Proof of a Conjecture on Online Ramsey Numbers of Stars versus Paths

Ruyu Song[a,b], Sha Wang[a,b], Yanbo Zhang[a,b,*]

[a]School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, 050024, Hebei, China
[b]Hebei International Joint Research Center for Mathematics and Interdisciplinary Science, Shijiazhuang, 050024, Hebei, China

Abstract

Given two graphs G and H, the online Ramsey number $\tilde{r}(G,H)$ is defined to be the minimum number of rounds that Builder can always guarantee a win in the following (G,H)-online Ramsey game between Builder and Painter. Starting from an infinite set of isolated vertices, in each round Builder draws an edge between some two vertices, and Painter immediately colors it red or blue. Builder's goal is to force either a red copy of G or a blue copy of H in as few rounds as possible, while Painter's goal is to delay it for as many rounds as possible. Let $K_{1,3}$ denote a star with three edges and P_ℓ a path with ℓ vertices. Latip and Tan conjectured that $\tilde{r}(K_{1,3},P_\ell)=(3/2+o(1))\ell$ [Bull. Malays. Math. Sci. Soc. 44 (2021) 3511–3521]. We show that $\tilde{r}(K_{1,3},P_\ell)=\lfloor 3\ell/2\rfloor$ for $\ell\ge 2$, which verifies the conjecture in a stronger form.

Keywords: Ramsey number, Online Ramsey number, Path, Star
2020 MSC: 05C55, 05C57, 05D10

1. Introduction

We are concerned with a game-theoretic notion called online Ramsey number in this paper. Given two graphs G and H, the (G,H)-*online Ramsey game* is a combinatorial game played on an infinite set of vertices between two players, Builder and Painter. Starting from an edgeless graph, in each round Builder draws an edge between two nonadjacent vertices, and Painter colors it red or blue immediately. Builder's goal is to force either a red copy of G or a blue copy of H in as few rounds as possible, while Painter's goal is to delay it for as many rounds as possible. The *online Ramsey number* $\tilde{r}(G,H)$ is the smallest number of rounds needed to create either a red copy of G or a blue copy of H, assuming that both Builder and Painter play optimally.

The online Ramsey number was introduced by Beck [3] and owes its name to Kurek and Ruciński [12]. It can be viewed as the online version of size Ramsey number. Recall that the Ramsey number $r(G,H)$ and the size Ramsey number $\hat{r}(G,H)$ are the smallest number of vertices and edges, respectively, in a graph F such that for any red-blue edge coloring of F, there is either a red copy of G or a blue copy of H. Clearly $\tilde{r}(G,H)\le\hat{r}(G,H)\le\binom{r(G,H)}{2}$.

In the classical case where both G and H are complete graphs, we write m, n instead of K_m, K_n for simplicity of notation. It is well known that the Ramsey number $r(n,n)$ is between $2^{n/2}$ and 2^{2n}. Both bounds have seen no exponential improvements in decades [10]. For size Ramsey numbers, a result attributed to Chvátal shows that $\hat{r}(m,n)=\binom{r(m,n)}{2}$. While for online Ramsey numbers, Conlon [4] showed that for infinitely many n, $\tilde{r}(n,n)\le 1.001^{-n}\hat{r}(n,n)$, which is an exponential improvement for infinite numbers.

*Corresponding author.
Email address: `ybzhang@hebtu.edu.cn` (Yanbo Zhang)

For the lower bound, Beck [3] described an elegant proof of $\tilde{r}(n,n) \geq r(n,n)/2$ which was found by Alon. Recently Conlon, Fox, Grinshpun, and He [5] proved that $\tilde{r}(n,n) \geq 2^{(2-\sqrt{2})n+O(1)}$, which is an exponential improvement to the lower bound. The basic conjecture, attributed by Kurek and Ruciński [12], is to show that $\tilde{r}(n,n) = o(\hat{r}(n,n))$. This conjecture is at present far from being solved. For the exact values, only two nontrivial ones were obtained: $\tilde{r}(3,3) = 8$ by Kurek and Ruciński [12], and $\tilde{r}(3,4) = 17$ by Prałat [15].

For sparse graphs, the online Ramsey numbers involving paths, stars, trees, and cycles have been studied [1, 2, 6, 7, 8, 9, 11, 13, 14, 16, 17]. In general, however, exact results are rare. Most results are upper or lower bounds, between which there is always a challenging gap. We are interested in exploring the following problem.

Question 1.1. *Determine the exact expressions of some online Ramsey functions $f(n) = \tilde{r}(G, H_n)$, where G is a small fixed graph, and H_n is a graph from a class of sparse graphs such as paths, stars, and cycles.*

Let P_n and C_n denote a path of order n and a cycle of order n, respectively. In 2015, Cyman, Dzido, Lapinskas, and Lo [7] obtained the exact online Ramsey numbers of P_3 versus all paths and all cycles, which are $\tilde{r}(P_3, P_n) = \lceil 5(n-1)/4 \rceil$ for $n \geq 3$ and $\tilde{r}(P_3, C_n) = \lceil 5n/4 \rceil$ for $n \geq 5$. In the same paper they showed that $2n - 2 \leq \tilde{r}(C_4, P_n) \leq 4n - 8$ for $n \geq 5$. Dybizbański, Dzido, and Zakrzewska [8] improved the upper bound of $\tilde{r}(C_4, P_n)$ from $4n - 8$ to $3n - 5$. Finally, Adamski and Bednarska-Bzdęga [2] closed the gap by proving that $\tilde{r}(C_4, P_n) = 2n-2$ for $n \geq 8$. Their proof involves a technical inductive argument, in which the base case is computer-assisted. In 2022, Song and Zhang [17] showed that $\tilde{r}(P_3, nK_2) = \lceil 3n/2 \rceil$ for $n \geq 2$; $\tilde{r}(P_4, nK_2) = \lceil 9n/5 \rceil$ for $n \geq 2$; $\tilde{r}(mK_2, P_n) = n + 2m - 4$ for $m = 2, 3$ and $n \geq 5$. Here, nK_2 denotes a matching with n edges.

In 2021, Latip and Tan [13] studied the function $\tilde{r}(K_{1,3}, P_\ell)$ and obtained that $\lceil 3(\ell-1)/2 \rceil \leq \tilde{r}(K_{1,3}, P_\ell) \leq 5\ell/3 + 2$. Here we use the clearer notation $K_{1,3}$ rather than S_3 to denote a star with three edges, because S_n may denote a star with n vertices or a star with n edges in different literatures. Latip and Tan believed that the lower bound is close to the exact expression and posed the following conjecture.

Conjecture 1.2. *[13] $\tilde{r}(K_{1,3}, P_\ell) = (3/2 + o(1))\ell$.*

In this paper, we determine the exact value of $\tilde{r}(K_{1,3}, P_\ell)$ for all $\ell \geq 2$, which verifies the above conjecture in a stronger form.

Theorem 1.3. $\tilde{r}(K_{1,3}, P_\ell) = \lfloor 3\ell/2 \rfloor$ *for* $\ell \geq 2$.

The remainder of this paper is organized as follows. Section 2 shows the lower bound of $\tilde{r}(K_{1,3}, P_\ell)$. We will prove the upper bound by induction for $\ell \geq 7$. Thus we divide the proof of the upper bound into two sections: Section 3 for $2 \leq \ell \leq 6$ and Section 4 for $\ell \geq 7$.

2. The lower bound

During the whole game Painter uses a blocking strategy: she always colors an edge red unless doing so would create either a red $K_{1,3}$ or a red cycle C_k for $3 \leq k \leq \lfloor \ell/2 \rfloor$. More specifically, let R_i be the graph induced by all red edges before the ith round, and let e_i be the edge chosen by Builder in his ith move. If $R_i + e_i$ contains a $K_{1,3}$ or a cycle whose length is at most $\lfloor \ell/2 \rfloor$, then Painter colors e_i blue. Otherwise she colors e_i red.

Let G, R, and B be the graphs induced by all edges, all red edges, and all blue edges before the $\lfloor 3\ell/2 \rfloor$th round, respectively. We will prove that G contains neither a red $K_{1,3}$ nor a blue P_ℓ. Hence Builder can not win the game in $\lfloor 3\ell/2 \rfloor - 1$ rounds, and the lower bound follows.

If R has at least $\lfloor \ell/2 \rfloor + 1$ edges, then B has at most $\lfloor 3\ell/2 \rfloor - 1 - (\lfloor \ell/2 \rfloor + 1)$ edges, which is $\ell - 2$. In this way, B can not contain a path P_ℓ. So we assume that R contains no cycle with length at least $\lfloor \ell/2 \rfloor + 1$.

Combining it with the Painter's strategy, we see that R contains no cycle, and has maximum degree at most two. Consequently, R consists of a disjoint union of paths. Assume that the number of components in R is s. Let X be the set of vertices which have degree two, and $\{y_{2i-1}, y_{2i}\}$ the ends of the ith path for $1 \leq i \leq s$. Thus, R has $|X| + s$ edges, which is at most $\lfloor \ell/2 \rfloor$.

Let P be a longest blue path in G. Since each vertex is incident to at most two edges of a path, P contains at most $2|X|$ edges incident to X. Recall that each blue edge is forced to appear, to avoid either a red $K_{1,3}$ or a red cycle. In other words, each blue edge is either incident to a vertex of X, or $y_{2i-1}y_{2i}$ for some i with $1 \leq i \leq s$. Therefore, the total number of edges in P is at most $2|X| + s$. Since $|E(R)| + |E(P)| \leq |E(G)|$, we have

$$3|X| + 2s \leq \lfloor 3\ell/2 \rfloor - 1.$$

Suppose to the contrary that there is a blue path of order ℓ, then

$$2|X| + s \geq \ell - 1. \tag{1}$$

If $s \geq 2$, then we have

$$\begin{aligned}\lfloor 3\ell/2 \rfloor - 1 &\geq 3|X| + 2s \\ &\geq 3(2|X| + s)/2 + 1 \\ &\geq 3(\ell - 1)/2 + 1 \\ &= (3\ell - 1)/2,\end{aligned}$$

which is a contradiction. So we have $s = 1$. The inequality (1) implies that $2|X| \geq \ell - 2$. It follows that $|X| + s \geq \ell/2$. Since R has $|X| + s$ edges, we see that $\ell/2 \leq |X| + s \leq \lfloor \ell/2 \rfloor$. Thus, ℓ is even, and R is a red path with $\ell/2$ edges (and $\ell/2 + 1$ vertices). If Builder joins the two end vertices of this path, he creates a cycle of length $\ell/2 + 1$. Painter will color this edge red by her strategy. Thus, the blue edges can only be forced by the vertices with degree two in R. Hence G contains only $\ell - 2$ blue edges, which contradicts our assumption that there is a blue P_ℓ. Thus we have the lower bound.

3. The upper bound for $2 \leq \ell \leq 6$

We start with the following simple lemma.

Lemma 3.1. *If there is a blue P_k in the online Ramsey game, then in the next three rounds, there is either a blue P_{k+1} or a red $K_{1,3}$.*

Proof. Let v be an end vertex of the blue P_k. Builder joins v to three new vertices in the next three rounds. If none of the three edges is blue, we have a red $K_{1,3}$. Otherwise, we have a blue P_{k+1}. □

For $\ell = 2, 3$, Builder draws a star with $\ell + 1$ edges. Then there is either a red $K_{1,3}$ or a blue P_ℓ. Hence $\tilde{r}(K_{1,3}, P_\ell) \leq \ell + 1 = \lfloor 3\ell/2 \rfloor$ for $\ell = 2, 3$. For $\ell = 4$, Builder joins v_0 to $v_1, v_2, \ldots$ one by one, until a blue $K_{1,2}$ appears. This blue $K_{1,2}$ can be obtained in at most four rounds, since otherwise a red $K_{1,3}$ shows up and our proof is done. If the blue $K_{1,2}$ is obtained in three rounds, by Lemma 3.1, our proof is done. If the blue $K_{1,2}$ shows up in the fourth round, we assume that v_0v_1, v_0v_2 are red, and v_0v_3, v_0v_4 are blue. Builder chooses v_1v_3 and v_1v_4 in the next two moves. If neither of them is blue, then there is a red $K_{1,3}$. Otherwise, there is a blue P_4. Thus $\tilde{r}(K_{1,3}, P_4) \leq 6$.

For $5 \leq \ell \leq 6$, Builder first draws a path P_4, denoted by $v_1v_2v_3v_4$. It has five color patterns up to symmetry: *bbb*, *bbr*, *brb*, *brr*, *rrr*. In fact, the path may have another color pattern *rbr*, which is avoided as follows. If the first two edges v_1v_2 and v_2v_3 are colored blue and red respectively, then Builder joins v_4 to

v_3. On the other hand, if v_1v_2 is red and v_2v_3 is blue, then Builder joins v_4 to v_1. Thus rbr can not show up in the first three rounds.

We assume that there is no red $K_{1,3}$ in $\lfloor 3\ell/2 \rfloor$ rounds, since otherwise our proof is done. If $v_1v_2v_3v_4$ has color pattern bbb, by Lemma 3.1, $\tilde{r}(K_{1,3}, P_\ell) \leq \lfloor 3\ell/2 \rfloor$ for $\ell = 5, 6$. If it has color pattern brb, Builder joins a new vertex v_5 to v_2 and v_3 in the next two moves. If both v_2v_5 and v_3v_5 are blue, then $v_1v_2v_5v_3v_4$ is a blue P_5. By Lemma 3.1, Builder forces a blue P_6 in the next three rounds. If both v_2v_5 and v_3v_5 are red, then for $\ell = 5$, Builder chooses two edges v_1v_5 and v_4v_5, both of which are forced to be blue. Hence $v_2v_1v_5v_4v_3$ is a blue P_5. For $\ell = 6$, Builder chooses three edges v_2v_6, v_5v_6, and v_4v_5, all of which are forced to be blue. Hence $v_1v_2v_6v_5v_4v_3$ is a blue P_6. If v_2v_5 and v_3v_5 have different colors, say, v_2v_5 is red and v_3v_5 is blue, then Builder joins v_2 to v_4 and hence $v_1v_2v_4v_3v_5$ is a blue P_5. By Lemma 3.1, Builder forces a blue P_6 in the next three rounds.

If $v_1v_2v_3v_4$ has color pattern bbr, Builder joins v_3 to a new vertex v_5. If v_3v_5 is blue, then Builder chooses v_1v_4 and v_4v_5 in the next two moves. At least one of the two edges is blue. Thus, we have a blue P_5 in six rounds. By Lemma 3.1, Builder forces a blue P_6 in the next three rounds. If v_3v_5 is red, then Builder joins v_4 to two new vertices v_6 and v_7. At least one of the two new edges is blue. If exactly one of them is blue, say, v_4v_6 is blue and v_4v_7 is red, then Builder chooses v_3v_6 for $\ell = 5$, and then joins v_4 to a new vertex v_8 for $\ell = 6$. As a result, $v_1v_2v_3v_6v_4$ is a blue P_5 and $v_1v_2v_3v_6v_4v_8$ is a blue P_6. If both v_4v_6 and v_4v_7 are blue, then Builder chooses v_3v_6, which has to be blue. Hence we obtain a blue P_6 (and also a blue P_5) in seven rounds, which is $v_1v_2v_3v_6v_4v_7$.

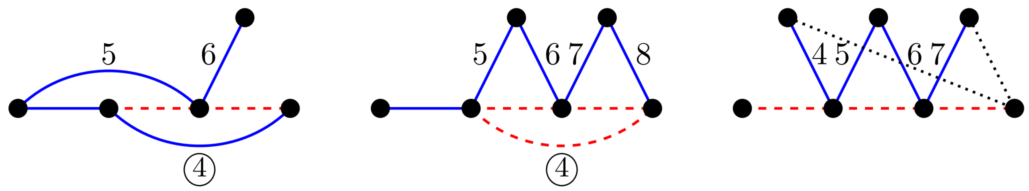

Figure 1: The color patterns brr and rrr of the first three edges.

For the other two color patterns brr and rrr, we illustrate Builder's strategy as in Figure 1. In the pattern brr, we use a circled number to denote that Painter has a choice in that move. Since there is a blue P_5 in six rounds in the left graph, it follows that a blue P_6 can be forced in nine rounds. In the pattern rrr, at least one of the black dotted edges is blue. To summarize, Builder can always force a blue P_5 in seven rounds and a blue P_6 in nine rounds.

4. The upper bound for $\ell \geq 7$

In this section we show that $\tilde{r}(K_{1,3}, P_\ell) \leq \lfloor 3\ell/2 \rfloor$ for $\ell \geq 7$. Assume that Painter always avoids a red $K_{1,3}$. When an edge is 'forced' to be blue in the following, it means that there is a red $K_{1,3}$ if the edge is colored red. We shall find a blue P_ℓ in $\lfloor 3\ell/2 \rfloor$ rounds. In the first two moves Builder draws a star $K_{1,2}$, which has three possible color patterns up to symmetry: bb, br, rr. For the last two patterns, Builder then joins the center of $K_{1,2}$ to a new vertex. Thus, there are three cases in total: the first two rounds form a blue path of order three; the first three rounds form a star with two edges blue and one edge red; the first three rounds form a star with two edges red and one edge blue. In most cases, the Builder's strategy is as follows. First he creates a small graph H. Then for some integer k with $4 \leq k \leq \ell - 2$, Builder forces a blue path $P_{\ell-k}$ that is vertex-disjoint with H by induction. Finally, he combines H and $P_{\ell-k}$ to a blue path P_ℓ.

Case 1. The first two rounds form a blue path of order three.

Assume that the blue path is $v_1v_2v_3$. Builder extends it to a longer path $v_1v_2v_3v_4v_5$ in the next two moves. We distinguish three subcases by the colors of v_3v_4 and v_4v_5.

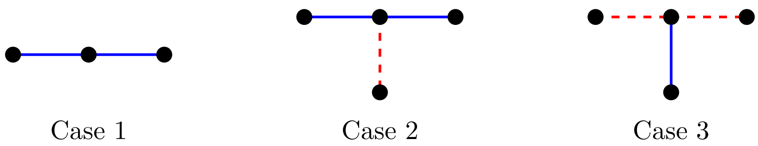

Figure 2: We distinguish three cases.

Subcase 1.1. The edge v_4v_5 is red.

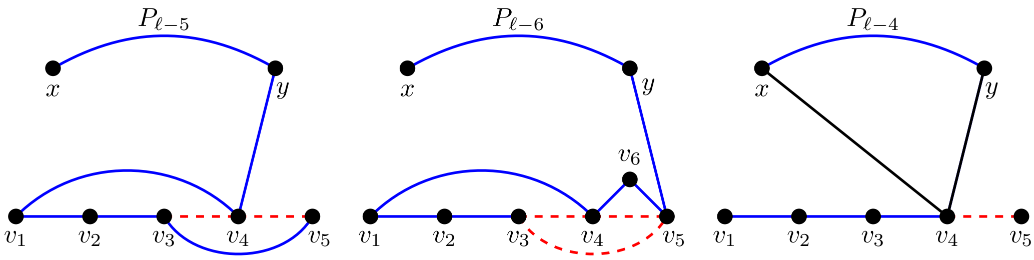

Figure 3: Force a P_ℓ in Subcase 1.1.

If v_3v_4 is red, Builder then chooses v_3v_5. If v_3v_5 is blue, Builder forces a blue $P_{\ell-5}$ in $\lfloor 3\ell/2\rfloor - 7$ rounds by induction, whose end vertices are denoted by x and y. Next Builder draws two edges yv_4 and v_4v_1, both of which are forced to be blue. Thus, there is a blue path $xP_{\ell-5}yv_4v_1v_2v_3v_5$ of order ℓ in $\lfloor 3\ell/2\rfloor$ rounds. If v_3v_5 is red and $\ell = 7$, Builder draws four edges yv_5, v_5v_6, v_6v_4, and v_4v_1, all of which are blue. Here, v_6 and y are two new vertices. Thus, $yv_5v_6v_4v_1v_2v_3$ is a blue P_7. If v_3v_5 is red and $\ell \geq 8$, Builder forces a blue $P_{\ell-6}$ with two ends x and y in $\lfloor 3\ell/2\rfloor - 9$ rounds by induction. Next Builder draws four edges yv_5, v_5v_6, v_6v_4, and v_4v_1. Here, v_6 is a new vertex, and the four edges are forced to be blue. Thus, there is a blue path $xP_{\ell-6}yv_5v_6v_4v_1v_2v_3$ of order ℓ in $\lfloor 3\ell/2\rfloor$ rounds.

If v_3v_4 is blue, Builder then forces a blue $P_{\ell-4}$ with two ends x and y in $\lfloor 3\ell/2\rfloor - 6$ rounds by induction. Next Builder draws two edges xv_4 and yv_4, at least one of which is blue. Without loss of generality, assume that yv_4 is blue. Thus, there is a blue path $xP_{\ell-4}yv_4v_3v_2v_1$ of order ℓ in $\lfloor 3\ell/2\rfloor$ rounds.

Subcase 1.2. The edge v_3v_4 is red and v_4v_5 is blue.

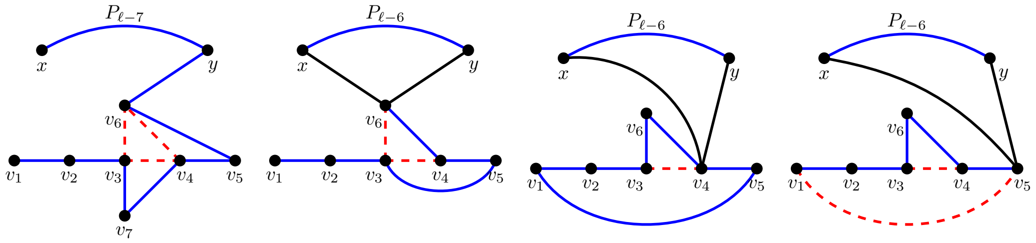

Figure 4: Force a P_ℓ in Subcase 1.2.

Builder joins both v_3 and v_4 to a new vertex v_6. If v_3v_6 and v_4v_6 are red, and $\ell = 7$, then $v_1v_2v_3v_7v_4v_5v_6$ is a blue P_7, where v_7 is a new vertex. If v_3v_6 and v_4v_6 are red, and $\ell = 8$, then $v_1v_2v_3v_7v_4v_5v_6y$ is a blue P_8, where v_7 and y are two new vertices. If v_3v_6 and v_4v_6 are red, and $\ell \geq 9$, then Builder forces a blue $P_{\ell-7}$ in $\lfloor 3\ell/2\rfloor - 10$ rounds by induction, whose end vertices are denoted by x and y. It follows that $v_1v_2v_3v_7v_4v_5v_6yP_{\ell-7}x$ is a blue P_ℓ, where v_7 is a new vertex. Thus, we obtain a blue P_ℓ in the required rounds.

If v_3v_6 is red and v_4v_6 is blue, and $\ell = 7$, then Builder chooses v_3v_5, which has to be blue. He then joins v_6 to two new vertices x and y. Either v_6x or v_6y is blue. Thus, we have a blue P_7, which is $v_1v_2v_3v_5v_4v_6x$ or $v_1v_2v_3v_5v_4v_6y$. If v_3v_6 is red and v_4v_6 is blue, and $\ell \geq 8$, then Builder forces a blue $P_{\ell-6}$ with two ends x and y in $\lfloor 3\ell/2 \rfloor - 9$ rounds by induction. Next Builder chooses v_3v_5, xv_6, and yv_6. The edge v_3v_5 has to be blue, and at least one of xv_6 and yv_6, say yv_6, is blue. As a result, $v_1v_2v_3v_5v_4v_6yP_{\ell-6}x$ is a blue P_ℓ. We obtain a blue P_ℓ in $\lfloor 3\ell/2 \rfloor$ rounds again. If v_3v_6 is blue and v_4v_6 is red, applying the same argument as above, we can obtain a blue P_ℓ.

If both v_3v_6 and v_4v_6 are blue, and $\ell = 7$, we have obtained a blue P_6 in six rounds, which is $v_1v_2v_3v_6v_4v_5$. Thus, it is easy to obtain a blue P_7 in nine rounds. If both v_3v_6 and v_4v_6 are blue, and $\ell \geq 8$, then Builder forces a blue $P_{\ell-6}$ with two ends x and y in $\lfloor 3\ell/2 \rfloor - 9$ rounds by induction. Next Builder chooses the edge v_1v_5. If v_1v_5 is blue, Builder joins v_4 to x and y. At least one of xv_4 and yv_4, say, yv_4 is blue. Accordingly, $v_5v_1v_2v_3v_6v_4yP_{\ell-6}x$ is a blue P_ℓ. Hence we obtain a blue P_ℓ in $\lfloor 3\ell/2 \rfloor$ rounds. If v_1v_5 is red, Builder then joins v_5 to x and y. At least one of xv_5 and yv_5, say, yv_5 is blue. Accordingly, $v_1v_2v_3v_6v_4v_5yP_{\ell-6}x$ is a blue P_ℓ. We obtain a blue P_ℓ in the required rounds.

Subcase 1.3. Both edges v_3v_4 and v_4v_5 are blue.

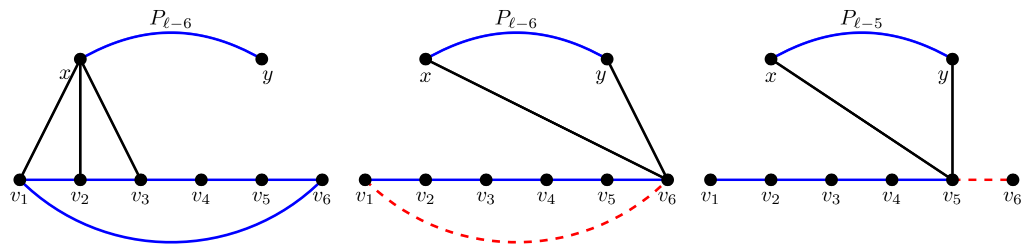

Figure 5: Force a P_ℓ in Subcase 1.3.

Builder joins v_5 to a new vertex v_6 in the next move. If v_5v_6 is blue and $\ell = 7$, Builder joins v_6 to three new vertices v_7, v_8, v_9. At least one of the three edges, say v_6v_7, is blue. Thus we obtain a blue P_7 in nine rounds. If v_5v_6 is blue and $\ell \geq 8$, Builder forces a blue $P_{\ell-6}$ in $\lfloor 3\ell/2 \rfloor - 9$ rounds by induction, whose end vertices are denoted by x and y. Builder joins v_1 and v_6 in the next move. If v_1v_6 is blue, then all v_i's for $1 \leq i \leq 6$ form a cycle. Builder joins x to v_1, v_2, v_3 respectively. At least one of xv_1, xv_2, xv_3 is blue. We may assume that xv_1 is blue without loss of generality. Accordingly, $v_2v_3v_4v_5v_6v_1xP_{\ell-6}y$ is a blue P_ℓ. If v_1v_6 is red, Builder chooses v_6x and v_6y in the last two moves. We may assume that v_6y is blue without loss of generality. It follows that $v_1v_2v_3v_4v_5v_6yP_{\ell-6}x$ is a blue P_ℓ.

If v_5v_6 is red, Builder forces a blue $P_{\ell-5}$ with end vertices x and y in $\lfloor 3\ell/2 \rfloor - 7$ rounds by induction. Builder chooses v_5x and v_5y in the last two moves. To avoid a red $K_{1,3}$, we may assume that v_5y is blue without loss of generality. It follows that $v_1v_2v_3v_4v_5yP_{\ell-5}x$ is a blue P_ℓ. In all three cases, we obtain a blue P_ℓ in at most $\lfloor 3\ell/2 \rfloor$ rounds.

Case 2. A star with two edges blue and one edge red appears in the first three rounds.

Assume that the blue path is $v_1v_2v_3$, and the red edge is v_2v_4. Since $\tilde{r}(K_{1,3}, P_{\ell-4}) \leq \lfloor 3\ell/2 \rfloor - 6$ by induction, Builder forces a blue $P_{\ell-4}$ in the next $\lfloor 3\ell/2 \rfloor - 6$ rounds, whose end vertices are denoted by x and y. Then he chooses v_3v_4 in the next move. If v_3v_4 is red, he draws two edges v_1v_4 and v_4x, both of which are forced to be blue. Hence there is a blue path $v_3v_2v_1v_4xP_{\ell-4}y$ of order ℓ in $\lfloor 3\ell/2 \rfloor$ rounds. If v_3v_4 is blue, he draws two edges v_4x and v_4y. To avoid a red $K_{1,3}$, at least one of v_4x and v_4y is blue, say, v_4x is blue. Thus, there is a blue path $v_1v_2v_3v_4xP_{\ell-4}y$ of order ℓ in $\lfloor 3\ell/2 \rfloor$ rounds.

Case 3. A star with two edges red and one edge blue appears in the first three rounds.

Assume that the red path is $v_1v_2v_3$, and the blue edge is v_2u_2. We extend this red path in the following

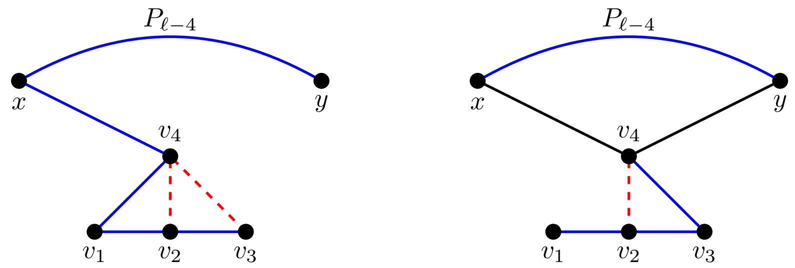

Figure 6: Force a P_ℓ in Case 2.

way. First Builder joins v_3 to two new vertices u_3 and v_4. If both v_3u_3 and v_3v_4 are blue, then we have found the required red path. If not, the two edges v_3u_3 and v_3v_4 have to be one red and one blue, since otherwise there is a red $K_{1,3}$, which contradicts our assumption. Without loss of generality, assume that v_3u_3 is blue and v_3v_4 is red. For each $i \geq 4$, if $v_{i-1}v_i$ is red, Builder joins v_i to two new vertices u_i and v_{i+1}. If both v_iu_i and v_iv_{i+1} are blue, then we stop the procedure. Otherwise, assume that v_iu_i is blue and v_iv_{i+1} is red. The procedure stops when either the red path has length $\lfloor \ell/2 \rfloor + 1$, or both v_tu_t and v_tv_{t+1} are blue for some t with $3 \leq t \leq \lfloor \ell/2 \rfloor + 1$.

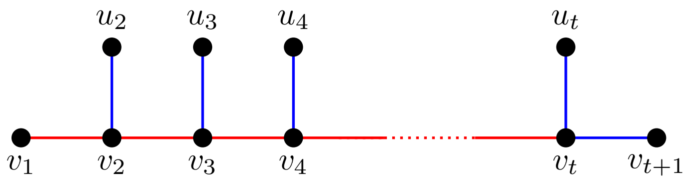

Figure 7: The graph that Builder constructs in the first phase of Case 3.

If the red path has length $\lfloor \ell/2 \rfloor + 1$, Builder joins u_i to v_{i+1} for each i with $2 \leq i \leq \lfloor \ell/2 \rfloor$. All edges u_iv_{i+1} have to be blue to avoid a red $K_{1,3}$. Thus, if ℓ is even, $v_2u_2v_3u_3 \cdots v_{\ell/2+1}u_{\ell/2+1}$ is a blue P_ℓ, and if ℓ is odd, $u_1v_2u_2v_3u_3 \cdots v_{(\ell-1)/2+1}u_{(\ell-1)/2+1}$ is a blue P_ℓ, where u_1 is a new vertex. In both cases, Builder can force a blue P_ℓ in $\ell - 1 + \lfloor \ell/2 \rfloor + 1$ rounds, which is $\lfloor 3\ell/2 \rfloor$ rounds.

Now we consider the other case that there exists an integer t with $3 \leq t \leq \lfloor \ell/2 \rfloor + 1$ such that both v_tu_t and v_tv_{t+1} are blue edges. Builder joins v_i to u_{i+1} for each i with $2 \leq i \leq t-1$. If $t = \lfloor \ell/2 \rfloor + 1$ and ℓ is odd, then $u_2v_2u_3v_3 \cdots u_tv_tv_{t+1}$ is a blue path of order ℓ. If $t = \lfloor \ell/2 \rfloor + 1$ and ℓ is even, then $u_2v_2u_3v_3 \cdots u_tv_t$ is a blue path of order ℓ. In both cases, a blue P_ℓ can be forced in $\lfloor 3\ell/2 \rfloor$ rounds.

If $t = \lfloor \ell/2 \rfloor$, Builder joins v_1 to v_{t+1}. If ℓ is even and v_1v_{t+1} is blue, then we can find a blue P_ℓ, which is $u_2v_2u_3v_3 \cdots u_tv_tv_{t+1}v_1$. If ℓ is even and v_1v_{t+1} is red, then Builder chooses v_1u_2, and $v_1u_2v_2u_3v_3 \cdots u_tv_tv_{t+1}$ is a blue P_ℓ. If ℓ is odd and v_1v_{t+1} is blue, then Builder draws two edges v_1x and v_1y, at least one of which is blue, say, v_1x is blue. It follows that $u_2v_2u_3v_3 \cdots u_tv_tv_{t+1}v_1x$ is a blue P_ℓ. If ℓ is odd and v_1v_{t+1} is red, then Builder draws two edges v_1x and v_1u_2, both of which are forced to be blue. Hence $xv_1u_2v_2u_3v_3 \cdots u_tv_tv_{t+1}$ is a blue P_ℓ. It is not difficult to check that the total number of rounds is at most $\lfloor 3\ell/2 \rfloor$. Thus, we assume that $3 \leq t \leq \lfloor \ell/2 \rfloor - 1$.

Since $\ell - 2t \geq 2$, we have $\tilde{r}(K_{1,3}, P_{\ell-2t}) \leq \lfloor 3(\ell - 2t)/2 \rfloor$ by induction. Thus, Builder can force a blue $P_{\ell-2t}$ in the next $\lfloor 3(\ell - 2t)/2 \rfloor$ rounds. Assume that the end vertices of this $P_{\ell-2t}$ is x and y. Builder joins v_1 to v_{t+1}. If v_1v_{t+1} is red, Builder chooses xv_1 and v_1u_2, which are forced to be blue. Hence $yP_{\ell-2t}xv_1u_2v_2u_3v_3 \cdots u_tv_tv_{t+1}$ is a blue P_ℓ. If v_1v_{t+1} is blue, Builder draws two edges v_1x and v_1y, at least one of which is blue, say, v_1x. It follows that $u_2v_2u_3v_3 \cdots u_tv_tv_{t+1}v_1xP_{\ell-2t}y$ is a blue P_ℓ. Extending the path $xP_{\ell-2t}y$ to a blue P_ℓ, we have used $2t$ blue edges and t red edges. Accordingly, the total number of rounds is at most $\lfloor 3(\ell-2t)/2 \rfloor + 3t$, which is $\lfloor 3\ell/2 \rfloor$. Therefore, $\tilde{r}(K_{1,3}, P_\ell) \le \lfloor 3\ell/2 \rfloor$.

References

[1] G. Adamski, M. Bednarska-Bzdęga, Online size Ramsey numbers: Odd cycles vs connected graphs (2022) `arXiv:2111.14147v2`.

[2] G. Adamski, M. Bednarska-Bzdęga, Online size Ramsey numbers: Path vs C_4 (2022) `arXiv:2211.12204`.

[3] J. Beck, Achievement games and the probabilistic method, in: Combinatorics, Paul Erdős is Eighty, volume 1, János Bolyai Mathematical Society, Budapest Hungary, 1993, pp. 51–78.

[4] D. Conlon, On-line Ramsey numbers, SIAM J. Discrete Math. 23 (2010) 1954–1963.

[5] D. Conlon, J. Fox, A. Grinshpun, X. He, Online Ramsey numbers and the subgraph query problem, in: Building Bridges II, Springer, Berlin Heidelberg, 2019, pp. 159–194.

[6] J. Cyman, T. Dzido, A note on on-line Ramsey numbers for quadrilaterals, Opuscula Math. 34 (2014) 463–468.

[7] J. Cyman, T. Dzido, J. Lapinskas, A. Lo, On-line Ramsey numbers of paths and cycles, Electron. J. Combin. 22 (2015) #P1.15.

[8] J. Dybizbański, T. Dzido, R. Zakrzewska, On-line Ramsey numbers for paths and short cycles, Discrete Appl. Math. 282 (2020) 265–270.

[9] T. Dzido, R. Zakrzewska, A note on on-line Ramsey numbers for some paths, Mathematics 9 (2021) 735.

[10] T. Gowers, The two cultures of mathematics, in: Mathematics: Frontiers and Perspectives, American Mathematical Society, Providence, RI, 2000, pp. 65–78.

[11] J.A. Grytczuk, H.A. Kierstead, P. Prałat, On-line Ramsey numbers for paths and stars, Discrete Math. Theor. Comput. Sci. 10 (3) (2008) 63–74.

[12] A. Kurek, A. Ruciński, Two variants of the size Ramsey number, Discuss. Math. Graph Theory 25 (2005) 141–149.

[13] F.N.N.B.M. Latip, T.S. Tan, A note on on-line Ramsey numbers of stars and paths, Bull. Malays. Math. Sci. Soc. 44 (2021) 3511–3521.

[14] P. Prałat, A note on small on-line Ramsey numbers for paths and their generalization, Australas. J. Combin. 40 (2008) 27–36.

[15] P. Prałat, $\overline{R}(3,4) = 17$, Electron. J. Combin. 15 (2008) #R67.

[16] P. Prałat, A note on off-diagonal small on-line Ramsey numbers for paths, Ars Combin. 107 (2012) 295–306.

[17] R. Song, Y. Zhang, Online and connected online Ramsey numbers of a matching versus a path, Symmetry 14 (2022) 2277.