lambda_1 + \lambda_k + 1)\sum_{\ell\neq 1,k} \lambda_\ell. \tag{3.11}$$

Therefore, it follows from (3.10) and (3.11) that

$$(\lambda_1 + \lambda_k - 1)\sum_{\ell\neq 1,k} \lambda_\ell + 2\lambda_1\lambda_k = 2m(d(v_i) + d(v_j)) - (d^2(v_i) + d^2(v_j) + d(v_i) + d(v_j)). \tag{3.12}$$

Noting that $\lambda_1 + \lambda_k = d(v_i) + d(v_j) + 1$ and $\sum\limits_{\ell\neq 1,k} \lambda_\ell = 2m - (\lambda_1 + \lambda_k)$, we can deduce that

$$\lambda_1\lambda_k = d(v_i)d(v_j),$$

as desired. □

As applications of Theorem 3.2, in the following two consequeces, we give some sufficent conditions for the LMRIV not occuring in two places.

Corollary 3.3. *For any tree T, the LMRIV will not occur in two places.*

Proof. If the $LMRIV$ occurs to T in two places by adding a new edge $e = v_iv_j$, by Theorem 2.4 and Theorem 2.6, the changed roots are λ_1 and $\lambda_n(= 0)$. By Theorem 3.2, we find that

$$d(v_i)d(v_j) = \lambda_1\lambda_n = 0,$$

which implies that $d(v_i) = 0$ or $d(v_j) = 0$, contradiction. □

Corollary 3.4. *For two nonadjacent vertices v_i and v_j, if $d(v_i) + d(v_j) \leq 3$, then the LMRIV will not occur to G in two places by adding a new edge $e = v_iv_j$.*

Proof. Assume that $LMRIV$ occurs to G in two places by adding the edge $e = v_iv_j$ with $d(v_i) + d(v_j) \leq 3$. It follows from Theorem 3.2 and the Vieta's formulas that

$$\lambda_1 = \frac{(d(v_i) + d(v_j) + 1) + \sqrt{(d(v_i) + d(v_j) + 1)^2 - 4d(v_i)d(v_j)}}{2} < d(v_i) + d(v_j) + 1 \leq 4.$$

Theorem 2.5 states that $\lambda_1 \geq \Delta(G) + 1$, which implies that $\Delta(G) < 3$. Since G contains two vertices v_i and v_j such that $d(v_i) + d(v_j) \leq 3$, we conclude that G is a path, which contradicts Corollary 3.3. □

Next, we will keep on discussing the LMRIV occuring in two places by combining another useful tool. By Theorem 2.3, we may let $\mathscr{LM}(G, x) = \sum_{i=0}^{n}(-1)^i b_i x^{n-i}$ and $\mathscr{LM}(G + e, x) = \sum_{i=0}^{n}(-1)^i \tilde{b}_i x^{n-i}$. If the $LMRIV$ occurs to G in two places by adding a new edge $e = v_iv_j$ and the changed roots of G are λ_1 and λ_k, then

$$\frac{\widetilde{b}_n}{b_n} = \frac{(\lambda_1 + 1)(\lambda_k + 1)}{\lambda_1\lambda_k}. \tag{3.13}$$

For convenience, denote by $\mathscr{H}(G)$ (or $\mathscr{H}$) the set of all the TU-subgraphs of G with n edges, and denote by $\mathscr{T}(G)$ the set of all spanning trees of G. By Theorem 2.3, we have

$$b_n = \omega(\mathscr{H}(G)) = \sum_{H\in\mathscr{H}(G)} \omega(H).$$

It should be mentioned here that for any $H \in \mathscr{H}(G)$, all components of H are unicylic because that H is a TU-subgraph of size n.

Lemma 3.5. *For two nonadjacent vertices v_i and v_j, if $d(v_i) = d(v_j) = 2$, then the LMRIV will not occur to G in two places by adding a new edge $e = v_iv_j$.*

Proof. Assume that $LMRIV$ occurs to G in two places by adding the edge $e = v_iv_j$ with $d(v_i) = d(v_j) = 2$. It follows from Theorem 3.2 that $\lambda_1 = 4$ and $\lambda_k = 1$. By Theorem 2.5, we can deduce that $\Delta(G) \le 3$. If $\Delta(G) = 3$, then $\lambda_1 = \Delta(G)+1$, and so Theorem 2.5 states that G is a star. This contradicts Corollary 3.3. If $\Delta(G) \le 2$, then G is a tree unless that G is a cylce C_n. By Corollary 3.3, we only consider the case that $G = C_n$ $(n \ge 3)$. It follows from Theorem 2.3 that $b_n = 2$ and $\widetilde{b_n} = 2(n+1)$. By (3.13),

$$\frac{2(n+1)}{2} = \frac{\widetilde{b}_n}{b_n} = \frac{(\lambda_1+1)(\lambda_k+1)}{\lambda_1\lambda_k} = \frac{5}{2}.$$

This contradiction completes the proof. ∎

Let us introduce more notations and definitions for later use. Recall that we always suppose that G is a connected graph of order n $(n \ge 3)$ and size m. The girth of G is denoted by $g(G)$, and the dimension of cycle space of G is denoted by $c(G)$, that is, $c(G) = m-n+1$. A connected graph G is unicylic (bicyclic, resp.) if $c(G) = 1$ $(c(G) = 2$, resp.) Let $\pi = \{V_1, \dots, V_{p(\pi)}\}$ be a partition of $V(G)$, and let Ω be the set of all these partions. Denote by G_i the subgraph of G induced by V_i. A partition π is called *TU-admissible* if for any i $(1 \le i \le p(\pi))$, G_i is a connected graph with $c(G_i) \ge 1$. We shall use $\mathscr{H}_\pi(G)$ to denote the subset of $\mathscr{H}(G)$ consisting of the TU-subgraphs of size n whose components are corresponding to π. So, $\mathscr{H}(G)$ can be partitioned as $\{\mathscr{H}_\pi(G)\}_{\pi\in\Omega}$. If $p(\pi) = 1$, we simply use $\mathscr{H}_1(G)$ instead of $\mathscr{H}_\pi(G)$, which denotes the set consisting of all unicyclic spanning subgraphs of G.

Lemma 3.6. *Let G be a connected graph with $c(G) \ge 1$. Then $\frac{|\mathscr{T}(G)|}{|\mathscr{H}_1(G)|} \ge \frac{g(G)}{c(G)}$.*

Proof. We are going to count pairs (T,U) consisting of unicyclic spanning subgraphs U and spanning trees T of G satisfying $E(T) \subset E(U)$. On the one hand, the number of such pairs is given by $|\mathscr{T}(G)|c(G)$. On the other hand, the number of pairs (T,U) is at least $|\mathscr{H}_1(G)|g(G)$. This completes the proof. ∎

Lemma 3.7. *Let G be a connected graph with $\frac{g(G)}{c(G)} > 1$. For two nonadjacent vertices v_i and v_j, if $d(v_i) = 1$ or $d(v_j) = 1$, then the LMRIV will not occur to G in two places by adding a new edge $e = v_iv_j$.*

Proof. Assume that $LMRIV$ occurs to G in two places by adding the edge $e = v_iv_j$ with $d(v_i) = 1$. Denote by f the edge incident to v_i. By Corollary 3.4, we can assume that $d(v_j) \ge 3$. It follows from (3.13) and Theorem 3.2 that

$$\frac{\widetilde{b}_n}{b_n} = \frac{(\lambda_1+1)(\lambda_k+1)}{\lambda_1\lambda_k} = 1 + \frac{1}{d(v_i)} + \frac{1}{d(v_j)} + \frac{2}{d(v_i)d(v_j)} \le 3.$$

To get a contradiction, we prove that $\frac{\widetilde{b}_n}{b_n} > 3$. Applying Theorem 2.3, we have

$$\frac{\widetilde{b}_n}{b_n} = \frac{\omega(\mathscr{H}(G+e))}{\omega(\mathscr{H}(G))} = \frac{\omega(\mathscr{H}^e(G+e)) + \omega(\mathscr{H}^f(G+e)) + \omega(\mathscr{H}^{e,f}(G+e))}{\omega(\mathscr{H}(G))},$$

where $\mathscr{H}^e(G+e)$ $(\mathscr{H}^f(G+e)$, $\mathscr{H}^{e,f}(G+e)$, resp.) is the subset of $\mathscr{H}(G+e)$ consisting of the TU-subgraphs of size n containing e but no f (containing e but no f, containing e and f, resp.). Clearly, $\omega(\mathscr{H}^f(G+e)) = \omega(\mathscr{H}(G))$. Note that any TU-subgraph in $\mathscr{H}^e(G+e)$ can be obtained from some TU-subgraph of G by replacing f by e, and vice versa. It is not hard to see that $\omega(\mathscr{H}^e(G+e)) = \omega(\mathscr{H}(G))$. Therefore, it is enough to show that

$$\frac{\omega(\mathscr{H}^{e,f}(G+e))}{\omega(\mathscr{H}(G))} > 1. \tag{3.14}$$

Let $\pi=\{V_1,\ldots,V_p\}$ be a TU-admissible partition of $V(G)$. We say that π is of Type I if v_i and v_j lie in the same element of π. Otherwise, π is called of Type II. Recall that $\mathscr{H}_\pi(G)$ is the subset of $\mathscr{H}(G)$ consisting of the TU-subgraphs of size n whose components are corresponding to π, and $\mathscr{H}(G)$ can be partitioned as $\{\mathscr{H}_\pi(G)\}_{\pi\in\Omega}$. For any $\mathscr{H}_\pi(G)$, we define the subset $\sigma(\mathscr{H}_\pi(G))$ of $\mathscr{H}^{e,f}(G+e)$ as follows.

If π is of Type I, without loss of generality, assume that $v_i, v_j\in V_1$. Denote by G_i ($\widetilde{G}_i$, resp.) the subgraph of G ($G+e$, resp.) induced by V_i for $i=1,\ldots,p$. In the situation, $\sigma(\mathscr{H}_\pi(G))$ is defined to be the subset of $\mathscr{H}^{e,f}(G+e)$ consisting of the TU-subgraphs of size n whose components are corresponding to π. Note that for any $\widetilde{H}\in\sigma(\mathscr{H}_\pi(G))$, the component $\widetilde{H}[V_1]$ of H corresponding to V_1 is comprised of a spanning tree $\widetilde{G}_1-v_i$ together with edge e and f, and the component $\widetilde{H}[V_i]$ of H corresponding to V_i is a unicyclic spanning subgraph of $\widetilde{G}_i$ for $i=2,\ldots,p$, which is preserved while we add e to G. Therefore, we can deduce that

$$\frac{\sum_{\widetilde{H}\in\sigma(\mathscr{H}_\pi(G))}\omega(\widetilde{H})}{\sum_{H\in\mathscr{H}_\pi(G)}\omega(H)}=\frac{2^p|\mathscr{T}(\widetilde{G}_1-v_i)|\Pi_{i=2}^p|\mathscr{H}_1(\widetilde{G}_i)|}{2^p\Pi_{i=1}^p|\mathscr{H}_1(G_i)|}=\frac{|\mathscr{T}(\widetilde{G}_1-v_i)|}{|\mathscr{H}_1(G_1)|}=\frac{|\mathscr{T}(G_1|}{|\mathscr{H}_1(G_1)|}>1, \tag{3.15}$$

where the last inequality follows from Lemma 3.6.

If π is of Type II, without loss of generality, assume that $v_i\in V_1$ and $v_j\in V_2$. We still use G_i ($\widetilde{G}_i$, resp.) to denote the subgraph of G ($G+e$, resp.) induced by V_i for $i=1,\ldots,p$. In the situation, $\sigma(\mathscr{H}_\pi(G))$ is defined to be the subset of $\mathscr{H}^{e,f}(G+e)$ consisting of the TU-subgraphs $\widetilde{H}$ of size n satisfying the following conditions:

- The components of $\widetilde{H}$ are corresponding to the partition $\{V_1\cup V_2, V_3,\ldots,V_p\}$;
- $\widetilde{H}[V_1\cup V_2]-e$ exactly have two compopents which correspond to V_1 and V_2 repectively. Equivalently, e connects a spanning tree of $\widetilde{G}_1$ and a unicyclic spanning subgraph of $\widetilde{G}_2$, or e connects a spanning tree of $\widetilde{G}_2$ and a unicyclic spanning subgraph of $\widetilde{G}_1$.

Note that $G_i=\widetilde{G}_i$ for $i=1,\ldots,p$ as π is of Type II. Therefore, we can deduce that

$$\begin{aligned}\frac{\sum_{\widetilde{H}\in\sigma(\mathscr{H}_\pi(G))}\omega(\widetilde{H})}{\sum_{H\in\mathscr{H}_\pi(G)}\omega(H)}&=\frac{2^{p-1}(|\mathscr{T}(\widetilde{G}_1)||\mathscr{H}_1(\widetilde{G}_2)|+|\mathscr{T}(\widetilde{G}_2)||\mathscr{H}_1(\widetilde{G}_1)|)\Pi_{i=3}^p|\mathscr{H}_1(\widetilde{G}_i)|}{2^p\Pi_{i=1}^p|\mathscr{H}_1(G_i)|}\\&=\frac{|\mathscr{T}(G_1)|}{2|\mathscr{H}_1(G_1)|}+\frac{|\mathscr{T}(G_2)|}{2|\mathscr{H}_1(G_2)|}>1,\end{aligned} \tag{3.16}$$

where the last inequality follows from Lemma 3.6.

We are now ready to prove (3.14). By the above definition of $\sigma(\mathscr{H}_\pi(G))$, it is clear that $\sigma(\mathscr{H}_\pi(G))\cap\sigma(\mathscr{H}_{\pi'}(G))=\emptyset$ if $\pi\neq\pi'$. Note that $\omega(\mathscr{H}(G))=\sum_{\pi\in\Omega}\sum_{H\in\mathscr{H}_\pi(G)}\omega(H)$. To establish (3.14), it suffices to show that $\frac{\sum_{\widetilde{H}\in\sigma(\mathscr{H}_\pi(G))}\omega(\widetilde{H})}{\sum_{H\in\mathscr{H}_\pi(G)}\omega(H)}>1$ for all TU-admissiable $\pi\in\Omega$, which has been provided by (3.15) and (3.16). The result follows. $\square$

We are now ready to present the main theorem in this paper.

Theorem 3.8. *Let G be a connected graph with $\frac{g(G)}{c(G)}>\frac{7}{6}$. Then, the LMRIV will not occur to G in two places by adding a new edge.*

Proof. Assume that $LMRIV$ occurs to G in two places by adding the edge $e=v_iv_j$. By combining Corollary 3.4, Corollary 3.5 and Lemma 3.7, we may assume that $2\le d(v_i)<d(v_j)$. It follows from (3.13) and Theorem 3.2 that

$$\frac{\widetilde{b}_n}{b_n}=\frac{(\lambda_1+1)(\lambda_k+1)}{\lambda_1\lambda_k}=1+\frac{1}{d(v_i)}+\frac{1}{d(v_j)}+\frac{2}{d(v_i)d(v_j)}\le\frac{13}{6}$$

with equality holds if and only if $d(v_i) = 2$ and $d(v_j) = 3$.

To get a contradiction, we prove that $\frac{\widetilde{b}_n}{b_n} > \frac{13}{6}$. Applying Theorem 2.3, we have

$$\frac{\widetilde{b}_n}{b_n} = \frac{\omega(\mathscr{H}(G+e))}{\omega(\mathscr{H}(G))} = \frac{\omega(\mathscr{H}^e(G+e)) + \omega(\mathscr{H}^{\hat{e}}(G+e))}{\omega(\mathscr{H}(G))},$$

where $\mathscr{H}^e(G+e)$ ($\mathscr{H}^{\hat{e}}(G+e)$, resp.) is the subset of $\mathscr{H}(G+e)$ consisting of the TU-subgraphs of size n containing e (containing no e, resp.). Clearly, $\omega(\mathscr{H}^{\hat{e}}(G+e)) = \omega(\mathscr{H}(G))$. Therefore, it is enough to show that

$$\frac{\omega(\mathscr{H}^e(G+e))}{\omega(\mathscr{H}(G))} > \frac{7}{6}.$$

The remaining proof is similar to the proof of Lemma 3.7, and the details are left to the reader. $\square$

The following consequence immediately follows from Theorem 3.8. We wonder that there is no connected graph G such that the LMRIV occuring to G in two places by adding a new edge.

Corollary 3.9. *Let G be a connected unicyclic or bicyclic graph. Then, the LMRIV will not occur to G in two places by adding a new edge.*

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