

On the Kemeny time for continuous-time reversible and irreversible Markov processes with applications to stochastic resetting and to conditioning towards forever-survival

Alain Mazzolo

Université Paris-Saclay, CEA, Service d'Études des Réacteurs et de Mathématiques Appliquées, 91191, Gif-sur-Yvette, France

Cécile Monthus

Université Paris-Saclay, CNRS, CEA, Institut de Physique Théorique, 91191 Gif-sur-Yvette, France

For continuous-time ergodic Markov processes, the Kemeny time τ_* is the characteristic time needed to converge towards the steady state $P_*(x)$: in real-space, the Kemeny time τ_* corresponds to the average of the Mean-First-Passage-Time $\tau(x, x_0)$ over the final configuration x drawn with the steady state $P_*(x)$, which turns out to be independent of the initial configuration x_0 ; in the spectral domain, the Kemeny time τ_* corresponds to the sum of the inverses of all the non-vanishing eigenvalues $\lambda_n \neq 0$ of the opposite generator. We describe many illustrative examples involving jumps and/or diffusion in one dimension, where the Kemeny time can be explicitly computed as a function of the system-size, via its real-space definition and/or via its spectral definition: we consider both reversible processes satisfying detailed-balance where the eigenvalues are real, and irreversible processes characterized by non-vanishing steady currents where the eigenvalues can be complex. In particular, we study the specific properties of the Kemeny times for Markov processes with stochastic resetting, and for absorbing Markov processes conditioned to survive forever.

I. INTRODUCTION

For discrete-time and continuous-time ergodic Markov processes, a very important characteristic time τ_* is known in the mathematical literature under the two main names 'Kemeny constant' [1–31] and 'Eigentime Identity' [32–42], while many other names have been also introduced by different authors rediscovering this essential observable again and again over the years since 1960. In the physics literature, this Kemeny time τ_* does not seem to have been much considered in its own right, except recently in the contexts of networks [44–47], of fractional laplacian [48, 49], and of lifted Monte-Carlo Markov Chains [50, 51]. One goal of the present paper is thus to give a self-contained unified description of the Kemeny time in a broad perspective for all physicists working in the field of Markov processes: we will emphasize its various interesting physical meanings as well as its nice mathematical properties. To simplify the discussions and the notations, we will consider only Markov processes in continuous-time, using the same notations as in the previous study [50], while the case of discrete-time Markov Chains is discussed in detail in the recent work [51]. Since we will consider both Markov chains with discrete configurations and diffusion processes with continuous configurations, it will be more convenient to first explain the properties of the Kemeny time for an ergodic Markov Chain with a finite number N of configurations.

A. Motivation: physical meanings of the Kemeny time for an ergodic Markov Chain on N configurations

For an ergodic continuous-time Markov Chain in a space of N configurations converging towards the steady state $P_*(x)$, the Kemeny time τ_* has two definitions, one in real-space and one in the spectral domain

$$\tau_* = \tau_*^{Space} = \tau_*^{Spectral} \quad (1)$$

that we recall below in Eqs 2 and 5 respectively.

1. τ_*^{Space} as the Mean-First-Passage-Time $\tau(x, x_0)$ averaged over the final configuration x drawn with the steady state $P_*(x)$

When the Mean-First-Passage-Time $\tau(x, x_0)$ at configuration x when starting at configuration x_0 is averaged over the final configuration x drawn with the steady state $P_*(x)$, the result $\tau_*(x_0)$ is actually independent of the initial configuration x_0 and is the real-space Kemeny time τ_*^{Space}

$$\tau_*(x_0) \equiv \sum_{x=1}^N P_*(x) \tau(x, x_0) \equiv \tau_*^{Space} \quad \text{independent of } x_0 \quad (2)$$

This independence with respect to the initial configuration x_0 is a very striking property at first and explains the name 'Kemeny constant' used the mathematical literature. From the physical point of view, Eq. 2 is a very appealing definition of the characteristic time τ_*^{Space} needed to converge towards the steady state $P_*(x)$ independently of the initial configuration x_0 . One of the most important issue is then to compute this characteristic time τ_*^{Space} as a function of the system size N , in particular for large N . As a consequence, we prefer to avoid the mathematical terminology 'Kemeny constant' that could be misleading and to use the name 'Kemeny time' in the present paper, where our goal will be to compute it in many different examples.

2. $\tau_*^{Spectral}$ as the sum of the inverses of the non-vanishing eigenvalues $\lambda_n \neq 0$ of the opposite generator $[-w]$

Let us first consider the case where the opposite generator $[-w]$ of the Markov chain is diagonalizable in terms of its N eigenvalues λ_n . The vanishing eigenvalue $\lambda_0 = 0$ is associated to the conservation of probability and to the steady state $P_*(x)$, while the other $(N - 1)$ eigenvalues have positive real parts

$$\text{Re}(\lambda_n) > 0 \quad \text{for } n = 1, \dots, N - 1 \quad (3)$$

and are associated to the $(N - 1)$ relaxation modes towards the steady state. The corresponding $(N - 1)$ relaxation times τ_n can be defined from the absolute values of the exponential factors $e^{-t\lambda_n}$ that appear in the propagator

$$|e^{-t\lambda_n}| = e^{-t\text{Re}(\lambda_n)} \equiv e^{-\frac{t}{\tau_n}} \quad \text{with } \tau_n = \frac{1}{\text{Re}(\lambda_n)} \quad \text{for } n = 1, \dots, N - 1 \quad (4)$$

As will be explained in detail in section II, the real-space Kemeny time τ_*^{Space} of Eq. 2 actually coincides with the spectral Kemeny time $\tau_*^{Spectral}$ defined as the sum of the inverses of the $(N - 1)$ non-vanishing eigenvalues $\lambda_n \neq 0$ with $n = 1, \dots, N - 1$

$$\tau_*^{Spectral} \equiv \sum_{n=1}^{N-1} \frac{1}{\lambda_n} \quad (5)$$

When the matrix $[-w]$ is not diagonalizable, its rewriting in the Jordan canonical form with the diagonal elements λ_n , and with 1 on the upper diagonal in the Jordan blocks, yields that the spectral expression of Eq. 5 for the Kemeny time remains valid.

From the physical point of view, the spectral expression Eq. 5 is very nice because it is a global characteristic time $\tau_*^{Spectral}$ that involves all the non-vanishing eigenvalues $\lambda_n \neq 0$, while the $(N - 1)$ relaxation-times τ_n of Eq. 4 are associated to specific relaxation eigenmodes. In particular, the maximal value among the $(N - 1)$ relaxation-times τ_n

$$\tau_{max} = \max_{1 \leq n \leq N-1} \tau_n = \frac{1}{\min_{1 \leq n \leq N-1} \text{Re}(\lambda_n)} \quad (6)$$

plays a major role as the relaxation-time of the slowest eigenmode, that governs the asymptotic convergence towards the steady state in $e^{-\frac{t}{\tau_{max}}}$ for large time $t \rightarrow +\infty$. In conclusion, the spectral Kemeny time $\tau_*^{Spectral}$ of Eq. 5 and the maximal relaxation time τ_{max} of Eq. 6 are two different characteristic times that are both essential to understand the convergence properties towards the steady state. Let us stress some of their similarities and differences in the next subsection.

3. Examples of comparison between the Kemeny time τ_* and the maximal relaxation time τ_{max}

The Kemeny time τ_* and the maximal relaxation time τ_{max} can display the same scaling with the system size L , either with two different prefactors or with the same prefactor, but that they can also be completely different. Let us illustrate these three possibilities with the following simple examples that will be discussed in more details later:

(i) In subsection IV F concerning the pure diffusion on the interval $[0, L]$ with reflecting boundary conditions, the maximal relaxation time $\tau_{max} = \frac{1}{\lambda_1}$ of Eq. 154 and the Kemeny time τ_* of Eq. 155

$$\begin{aligned} \tau_{max}^{Diffusion} &= \frac{1}{\lambda_1} = \frac{L^2}{\pi^2} \\ \tau_*^{Diffusion} &= \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} = \frac{L^2}{6} \end{aligned}$$

both display the same expected diffusive scaling in L^2 but with different prefactors.

(ii) In subsection IV G 3 containing the case of a symmetric saw-tooth potential barrier of slopes $\pm\mu$ of between two valleys minima at $x = 0$ and $x = L$, the Kemeny time τ_* of Eq. 179 and the maximal relaxation time $\tau_{max} = \frac{1}{\lambda_1}$ associated to the lowest non-vanishing eigenvalue λ_1 of Eq. F25 become asymptotically equal for large system-size L

$$\begin{aligned}\tau_1^{SawTooth} &= \frac{1}{\lambda_1} \underset{L \rightarrow +\infty}{\simeq} \frac{1}{\mu^2} e^{\frac{\mu}{2}L} \\ \tau_*^{SawTooth} &= \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} \underset{L \rightarrow +\infty}{\simeq} \frac{1}{\mu^2} e^{\frac{\mu}{2}L}\end{aligned}$$

with the exponential growth with respect to the barrier $B(L) = \frac{\mu}{2}L$ that diverges with L , and with exactly the same prefactor $\frac{1}{\mu^2}$.

(iii) In Appendix G, we will discuss why the Kemeny time τ_* can diverge for some diffusion processes when the system-size is infinite $L = +\infty$. The simplest example is the well-known Ornstein-Uhlenbeck on the whole line $]-\infty, +\infty[$ corresponding to a quadratic confining potential: its linear spectrum $\lambda_n = n\omega$ in n of Eq. G10 yields that the maximal relaxation time τ_{max} is finite while the corresponding spectral Kemeny time τ_* is infinite

$$\begin{aligned}\tau_{max}^{OU} &= \frac{1}{\lambda_1} = \frac{1}{\omega} \\ \tau_*^{OU} &= \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} = \frac{1}{\omega} \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty\end{aligned}$$

This example shows that these two characteristic times can be completely different.

Besides the scaling with respect to the system-size L that we have just discussed, it is also interesting to compare the Kemeny time $\tau_*^{Spectral}$ and the maximal relaxation time τ_{max} from the practical point of view of their computations in given Markov models:

(a) on one hand, it is clear that the spectral Kemeny time $\tau_*^{Spectral}$ is a priori more complicated than the maximal relaxation time τ_{max} , since one needs to know all the eigenvalues instead of only the dominant non-vanishing eigenvalue. And indeed from the numerical point of view (that will not be discussed in the present paper), the efforts required to compute the whole spectrum or only the dominant eigenvalue are completely different. Note however that from the analytical point of view that we will also consider in the present paper, the efforts are actually not different since it is the same calculation that leads to the whole list of explicit eigenvalues.

(b) on the other hand, the Kemeny time can be alternatively computed via its real-space definition τ_*^{Space} of Eq. 2, while we are not aware of any real-space expression for the maximal relaxation time τ_{max} . As a consequence, it is possible in some cases to obtain an explicit real-space expression for the Kemeny time, even if one is not able to compute any individual eigenvalue, i.e. even if one is not able to compute the maximal relaxation time τ_{max} .

B. Goals and organization of the paper

The goal of the present paper is to describe in a unified framework the general properties of the Kemeny time for continuous-time Markov processes, and to give explicit results in simple illustrative examples, for both reversible or irreversible dynamics, for both discrete configurations or continuous configurations, with a special attention to stochastic resetting, and to absorbing Markov processes conditioned to survive forever. In particular, we will stress the respective advantages of the real-space and of the spectral definitions of the Kemeny time that can be summarized as follows.

1. Main advantage of the real-space definition τ_*^{Space} of the Kemeny time

The main advantage of the real-space definition τ_*^{Space} is that one can write explicit expressions for general Markov models in simple geometries, in particular in dimension $d = 1$ on which we focus in the present paper:

(i) for Markov chains with arbitrary nearest-neighbor transitions rates $w(x \pm 1, x)$, the real-space Kemeny time τ_*^{Space} is explicit for one-dimensional intervals $x = 1, 2, \dots, N$ with reflecting boundary conditions (Eq. 73) and for one-dimensional rings with periodic boundary conditions (with the special directed case in Eq. 271), as well as for some generalizations with one absorbing boundary condition (Eq. 312).

(ii) for diffusion processes with an arbitrary force $F(x)$ and an arbitrary diffusion coefficient $D(x)$, the real-space Kemeny time τ_*^{Space} is explicit for one-dimensional intervals $x \in]0, L[$ with reflecting boundary conditions (Eq. 150) and for one-dimensional rings with periodic boundary conditions (Eq. 246), as well as for some generalizations for one absorbing boundary condition (Eq. 321).

2. Main advantage of the spectral definition $\tau_*^{Spectral}$ of the Kemeny time

Whenever all the eigenvalues λ_n of a Markov process are explicitly known, it is very easy to analyze the addition of stochastic resetting at rate γ : all the excited eigenvalues are simply shifted $\lambda_n^{[Reset]} = \lambda_n + \gamma$ (Eq. 50) and one directly obtains the corresponding Kemeny time $\tau_*^{[Reset]Spectral}$ via Eq. 53. Simple examples that will be discussed later in the present paper are given in the three Tables I, II and III.

Similarly, whenever all the eigenvalues $\lambda_n^{[Abs]}$ of an absorbing Markov process are explicitly known, it is very easy to take into account the conditioning towards forever-survival: the new excited eigenvalues are simply obtained via the differences $\lambda_m^{[Cond]} = \lambda_{1+m}^{[Abs]} - \lambda_1^{[Abs]}$ of Eq. 43, and one directly obtains the corresponding Kemeny time via Eq. 44. Simple examples that will be discussed later in the present paper can be found in Table II.

Markov chains on the interval of N sites $x = 1, 2, \dots, N$	Excited Eigenvalues λ_n with $n = 1, 2, \dots, N - 1$	Spectral Kemeny time $\tau_* \equiv \sum_{n=1}^{N-1} \frac{1}{\lambda_n}$
Symmetric random walk $w(x \pm 1, x) = 1$	$\lambda_n = 2 \left[1 - \cos \left(n \frac{\pi}{N} \right) \right]$	$\tau_* = \frac{N^2 - 1}{6}$
Symmetric random walk $w(x \pm 1, x) = 1$ with resetting rate γ	$\lambda_n^{[Reset]} = \gamma + 2 \left[1 - \cos \left(n \frac{\pi}{N} \right) \right]$	$\tau_* = \frac{1}{\sqrt{\gamma(4+\gamma)}} \left[N \frac{\left(\sqrt{\frac{\gamma}{4} + 1} + \sqrt{\frac{\gamma}{4} + 1} \right)^{4N} - 1}{\left(\sqrt{\frac{\gamma}{4} + 1} + \sqrt{\frac{\gamma}{4} + 1} \right)^{4N} + 1} - \frac{\gamma + 2}{\sqrt{\gamma(4+\gamma)}} \right]$
Asymmetric random walk $w(x \pm 1, x) = e^{\pm \frac{\mu}{2}}$	$\lambda_n = 2 \cosh \left(\frac{\mu}{2} \right) - 2 \cos \left(n \frac{\pi}{N} \right)$	$\tau_* = \frac{1}{2 \sinh \left(\frac{\mu}{2} \right)} \left[\frac{N}{\tanh \left(N \frac{\mu}{2} \right)} - \frac{1}{\tanh \left(\frac{\mu}{2} \right)} \right]$
Asymmetric random walk $w(x \pm 1, x) = e^{\pm \frac{\mu}{2}}$ with resetting rate γ	$\lambda_n^{[Reset]} = \gamma + 2 \cosh \left(\frac{\mu}{2} \right) - 2 \cos \left(n \frac{\pi}{N} \right)$	$\tau_* = \frac{\left[\left(\frac{\sqrt{\gamma + 4 \sinh^2 \left(\frac{\mu}{4} \right)} + \sqrt{\gamma + 4 \cosh^2 \left(\frac{\mu}{4} \right)}}{2} \right)^{4N} - 1 \right]}{N \left(\frac{\sqrt{\gamma + 4 \sinh^2 \left(\frac{\mu}{4} \right)} + \sqrt{\gamma + 4 \cosh^2 \left(\frac{\mu}{4} \right)}}{2} \right)^{4N} + 1} - \frac{\gamma + 2 \cosh \left(\frac{\mu}{2} \right)}{\sqrt{[\gamma + 4 \sinh^2 \left(\frac{\mu}{4} \right)][\gamma + 4 \cosh^2 \left(\frac{\mu}{4} \right)]}} \right] \frac{1}{\sqrt{[\gamma + 4 \sinh^2 \left(\frac{\mu}{4} \right)][\gamma + 4 \cosh^2 \left(\frac{\mu}{4} \right)]}}$

Table I. Examples of spectral Kemeny times $\tau_*^{Spectral}$ for some Markov chains with nearest-neighbor transition rates $w(x \pm 1, x)$ on the interval of N sites $x = 1, 2, \dots, N$.

3. Organization of the paper

The paper is organized as follows. In section II, the general properties of the Kemeny time τ_* are explained in detail for the case of an ergodic continuous-time Markov Chain in a space of N configurations converging towards the steady state $P_*(x)$. We also describe the specific properties of the Kemeny times for Markov processes in the presence of stochastic resetting at constant rate, and for absorbing Markov processes conditioned to survive forever. In all the other sections, this general framework is applied to various types of Markov processes as follows.

We first consider the simple geometry of one-dimensional finite intervals, where the zero-current condition at the two boundaries yields that the steady current has to vanish also everywhere in the bulk of the one-dimensional interval, so that detailed-balance is always satisfied: we give the explicit expressions of the real-space Kemeny time for Markov chains with arbitrary transitions rates $w(x \pm 1, x)$ in section III and for diffusion processes with an arbitrary force

Diffusion processes on $]0, L[$	Excited Eigenvalues λ_n with $n = 1, 2, \dots, +\infty$	Spectral Kemeny time $\tau_* \equiv \sum_{n=1}^{+\infty} \frac{1}{\lambda_n}$
Pure diffusion $D(x) = 1$ and $F(x) = 0$	$\lambda_n = \frac{\pi^2 n^2}{L^2}$	$\tau_* = \frac{L^2}{6}$
Pure diffusion $D(x) = 1$ and $F(x) = 0$ with resetting rate γ	$\lambda_n^{[Reset]} = \gamma + \frac{\pi^2 n^2}{L^2}$	$\tau_* = \frac{1}{2\gamma} \left[\frac{\sqrt{\gamma} L}{\tanh(\sqrt{\gamma} L)} - 1 \right]$
Biased diffusion $D(x) = 1$ and $F(x) = \mu$	$\lambda_n = \frac{\mu^2}{4} + \frac{\pi^2 n^2}{L^2}$	$\tau_* = \frac{2}{\mu^2} \left[\frac{\frac{\mu L}{2}}{\tanh(\frac{\mu L}{2})} - 1 \right]$
Biased diffusion $D(x) = 1$ and $F(x) = \mu$ with resetting rate γ	$\lambda_n^{[Reset]} = \gamma + \frac{\mu^2}{4} + \frac{\pi^2 n^2}{L^2}$	$\tau_* = \frac{1}{2(\gamma + \frac{\mu^2}{4})} \left[\frac{\sqrt{\gamma + \frac{\mu^2}{4}} L}{\tanh\left(\sqrt{\gamma + \frac{\mu^2}{4}} L\right)} - 1 \right]$
Pure diffusion $D(x) = 1$ and $F(x) = 0$ with two absorbing boundaries conditioned to survive forever	$\lambda_n^{[Cond]} = \frac{\pi^2}{L^2} (n^2 + 2n)$	$\tau_* = \frac{3L^2}{4\pi^2}$
Pure diffusion $D(x) = 1$ and $F(x) = 0$ with one absorbing boundary conditioned to survive forever	$\lambda_n^{[Cond]} = \frac{\pi^2}{L^2} (n^2 + n)$	$\tau_* = \frac{L^2}{\pi^2}$
All the diffusion processes with the Jacobi spectrum of parameters $A > 0$ and $B > -1$	$\lambda_n = A(n^2 + Bn)$	$\tau_* = \frac{1}{AB} \left[\gamma_{Euler} + \frac{\Gamma'(B+1)}{\Gamma(B+1)} \right]$

Table II. Examples of spectral Kemeny times $\tau_*^{Spectral}$ for some diffusion processes with diffusion coefficient $D(x)$ and force $F(x)$ on the interval $]0, L[$.

Irreversible Markov processes on periodic Rings	Excited Eigenvalues λ_n with imaginary parts	Spectral Kemeny time $\tau_* \equiv \sum_{n \neq 0} \frac{1}{\lambda_n}$
Biased diffusion $D(x) = 1$ and $F(x) = \mu$ on the ring of length L	$\lambda_n = \frac{4\pi^2}{L^2} \left[n^2 + in \frac{\mu L}{2\pi} \right]$ with $n = \pm 1, \pm 2, \dots \pm \infty$	$\tau_* = \frac{1}{\mu^2} \left[\frac{\frac{\mu L}{2}}{\tanh(\frac{\mu L}{2})} - 1 \right]$
Biased diffusion $D(x) = 1$ and $F(x) = \mu$ on the ring of length L with resetting rate γ	$\lambda_n^{[Reset]} = \gamma + \frac{4\pi^2}{L^2} \left[n^2 + in \frac{\mu L}{2\pi} \right]$ with $n = \pm 1, \pm 2, \dots \pm \infty$	$\tau_* = -\frac{1}{\gamma} + \frac{L \sinh\left(\frac{L}{2} \sqrt{\mu^2 + 4\gamma}\right)}{2\sqrt{\mu^2 + 4\gamma} \sinh\left(L \frac{\sqrt{\mu^2 + 4\gamma} + \mu}{4}\right) \sinh\left(L \frac{\sqrt{\mu^2 + 4\gamma} - \mu}{4}\right)}$
Directed Pure Trap model $w(x+1, x) = 2$ on the ring of N sites	$\lambda_n = 2 \left(1 - e^{-i2\pi \frac{n}{N}} \right)$ with $n = 1, \dots, N-1$	$\tau_* = \frac{N-1}{4}$
Directed Pure Trap model $w(x+1, x) = 2$ on the ring of N sites with resetting rate γ	$\lambda_n^{[Reset]} = \gamma + 2 \left(1 - e^{-i2\pi \frac{n}{N}} \right)$ with $n = 1, \dots, N-1$	$\tau_* = \frac{N}{(\gamma+2) \left[1 - \left(\frac{\gamma}{2} + 1 \right)^{-N} \right]} - \frac{1}{\gamma}$

Table III. Examples of spectral Kemeny time $\tau_*^{Spectral}$ for some irreversible processes on rings with complex eigenvalues.

$F(x)$ and an arbitrary diffusion coefficient $D(x)$ in section IV. We also consider simple examples where the eigenvalues are explicitly known in order to analyze the effect of stochastic resetting and of conditioning towards forever-survival.

We then focus on the periodic ring which is the simplest geometry where non-equilibrium steady states can exist

with a non-vanishing steady current along the ring: we give the explicit expression of the real-space Kemeny time for diffusion processes with an arbitrary force $F(x)$ and an arbitrary diffusion coefficient $D(x)$ in section V, while in section VI, we focus on the Directed Markov chains on the ring. Again, we also consider simple examples where the eigenvalues are explicitly known and possibly complex in order to analyze the effect of stochastic resetting.

Finally in section VII, we discuss the generalizations of the Kemeny time for absorbing Markov processes. Our conclusions are summarized in section VIII. The various appendices contain either complementary discussions or more technical computations.

II. PROPERTIES OF THE KEMENY TIME τ_* EXPLAINED VIA THE GREEN FUNCTION G

In this section, we describe the general properties of the Kemeny time τ_* of Eq. 1 for an ergodic continuous-time Markov chain in a space of N configurations converging towards the steady state $P_*(x)$. The generator w is then an $N \times N$ matrix with the following properties: the positive off-diagonal $x \neq y$ matrix element $w(x, y) \geq 0$ represents the transition rate from configuration y to configuration x , while the diagonal elements are negative $w(x, x) \leq 0$ and are fixed by the conservation of probability to be

$$w(x, x) \equiv - \sum_{y \neq x} w(y, x) \quad (7)$$

A. Properties of the propagator $P_t(x|x_0)$ converging towards the steady state $P_*(x)$

The propagator $P_t(x|x_0)$, i.e. the probability to be in configuration x at time t when starting in configuration x_0 at time 0

$$P_t(x|x_0) \equiv \langle x | e^{wt} | x_0 \rangle \quad (8)$$

satisfies the forward master equation with respect to the final configuration x

$$\partial_t P_t(x|x_0) = \langle x | w e^{wt} | x_0 \rangle = \sum_y w(x, y) P_t(y|x_0) \quad (9)$$

and the backward master equation with respect to the initial configuration x_0

$$\partial_t P_t(x|x_0) = \langle x | e^{wt} w | x_0 \rangle = \sum_{y_0} P_t(x|y_0) w(y_0, x_0) \quad (10)$$

The steady state $P_*(x)$ satisfies the steady version of the forward master Eq. 9

$$0 = \sum_y w(x, y) P_*(y) \quad (11)$$

Eqs 7 and 11 mean that the highest eigenvalue of the Markov matrix $w(., .)$ is $\lambda_0 = 0$

$$\begin{aligned} 0 &= \langle l_0 | w \\ 0 &= w | r_0 \rangle \end{aligned} \quad (12)$$

where the positive left eigenvector is trivial

$$l_0(x) = 1 \quad (13)$$

while the positive right eigenvector is given by the steady state

$$r_0(y) = P_*(y) \quad (14)$$

B. Properties of the Green function $G(x, x_0)$ inherited from the propagator $P_t(x|x_0)$

To characterize the convergence of the propagator $P_t(x|x_0)$ towards the steady state $P_*(x)$ for large time $t \rightarrow +\infty$

$$P_t(x|x_0) \underset{t \rightarrow +\infty}{\simeq} P_*(x) \quad (15)$$

it is useful to introduce the Green function corresponding to the integration over time $t \in [0, +\infty[$ of the difference $[P_t(x|x_0) - P_*(x)]$

$$G(x, x_0) \equiv \int_0^{+\infty} dt [P_t(x|x_0) - P_*(x)] = \int_0^{+\infty} dt [\langle x|e^{wt}|x_0\rangle - P_*(x)] \quad (16)$$

while the many different names used in the mathematical literature include 'the fundamental matrix' [2], 'the ergodic potential' [52], 'the deviation matrix' [19, 53], 'the deviation kernel' [54], 'the centered resolvent' [55].

The forward and backward equations of Eqs 9 and 10 for the propagator $P_t(x|x_0)$ translate into the following forward and backwards equations for the Green function

$$\begin{aligned} \sum_y w(x, y)G(y, x_0) &= \int_0^{+\infty} dt \partial_t P_t(x|x_0) = [P_t(x|x_0)]_{t=0}^{t=+\infty} = P_*(x) - \delta_{x, x_0} \\ \sum_{y_0} G(x, y_0)w(y_0, x_0) &= \int_0^{+\infty} dt \partial_t P_t(x|x_0) = [P_t(x|x_0)]_{t=0}^{t=+\infty} = P_*(x) - \delta_{x, x_0} \end{aligned} \quad (17)$$

In addition, the orthogonality relations $\langle l_0|G = 0$ and $G|r_0\rangle = 0$ with respect to the left and right eigenvectors associated to the vanishing eigenvalues $\lambda_0 = 0$ yield

$$\begin{aligned} \text{for any } x_0 : \quad 0 &= \langle l_0|G|x_0\rangle = \sum_x \langle l_0|x\rangle G(x, x_0) = \sum_x G(x, x_0) \\ \text{for any } x : \quad 0 &= \langle x|G|r_0\rangle = \sum_{x_0} G(x, x_0)\langle x_0|r_0\rangle = \sum_{x_0} G(x, x_0)P_*(x_0) \end{aligned} \quad (18)$$

All these properties can be summarized at the matrix level as follows: the Green function G is the inverse of the matrix $[-w]$ within the subspace $(\mathbb{1} - |r_0\rangle\langle l_0|)$ orthogonal to the subspace $(|r_0\rangle\langle l_0|)$ associated to the zero eigenvalue of the generator w

$$G = (\mathbb{1} - |r_0\rangle\langle l_0|) \frac{1}{[-w]} (\mathbb{1} - |r_0\rangle\langle l_0|) \quad (19)$$

This form is reminiscent of the second-order perturbation theory of energy levels in quantum mechanics, and indeed, the Green function G of Eq. 19 governs the perturbation theory of the lowest eigenvalue λ_0 when the unperturbed Markov matrix $w(x, y)$ is changed into the new matrix $[w(x, y) + \epsilon w_{per}(x, y)]$ involving some perturbation $\epsilon w_{per}(x, y)$ of small amplitude $\epsilon \rightarrow 0$ that allows to construct the perturbation theory in powers of ϵ (see Appendix A of [56] for details).

C. Link with the averaged time $t_{[0, T]}(x, x_0)$ spent in configuration x during $[0, T]$ when starting at x_0

The averaged time $t_{[0, T]}(x, x_0)$ spent in configuration x during the time-window $[0, T]$ when starting at configuration x_0 can be computed from the propagator of Eq. 8

$$t_{[0, T]}(x, x_0) \equiv \int_0^T dt P_t(x|x_0) \quad (20)$$

Its large-time behavior is extensive in T and involves the steady-state $P_*(x)$ of Eq. 15

$$t_{[0, T]}(x, x_0) \underset{T \rightarrow +\infty}{\simeq} TP_*(x) + o(T) \quad (21)$$

So the Green function of Eq. 16 represents the remaining finite contribution to $t_{[0, T]}(x, x_0)$ when the leading extensive contribution of Eq. 21 is subtracted from Eq. 20

$$G(x, x_0) = \lim_{T \rightarrow +\infty} \left[\int_0^T dt P_t(x|x_0) - TP_*(x) \right] = \lim_{T \rightarrow +\infty} [t_{[0, T]}(x, x_0) - TP_*(x)] \quad (22)$$

As a consequence, the averaged time $t_{[0,T]}(x, x_0)$ spent at configuration x for large T of Eq. 21 displays the extensive leading term $TP_*(x)$ and the subleading finite term $G(x, x_0)$

$$t_{[0,T]}(x, x_0) \underset{T \rightarrow +\infty}{\simeq} TP_*(x) + G(x, x_0) + o(T^0) \quad (23)$$

D. Link with the Mean-First-Passage-Time $\tau(x, x_0)$ at configuration x when starting at configuration x_0

To analyze further the averaged time $t_{[0,T]}(x, x_0)$ spent in configuration x during $[0, T]$ when starting at x_0 discussed in the previous subsection, it is useful to introduce the Mean-First-Passage-Time $\tau(x, x_0)$ at configuration x when starting at x_0 and to decompose the large time-window $[0, T]$ into two intervals:

- (i) the initial interval $[0, \tau(x, x_0)]$ during which the particle is not yet able to visit the configuration x ;
- (ii) the remaining interval $[\tau(x, x_0), T]$, during which the time spent at configuration x can be evaluated with x as the initial configuration instead of x_0 .

This decomposition thus yields the very simple relation

$$t_{[0,T]}(x, x_0) = 0 + t_{[0, T - \tau(x, x_0)]}(x, x) \quad (24)$$

Since $\tau(x, x_0)$ is a finite Mean-First-Passage-Time independent of the large time-window $[0, T]$, one can plug the large-time behavior of Eq. 23 on the two sides of Eq. 24 to obtain

$$TP_*(x) + G(x, x_0) + o(T^0) = [T - \tau(x, x_0)]P_*(x) + G(x, x) + o(T^0) \quad (25)$$

so that the Green function $G(x, x_0)$ and the Mean-First-Passage-Time $\tau(x, x_0)$ are directly related via

$$G(x, x_0) = G(x, x) - P_*(x)\tau(x, x_0) \quad (26)$$

This relation can be alternatively obtained by considering the whole normalized probability distribution $F(t_1; x, x_0)$ of the first-passage-time $t_1 \in [0, +\infty[$ at x when starting at x_0 in order to decompose the propagator via the convolution $P_t(x|x_0) = \int_0^{+\infty} dt_1 P_{t-t_1}(x|x)F(t_1; x, x_0)$ (see [50] for more details on this alternative derivation).

Plugging Eq. 26 into the backward Eq. 17 for the Green function yields

$$-1 + \frac{\delta_{x, x_0}}{P_*(x)} = \sum_{y_0} \tau(x, y_0) w(y_0, x_0) = \sum_{y_0 \neq x_0} [\tau(x, y_0) - \tau(x, x_0)] w(y_0, x_0) \quad (27)$$

This backward equation for the MFPT $\tau(x, x_0)$ for $x_0 \neq x$ is indeed a standard method to compute $\tau(x, x_0)$ with the boundary condition $\tau(x, x) = 0$ (see for instance the textbooks [57–60]).

Plugging Eq. 26 into the two orthogonality conditions of Eq. 18 for the Green function yield

$$\begin{aligned} \text{for any } x_0 : \quad & \sum_x G(x, x) = \sum_x P_*(x)\tau(x, x_0) \\ \text{for any } x : \quad & G(x, x) = P_*(x) \sum_{x_0} \tau(x, x_0)P_*(x_0) \end{aligned} \quad (28)$$

The second equation valid for any x provides an interesting expression for the diagonal element $G(x, x) > 0$ that is found to be proportional to the steady state $P_*(x)$ and to the average of the MFPT $\tau(x, x_0)$ at position x when the initial condition x_0 is drawn with the steady state $P_*(x_0)$. As a consequence, the Green function $G(x, x_0)$ of Eq. 26 becomes

$$G(x, x_0) = P_*(x) \left[\sum_{y_0} \tau(x, y_0)P_*(y_0) - \tau(x, x_0) \right] \quad (29)$$

In particular, the sign of the Green function $G(x, x_0)$ is determined by the sign of the parenthesis, i.e. whether the MFPT $\tau(x, x_0)$ is bigger or smaller than the average $\left[\sum_{y_0} \tau(x, y_0)P_*(y_0) \right]$ over the initial point y_0 drawn with the steady state $P_*(y_0)$.

E. The real-space Kemeny time τ_*^{Space} as the trace of the Green function

The first equation of Eq. 28 valid for any x_0 corresponds to the fact that the real-space Kemeny time of Eq. 2 is independent of the initial point x_0 and coincides with the sum of the N diagonal elements $G(x, x)$, i.e. with the trace of Green function G

$$\tau_*^{Space} \equiv \sum_x P_*(x) \tau(x, x_0) = \sum_{x=1}^N G(x, x) \equiv \text{Trace}(G) \quad (30)$$

This trace can be also rewritten using Eq. 16

$$\tau_*^{Space} = \text{Trace}(G) = \sum_{x=1}^N G(x, x) = \int_0^{+\infty} dt \left[\sum_{x=1}^N \langle x | e^{wt} | x \rangle - \sum_{x=1}^N P_*(x) \right] = \int_0^{+\infty} dt [\text{Trace}(e^{wt}) - 1] \quad (31)$$

in terms of the trace of the evolution operator $\text{Trace}(e^{wt})$.

F. Link with the spectral Kemeny time $\tau_*^{Spectral}$ involving the eigenvalues λ_n of the opposite generator $[-w]$

1. When the generator w can be diagonalized in terms of right and left eigenvectors

When the Markov matrix w is diagonalizable, its spectral decomposition

$$w = - \sum_{n=1}^{N-1} \lambda_n |r_n\rangle \langle l_n| \quad (32)$$

involves the eigenvalues $(-\lambda_n)$ with $\text{Re}(\lambda_n) > 0$ labeled by $n = 1, \dots, N-1$ (while the vanishing highest eigenvalue $\lambda_0 = 0$ has been discussed in Eq. 12) with their right eigenvectors $|r_n\rangle$ and their left eigenvectors $\langle l_n|$

$$\begin{aligned} w|r_n\rangle &= -\lambda_n |r_n\rangle \\ \langle l_n|w &= -\lambda_n \langle l_n| \end{aligned} \quad (33)$$

satisfying the orthonormalization and closure relations

$$\begin{aligned} \langle l_n | r_m \rangle &= \delta_{nm} \\ \mathbb{1} &= |r_0\rangle \langle l_0| + \sum_{n=1}^{N-1} |r_n\rangle \langle l_n| \end{aligned} \quad (34)$$

The spectral decomposition of Eq. 32 allows to write the decomposition into eigenmodes of the propagator of Eq. 8

$$P_t(x|x_0) = \langle x | e^{wt} | x_0 \rangle = P_*(x) + \sum_{n=1}^{N-1} e^{-\lambda_n t} \langle x | r_n \rangle \langle l_n | x_0 \rangle \quad (35)$$

and of the Green function $G(x, x_0)$ of Eq. 16

$$G(x, x_0) = \int_0^{+\infty} dt \left[\sum_{n=1}^{N-1} e^{-t\lambda_n} \langle x | r_n \rangle \langle l_n | x_0 \rangle \right] = \sum_{n=1}^{N-1} \frac{\langle x | r_n \rangle \langle l_n | x_0 \rangle}{\lambda_n} \quad (36)$$

Its trace computed using Eq. 34 allows to rewrite the real-space Kemeny time

$$\tau_*^{Space} = \sum_{x=1}^N G(x, x) = \sum_{n=1}^{N-1} \frac{\sum_x \langle l_n | x \rangle \langle x | r_n \rangle}{\lambda_n} = \sum_{n=1}^{N-1} \frac{\langle l_n | r_n \rangle}{\lambda_n} = \sum_{n=1}^{N-1} \frac{1}{\lambda_n} \equiv \tau_*^{Spectral} \quad (37)$$

as the sum of the inverses of the $(N-1)$ non-vanishing eigenvalues $\lambda_n \neq 0$ that corresponds to the spectral-Kemeny time introduced in Eq. 5.

2. When the generator is not diagonalizable and involves Jordan blocks

As already mentioned in the introduction around Eq. 5, the expression of Eq. 37 remains valid even if the generator w is not diagonalizable and involves Jordan blocks, as can be seen from the evaluation of the trace $\text{Trace}(e^{wt})$ of Eq. 31 when the generator is written in the canonical Jordan form with λ_n on the diagonal

$$\tau_*^{Space} = \int_0^{+\infty} dt [\text{Trace}(e^{wt}) - 1] = \int_0^{+\infty} dt \left[\sum_{n=1}^{N-1} e^{-\lambda_n t} \right] = \sum_{n=1}^{N-1} \frac{1}{\lambda_n} \equiv \tau_*^{Spectral} \quad (38)$$

It should be stressed however that the presence of Jordan blocks will explicitly change other observables, in particular:

- (i) the propagator of Eq. 35 will contain powers of the time t in front of the exponential functions associated to Jordan blocks;
- (ii) the Green function of Eq. 36 will contain higher-inverse-powers of the eigenvalues associated to Jordan blocks.

G. Complementary discussions on the general properties of the Kemeny time τ_* in Appendices

Further discussions on the general properties of the Kemeny time τ_* can be found in the three Appendices:

- Appendix A mentions the relations with the spectral zeta-function and the spectral partition function.
- Appendix B describes the specific spectral properties of reversible Markov processes satisfying detailed-balance.
- Appendix C recalls how the Green function allows to analyze the statistical properties of all the time-additives observables of the Markov process.

H. Kemeny time for Markov processes with killing or absorption conditioned to survive forever

For a Markov jump process with $x = 1, 2, \dots, N$ living configurations and one absorbing 'dead' configuration that will be denoted by 0, the spectral analysis [61–69] can be summarized as follows. In the propagator of Eq. 35, the steady state is replaced by a delta function on the dead configuration $x = 0$

$$P_t^{[Abs]}(x|x_0) \equiv \langle x | e^{tw^{[Abs]}} | x_0 \rangle = \delta_{x,0} + \sum_{n=1}^N e^{-t\lambda_n^{[Abs]}} \langle x | r_n \rangle \langle l_n | x_0 \rangle \quad (39)$$

It is then interesting to consider the process conditioned to survive forever, i.e. that remains forever among the N living configurations $x = 1, \dots, N$. The corresponding conditioned Markov matrix $w^{[Cond]}$ can be computed from the initial Markov matrix $w^{[Abs]}$ with absorption and from the lowest relaxation rate $\lambda_{n=1}^{[Abs]}$ with its left eigenvector $\langle l_{n=1} |$ (see [69] for more details with the same notations)

$$w^{[Cond]}(x, y) = \frac{\langle l_1 | x \rangle}{\langle l_1 | y \rangle} w^{[Abs]}(x, y) + \lambda_1^{[Abs]} \delta_{x,y} \quad (40)$$

The propagator for this process conditioned to survive forever on the living configurations $x = 1, \dots, N$

$$\begin{aligned} P_t^{[Cond]}(x|x_0) &= \frac{\langle l_1 | x \rangle}{\langle l_1 | x_0 \rangle} e^{t\lambda_1^{[Abs]}} P_t^{[Abs]}(x|x_0) \\ &= \langle l_1 | x \rangle \langle x | r_1 \rangle + \sum_{n=2}^N e^{-t[\lambda_n^{[Abs]} - \lambda_1^{[Abs]}]} [\langle l_1 | x \rangle \langle x | r_n \rangle] \left[\frac{\langle l_n | x_0 \rangle}{\langle l_1 | x_0 \rangle} \right] \end{aligned} \quad (41)$$

will converge towards the steady state

$$P_*^{[Cond]}(x) = \langle l_1 | x \rangle \langle x | r_1 \rangle \quad (42)$$

that is indeed normalized on the living configurations (again, see [69] for more details with the same notations). The $(N - 1)$ non-vanishing eigenvalues that govern the convergence of the propagator of Eq. 41 towards its steady state are simply given by the following differences of the initial eigenvalues $\lambda_n^{[Abs]}$

$$\lambda_m^{[Cond]} = \lambda_{1+m}^{[Abs]} - \lambda_1^{[Abs]} \quad \text{for } m = 1, \dots, N - 1 \quad (43)$$

For this process conditioned to survive forever, the spectral Kemeny time of Eq. 37

$$\tau_*^{[Cond]Spectral} = \sum_{m=1}^{N-1} \frac{1}{\lambda_m^{[Cond]}} = \sum_{m=1}^{N-1} \frac{1}{\lambda_{1+m}^{[Abs]} - \lambda_1^{[Abs]}} \quad (44)$$

can be thus computed from the absorbing eigenvalues $\lambda_n^{[Abs]}$ of the initial absorbing Markov process, without even constructing the conditioned process. Examples of Eq. 43 will be given in Eqs 218 and 221 with their corresponding Kemeny times.

Besides the Kemeny time of Eq. 44 for the process conditioned to survive forever, it is also interesting to consider the analogs of the Kemeny time for the initial unconditioned Markov process that gets absorbed in the dead configuration 0, but this discussion will be postponed to the very last section VII of the main text, in order to avoid interferences with all the other sections of the present paper that concern ergodic Markov processes.

I. Kemeny time for Markov processes with stochastic resetting at rate γ towards the distribution Π

Since stochastic resetting has been much studied recently (see the review [70] and references therein), in particular from the point of view of their spectral properties [71], it is interesting to stress here the consequences for the spectral Kemeny time.

When the initial Markov jump process described by the Markov matrix w is modified by the stochastic resetting at rate γ towards some normalized distribution Π , the process with resetting is described by the new Markov matrix

$$w^{[Reset]}(x, y) \equiv w(x, y) + \gamma [\Pi(x) - \delta_{x,y}] \quad (45)$$

The propagator $P_t^{[Reset]}(x|x_0)$ in the presence of resetting can be written in terms of the propagator $P_t(x|x_0) = \langle x|e^{tw}|x_0\rangle$ without resetting as

$$P_t^{[Reset]}(x|x_0) \equiv \langle x|e^{tw^{[Reset]}}|x_0\rangle = e^{-\gamma t} P_t(x|x_0) + \gamma \int_0^t d\tau e^{-\gamma\tau} \sum_y P_\tau(x|y) \Pi(y) \quad (46)$$

with the following interpretation. The first contribution involves the probability $e^{-\gamma t}$ of no resetting during the time-window $[0, t]$ and the initial propagator $P_t(x|x_0)$ from x_0 to x in time t . The second contribution takes into account the last resetting event during the time-window $[0, t]$, that occurs at some time $(t - \tau)$ and towards the position y drawn with $\Pi(y)$, so that it involves the initial propagator $P_\tau(x|y)$ from y to x in time τ . Plugging the spectral decomposition of Eq. 35 for $P_t(x|x_0)$ and for $P_\tau(x|y)$ into Eq. 46

$$\begin{aligned} P_t^{[Reset]}(x|x_0) &= e^{-\gamma t} \left[P_*(x) + \sum_{n=1}^{N-1} e^{-\lambda_n t} \langle x|r_n \rangle \langle l_n|x_0 \rangle \right] + \gamma \int_0^t d\tau e^{-\gamma\tau} \sum_y \left[P_*(x) + \sum_{n=1}^{N-1} e^{-\lambda_n \tau} \langle x|r_n \rangle \langle l_n|y \rangle \right] \Pi(y) \\ &= e^{-\gamma t} P_*(x) + \sum_{n=1}^{N-1} e^{-(\gamma+\lambda_n)t} \langle x|r_n \rangle \langle l_n|x_0 \rangle + [1 - e^{-\gamma t}] P_*(x) + \gamma \sum_{n=1}^{N-1} \frac{1 - e^{-(\gamma+\lambda_n)t}}{\gamma + \lambda_n} \langle x|r_n \rangle \sum_y \langle l_n|y \rangle \Pi(y) \\ &= \left(P_*(x) + \gamma \sum_{n=1}^{N-1} \frac{\langle x|r_n \rangle}{\gamma + \lambda_n} \left[\sum_y \langle l_n|y \rangle \Pi(y) \right] \right) + \sum_{n=1}^{N-1} e^{-(\gamma+\lambda_n)t} \langle x|r_n \rangle \left(\langle l_n|x_0 \rangle - \frac{\gamma}{\gamma + \lambda_n} \left[\sum_y \langle l_n|y \rangle \Pi(y) \right] \right) \end{aligned} \quad (47)$$

allows to identify the spectral properties of the process with resetting as follows.

(i) The vanishing eigenvalue $\lambda_0^{[Reset]} = 0$ is associated to the left trivial eigenvector $\langle l_0^{[Reset]}|x_0 \rangle = 1$ as it should, and to the following right eigenvector $\langle x|r_0^{[Reset]} \rangle$ corresponding to the new steady state

$$P_*^{[Reset]}(x) \equiv \langle x|r_0^{[Reset]} \rangle = P_*(x) + \gamma \sum_{n=1}^{N-1} \frac{\langle x|r_n \rangle}{\gamma + \lambda_n} \left[\sum_y \langle l_n|y \rangle \Pi(y) \right] \quad (48)$$

Its physical meaning can be understood from the limit $t \rightarrow +\infty$ of Eq. 46

$$P_*^{[Reset]}(x) = \gamma \int_0^{+\infty} d\tau e^{-\gamma\tau} \sum_y P_\tau(x|y) \Pi(y) \quad (49)$$

where the initial propagator $P_\tau(x|y)$ is averaged over the time τ drawn with the exponential distribution $(\gamma e^{-\gamma\tau})$ and over the initial point y drawn with the resetting probability $\Pi(y)$.

(ii) All the non-vanishing eigenvalues $\lambda_n^{[Reset]} \neq 0$ are obtained from the initial non-vanishing eigenvalues $\lambda_n \neq 0$ that are all increased by the same amount given by the resetting rate γ

$$\lambda_n^{[Reset]} = \lambda_n + \gamma \quad \text{for } n = 1, 2, \dots, N-1 \quad (50)$$

The corresponding right eigenvectors are unchanged

$$\langle x | r_n^{[Reset]} \rangle = \langle x | r_n \rangle \quad \text{for } n = 1, 2, \dots, N-1 \quad (51)$$

while the new left eigenvectors read

$$\langle l_n^{[Reset]} | x_0 \rangle = \langle l_n | x_0 \rangle - \frac{\gamma}{\gamma + \lambda_n} \left[\sum_y \langle l_n | y \rangle \Pi(y) \right] \quad \text{for } n = 1, 2, \dots, N-1 \quad (52)$$

Equivalently, all these spectral properties can be obtained from the corresponding eigenvalues equations for the Markov matrix $w^{[Reset]}$ of Eq. 45 using Eqs 33 34, as described in [71].

The spectral Kemeny time of Eq. 37 that involves the non-vanishing eigenvalues of Eq. 50

$$\tau_*^{[Reset]Spectral} = \sum_{n=1}^{N-1} \frac{1}{\lambda_n^{[Reset]}} = \sum_{n=1}^{N-1} \frac{1}{\lambda_n + \gamma} \quad (53)$$

can be thus computed from the initial eigenvalues $\lambda_n \neq 0$ and from the resetting rate γ . Note that the resetting probability $\Pi(y)$ does not appear in the Kemeny time of Eq. 53: this property might seem somewhat surprising at first, since the resetting probability $\Pi(y)$ is essential to determine the new steady state $P_*^{[Reset]}(x)$ of Eqs 48 49, towards which the convergence time is measured. Examples of the formula of Eq. 53 are given in the three Tables I II III and will be discussed in more details around Eqs 88 103 156 173 261 263 280.

III. MARKOV CHAINS ON THE INTERVAL OF N SITES $x = 1, 2, \dots, N$ (ALWAYS REVERSIBLE)

In this section, we focus on Markov chains on the interval of N sites $x = 1, 2, \dots, N$, where the dynamics always satisfy the detailed-balance condition as a consequence of the vanishing steady current at the two boundaries.

A. Forward master equation in terms of the flows and the currents of the links

On each of the $(N-1)$ links $(x, x+1)$ with $x = 1, \dots, N-1$, it is useful to introduce the two flows $\mathcal{F}_t^\pm(x+1/2)$ in the two directions produced by the two rates $w(x+1, x)$ and $w(x, x+1)$, and by the probability $P_t(y)$ to be at position y at t

$$\begin{aligned} \mathcal{F}_t^+ \left(x + \frac{1}{2} \right) &\equiv w(x+1, x) P_t(x) \\ \mathcal{F}_t^- \left(x + \frac{1}{2} \right) &\equiv w(x, x+1) P_t(x+1) \end{aligned} \quad (54)$$

so that their difference represents the current on this link

$$J_t \left(x + \frac{1}{2} \right) \equiv \mathcal{F}_t^+ \left(x + \frac{1}{2} \right) - \mathcal{F}_t^- \left(x + \frac{1}{2} \right) = w(x+1, x) P_t(x) - w(x, x+1) P_t(x+1) \quad (55)$$

The forward master Eq. 9 for the probability $P_t(x)$ to be as position x at time t can be rewritten as continuity equations involving the currents of Eq. 55

$$\begin{aligned} x = 1 : \quad \partial_t P_t(1) &= -J_t \left(1 + \frac{1}{2} \right) = w(1, 2) P_t(2) - w(2, 1) P_t(1) \\ x = 2, \dots, N-1 : \quad \partial_t P_t(x) &= J_t \left(x - \frac{1}{2} \right) - J_t \left(x + \frac{1}{2} \right) = w(x, x-1) P_t(x-1) + w(x, x+1) P_t(x+1) - [w(x-1, x) + w(x+1, x)] P_t(x) \\ x = N : \quad \partial_t P_t(N) &= J_t \left(N - \frac{1}{2} \right) = w(N, N-1) P_t(N-1) - w(N-1, N) P_t(N) \end{aligned} \quad (56)$$

B. Steady-state $P_*(x)$ with its vanishing steady current $J_* = 0$ and its steady flows $\mathcal{F}_*(x + \frac{1}{2})$

For the steady state $P_*(x)$, Eq. 56 yields that the steady current $J_*(x + \frac{1}{2})$ should vanish on all the links $(x, x + 1)$

$$\begin{aligned} \text{for } x = 1, 2, \dots, N-1 : \quad 0 &= J_*\left(x + \frac{1}{2}\right) = \mathcal{F}_*^+\left(x + \frac{1}{2}\right) - \mathcal{F}_*^-\left(x + \frac{1}{2}\right) \\ &= w(x+1, x)P_*(x) - w(x, x+1)P_*(x+1) \end{aligned} \quad (57)$$

i.e. the detailed-balance condition of Eq. B1 is satisfied with the many consequences summarized in Appendix B. The steady state $P_*(x)$ can be obtained from the recurrence of Eq. 57 for arbitrary transition rates $w(.,.)$

$$\text{for } x = 2, \dots, N : \quad P_*(x) = P_*(1) \prod_{y=1}^{x-1} \frac{w(y+1, y)}{w(y, y+1)} \equiv \frac{e^{-U(x)}}{Z} \quad (58)$$

where the potential $U(x)$ can be computed with respect to the reference $U(x=1) = 0$ to obtain for $x = 2, \dots, N$

$$U(x) = \sum_{y=1}^{x-1} \ln \left(\frac{w(y, y+1)}{w(y+1, y)} \right) \quad (59)$$

The partition function Z ensuring the normalization of the steady state of Eq. 58 reads

$$Z = \sum_{x=1}^N e^{-U(x)} = 1 + \sum_{x=2}^N \prod_{y=1}^{x-1} \frac{w(y+1, y)}{w(y, y+1)} \quad (60)$$

As explained around Eq. B5, it is useful to introduce the diffusion coefficients associated to the $(N-1)$ links $(x, x+1)$

$$D\left(x + \frac{1}{2}\right) \equiv \sqrt{w(x, x+1)w(x+1, x)} \quad (61)$$

that will determine the steady flows of Eq. B7 associated to the zero-current of Eq. 57

$$\begin{aligned} \mathcal{F}_*\left(x + \frac{1}{2}\right) &= w(x, x+1)P_*(x+1) = w(x+1, x)P_*(x) \\ &= D\left(x + \frac{1}{2}\right) \sqrt{P_*(x)P_*(x+1)} = D\left(x + \frac{1}{2}\right) \frac{e^{-\frac{U(x)+U(x+1)}{2}}}{Z} \end{aligned} \quad (62)$$

C. Mean-First-Passage-Time $\tau(x, x_0)$

In the one-dimensional geometry, in order to go from x_0 to $x > x_0$, one needs to visit sequentially all the sites between them, i.e the first neighbor $(x_0 + 1)$, then the second neighbor $(x_0 + 2)$, etc up to the site $(x-1)$. As a consequence, the Mean-First-Passage-Time $\tau(x, x_0)$ at x when starting at x_0 can be decomposed in the two regions $x > x_0$ and $x < x_0$ into the following sums over the links between them

$$\begin{aligned} \text{for } x > x_0 : \quad \tau(x, x_0) &= \tau(x, x-1) + \tau(x-1, x-2) + \dots + \tau(x_0+1, x_0) = \sum_{z=x_0}^{x-1} \tau(z+1, z) \\ \text{for } x < x_0 : \quad \tau(x, x_0) &= \tau(x, x+1) + \tau(x+1, x+2) + \dots + \tau(x_0-1, x_0) = \sum_{z=x}^{x_0-1} \tau(z, z+1) \end{aligned} \quad (63)$$

that involve the elementary Mean-First-Passage-Times $\tau(z+1, z)$ and $\tau(z, z+1)$ between two neighboring sites.

The backward equation of Eq. 27 for the Mean-First-Passage-Time $\tau(x, x_0)$ at x when starting at x_0 can be then rewritten in these two regions as follows:

(i) for $x > x_0$, Eq. 27 reads

$$\begin{aligned} \text{for } x_0 = 1 : \quad -1 &= [\tau(x, 2) - \tau(x, 1)]w(2, 1) = -\tau(2, 1)w(2, 1) \\ \text{for } x_0 = 2, \dots, x-1 : \quad -1 &= [\tau(x, x_0+1) - \tau(x, x_0)]w(x_0+1, x_0) + [\tau(x, x_0-1) - \tau(x, x_0)]w(x_0-1, x_0) \\ &= -\tau(x_0+1, x_0)w(x_0+1, x_0) + \tau(x_0, x_0-1)w(x_0-1, x_0) \end{aligned} \quad (64)$$

The solution reads in terms of the potential $U(\cdot)$ of Eq. 59 and the diffusion coefficients $D(\cdot)$ of Eq. 61

$$\tau(z+1, z) = \frac{1}{D\left(z + \frac{1}{2}\right) e^{-\frac{U(z)+U(z+1)}{2}}} \sum_{y=1}^z e^{-U(y)} \quad (65)$$

or equivalently in terms of the steady state of Eq. 58 and of the steady flow $\mathcal{F}_*(z + \frac{1}{2})$ of Eq. 62

$$\tau(z+1, z) = \frac{1}{\mathcal{F}_*\left(z + \frac{1}{2}\right)} \sum_{y=1}^z P_*(y) \equiv \frac{P_*([1, z])}{\mathcal{F}_*\left(z + \frac{1}{2}\right)} \quad (66)$$

where the numerator $P_*([1, z])$ corresponds to the steady weight of the interval $[1, z]$, i.e. to the cumulative distribution associated to the steady state

$$P_*([1, z]) \equiv \sum_{y=1}^z P_*(y) \quad (67)$$

(ii) for $x < x_0$, Eq. 27 reads

$$\begin{aligned} x_0 = x+1, \dots, N-1 : \quad -1 &= [\tau(x, x_0+1) - \tau(x, x_0)] w(x_0+1, x_0) + [\tau(x, x_0-1) - \tau(x, x_0)] w(x_0-1, x_0) \\ &= \tau(x_0, x_0+1) w(x_0+1, x_0) - \tau(x_0-1, x_0) w(x_0-1, x_0) \\ x_0 = N : \quad -1 &= [\tau(x, N-1) - \tau(x, N)] w(N-1, N) = -\tau(N-1, N) w(N-1, N) \end{aligned} \quad (68)$$

The solution reads in terms of the potential $U(\cdot)$ of Eq. 59 and the diffusion coefficients $D(\cdot)$ of Eq. 61 as

$$\tau(z, z+1) = \frac{1}{D\left(z + \frac{1}{2}\right) e^{-\frac{U(z)+U(z+1)}{2}}} \sum_{y=z+1}^N e^{-U(y)} \quad (69)$$

or equivalently in terms of the steady state of Eq. 58 and of the steady flow $\mathcal{F}_*(z + \frac{1}{2})$ of Eq. 62

$$\tau(z, z+1) = \frac{1}{\mathcal{F}_*\left(z + \frac{1}{2}\right)} \sum_{y=z+1}^N P_*(y) = \frac{P_*([z+1, N])}{\mathcal{F}_*\left(z + \frac{1}{2}\right)} \quad (70)$$

where the numerator $P_*([z+1, N])$ corresponds to the steady weight of the interval $[z+1, N]$, i.e. to the complementary cumulative distribution with respect to Eq. 67

$$P_*([z+1, N]) \equiv \sum_{y=z+1}^N P_*(y) = 1 - \sum_{y=1}^z P_*(y) = 1 - P_*([1, z]) \quad (71)$$

Plugging Eqs 66 and 70 into Eq. 63 yields the final result for the Mean-First-Passage-Times $\tau(x, x_0)$

$$\begin{aligned} \text{for } x_0 < x : \quad \tau(x, x_0) &= \sum_{z=x_0}^{x-1} \tau(z+1, z) = \sum_{z=x_0}^{x-1} \frac{1}{\mathcal{F}_*\left(z + \frac{1}{2}\right)} \sum_{y=1}^z P_*(y) \\ \text{for } x_0 > x : \quad \tau(x, x_0) &= \sum_{z=x}^{x_0-1} \tau(z, z+1) = \sum_{z=x}^{x_0-1} \frac{1}{\mathcal{F}_*\left(z + \frac{1}{2}\right)} \sum_{y=z+1}^N P_*(y) \end{aligned} \quad (72)$$

D. Real-space Kemeny time τ_*^{Space}

The Real-space Kemeny time τ_*^{Space} of Eq. 2 does not depend on the initial configuration x_0 and can be thus evaluated for the special case $x_0 = 1$ using Eq. 72

$$\begin{aligned} \tau_*^{Space} &= \sum_{x=1}^N P_*(x) \tau(x, x_0 = 1) = \sum_{x=2}^N P_*(x) \tau(x, x_0 = 1) = \sum_{x=2}^N P_*(x) \sum_{z=1}^{x-1} \frac{1}{\mathcal{F}_*\left(z + \frac{1}{2}\right)} \sum_{y=1}^z P_*(y) \\ &= \sum_{z=1}^{N-1} \frac{1}{\mathcal{F}_*\left(z + \frac{1}{2}\right)} \left[\sum_{y=1}^z P_*(y) \right] \left[\sum_{x=z+1}^N P_*(x) \right] \equiv \sum_{z=1}^{N-1} \Upsilon(z+1, z) \end{aligned} \quad (73)$$

In this sum over the $(N - 1)$ links, the elementary contribution $\Upsilon(z + 1, z)$ of the link $(z, z + 1)$ reads

$$\Upsilon(z + 1, z) \equiv \frac{1}{\mathcal{F}_*(z + \frac{1}{2})} \left[\sum_{y=1}^z P_*(y) \right] \left[\sum_{x=z+1}^N P_*(x) \right] = \frac{P_*([1, z])P_*([z + 1, N])}{\mathcal{F}_*(z + \frac{1}{2})} \quad (74)$$

The numerator contains the product of the cumulative distribution $P_*([1, z])$ of Eq. 67 and of the complementary cumulative distribution $P_*([z + 1, N])$ of Eq. 71, while the denominator corresponds to the link steady flow $\mathcal{F}_*(z + \frac{1}{2})$ of Eq. 62. The result of Eqs 73 and 74 is thus in direct correspondence with the analog result for discrete-time Markov chains described in [51]. If one replaces the steady state by Eq. 58 and the steady flow by Eq. 62, the elementary contribution $\Upsilon(z + 1, z)$ of the link $(z, z + 1)$

$$\Upsilon(z + 1, z) \equiv \frac{1}{ZD(z + \frac{1}{2})e^{-\frac{U(z)+U(z+1)}{2}}} \left[\sum_{y=1}^z e^{-U(y)} \right] \left[\sum_{x=z+1}^N e^{-U(x)} \right] \quad (75)$$

involves the diffusion coefficient $D(\cdot)$ of Eq. 61, the potential $U(\cdot)$ of Eq. 59 and its partition function Z of Eq. 60.

In summary, for any Markov chain defined on the one-dimensional interval of N sites $x = 1, \dots, N$, the explicit steady state $P_*(x)$ is given in Eq. 58 and the Kemeny time can be computed via the real-space expression of Eq. 73. In particular, this gives an explicit result for the spectral Kemeny time of Eq. 5 even when the $(N - 1)$ individual eigenvalues λ_n are not explicit.

E. Example of the continuous-time symmetric random walk $w(x \pm 1, x) = 1$

Let us consider the simplest example where all the rates have the same value unity

$$w(x + 1, x) = 1 = w(x, x + 1) \quad \text{for } x = 1, 2, \dots, N - 1 \quad (76)$$

so the diffusion coefficient of Eq. 61 is unity on all the links

$$D\left(x + \frac{1}{2}\right) \equiv \sqrt{w(x, x + 1)w(x + 1, x)} = 1 \quad \text{for } x = 1, 2, \dots, N - 1 \quad (77)$$

and the potential $U(x)$ of Eq. 59 vanishes on all the sites

$$U(x) = 0 \quad \text{for } x = 1, 2, \dots, N \quad (78)$$

1. Real-space Kemeny time

The steady state of Eqs 58 is uniform with $Z = N$

$$P_*(x) = \frac{1}{N} \quad \text{for } x = 1, 2, \dots, N \quad (79)$$

so the real-space Kemeny time of Eq. 73 75 reduces to

$$\tau_*^{Space} = \frac{1}{N} \sum_{z=1}^{N-1} \left[\sum_{y=1}^z 1 \right] \left[\sum_{x=z+1}^N 1 \right] = \frac{1}{N} \sum_{z=1}^{N-1} z(N - z) = \frac{N^2 - 1}{6} \quad (80)$$

and displays the expected diffusive scaling N^2 for large size N .

2. Spectral Kemeny time

For the Markov chain of Eq. 56 with the rates of Eq. 76, the eigenvalue equation of Eq. 33 reads for the right eigenvector $r(x)$ associated to the eigenvalue λ

$$\begin{aligned} x = 1 : & \quad -\lambda r(1) = r(2) - r(1) \\ x = 2, \dots, N - 1 : & \quad -\lambda r(x) = r(x - 1) + r(x + 1) - 2r(x) \\ x = N : & \quad -\lambda r(N) = r(N - 1) - r(N) \end{aligned} \quad (81)$$

In the bulk $x = 2, \dots, N-1$ where the recurrence is translation-invariant, the general solution is a linear combination of the two Fourier solutions $e^{\pm i k x}$, where the eigenvalue λ is related to the wavevector k via

$$\lambda(k) = 2 - e^{-ik} - e^{ik} = 2[1 - \cos(k)] \quad (82)$$

If one takes into account the satisfied bulk recurrence, the two boundary conditions of Eq. 56 for $x = 1$ and $x = N$ are equivalent to $r(0) = r(1)$ and $r(N) = r(N+1)$. The not-normalized linear combination $r_k(x)$ of the two Fourier solutions $e^{\pm i k x}$ satisfying $r(N) = r(N+1)$ can be written as

$$r_k(x) = \cos \left[k \left(N + \frac{1}{2} - x \right) \right] \quad \text{for } x = 1, \dots, N \quad (83)$$

while the other boundary condition $r(0) = r(1)$

$$0 = r_k(0) - r_k(1) = \cos \left[k \left(N + \frac{1}{2} \right) \right] - \cos \left[k \left(N - \frac{1}{2} \right) \right] = -2 \sin(kN) \sin \left(\frac{k}{2} \right) \quad (84)$$

selects the N possible wavevectors k as a function of the size N via the condition $\sin(kN) = 0$ leading to

$$k_n = n \frac{\pi}{N} \quad \text{with } n = 0, 1, \dots, N-1 \quad (85)$$

with the corresponding eigenvalues of Eq. 82

$$\lambda_n = 2[1 - \cos(k)] = 2 \left[1 - \cos \left(n \frac{\pi}{N} \right) \right] \quad \text{with } n = 0, 1, \dots, N-1 \quad (86)$$

The spectral Kemeny time of Eq. 5 that involves the $(N-1)$ non-vanishing eigenvalues λ_n that can be computed using Eq. I13

$$\tau_*^{Spectral} \equiv \sum_{n=1}^{N-1} \frac{1}{\lambda_n} = \frac{1}{2} \sum_{n=1}^{N-1} \frac{1}{1 - \cos \left(n \frac{\pi}{N} \right)} = \frac{N^2 - 1}{6} \quad (87)$$

is in agreement with the real-space Kemeny time of Eq. 80 as it should.

3. Spectral Kemeny time in the presence of stochastic resetting at rate γ

In the presence of stochastic resetting at rate γ , the spectral Kemeny time of Eq. 87 is modified into Eq. 53 that can be computed using Eq. I25

$$\begin{aligned} \tau_*^{[Reset]Spectral} &= \sum_{n=1}^{N-1} \frac{1}{\lambda_n + \gamma} = \sum_{n=1}^{N-1} \frac{1}{(\gamma + 2) - 2 \cos \left(n \frac{\pi}{N} \right)} \\ &= \frac{1}{\sqrt{\gamma(4 + \gamma)}} \left[N \frac{(\sqrt{\frac{\gamma}{4}} + \sqrt{\frac{\gamma}{4} + 1})^{4N} - 1}{(\sqrt{\frac{\gamma}{4}} + \sqrt{\frac{\gamma}{4} + 1})^{4N} + 1} - \frac{\gamma + 2}{\sqrt{\gamma(4 + \gamma)}} \right] \end{aligned} \quad (88)$$

In particular, the behavior for large N of Eq. 88 is linear for any non-vanishing resetting rate $\gamma > 0$

$$\tau_*^{[Reset]Spectral} \underset{N \rightarrow +\infty}{\simeq} \frac{N}{\sqrt{\gamma(4 + \gamma)}} \quad (89)$$

in contrast to the diffusive scaling of Eq. 87 when there is no resetting $\gamma = 0$. The crossover between these two scaling behaviors for large N is analyzed in Eq. I26 of Appendix I.

F. Example of the continuous-time asymmetric random walk $w(x \pm 1, x) = e^{\pm \frac{\mu}{2}}$

Let us consider the biased Markov chain

$$\begin{aligned} w(x+1, x) &= e^{\frac{\mu}{2}} \quad \text{for } x = 1, 2, \dots, N-1 \\ w(x, x+1) &= e^{-\frac{\mu}{2}} \quad \text{for } x = 1, 2, \dots, N-1 \end{aligned} \quad (90)$$

so that the diffusion coefficient of Eq. 61 is unity on all the links

$$D\left(x + \frac{1}{2}\right) \equiv \sqrt{w(x, x+1)w(x+1, x)} = 1 \quad \text{for } x = 1, 2, \dots, N-1 \quad (91)$$

while the potential $U(x)$ of Eq. 59 is linear

$$U(x) = -\mu(x-1) \quad (92)$$

The steady state of Eqs 58-60 is then exponential for $\mu \neq 0$

$$\begin{aligned} P_*(x) &= \frac{e^{-U(x)}}{Z} = \frac{e^{\mu(x-1)}}{Z} \\ Z &= \sum_{x=1}^N e^{\mu(x-1)} = \frac{e^{\mu N} - 1}{e^{\mu} - 1} \end{aligned} \quad (93)$$

instead of the uniform steady state of Eq. 79 for $\mu = 0$.

1. Real-space Kemeny time

The real-space Kemeny time of Eq. 73-75 reads

$$\begin{aligned} \tau_*^{Space} &= \sum_{z=1}^{N-1} \frac{1}{ZD\left(z + \frac{1}{2}\right) e^{-\frac{U(z)+U(z+1)}{2}}} \left[\sum_{y=1}^z e^{-U(y)} \right] \left[\sum_{x=z+1}^N e^{-U(x)} \right] = \frac{1}{Z} \sum_{z=1}^{N-1} e^{\frac{\mu}{2} - \mu z} \left[\sum_{y=1}^z e^{\mu(y-1)} \right] \left[\sum_{x=z+1}^N e^{\mu(x-1)} \right] \\ &= \frac{e^{\frac{\mu}{2}}}{Z(e^{\mu} - 1)^2} \sum_{z=1}^{N-1} [e^{\mu N} + 1 - e^{\mu z} - e^{\mu N} e^{-\mu z}] = \frac{e^{\frac{\mu}{2}}}{(e^{\mu N} - 1)(e^{\mu} - 1)} \left[(e^{\mu N} + 1)(N-1) - 2 \frac{e^{\mu N} - e^{\mu}}{e^{\mu} - 1} \right] \\ &= \frac{1}{2 \sinh\left(\frac{\mu}{2}\right)} \left[\frac{N}{\tanh\left(N\frac{\mu}{2}\right)} - \frac{1}{\tanh\left(\frac{\mu}{2}\right)} \right] \end{aligned} \quad (94)$$

The scaling for large N is linear for any non-vanishing bias $\mu \neq 0$

$$\tau_*^{Space} \underset{N \rightarrow +\infty}{\simeq} \frac{N}{2 \sinh\left(\frac{|\mu|}{2}\right)} \quad (95)$$

in contrast to the diffusive scaling of Eq. 80 for $\mu = 0$. The crossover between these two scaling behaviors for large N is analyzed in Eq. I22 of Appendix I concerning spectral Kemeny times that we now describe.

2. Spectral Kemeny time

For the transition rates of Eq. 90, the eigenvalue equation of Eq. 33 reads for the right eigenvector $r(x)$ associated to the eigenvalue λ

$$\begin{aligned} x = 1 : \quad & -\lambda r(1) = e^{-\frac{\mu}{2}} r(2) - e^{\frac{\mu}{2}} r(1) \\ x = 2, \dots, N-1 : \quad & -\lambda r(x) = e^{\frac{\mu}{2}} r(x-1) + e^{-\frac{\mu}{2}} r(x+1) - \left[e^{\frac{\mu}{2}} + e^{-\frac{\mu}{2}} \right] r(x) \\ x = N : \quad & -\lambda r(N) = e^{\frac{\mu}{2}} r(N-1) - e^{-\frac{\mu}{2}} r(N) \end{aligned} \quad (96)$$

In the bulk $x = 2, \dots, N-1$ where the recurrence is translation-invariant, the general solution is a linear combination of the solutions $e^{\frac{\mu}{2}x \pm ikx}$, where the eigenvalue λ is related to the bias and to the wavevector k via

$$\lambda(k) = 2 \cosh\left(\frac{\mu}{2}\right) - 2 \cos(k) \quad (97)$$

The not-normalized eigenstates $r_k(x)$ can be written as the linear combination

$$r_k(x) = e^{\frac{\mu}{2}x} [A \cos(kx) + B \sin(kx)] \quad \text{for } x = 1, \dots, N \quad (98)$$

When one takes into account the satisfied bulk recurrence, the two boundary conditions of Eq. 96 for $x = 1$ and $x = N$ are equivalent to $e^{\frac{\mu}{2}}r(0) = e^{-\frac{\mu}{2}}r(1)$ and $e^{\frac{\mu}{2}}r(N) = e^{-\frac{\mu}{2}}r(N+1)$, so that the coefficients A and B should satisfy

$$\begin{aligned} 0 &= r(0) - e^{-\mu}r(1) = A - e^{-\frac{\mu}{2}} [A \cos(k) + B \sin(k)] \\ 0 &= r(N) - e^{-\mu}r(N+1) = e^{\frac{\mu}{2}N} [A \cos(kN) + B \sin(kN)] - e^{\frac{\mu}{2}N} e^{-\frac{\mu}{2}} [A \cos(k(N+1)) + B \sin(k(N+1))] \end{aligned} \quad (99)$$

The possible wavevectors k are selected via the condition of vanishing determinant for this system, and one finally recovers exactly the same condition $\sin(kN) = 0$ leading to the same values of Eq. 85 of the case $\mu = 0$, so that the corresponding eigenvalues of Eq. 97 read

$$\lambda_n = 2 \cosh\left(\frac{\mu}{2}\right) - 2 \cos\left(n \frac{\pi}{N}\right) \quad \text{with } n = 0, 1, \dots, N-1 \quad (100)$$

The spectral-Kemeny time of Eq. 5 that can be computed using Eq. 118

$$\tau_*^{Spectral} \equiv \sum_{n=1}^{N-1} \frac{1}{\lambda_n} = \sum_{n=1}^{N-1} \frac{1}{2 \cosh\left(\frac{\mu}{2}\right) - 2 \cos\left(n \frac{\pi}{N}\right)} = \frac{1}{2 \sinh\left(\frac{\mu}{2}\right)} \left[\frac{N}{\tanh\left(N \frac{\mu}{2}\right)} - \frac{1}{\tanh\left(\frac{\mu}{2}\right)} \right] \quad (101)$$

is in agreement with the real-space-Kemeny time of Eq. 94.

3. Spectral Kemeny time in the presence of stochastic resetting at rate γ

In the presence of stochastic resetting at rate γ , the excited eigenvalues of Eq. 100 are shifted via Eq. 50

$$\lambda_n[Reset] = \gamma + 2 \cosh\left(\frac{\mu}{2}\right) - 2 \cos\left(n \frac{\pi}{N}\right) \quad \text{with } n = 1, \dots, N-1 \quad (102)$$

where the parameter μ and the resetting rate γ only appear via the global variable $[\gamma + 2 \cosh(\frac{\mu}{2})]$ and lead to the spectral Kemeny time

$$\tau_*^{[Reset]Spectral} = \sum_{n=1}^{N-1} \frac{1}{\lambda_n[Reset]} = \frac{\left[N \frac{\left(\frac{\sqrt{\gamma+4 \sinh^2(\frac{\mu}{4})} + \sqrt{\gamma+4 \cosh^2(\frac{\mu}{4})} \right)^{4N} - 1}{\left(\frac{\sqrt{\gamma+4 \sinh^2(\frac{\mu}{4})} + \sqrt{\gamma+4 \cosh^2(\frac{\mu}{4})} \right)^{4N} + 1} - \frac{\gamma + 2 \cosh(\frac{\mu}{2})}{\sqrt{[\gamma+4 \sinh^2(\frac{\mu}{4})][\gamma+4 \cosh^2(\frac{\mu}{4})]}} \right]}{\sqrt{[\gamma+4 \sinh^2(\frac{\mu}{4})][\gamma+4 \cosh^2(\frac{\mu}{4})]}} \quad (103)$$

G. Example $w(x \pm 1, x) = \frac{1}{2\vartheta_x}$ of the symmetric trap model where each site x has its own trapping time ϑ_x

Trap models have been much studied in the context of anomalously slow glassy behaviors [56, 73–82, 143]. The symmetric trap model on the one-dimensional interval $\{1, \dots, N\}$ is defined by the transition rates

$$\begin{aligned} w(x+1, x) &= \frac{1}{2\vartheta_x} \quad \text{for } x = 1, 2, \dots, N-1 \\ w(x-1, x) &= \frac{1}{2\vartheta_x} \quad \text{for } x = 2, 3, \dots, N \end{aligned} \quad (104)$$

So when the particle is on a bulk site $x = 2, \dots, N-1$ at time t , the escape-time $t \in [0, +\infty[$ from the site x follows the exponential probability distribution

$$p_x^{escape}(t) = \frac{1}{\vartheta_x} e^{-\frac{t}{\vartheta_x}} \quad \text{with the normalization} \quad \int_0^{+\infty} dt p_x^{escape}(t) = 1 \quad (105)$$

whose averaged value is directly ϑ_x that will be called the trapping time of site x

$$\int_0^{+\infty} dt t p_x^{escape}(t) = \vartheta_x \quad (106)$$

and the particle jumps towards one of the two neighbors ($x \pm 1$) with equal probabilities $(\frac{1}{2}, \frac{1}{2})$.

The steady state $P_*(x)$ at site x is simply proportional to its trapping time ϑ_x

$$P_*(x) = \frac{\vartheta_x}{Z}$$

$$Z = \sum_{y=1}^L \vartheta_y \quad (107)$$

while the corresponding steady flows of Eq. 62 take the same value on all the links $x = 1, \dots, N-1$

$$\mathcal{F}_* \left(x + \frac{1}{2} \right) = w(x, x+1)P_*(x+1) = w(x+1, x)P_*(x) = \frac{1}{2Z} \quad (108)$$

The real-space Kemeny time of Eq. 73 reads

$$\begin{aligned} \tau_*^{Space} &= \sum_{z=1}^{N-1} \frac{1}{\mathcal{F}_* \left(z + \frac{1}{2} \right)} \left[\sum_{y=1}^z P_*(y) \right] \left[\sum_{x=z+1}^N P_*(x) \right] \\ &= \frac{2}{Z} \sum_{z=1}^{N-1} \left[\sum_{y=1}^z \vartheta_y \right] \left[\sum_{x=z+1}^N \vartheta_x \right] = \frac{2}{Z} \sum_{y=1}^{N-1} \vartheta_y \sum_{x=y+1}^N \vartheta_x \sum_{z=y}^{x-1} 1 \\ &= \frac{\sum_{y=1}^{N-1} \sum_{x=y+1}^N 2(x-y)\vartheta_y\vartheta_x}{\sum_{z=1}^N \vartheta_z} \end{aligned} \quad (109)$$

In the numerator, each pair $x < y$ of sites contributes with their trapping times ϑ_y and ϑ_x , but also with their distance $(x-y)$, so that the positions of the trapping times within the sample is important, in contrast to the Directed version of the model that will be discussed later in Eqs 271–272.

H. Markov chains with explicit eigenvalues λ_n associated to discrete orthogonal polynomials

As a final remark, let us mention that the spectral Kemeny time $\tau_*^{Spectral}$ can be computed for the Markov chains on intervals $x \in \{1, \dots, N\}$ with explicit eigenvalues λ_n associated to discrete orthogonal polynomials with the following possible dependencies with respect to n (see [72] for more details):

(i) a first family of models is characterized by the quadratic and/or linear dependence in n

$$\lambda_n = c_2 n^2 + c_1 n \quad (110)$$

The simplest examples involve rates $w(x \pm 1, x)$ that are linear or quadratic with respect to x .

(ii) a second family of models is characterized by the $(q^{-n} - 1)$ and/or $(1 - q^n)$ dependence in n involving some parameter q

$$\lambda_n = c_-(q^{-n} - 1) + c_+(1 - q^n) \quad (111)$$

The simplest examples involve rates $w(x \pm 1, x)$ that are linear or quadratic with respect to q^x .

IV. DIFFUSION PROCESSES ON INTERVALS $]x_L, x_R[$ (ALWAYS REVERSIBLE)

In this section, we focus on diffusion processes on intervals $]x_L, x_R[$, where the dynamics always satisfies the detailed-balance as a consequence of the vanishing steady current at the two boundaries. The important technical difference with the previous section is that the space of configurations is continuous instead of discrete, while the physical results are in direct correspondence as already discussed on a specific case in [50]. Previous mathematical references on the Kemeny time for one-dimensional diffusion processes include [24, 38, 40–42] (see Appendix D for the translation of some notations and terminology).

A. Forward and backward equations for the propagator $p_t(x|x_0)$

The propagator $p_t(x|x_0)$ satisfies the Fokker-Planck equation with respect to the final position x

$$\partial_t p_t(x|x_0) = \partial_x [-F(x) + D(x)\partial_x] p_t(x|x_0) \equiv \mathcal{L}_x p_t(x|x_0) \quad (112)$$

where the differential operator

$$\mathcal{L}_x = \partial_x [-F(x) + D(x)\partial_x] \quad (113)$$

replaces the Markov matrix $w(.,.)$ of the previous sections. The backward dynamics with respect to the initial position x_0

$$\partial_t p_t(x|x_0) = \mathcal{L}_{x_0}^\dagger p_t(x|x_0) \quad (114)$$

involves the adjoint operator of Eq. 113

$$\mathcal{L}_{x_0}^\dagger = [F(x_0) + \partial_{x_0} D(x_0)] \partial_{x_0} \quad (115)$$

The Fokker-Planck Eq. 112 can be rewritten as a continuity equation analog to Eq. 56

$$\partial_t p_t(x|x_0) = -\partial_x j_t(x|x_0) \quad (116)$$

where the current $j_t(x|x_0)$ associated to $p_t(x|x_0)$ involves the force $F(x)$ and the diffusion coefficient $D(x)$

$$j_t(x|x_0) = F(x)p_t(x|x_0) - D(x)\partial_x p_t(x|x_0) \quad (117)$$

The conservation of the probability on the interval $]x_L, x_R[$

$$1 = \int_{x_L}^{x_R} dx p_t(x|x_0) \quad (118)$$

imposes the following boundary conditions. The derivative of Eq. 118 with respect to time and the forward dynamics of Eqs 112 116

$$0 = \int_{x_L}^{x_R} dx \partial_t p_t(x|x_0) = \int_{x_L}^{x_R} dx \mathcal{L}_x p_t(x|x_0) = - \int_{x_L}^{x_R} dx \partial_x j_t(x|x_0) = j_t(x_L|x_0) - j_t(x_R|x_0) \quad (119)$$

yields that the current $j_t(x|x_0)$ should vanish at the two boundaries at $x = x_L$ and $x = x_R$

$$j_t(x_L|x_0) = 0 = j_t(x_R|x_0) \quad (120)$$

as expected from the physical point of view. At the technical level, this means that the second-order differential operator \mathcal{L}_x of Eq. 113 acting on any function $r(x)$ on the interval $x \in]x_L, x_R[$ has to be supplemented by the following vanishing-current boundary conditions at $x = x_L$ and at $x = x_R$

$$\text{B.C. for } \mathcal{L}_x : \quad [F(x)r(x) - D(x)r'(x)]_{x=x_L} = 0 = [F(x)r(x) - D(x)r'(x)]_{x=x_R} \quad (121)$$

in order to be fully defined. Then the evaluation via two integrations by parts of the following integral involving another function $l(x)$

$$\begin{aligned} \int_{x_L}^{x_R} dx l(x) \mathcal{L}_x r(x) &= - \int_{x_L}^{x_R} dx l(x) \frac{d}{dx} [F(x)r(x) - D(x)r'(x)] = \int_{x_L}^{x_R} dx [F(x)r(x) - D(x)r'(x)] l'(x) \\ &= \int_{x_L}^{x_R} dx r(x) [F(x)l'(x) + \partial_x (D(x)l'(x))] - [r(x)D(x)l'(x)]_{x=x_L}^{x=x_R} \\ &= \int_{x_L}^{x_R} dx r(x) \mathcal{L}_x^\dagger l(x) + r(x_L)D(x_L)l'(x_L) - r(x_R)D(x_R)l'(x_R) \end{aligned} \quad (122)$$

yields that the proper definition of the action of the adjoint operator \mathcal{L}_x^\dagger on some function $l(x)$ should include, besides the explicit differential expression of Eq. 113 on the interval $x \in]x_L, x_R[$, the following boundary conditions at $x = x_L$ and $x = x_R$

$$\text{B.C. for } \mathcal{L}_x^\dagger : \quad [D(x)l'(x)]_{x=x_L} = 0 = [D(x)l'(x)]_{x=x_R} \quad (123)$$

B. Steady-state $p_*(x)$ with its vanishing steady current $j_* = 0$

The steady-state $p_*(x)$ of the Fokker-Planck Eq. 116 with the zero-current boundary conditions of Eq. 120 can be found by the vanishing of the steady-current $j_*(x) = 0$ on the interval $x \in]x_L, x_R[$

$$0 = j_*(x) = F(x)p_*(x) - D(x)p'_*(x) \quad (124)$$

i.e. as already mentioned for the Markov chain around Eq. 57, on a one-dimensional interval with boundaries one cannot avoid detailed-balance. The steady state $p_*(x)$ can be rewritten as the Boltzmann distribution of Eq. B2

$$p_*(x) = \frac{e^{-U(x)}}{Z} \quad (125)$$

where the potential $U(x)$ involves some arbitrary reference position $x_{ref} \in [x_L, x_R]$ that can be chosen as one wishes in specific examples

$$\begin{aligned} U(x) &\equiv - \int_{x_{ref}}^x dy \frac{F(y)}{D(y)} \\ U'(x) &= - \frac{F(x)}{D(x)} \end{aligned} \quad (126)$$

while the normalization Z corresponds to the partition function of the interval $]x_L, x_R[$

$$Z = \int_{x_L}^{x_R} dx e^{-U(x)} \quad (127)$$

C. Associated quantum supersymmetric Hamiltonian H

The change of variables involving the steady state $p_*(.)$ of Eq. 125 with its potential $U(.)$ of Eq. 126

$$p_t(x|x_0) = \sqrt{\frac{p_*(x)}{p_*(x_0)}} \psi_t(x|x_0) = e^{\frac{U(x_0) - U(x)}{2}} \psi_t(x|x_0) \quad (128)$$

transforms the forward dynamics of Eq. 112 for the propagator $p_t(x, x_0)$ into an Euclidean Schrödinger equation for the quantum propagator $\psi_t(x|x_0)$

$$-\partial_t \psi_t(x|x_0) = H \psi_t(x|x_0) \quad (129)$$

The corresponding quantum Hermitian Hamiltonian

$$H = H^\dagger = -\frac{\partial}{\partial x} D(x) \frac{\partial}{\partial x} + V(x) \quad (130)$$

involves the scalar potential

$$\begin{aligned} V(x) &\equiv D(x) \frac{[U'(x)]^2}{4} - D(x) \frac{U''(x)}{2} - D'(x) \frac{U'(x)}{2} \\ &= \frac{F^2(x)}{4D(x)} + \frac{F'(x)}{2} \end{aligned} \quad (131)$$

that allows to factorize the Hamiltonian of Eq. 130 into the well-known supersymmetric form (see the review [83] and references therein)

$$H \equiv Q^\dagger Q \quad (132)$$

involving the two first-order operators

$$\begin{aligned} Q &\equiv \sqrt{D(x)} \left(\frac{d}{dx} + \frac{U'(x)}{2} \right) \\ Q^\dagger &\equiv \left(-\frac{d}{dx} + \frac{U'(x)}{2} \right) \sqrt{D(x)} \end{aligned} \quad (133)$$

The factorization of Eq. 132 can be considered as the continuous-space counterpart of Eq. B14 concerning Markov chains with discrete configurations.

The consequences for the spectral properties have already been discussed in Appendix B for the case of discrete configurations. In particular, the positive groundstate wavefunction $\phi_0(x)$ associated to the zero-eigenvalue $\lambda_0 = 0$ corresponds to the square-root of the steady state $p_*(x)$ as in Eq. B22

$$\phi_0(x) = \sqrt{p_*(x)} = \frac{e^{-\frac{U(x)}{2}}}{\sqrt{Z}} \quad (134)$$

and is annihilated by the first-order differential operator Q of Eq. 133

$$Q\phi_0(x) = \sqrt{D(x)} \left(\frac{d}{dx} + \frac{U'(x)}{2} \right) \frac{e^{-\frac{U(x)}{2}}}{\sqrt{Z}} = 0 \quad (135)$$

The zero-current boundary conditions of Eqs 120 121 mean that the current $j_n(x)$ associated to any right-eigenvector $r_n(x)$ of the Fokker-Planck generator

$$j_n(x) \equiv F(x)r_n(x) - D(x)r'_n(x) = D(x)[-U'(x)r_n(x) - r'_n(x)] \quad (136)$$

should vanish at the two boundaries

$$j_n(x_L) = 0 = j_n(x_R) \quad (137)$$

Via the transformation of Eq. B21

$$\begin{aligned} r_n(x) &= \sqrt{p_*(x)}\phi_n(x) = \frac{e^{-\frac{U(x)}{2}}}{\sqrt{Z}}\phi_n(x) \\ l_n(x) &= \frac{\phi_n(x)}{\sqrt{p_*(x)}} = \frac{e^{\frac{U(x)}{2}}}{\sqrt{Z}}\phi_n(x) \end{aligned} \quad (138)$$

the current of Eq. 136 that should vanish at the two boundaries reads in terms of the quantum eigenvector $\phi_n(x)$

$$j_n(x) = F(x)r_n(x) - D(x)r'_n(x) = D(x)\frac{e^{-\frac{U(x)}{2}}}{\sqrt{Z}} \left[-\frac{U'(x)}{2}\phi_n(x) - \phi'_n(x) \right] \quad (139)$$

and in terms of the left eigenvector $l_n(x)$

$$j_n(x) = F(x)r_n(x) - D(x)r'_n(x) = -D(x)l'_n(x) \quad (140)$$

so that one recovers the boundary conditions of Eq. 123 for the adjoint operator \mathcal{L}_x^\dagger as it should for consistency.

D. Mean-First-Passage-Time $\tau(x, x_0)$

The backward equation of Eq. 27 for the Mean-First-Passage-Time $\tau(x, x_0)$ at x when starting at x_0 now involves the adjoint operator $\mathcal{L}_{x_0}^\dagger$ of Eq. 115 that can be rewritten using $F(x_0) = -U'(x_0)D(x_0)$ of Eq. 126 as

$$\begin{aligned} -1 + \frac{\delta(x_0 - x)}{p_*(x)} &= \mathcal{L}_{x_0}^\dagger \tau(x, x_0) = [F(x_0) + \partial_{x_0}D(x_0)] \partial_{x_0} \tau(x, x_0) \\ &= [-U'(x_0) + \partial_{x_0}] D(x_0) \partial_{x_0} \tau(x, x_0) \end{aligned} \quad (141)$$

while the boundary conditions of Eq. 123 for the adjoint operator $\mathcal{L}_{x_0}^\dagger$ reads

$$[D(x_0)\partial_{x_0}\tau(x, x_0)]_{x_0=x_L} = 0 = [D(x_0)\partial_{x_0}\tau(x, x_0)]_{x_0=x_R} \quad (142)$$

Since the final position x plays the role of a parameter, it is convenient to introduce the auxiliary function

$$\chi^{[x]}(x_0) \equiv D(x_0)\partial_{x_0}\tau(x, x_0) \quad (143)$$

that should vanish at the two boundaries $x_0 = x_L$ and $x = x_R$ as a consequence of Eq. 142, while Eq. 141 becomes the first order differential equation for $\chi^{[x]}(x_0)$

$$-1 + \frac{\delta(x_0 - x)}{p_*(x)} = [-U'(x_0) + \partial_{x_0}] \chi^{[x]}(x_0) \quad (144)$$

In the two regions $x_0 < x$ and $x_0 > x$, the solutions that vanish at the boundaries $x_0 = x_L$ and $x_0 = x_R$ respectively can be rewritten in terms of the potential $U(\cdot)$ or in terms of the steady state $p_*(\cdot)$ of Eq. 125

$$\begin{aligned}\chi^{[x]}(x_0) &= -e^{U(x_0)} \int_{x_L}^{x_0} dz e^{-U(z)} = -\frac{1}{p_*(x_0)} \int_{x_L}^{x_0} dz p_*(z) \quad \text{for } x_0 \in]x_L, x[\\ \chi^{[x]}(x_0) &= e^{U(x_0)} \int_{x_0}^{x_R} dz e^{-U(z)} = \frac{1}{p_*(x_0)} \int_{x_0}^{x_R} dz p_*(z) \quad \text{for } x_0 \in]x, x_R[\end{aligned} \quad (145)$$

The integration of Eq. 144 over x_0 between $(x - \epsilon)$ and $(x + \epsilon)$ for $\epsilon \rightarrow 0^+$ corresponds to the discontinuity

$$\begin{aligned}\frac{1}{p_*(x)} &= \int_{x-\epsilon}^{x+\epsilon} dx_0 [-U'(x_0) + \partial_{x_0}] \chi^{[x]}(x_0) = \chi^{[x]}(x + \epsilon) - \chi^{[x]}(x - \epsilon) \\ &= \frac{1}{p_*(x)} \int_x^{x_R} dz p_*(z) + \frac{1}{p_*(x)} \int_{x_L}^x dz p_*(z) = \frac{1}{p_*(x)} \int_{x_L}^{x_R} dz p_*(z) \end{aligned} \quad (146)$$

which is satisfied using the normalization of the steady state.

We may now use Eq. 143

$$\partial_{x_0} \tau(x, x_0) = \frac{\chi^{[x]}(x_0)}{D(x_0)} \quad (147)$$

with the solution of Eq. 145 for $\chi^{[x]}(x_0)$ to obtain the Mean-First-Passage-Time $\tau(x, x_0)$ in the two regions $x_0 < x$ and $x_0 > x$

$$\begin{aligned}\text{for } x_0 \in]x_L, x[: \quad \tau(x, x_0) &= \int_{x_0}^x dy \frac{1}{D(y)e^{-U(y)}} \int_{x_L}^y dz e^{-U(z)} = \int_{x_0}^x dy \frac{1}{D(y)p_*(y)} \int_{x_L}^y dz p_*(z) \equiv \int_{x_0}^x dy \frac{p_*([x_L, y])}{D(y)p_*(y)} \\ \text{for } x_0 \in]x, x_R[: \quad \tau(x, x_0) &= \int_{x_0}^{x_R} dy \frac{1}{D(y)e^{-U(y)}} \int_y^{x_R} dz e^{-U(z)} = \int_{x_0}^{x_R} dy \frac{1}{D(y)p_*(y)} \int_y^{x_R} dz p_*(z) \equiv \int_{x_0}^{x_R} dy \frac{p_*([y, x_R])}{D(y)p_*(y)} \end{aligned} \quad (148)$$

where one recognizes the cumulative distribution $p_*([x_L, y])$ and the complementary cumulative distribution $p_*([y, x_R])$ analog to Eqs 67 71

$$\begin{aligned}p_*([x_L, y]) &\equiv \int_{x_L}^y dz p_*(z) \\ p_*([y, x_R]) &\equiv \int_y^{x_R} dz p_*(z) = 1 - \int_{x_L}^y dz p_*(z) \end{aligned} \quad (149)$$

while the denominator $D(y)p_*(y)$ is the continuous analog of the discrete-space flow of Eq. 62. The final results of Eqs 148 are thus the direct continuous analogs of the discrete-space expressions of Eqs 65 66 69 70.

E. Real-space Kemeny time

The real-space Kemeny time of Eq. 2 can be evaluated for the initial position at the left boundary $x_0 = x_L$ using Eq. 148

$$\begin{aligned}\tau_*^{Space} &= \int_{x_L}^{x_R} dx p_*(x) \tau(x, x_0 = x_L) = \int_{x_L}^{x_R} dx p_*(x) \int_{x_L}^x dy \frac{1}{D(y)p_*(y)} \int_{x_L}^y dz p_*(z) \\ &= \int_{x_L}^{x_R} dy \frac{1}{D(y)p_*(y)} \left[\int_{x_L}^y dz p_*(z) \right] \left[\int_y^{x_R} dx p_*(x) \right] \\ &\equiv \int_{x_L}^{x_R} dy \frac{p_*([x_L, y])p_*([y, x_R])}{D(y)p_*(y)} \end{aligned} \quad (150)$$

where the numerator involves the product of the cumulative distribution $p_*([x_L, y])$ and the complementary cumulative distribution $p_*([y, x_R])$ of Eq. 149, while the denominator $D(y)p_*(y)$ is the continuous analog of the discrete-space flow of Eq. 62. The final result of Eq. 150 is thus the direct continuous analog of the discrete-space expression of Eqs 73 74.

The conclusion is thus the same as at the end of subsection III D: for any diffusion process defined on the interval $]x_L, x_R[$, the steady state $p_*(x)$ is known via Eq. 125 126 and the Kemeny time can be computed by the real-space expression of Eq. 150, even when the individual eigenvalues λ_n are not explicit.

From the physical point of view, one of the most important issue is the scaling of the Kemeny time τ_* with respect to the size $L = x_R - x_L$ of the interval. Let us now describe various examples.

F. Example: diffusion coefficient $D(x) = 1$ and no force $F(x) = 0$ on the interval $[0, L]$

1. Real-space Kemeny time

The steady state of Eq. 125 is uniform

$$p_*(x) = \frac{1}{L} \quad \text{for } 0 \leq x \leq L \quad (151)$$

So the real-space Kemeny time Eq. 150 reduces to

$$\tau_*^{Space} = \int_0^L dy \frac{1}{p_*(y)} \int_0^y dz p_*(z) \int_y^L dx p_*(x) = \frac{1}{L} \int_0^L dy y(L-y) = \frac{L^2}{6} \quad (152)$$

and displays the expected diffusive scaling in L^2 with respect to the size L .

2. Spectral Kemeny time

The Fourier modes $r(x) = \cos(kx)$ that have a vanishing derivative at $x = 0$ should also have a vanishing derivative at $x = L$, so that the condition $\sin(kL) = 0$ selects the series of possible wavevectors as a function of the size L

$$k_n = \frac{\pi n}{L} \quad \text{with } n = 0, 1, \dots, +\infty \quad (153)$$

with the corresponding eigenvalues

$$\lambda_n = k_n^2 = \frac{\pi^2 n^2}{L^2} \quad \text{with } n = 0, 1, \dots, +\infty \quad (154)$$

The spectral Kemeny time of Eq. 5 involving the non-vanishing eigenvalues

$$\tau_*^{Spectral} = \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} = \frac{L^2}{\pi^2} \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{L^2}{6} \quad (155)$$

is in agreement with the real-space Kemeny time of Eq. 152 as it should.

3. Spectral Kemeny time in the presence of stochastic resetting at rate γ

In the presence of stochastic resetting at rate γ , the spectral Kemeny time of Eq. 155 is modified into Eq. 53

$$\tau_*^{[Reset]Spectral} = \sum_{n=1}^{+\infty} \frac{1}{\lambda_n + \gamma} = \sum_{n=1}^{+\infty} \frac{1}{\gamma + \frac{\pi^2 n^2}{L^2}} = \frac{1}{2\gamma} \left[\frac{\sqrt{\gamma} L}{\tanh(\sqrt{\gamma} L)} - 1 \right] \quad (156)$$

In particular, the behavior for large L is linear for any non-vanishing resetting rate $\gamma > 0$

$$\tau_*^{[Reset]Spectral} \underset{L \rightarrow +\infty}{\simeq} \frac{L}{2\sqrt{\gamma}} \quad (157)$$

in contrast to the diffusive scaling of Eq. 152 when there is no resetting $\gamma = 0$.

In order to describe the crossover between the two scaling behaviors as L^2 in Eq. 152 and as L in Eq. 157 for large L , one can divide Eq. 156 by its value of Eq. 152 without resetting $\gamma = 0$

$$\frac{\tau_*^{[Reset]Spectral}}{\tau_*^{Spectral}} = \frac{3}{(\sqrt{\gamma} L)^2} \left[\frac{\sqrt{\gamma} L}{\tanh(\sqrt{\gamma} L)} - 1 \right] \equiv \mathcal{T}(\hat{L} = \sqrt{\gamma} L) \quad (158)$$

in order to obtain the following scaling function for the rescaled variable $\hat{L} \equiv \sqrt{\gamma}L$

$$\mathcal{T}(\hat{L}) = \frac{3}{\hat{L}^2} \left[\frac{\hat{L}}{\tanh \hat{L}} - 1 \right] \quad (159)$$

G. Example: diffusion coefficient $D(x) = 1$ and force $F(x) = \pm\mu$ on two intervals $]0, \rho L[$ and $] \rho L, L[$

Let us now focus on the simple example on the interval $[0, L]$, where the diffusion coefficient is constant $D(x) = 1$ while the force $F(x)$ takes only two values $(\pm\mu)$ and changes at the position (ρL) parametrized by ρ

$$\begin{aligned} F(x) &= -\mu & \text{for } 0 < x < \rho L \\ F(x) &= \mu & \text{for } \rho L < x < L \end{aligned} \quad (160)$$

The potential of Eq. 126 is then the saw-tooth potential

$$\begin{aligned} U(x) &= \mu x & \text{for } 0 \leq x \leq \rho L \\ U(x) &= 2\mu\rho L - \mu x & \text{for } \rho L \leq x \leq L \end{aligned} \quad (161)$$

with the three important extremal values

$$\begin{aligned} U(x=0) &= 0 \\ U(x=\rho L) &= \mu\rho L \\ U(x=L) &= \mu(2\rho - 1)L \end{aligned} \quad (162)$$

that will allow to analyze different physical interesting cases depending on the two parameters μ and ρ .

The steady state of Eq. 125 reads

$$\begin{aligned} p_*(x) &= \frac{e^{-\mu x}}{Z} & \text{for } 0 \leq x \leq \rho L \\ p_*(x) &= \frac{e^{\mu x - 2\mu\rho L}}{Z} & \text{for } \rho L \leq x \leq L \end{aligned} \quad (163)$$

with the partition function

$$Z = \int_0^L dx e^{-U(x)} = \int_0^{\rho L} dx e^{-\mu x} + e^{-2\mu\rho L} \int_{\rho L}^L dx e^{\mu x} = \frac{1 + e^{\mu(1-2\rho)L} - 2e^{-\mu\rho L}}{\mu} \quad (164)$$

The corresponding cumulative distributions of Eq. 149

$$\begin{aligned} \text{for } 0 \leq y \leq \rho L : \quad p_*([0, y]) &= \int_0^y dx p_*(x) = \int_0^y dx \frac{e^{-\mu x}}{Z} = \frac{1 - e^{-\mu y}}{\mu Z} \\ \text{for } \rho L \leq y \leq L : \quad p_*([y, L]) &= \int_y^L dx p_*(x) = e^{-2\mu\rho L} \frac{e^{\mu L} - e^{\mu y}}{\mu Z} \end{aligned} \quad (165)$$

can be plugged into Eq. 150 to compute the real-space Kemeny time

$$\begin{aligned} \tau_*^{Space} &= \int_0^{\rho L} dy \frac{p_*([0, y]) [1 - p_*([0, y])]}{p_*(y)} + \int_{\rho L}^L dy \frac{p_*([y, L]) [1 - p_*([y, L])]}{p_*(y)} \\ &= \int_0^{\rho L} dy \frac{\frac{1 - e^{-\mu y}}{\mu Z} \left[1 - \frac{1 - e^{-\mu y}}{\mu Z} \right]}{\frac{e^{-\mu y}}{Z}} + \int_{\rho L}^L dy \frac{e^{-2\mu\rho L} \frac{e^{\mu L} - e^{\mu y}}{\mu Z} \left[1 - e^{-2\mu\rho L} \frac{e^{\mu L} - e^{\mu y}}{\mu Z} \right]}{\frac{e^{\mu y - 2\mu\rho L}}{Z}} \end{aligned} \quad (166)$$

These elementary integrals and the value of the partition function Z of Eq. 164 leads to the final result

$$\tau_* = \frac{2e^{\mu(1-\rho)L} + e^{\mu(1-2\rho)L} [\mu(1-2\rho)L - 4] + e^{-\mu\rho L} [6 + 2\mu L] + \mu(2\rho - 1)L - 4}{\mu^2 [1 + e^{\mu(1-2\rho)L} - 2e^{-\mu\rho L}]} \quad (167)$$

It is now interesting to discuss the scaling of the Kemeny time τ_* for large size L as a function of the two parameters (ρ, μ) defining the saw-tooth potential of Eq. 162.

1. Case $\rho = 0$: Diffusion $D(x) = 1$ and constant force $F(x) = \mu \neq 0$ on the interval $[0, L]$

For $\rho = 0$, the real-space Kemeny time of Eq. 167 reduces to

$$\tau_*^{Space} = \frac{e^{\mu L}[\mu L - 2] + [\mu L + 2]}{\mu^2 [e^{\mu L} - 1]} = \frac{2}{\mu^2} \left[\frac{\frac{\mu L}{2}}{\tanh\left(\frac{\mu L}{2}\right)} - 1 \right] \quad (168)$$

so the leading behavior for large L

$$\tau_*^{Space} \underset{L \rightarrow +\infty}{\simeq} \frac{2}{\mu^2} \left[\frac{\frac{\mu L}{2}}{\text{sgn}(\mu)} \right] = \frac{L}{|\mu|} \quad (169)$$

is linear for any non-vanishing force $\mu \neq 0$ instead of the diffusive scaling of Eq. 152 for vanishing force $\mu = 0$. In order to describe the crossover between these two scaling behaviors for large L , one can divide Eq. 168 by its value of Eq. 152 without force $\mu = 0$

$$\frac{\tau_*^{Space}}{\frac{L^2}{6}} = \frac{3}{\left(\frac{\mu L}{2}\right)^2} \left[\frac{\frac{\mu L}{2}}{\tanh\left(\frac{\mu L}{2}\right)} - 1 \right] \equiv \mathcal{T}\left(\hat{L} = \frac{|\mu|}{2}L\right) \quad (170)$$

where one recognizes the scaling function $\mathcal{T}(\hat{L})$ already encountered in Eq. 159 but for the different rescaled variable $\hat{L} \equiv \frac{|\mu|}{2}L$, that involves the force μ instead of the resetting rate γ . In order to understand why the force μ and the resetting rate γ have similar effects on the Kemeny time, whereas they correspond to completely different processes with completely different steady states, it is useful to consider the excited eigenvalues in the presence of the force μ computed in Eq. E11 of Appendix E

$$\lambda_n = \frac{\mu^2}{4} + k_n^2 = \frac{\mu^2}{4} + \frac{\pi^2 n^2}{L^2} \quad \text{with } n = 1, 2, \dots, +\infty \quad (171)$$

that lead to the spectral Kemeny time

$$\tau_*^{Spectral} = \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} = \frac{L^2}{\pi^2} \sum_{n=1}^{+\infty} \frac{1}{n^2 + \left(\frac{\mu L}{2\pi}\right)^2} = \frac{2}{\mu^2} \left[\frac{\frac{\mu L}{2}}{\tanh\left(\frac{\mu L}{2}\right)} - 1 \right] \quad (172)$$

in agreement with the real-space Kemeny time of Eq. 168.

In the presence of stochastic resetting at rate γ in the presence of the force μ , the excited eigenvalues of Eq. 171 are shifted via Eq. 50

$$\lambda_n^{[Reset]} = \gamma + \frac{\mu^2}{4} + \frac{\pi^2 n^2}{L^2} \quad \text{with } n = 1, 2, \dots, +\infty \quad (173)$$

and lead to the spectral Kemeny time

$$\tau_*^{[Reset]Spectral} = \sum_{n=1}^{+\infty} \frac{1}{\lambda_n^{[Reset]}} = \frac{1}{2\left(\gamma + \frac{\mu^2}{4}\right)} \left[\frac{\sqrt{\gamma + \frac{\mu^2}{4}}L}{\tanh\left(\sqrt{\gamma + \frac{\mu^2}{4}}L\right)} - 1 \right] \quad (174)$$

where the force μ and the resetting rate γ only appear via the global variable $\sqrt{\gamma + \frac{\mu^2}{4}}$.

2. Case $\mu = -|\mu| < 0$: one valley with minimum $U(x = \rho L) = -|\mu|\rho L$

For $\mu = -|\mu| < 0$, the leading behavior of Eq. 167 for large L

$$\tau_*^{[\mu < 0]} \underset{L \rightarrow +\infty}{\simeq} \frac{e^{|\mu|\rho L}[-2|\mu|L]}{\mu^2 [-2e^{|\mu|\rho L}]} = \frac{L}{|\mu|} \quad (175)$$

is linear and does not depend on ρ , so that it coincides with Eq. 168 corresponding to $\rho = 0$.

3. Case $\mu > 0$ and $0 < \rho < 1$: maximum $U(x = \rho L) = \mu \rho L$ separating two valleys with minima at $x = 0$ and $x = L$

Here one expects that the result will depend on the smallest of the two barriers

$$\begin{aligned} B_1(L) &\equiv U(x = \rho L) - U(x = 0) = \mu \rho L \\ B_2(L) &\equiv U(x = \rho L) - U(x = L) = \mu(1 - \rho)L \end{aligned} \quad (176)$$

i.e. that it will depend on the sign of $(\rho - \frac{1}{2})$ as follows:

(i) For $0 < \rho < \frac{1}{2}$, the leading behavior of Eq. 167 for large L

$$\tau_*^{[\mu > 0; 0 < \rho < \frac{1}{2}]} \underset{L \rightarrow +\infty}{\simeq} \frac{2e^{\mu(1-\rho)L}}{\mu^2 [e^{\mu(1-2\rho)L}]} = \frac{2}{\mu^2} e^{\mu\rho L} = \frac{2}{\mu^2} e^{B_1(L)} \quad (177)$$

is exponentially large in L and is dominated by the barrier $B_1(L) = \mu\rho L$ of Eq. 176.

(ii) For $\frac{1}{2} < \rho < 1$, the leading behavior of Eq. 167 for large L

$$\tau_*^{[\mu > 0; \frac{1}{2} < \rho < 1]} \underset{L \rightarrow +\infty}{\simeq} \frac{2e^{\mu(1-\rho)L}}{\mu^2 [1]} = \frac{2}{\mu^2} e^{\mu(1-\rho)L} = \frac{2}{\mu^2} e^{B_2(L)} \quad (178)$$

is exponentially large in L and is dominated by the barrier $B_2(L) = \mu(1 - \rho)L$ of Eq. 176.

(iii) For the symmetric case $\rho = \frac{1}{2}$ where the two barriers of Eq. 176 coincide $B_1(L) = B_2(L) = \frac{\mu}{2}L$, Eq. 167 reduces to

$$\tau_*^{[\mu > 0; \rho = \frac{1}{2}]} = \frac{e^{\frac{\mu}{2}L} + e^{-\frac{\mu}{2}L} [3 + 1\mu L] - 4}{\mu^2 [1 - e^{-\frac{\mu}{2}L}]} \underset{L \rightarrow +\infty}{\simeq} \frac{1}{\mu^2} e^{\frac{\mu}{2}L} \quad (179)$$

In Appendix F, these three exponentially-large asymptotic behaviors of the real-space Kemeny times of Eqs 177 178 and 179 are explained by considering only the relaxation time $\tau_1 \equiv \frac{1}{\lambda_1}$ associated to the lowest non-vanishing eigenvalue λ_1 (see Eqs F25 and F29).

H. Spectral Kemeny times for the diffusion processes with the explicit Jacobi spectra $\lambda_n = An(n + B)$

The Pearson family of ergodic diffusions (see the mini review in section 2 of the mathematical paper [84] and the recent physics preprint [85] with references therein) contains the one-dimensional diffusions with a positive quadratic diffusion coefficient $D(x)$ and a linear force $F(x)$, i.e. equivalently a linear Ito force $f^I(x)$ or a linear Stratonovich force $f^S(x)$ as a consequence of Eq. D2. These technical simplifications produce six types of simple analytical steady states and allow to compute the corresponding spectrum of eigenvalues [84–86]. The only case with a finite Kemeny time corresponds to the Jacobi spectrum of eigenvalues

$$\lambda_n = An(n + B) \quad \text{for } n = 0, 1, \dots + \infty \quad \text{with parameters } A > 0 \quad B > -1 \quad (180)$$

and to the only Pearson diffusion defined on a finite interval. Indeed, the other cases defined on $] - \infty, +\infty[$ or $[0, +\infty[$ are associated to the linear Ornstein-Uhlenbeck spectrum discussed in Eq. G10 or contain a continuum part in the spectrum besides a finite number of discrete states [84–86], so that their Kemeny times diverge as explained in Appendix G, where the criteria for the convergence or the divergence for the Kemeny times of diffusion processes defined on infinite intervals $(x_R - x_L) \rightarrow +\infty$ are discussed in detail in real-space and in the spectral domain.

The spectral Kemeny time associated to the Jacobi spectrum of Eq. 180

$$\tau_*^{Spectral[Jacobi]} = \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} = \frac{1}{A} \sum_{n=1}^{+\infty} \frac{1}{n(n + B)} = \frac{H(B)}{AB} \quad (181)$$

involves the Harmonic number function $H(B)$ that can be written as an integral or in terms of the Γ function and of the Euler constant γ_{Euler}

$$H(B) = \int_0^1 du \frac{1 - u^B}{1 - u} = \gamma_{Euler} + \frac{\Gamma'(B + 1)}{\Gamma(B + 1)} \quad (182)$$

and that has a simple fractional value when the parameter B is an integer

$$\text{Integer } B = 1, 2, \dots, +\infty : H(B) = \int_0^1 du \left[\sum_{k=0}^{B-1} u^k \right] = \sum_{k=0}^{B-1} \frac{1}{k + 1} = \sum_{m=1}^B \frac{1}{m} \quad (183)$$

Let us now describe some examples characterized by the Jacobi spectrum of Eq. 180 starting with the Jacobi process itself.

1. *Jacobi process on $]0, 1[$ converging towards the Beta-distribution of parameters $\alpha > 0$ and $\beta > 0$ with $D(x) = x(1 - x)$*

For the Jacobi process [84, 85] defined on the finite interval $x \in]0, 1[$, the diffusion coefficient is a positive quadratic polynomial vanishing at the two boundaries $x = 0$ and $x = 1$

$$D(x) = Ax(1 - x) \quad \text{with } A > 0 \quad (184)$$

while the normalized steady state is the Beta-distribution of parameters $\alpha > 0$ and $\beta > 0$

$$p_*(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{Z(\alpha, \beta)} \quad \text{for } x \in]0, 1[$$

$$Z(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (185)$$

Applications of the Jacobi process can be found in mathematical finance [87], genetics [88] as well as in neuroscience [89]. The corresponding potential of Eq. 125 involves a sum of logarithms

$$U(x) = -(\alpha - 1)\ln(x) - (\beta - 1)\ln(1 - x) \quad (186)$$

while the force of Eq. 126 is linear

$$F(x) = -U'(x)D(x) = (\alpha - 1)A + (2 - \alpha - \beta)Ax \quad (187)$$

The eigenvalues correspond to the parameter $B = \alpha + \beta - 1$ in the Jacobi spectrum of Eq. 180

$$\lambda_n = An(n + \alpha + \beta - 1) \quad \text{for } n = 0, 1, \dots + \infty \quad (188)$$

while the corresponding eigenvectors involve the Jacobi family of orthogonal polynomials.

The simplest example corresponds to the parameters $\alpha = 1 = \beta$ where the steady state $p_*(x)$ of Eq. 185 is uniform on $[0, 1]$

$$p_*(x) = 1 \quad \text{for } 0 \leq x \leq 1 \quad (189)$$

and where the force of Eq. 187 vanishes $F(x) = 0$. Then the spectral Kemeny time of Eq. 181 with $B = 1$ and $H(1) = 1$ of Eq. 183

$$\tau_*^{Spectral[\alpha=1=\beta]} = \frac{1}{A} \sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = \frac{1}{A} \sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = \frac{H(1)}{A} = \frac{1}{A} \quad (190)$$

coincides with the real-space Kemeny time of Eq. 150

$$\tau_*^{Space[\alpha=1=\beta]} = \int_0^1 dy \frac{1}{D(y)} \left[\int_0^y dz \right] \left[\int_y^1 dx \right] = \int_0^1 dy \frac{1}{Ay(1-y)} y(1-y) = \frac{1}{A} \quad (191)$$

As stressed in [84, 85], the Jacobi eigenvalues of Eq. 188 will be conserved via any change of variables $x \rightarrow y(x)$ that is twice-differentiable and invertible. Among the many new processes $y(t)$ with the same spectrum that can be constructed via this procedure, one of the most natural is the process $y(t)$ that has a constant diffusion coefficient instead of the quadratic polynomial of Eq. 184, as described in the next subsection.

2. *Change of variables $x \rightarrow y$ towards the process $y(t)$ on $[-\frac{L}{2}, +\frac{L}{2}]$ with constant diffusion coefficient $\mathcal{D}(y) = 1$*

The change of variables from $x \in [0, 1]$ to $y \in [-\frac{L}{2}, +\frac{L}{2}]$

$$x = \frac{1 + \sin\left(\pi \frac{y}{L}\right)}{2}$$

$$dx = dy \frac{\pi}{2L} \cos\left(\pi \frac{y}{L}\right) = dy \frac{\pi}{2L} \sqrt{1 - \sin^2\left(\pi \frac{y}{L}\right)} = dy \sqrt{\frac{\pi^2}{L^2} x(1-x)} \quad (192)$$

allows to transform the quadratic diffusion coefficient $D(x) = Ax(1-x)$ of Eq. 184 into the constant diffusion coefficient $\mathcal{D}(y) = 1$ for the new process $y(t)$ if one chooses the following value for the parameter A

$$A = \frac{\pi^2}{L^2} \quad (193)$$

The normalized steady state $\mathcal{P}_*(y)$ $y \in [-\frac{L}{2}, +\frac{L}{2}]$ obtained from Eq. 185 reads

$$\begin{aligned}\mathcal{P}_*(y) &= p_*(x) \frac{dx}{dy} = \frac{\pi}{LZ(\alpha, \beta)} x^{\alpha-\frac{1}{2}} (1-x)^{\beta-\frac{1}{2}} \\ &= \frac{\pi \Gamma(\alpha + \beta)}{L \Gamma(\alpha) \Gamma(\beta) 2^{\alpha+\beta-1}} \left[1 + \sin\left(\pi \frac{y}{L}\right)\right]^{\alpha-\frac{1}{2}} \left[1 - \sin\left(\pi \frac{y}{L}\right)\right]^{\beta-\frac{1}{2}}\end{aligned}\quad (194)$$

The potential of Eq. 125 associated to this steady state is

$$\mathcal{U}(y) = -\left(\alpha - \frac{1}{2}\right) \ln \left[1 + \sin\left(\pi \frac{y}{L}\right)\right] - \left(\beta - \frac{1}{2}\right) \ln \left[1 - \sin\left(\pi \frac{y}{L}\right)\right] \quad (195)$$

The force of Eq. 126

$$\begin{aligned}\mathcal{F}(y) = -\mathcal{U}'(y) &= \left(\alpha - \frac{1}{2}\right) \frac{\frac{\pi}{L} \cos\left(\pi \frac{y}{L}\right)}{1 + \sin\left(\pi \frac{y}{L}\right)} - \left(\beta - \frac{1}{2}\right) \frac{\frac{\pi}{L} \cos\left(\pi \frac{y}{L}\right)}{1 - \sin\left(\pi \frac{y}{L}\right)} \\ &= \frac{\pi \left[(\alpha - \beta) - (\alpha + \beta - 1) \sin\left(\pi \frac{y}{L}\right)\right]}{L \cos\left(\pi \frac{y}{L}\right)}\end{aligned}\quad (196)$$

and its derivative

$$\mathcal{F}'(y) = \frac{\pi^2 \left[(\alpha - \beta) \sin\left(\pi \frac{y}{L}\right) - (\alpha + \beta - 1)\right]}{L^2 \cos^2\left(\pi \frac{y}{L}\right)} \quad (197)$$

allow to compute the quantum potential of Eq. 131

$$\begin{aligned}\mathcal{V}(y) &= \frac{\mathcal{F}^2(y)}{4} + \frac{\mathcal{F}'(y)}{2} \\ &= \frac{\pi^2}{2L^2} \left[\frac{\left(\alpha - \frac{1}{2}\right) \left(\alpha - \frac{3}{2}\right)}{1 + \sin\left(\pi \frac{y}{L}\right)} + \frac{\left(\beta - \frac{1}{2}\right) \left(\beta - \frac{3}{2}\right)}{1 - \sin\left(\pi \frac{y}{L}\right)} - \frac{(\alpha + \beta - 1)^2}{2} \right]\end{aligned}\quad (198)$$

The Jacobi eigenvalues are given by Eq. 188 for the parameter $A = \frac{\pi^2}{L^2}$ of Eq. 193

$$\lambda_n = \frac{\pi^2}{L^2} n(n + \alpha + \beta - 1) \quad \text{for } n = 0, 1, \dots + \infty \quad (199)$$

so that the spectral Kemeny time of Eq. 181

$$\tau_*^{Spectral} = \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} = L^2 \frac{H(\alpha + \beta - 1)}{\pi^2(\alpha + \beta - 1)} \quad (200)$$

grows as L^2 with respect to the system size L , while the prefactor depends on the sum $(\alpha + \beta)$.

For the symmetric case $\beta = \alpha$ studied in detail in [90], the steady state of Eq. 194 simplifies into a power of $\cos\left(\pi \frac{y}{L}\right)$

$$\mathcal{P}_*^{[\beta=\alpha]}(y) = \frac{\pi \Gamma(2\alpha)}{L \Gamma^2(\alpha) 2^{2\alpha-1}} \left[1 - \sin^2\left(\pi \frac{y}{L}\right)\right]^{\alpha-\frac{1}{2}} = \frac{\sqrt{\pi} \Gamma\left(\alpha + \frac{1}{2}\right)}{L \Gamma(\alpha)} \left[\cos\left(\pi \frac{y}{L}\right)\right]^{2\alpha-1} \quad (201)$$

while the force of Eq. 196 reduces to the tangent force

$$\mathcal{F}^{[\beta=\alpha]}(y) = -(2\alpha - 1) \frac{\pi}{L} \tan\left(\pi \frac{y}{L}\right) \quad (202)$$

and the quantum potential of Eq. 198 becomes

$$\mathcal{V}^{[\beta=\alpha]}(y) = \frac{\pi^2}{2L^2} \left[\frac{2 \left(\alpha - \frac{1}{2}\right) \left(\alpha - \frac{3}{2}\right)}{\cos^2\left(\pi \frac{y}{L}\right)} - \frac{(2\alpha - 1)^2}{2} \right] \quad (203)$$

Let us now discuss the simplest values of the parameters (α, β) that produce a constant quantum potential $\mathcal{V}(y)$ in Eq. 198:

(i) the symmetric case $\beta = \alpha = \frac{1}{2}$ corresponds to the uniform steady state in Eq. 201

$$\mathcal{P}_*^{[\beta=\alpha=\frac{1}{2}]}(y) = \frac{1}{L} \quad (204)$$

and one recovers via Eq. 199 the eigenvalues

$$\lambda_n^{[\beta=\alpha=\frac{1}{2}]} = \frac{\pi^2}{L^2} n^2 \quad \text{for } n = 0, 1, \dots + \infty \quad (205)$$

discussed in Eq. 154 for the pure diffusion $D(x) = 1$ with reflecting boundaries separated by the distance L .

(ii) the symmetric case $\beta = \alpha = \frac{3}{2}$ corresponds to the steady state in Eq. 201

$$\mathcal{P}_*^{[\beta=\alpha=\frac{3}{2}]}(y) = \frac{2 \cos^2(\pi \frac{y}{L})}{L} = \frac{1 + \cos(2\pi \frac{y}{L})}{L} \quad (206)$$

where one recognizes the steady state of the Taboo process, i.e. the pure diffusion $D(x) = 1$ with absorbing boundaries at $x = -\frac{L}{2}$ and at $x = +\frac{L}{2}$ conditioned to survive forever (see [90–94] and the next subsection for more explanations), with its eigenvalues of Eq. 199

$$\lambda_n^{[\beta=\alpha=\frac{3}{2}]} = \frac{\pi^2}{L^2} n(n+2) \quad \text{for } n = 0, 1, \dots + \infty \quad (207)$$

The spectral Kemeny time of Eq. 200

$$\tau_*^{Spectral[\beta=\alpha=\frac{3}{2}]} = \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} = L^2 \frac{H(2)}{2\pi^2} = \frac{3}{4\pi^2} L^2 \quad (208)$$

is in agreement with the real-space Kemeny time of Eq. 150 based on the steady state of Eq. 206

$$\begin{aligned} \tau_*^{[\beta=\alpha=\frac{3}{2}]} &= \int_{-\frac{L}{2}}^{-\frac{L}{2}} dy \frac{1}{\mathcal{P}_*(y)} \left[\int_{-\frac{L}{2}}^y dz \mathcal{P}_*(z) \right] \left[\int_y^{\frac{L}{2}} dx \mathcal{P}_*(x) \right] \\ &= \int_{-\frac{L}{2}}^{-\frac{L}{2}} dy \frac{L}{2 \cos^2(\pi \frac{y}{L})} \left[\frac{\frac{L}{2} + y + \frac{L}{2\pi} \sin(2\pi \frac{y}{L})}{L} \right] \left[\frac{\frac{L}{2} - y - \frac{L}{2\pi} \sin(2\pi \frac{y}{L})}{L} \right] = \frac{3}{4\pi^2} L^2 \end{aligned} \quad (209)$$

(iii) the asymmetric case $[\alpha = \frac{3}{2}; \beta = \frac{1}{2}]$ corresponds to the steady state in Eq. 194

$$\mathcal{P}_*^{[\alpha=\frac{3}{2};\beta=\frac{1}{2}]}(y) = \frac{1 + \sin(\pi \frac{y}{L})}{L} \quad (210)$$

and to the eigenvalues of Eq. 199

$$\lambda_n^{[\alpha=\frac{3}{2};\beta=\frac{1}{2}]} = \frac{\pi^2}{L^2} n(n+1) \quad \text{for } n = 0, 1, \dots + \infty \quad (211)$$

The spectral Kemeny time of Eq. 200

$$\tau_*^{Spectral[\alpha=\frac{3}{2};\beta=\frac{1}{2}]} = \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} = L^2 \frac{H(1)}{\pi^2} = \frac{L^2}{\pi^2} \quad (212)$$

is in agreement with the real-space Kemeny time of Eq. 150 based on the steady state of Eq. 210

$$\begin{aligned} \tau_*^{Space[\alpha=\frac{3}{2};\beta=\frac{1}{2}]} &= \int_{-\frac{L}{2}}^{-\frac{L}{2}} dy \frac{1}{\mathcal{P}_*(y)} \left[\int_{-\frac{L}{2}}^y dz \mathcal{P}_*(z) \right] \left[\int_y^{\frac{L}{2}} dx \mathcal{P}_*(x) \right] \\ &= \int_{-\frac{L}{2}}^{-\frac{L}{2}} dy \frac{L}{1 + \sin(\pi \frac{y}{L})} \left[\frac{\frac{L}{2} + y - \frac{L}{\pi} \cos(\pi \frac{y}{L})}{L} \right] \left[\frac{\frac{L}{2} - y + \frac{L}{\pi} \cos(\pi \frac{y}{L})}{L} \right] = \frac{L^2}{\pi^2} \end{aligned} \quad (213)$$

The physical interpretation of the process $[\alpha = \frac{3}{2}; \beta = \frac{1}{2}]$ can be guessed from the previous cases (i) and (ii): it should be the diffusion $D(x) = 1$ with a reflecting boundary at $y = +\frac{L}{2}$ and an absorbing boundary at $y = -\frac{L}{2}$ conditioned to survive forever, as will be checked in the next subsection.

(iv) the asymmetric case $[\alpha = \frac{1}{2}; \beta = \frac{3}{2}]$ is similar to (iii) since it amounts to exchange the role of the two boundaries.

I. Kemeny time for diffusion processes with killing or absorption conditioned to survive forever

For a diffusion process with killing or absorption characterized by the series of eigenvalues $\lambda_n^{[Abs]} > 0$ with $n = 1, \dots, +\infty$, the process conditioned to survive forever will have the series of eigenvalues given by Eq. 42

$$\lambda_m^{[Cond]} = \lambda_{1+m}^{[Abs]} - \lambda_1^{[Abs]} \quad \text{for } m = 1, \dots, +\infty \quad (214)$$

with the corresponding spectral Kemeny time of Eq. 44

$$\tau_*^{Spectral[Cond]} = \sum_{m=1}^{+\infty} \frac{1}{\lambda_m^{[Cond]}} = \sum_{m=1}^{+\infty} \frac{1}{\lambda_{1+m}^{[Abs]} - \lambda_1^{[Abs]}} \quad (215)$$

As simple examples, let us consider the Taboo processes already mentioned in (ii) and (iii) of the previous subsection.

1. Diffusion $D(x) = 1$ with absorbing boundaries at $x = -\frac{L}{2}$ and at $x = +\frac{L}{2}$ conditioned to survive forever

When the two boundaries at $x = \pm\frac{L}{2}$ are absorbing, the Fourier mode $r(x) = \sin[k(x + \frac{L}{2})]$ that already vanishes at $x = -\frac{L}{2}$ should also vanish at $x = +\frac{L}{2}$

$$\sin[kL] = 0 \quad (216)$$

The corresponding wavevectors and eigenvalues for this process with absorption

$$\begin{aligned} k_n^{[Abs]} &= n \frac{\pi}{L} \quad \text{for } n = 1, 2, \dots, +\infty \\ \lambda_n^{[Abs]} &= \left[k_n^{[Abs]} \right]^2 = n^2 \frac{\pi^2}{L^2} \end{aligned} \quad (217)$$

lead to the eigenvalues of Eq. 214 for the process conditioned to survive forever

$$\text{for } m = 1, \dots, +\infty \quad \lambda_m^{[Cond]} = \lambda_{1+m}^{[Abs]} - \lambda_1^{[Abs]} = \frac{\pi^2}{L^2} [(m+1)^2 - 1] = \frac{\pi^2}{L^2} (m^2 + 2m) \quad (218)$$

in agreement with the spectrum of Eq. 207 with its Kemeny time of Eq. 208.

2. Diffusion $D(x) = 1$ with reflecting boundary at $x = +\frac{L}{2}$ and absorbing boundary at $x = -\frac{L}{2}$ conditioned to survive forever

When the boundary at $x = -\frac{L}{2}$ is absorbing, while the boundary at $y = +\frac{L}{2}$ is reflecting, the Fourier mode $\sin[k(x + \frac{L}{2})]$ that vanishes at $x = -\frac{L}{2}$ should have a vanishing derivative at $x = +\frac{L}{2}$

$$\cos[kL] = 0 \quad (219)$$

The corresponding wavevectors and eigenvalues for this process with absorption

$$\begin{aligned} k_n^{[Abs]} &= \left(n - \frac{1}{2} \right) \frac{\pi}{L} \quad \text{for } n = 1, 2, \dots, +\infty \\ \lambda_n^{[Abs]} &= \left[k_n^{[Abs]} \right]^2 = \left(n - \frac{1}{2} \right)^2 \frac{\pi^2}{L^2} \end{aligned} \quad (220)$$

lead to the eigenvalues of Eq. 214 for the process conditioned to survive forever

$$\text{for } m = 1, \dots, +\infty \quad \lambda_m^{[Cond]} = \lambda_{1+m}^{[Abs]} - \lambda_1^{[Abs]} = \frac{\pi^2}{L^2} \left[\left(m + \frac{1}{2} \right)^2 - \left(\frac{1}{2} \right)^2 \right] = \frac{\pi^2}{L^2} [m^2 + m] \quad (221)$$

in agreement with the spectrum of Eq. 211 with its Kemeny time of Eq. 212.

V. IRREVERSIBLE DIFFUSION PROCESSES ON THE PERIODIC RING OF LENGTH L

Instead of the interval $]x_L, x_R[$ with two boundaries considered in the previous section, where one cannot avoid the detailed-balance, we focus in the present section on diffusion processes on the periodic ring of length L with $L + x \equiv x$ in order to have a non-equilibrium steady state with a non-vanishing steady-current along the ring and to compute the corresponding real-space Kemeny time.

A. Steady-state $p_*(x)$ and steady-current j_* along the ring

On the periodic ring, the steady-state $p_*(x)$ of the Fokker-Planck Eq. 116 corresponds to a uniform steady-current $j_*(x) = j_*$ along the ring

$$j_* = F(x)p_*(x) - D(x)p'_*(x) \quad (222)$$

It is still convenient to use the potential of Eq. 126 with reference $x_{ref} = 0$ to have $U(0) = 0$

$$U(x) \equiv - \int_0^x dy \frac{F(y)}{D(y)} \quad \text{for } x \in [0, L] \quad (223)$$

while the value $U(L)$ at $x = L$ depends on the force $F(y)$ and on the diffusion coefficient $D(y)$ on the whole ring $0 \leq y \leq L$

$$U(L) \equiv - \int_0^L dy \frac{F(y)}{D(y)} \quad (224)$$

Using $F(x) = -D(x)U'(x)$, Eq. 222 becomes

$$p'_*(x) + U'(x)p_*(x) = -\frac{j_*}{D(x)} \quad (225)$$

with the solution

$$p_*(x) = e^{-U(x)} \left[K - j_* \int_0^x dy \frac{e^{U(y)}}{D(y)} \right] \quad (226)$$

where the integration constant K is fixed by the periodicity requirement for the steady state $p_*(x=0) = p_*(x=L)$

$$K = e^{-U(L)} \left[K - j_* \int_0^L dy \frac{e^{U(y)}}{D(y)} \right] \quad (227)$$

So one needs to distinguish whether the value $U(L)$ of Eq. 224 vanishes or not.

1. Case of periodic potential $U(L) = U(0) = 0$: equilibrium steady-state with vanishing steady current $j_* = 0$

When the potential $U(L)$ of Eq. 224 vanishes $U(L) = 0$, Eq. 227 yields that the steady state current vanishes $j_* = 0$, so the detailed-balance condition is satisfied. Then the steady state reduces to the Boltzmann distribution in the potential $U(x)$ as in Eq. 125

$$p_*^{DB}(x) = \frac{e^{-U(x)}}{Z} \quad (228)$$

where the normalization Z corresponds to the partition function on the ring

$$Z = \int_0^L dx e^{-U(x)} \quad (229)$$

2. *Case of non-periodic potential $U(L) \neq U(0) = 0$: non-equilibrium steady-state with a finite steady current $j_* \neq 0$*

When the potential $U(L)$ of Eq. 224 does not vanish $U(L) \neq 0$, Eq. 227 yields the value of the constant K

$$K = j_* \frac{\int_0^L dy \frac{e^{U(y)}}{D(y)}}{1 - e^{U(L)}} \quad (230)$$

so that the steady state of Eq. 226 reads

$$p_*(x) = j_* \frac{e^{-U(x)}}{e^{-U(L)} - 1} \left[\int_0^x dy \frac{e^{U(y)}}{D(y)} + e^{-U(L)} \int_x^L dy \frac{e^{U(y)}}{D(y)} \right] \quad (231)$$

The steady-state current j_* is determined by the normalization of the steady state

$$\begin{aligned} 1 = \int_0^L dx p_*(x) &= \frac{j_*}{e^{-U(L)} - 1} \left[\int_0^L dx e^{-U(x)} \int_0^x dy \frac{e^{U(y)}}{D(y)} + e^{-U(L)} \int_0^L dx e^{-U(x)} \int_x^L dy \frac{e^{U(y)}}{D(y)} \right] \\ &= \frac{j_*}{e^{-U(L)} - 1} \int_0^L dy \frac{e^{U(y)}}{D(y)} \left[\int_y^L dx e^{-U(x)} + e^{-U(L)} \int_0^y dx e^{-U(x)} \right] \end{aligned} \quad (232)$$

leading to

$$j_* = \frac{e^{-U(L)} - 1}{\int_0^L dy \frac{e^{U(y)}}{D(y)} \left[\int_y^L dx e^{-U(x)} + e^{-U(L)} \int_0^y dx e^{-U(x)} \right]} \quad (233)$$

B. Interpretation of the non-equilibrium steady-state $p_*(x)$ via non-hermitian quantum mechanics

As explained in [152], it is interesting to analyze the meaning of the non-equilibrium steady-state $p_*(x)$ associated to the non-vanishing steady current $j_* \neq 0$ in the quantum mechanical language of Eq. 128. Equation B21 means that one should now distinguish the left eigenvector $\phi_0^{[l]}(x_0)$ and the left eigenvector $\phi_0^{[r]}(x)$ for the quantum supersymmetric Hamiltonian H of Eq. 130 131 132 133:

(i) the left eigenvector of Eq. B21 (dropping out the constant factor involving Z)

$$\phi_0^{[l]}(x) = e^{-\frac{U(x)}{2}} l_0(x) = e^{-\frac{U(x)}{2}} \quad (234)$$

is still annihilated by the operator Q of Eq. 133

$$Q\phi_0^{[l]}(x) = \sqrt{D(x)} \left(\frac{d}{dx} + \frac{U'(x)}{2} \right) e^{-\frac{U(x)}{2}} = 0 \quad (235)$$

(ii) the right eigenvector of Eq. B21 (dropping out the constant factor involving Z)

$$\begin{aligned} \phi_0^{[r]}(x) &= e^{\frac{U(x)}{2}} r_0(x) = e^{\frac{U(x)}{2}} p_*(x) \\ &= j_* \frac{e^{-\frac{U(x)}{2}}}{e^{-U(L)} - 1} \left[\int_0^x dy \frac{e^{U(y)}}{D(y)} + e^{-U(L)} \int_x^L dy \frac{e^{U(y)}}{D(y)} \right] \end{aligned} \quad (236)$$

is not annihilated by the operator Q of Eq. 133 anymore

$$Q\phi_0^{[r]}(x) = \sqrt{D(x)} \left(\frac{d}{dx} + \frac{U'(x)}{2} \right) \phi_0^{[r]}(x) = \frac{(-j_*)}{\sqrt{D(x)}} e^{\frac{U(x)}{2}} \quad (237)$$

but this state is annihilated by the operator Q^\dagger of Eq. 133

$$Q^\dagger \left(Q\phi_0^{[r]}(x) \right) = \left(-\frac{d}{dx} + \frac{U'(x)}{2} \right) \sqrt{D(x)} \left(\frac{(-j_*)}{\sqrt{D(x)}} e^{\frac{U(x)}{2}} \right) = 0 \quad (238)$$

as it should to produce the ground-state of energy zero of the Hamiltonian of Eq. 132. So even if the quantum Hamiltonian of Eq. 132 is Hermitian, the periodic boundary conditions on the Fokker-Planck right and left eigenvectors

$l_0(L) = l_0(0)$ and $r_0(L) = r_0(0)$ induces a difference between the associated left and right quantum eigenvectors of Eqs 234 and 236 that are submitted to the following different boundary conditions whenever the potential $U(\cdot)$ is not periodic $U(L) \neq U(0) = 0$

$$\begin{aligned} \frac{\phi_0^{[l]}(x=L)}{\phi_0^{[l]}(x=0)} &= e^{-\frac{U(L)}{2}} \\ \frac{\phi_0^{[r]}(x=L)}{\phi_0^{[r]}(x=0)} &= e^{+\frac{U(L)}{2}} \end{aligned} \quad (239)$$

These boundary conditions make the quantum problem non-hermitian and can be interpreted as the Aharonov-Bohm effect for an imaginary magnetic field that produces an imaginary flux through the ring (see [152] for more details). In higher dimensions $d \geq 2$, the link between non-equilibrium diffusion processes and non-hermitian quantum mechanics associated to imaginary magnetic fields is described in detail in [159].

C. Mean-First-Passage-Times $\tau(x, x_0)$

The backward equation of Eq. 141 for the Mean-First-Passage-Time $\tau(x, x_0)$ at x when starting at x_0 can be again rewritten for the auxiliary function $\chi^{[x]}(x_0)$ of Eq. 143 as Eq. 144

$$-1 + \frac{\delta(x_0 - x)}{p_*(x)} = [-U'(x_0) + \partial_{x_0}] \chi^{[x]}(x_0) \quad (240)$$

that should now be solved with the periodic boundary condition $\chi^{[x]}(x_0 = 0) = \chi^{[x]}(x_0 = L)$ so that one obtains

$$\begin{aligned} \text{for } x_0 \in [0, x]: \quad \chi^{[x]}(x_0) &= \frac{e^{U(x_0)}}{e^{-U(L)} - 1} \left[\frac{e^{-U(x)}}{p_*(x)} - e^{-U(L)} \int_0^{x_0} dz e^{-U(z)} - \int_{x_0}^L dz e^{-U(z)} \right] \\ \text{for } x_0 \in]x, L]: \quad \chi^{[x]}(x_0) &= \frac{e^{U(x_0)}}{e^{-U(L)} - 1} \left[e^{-U(L)} \frac{e^{-U(x)}}{p_*(x)} - e^{-U(L)} \int_0^{x_0} dz e^{-U(z)} - \int_{x_0}^L dz e^{-U(z)} \right] \end{aligned} \quad (241)$$

Plugging this solution into Eq. 143

$$\partial_{x_0} \tau(x, x_0) = \frac{\chi^{[x]}(x_0)}{D(x_0)} \quad (242)$$

yields the Mean-First-Passage-Time $\tau(x, x_0)$ vanishing at $x_0 = x$

$$\begin{aligned} \text{for } x_0 \in [0, x]: \quad \tau(x, x_0) &= - \int_{x_0}^x dy \frac{\chi^{[x]}(y)}{D(y)} \\ &= \frac{1}{e^{-U(L)} - 1} \left[- \frac{e^{-U(x)}}{p_*(x)} \int_{x_0}^x dy \frac{e^{U(y)}}{D(y)} + \int_{x_0}^x dy \frac{e^{U(y)}}{D(y)} \left(e^{-U(L)} \int_0^y dz e^{-U(z)} + \int_y^L dz e^{-U(z)} \right) \right] \end{aligned} \quad (243)$$

and

$$\begin{aligned} \text{for } x_0 \in [x, L]: \quad \tau(x, x_0) &= \int_x^{x_0} dy \frac{\chi^{[x]}(y)}{D(y)} \\ &= \frac{1}{e^{-U(L)} - 1} \left[e^{-U(L)} \frac{e^{-U(x)}}{p_*(x)} \int_x^{x_0} dy \frac{e^{U(y)}}{D(y)} - \int_x^{x_0} dy \frac{e^{U(y)}}{D(y)} \left(e^{-U(L)} \int_0^y dz e^{-U(z)} + \int_y^L dz e^{-U(z)} \right) \right] \end{aligned} \quad (244)$$

D. Real-space Kemeny time

The real-space Kemeny time of Eq. 2 can be evaluated for the initial position at $x_0 = 0$ using Eq. 243

$$\begin{aligned}\tau_*^{Space} &= \int_0^L dx p_*(x) \tau(x, x_0 = 0) \\ &= -\frac{1}{e^{-U(L)} - 1} \int_0^L dx e^{-U(x)} \int_0^x dy \frac{e^{U(y)}}{D(y)} + \frac{1}{e^{-U(L)} - 1} \int_0^L dx p_*(x) \int_0^x dy \frac{e^{U(y)}}{D(y)} \left[e^{-U(L)} \int_0^y dz e^{-U(z)} + \int_y^L dz e^{-U(z)} \right] \\ &= -\frac{1}{e^{-U(L)} - 1} \int_0^L dy \frac{e^{U(y)}}{D(y)} \int_y^L dx e^{-U(x)} + \frac{1}{e^{-U(L)} - 1} \int_0^L dy \frac{e^{U(y)}}{D(y)} \left[e^{-U(L)} \int_0^y dz e^{-U(z)} + \int_y^L dz e^{-U(z)} \right] \int_y^L dx p_*(x)\end{aligned}\quad (245)$$

In the last contribution, it is convenient to replace $\int_y^L dx p_*(x) = 1 - \int_0^y dx p_*(x)$ in order to obtain the final result

$$\tau_*^{Space} = \frac{1}{e^{-U(L)} - 1} \int_0^L dy \frac{e^{U(y)}}{D(y)} \left[e^{-U(L)} \left(\int_0^y dz e^{-U(z)} \right) \left(\int_y^L dx p_*(x) \right) - \left(\int_y^L dz e^{-U(z)} \right) \left(\int_0^y dx p_*(x) \right) \right] \quad (246)$$

One may also change the order of the integrals to obtain the alternative form

$$\tau_*^{Space} = \frac{1}{e^{-U(L)} - 1} \int_0^L dx p_*(x) \left[e^{-U(L)} \int_0^x dy \frac{e^{U(y)}}{D(y)} \int_0^y dz e^{-U(z)} - \int_x^L dy \frac{e^{U(y)}}{D(y)} \int_y^L dz e^{-U(z)} \right] \quad (247)$$

So once again, the real-space expressions Eqs 246–247 give explicit results for the spectral Kemeny time even if the individual eigenvalues λ_n are not known.

E. Example with diffusion coefficient $D(x) = 1$ and constant force $F(x) = \mu$ on the ring of length L

The simplest example corresponds to the diffusion coefficient $D(x) = 1$ and to the constant force $F(x) = \mu$ on the ring. Then the potential of Eq. 223 is linear

$$U(x) \equiv -\mu x \quad \text{for } x \in [0, L] \quad (248)$$

The steady current of Eq. 233 reduces to

$$j_* = \frac{\mu}{L} \quad (249)$$

and the steady state of Eq. 231 is uniform

$$p_*(x) = \frac{1}{L} \quad (250)$$

1. Real-space Kemeny time

The real-space Kemeny time of Eq. 246 reduces to

$$\begin{aligned}\tau_*^{Space} &= \frac{1}{\mu L(1 - e^{-\mu L})} \int_0^L dy [L - L e^{-\mu y} - (1 - e^{-\mu L})y] = \frac{\left(\frac{\mu L}{2} - 1\right) + e^{-\mu L} \left(\frac{\mu L}{2} + 1\right)}{\mu^2(1 - e^{-\mu L})} \\ &= \frac{1}{\mu^2} \left[\frac{\frac{\mu L}{2}}{\tanh\left(\frac{\mu L}{2}\right)} - 1 \right]\end{aligned}\quad (251)$$

For any non-vanishing $\mu \neq 0$, the Kemeny time τ_* displays the linear scaling for large L

$$\tau_*^{[\mu \neq 0]} \underset{L \rightarrow +\infty}{\simeq} \frac{L}{2|\mu|} \quad (252)$$

while for $\mu \rightarrow 0$ corresponding to detailed-balance, one recovers the diffusive scaling

$$\tau_*^{[\mu \rightarrow 0]} = \frac{L^2}{12} \quad (253)$$

The present prefactor $1/12$ corresponding to periodic boundary conditions on the ring is different from the constant $1/6$ found in Eq. 152 corresponding to reflecting boundary conditions.

2. Spectral Kemeny time

The generator \mathcal{L}_x of Eq. 113 has for right eigenvectors the periodic Fourier modes

$$r_n(x) = e^{in2\pi \frac{x}{L}} \quad \text{with } n \in \mathbb{Z} \quad (254)$$

The eigenvalues λ_n computed from

$$-\lambda_n e^{in2\pi \frac{x}{L}} = \mathcal{L}_x e^{in2\pi \frac{x}{L}} = \partial_x [-\mu + \partial_x] e^{in2\pi \frac{x}{L}} = in \frac{2\pi}{L} \left[-\mu + in \frac{2\pi}{L} \right] e^{in2\pi \frac{x}{L}} \quad (255)$$

read

$$\lambda_n = n \frac{2\pi}{L} \left[+i\mu + n \frac{2\pi}{L} \right] = \frac{4\pi^2}{L^2} \left[n^2 + in \frac{\mu L}{2\pi} \right] \quad (256)$$

Besides the vanishing eigenvalue $\lambda_{n=0} = 0$, all the other eigenvalues appear in complex-conjugate pairs

$$\lambda_{(-n)} = \frac{4\pi^2}{L^2} \left[n^2 - in \frac{\mu L}{2\pi} \right] = \lambda_n^* \quad (257)$$

The spectral Kemeny time of Eq. 5

$$\begin{aligned} \tau_*^{Spectral} &= \sum_{n=1}^{+\infty} \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_n^*} \right) = \sum_{n=1}^{+\infty} \frac{\lambda_n + \lambda_n^*}{\lambda_n \lambda_n^*} = \frac{L^2}{2\pi^2} \sum_{n=1}^{+\infty} \frac{1}{n^2 + \left(\frac{\mu L}{2\pi} \right)^2} \\ &= \frac{1}{\mu^2} \left[\frac{\frac{\mu L}{2}}{\tanh\left(\frac{\mu L}{2}\right)} - 1 \right] \end{aligned} \quad (258)$$

is in agreement with the real-space Kemeny time of Eq. 251.

3. Spectral Kemeny time in the presence of stochastic resetting at rate γ

The eigenvalues λ_n without resetting computed in Eq. 256 leads to the following non-vanishing eigenvalues of Eq. 50 for the process with resetting

$$\lambda_n^{[Reset]} = \lambda_n + \gamma = \frac{4\pi^2}{L^2} n^2 + i\mu \frac{2\pi}{L} n + \gamma \quad \text{with } n = \pm 1, \pm 2, \pm 3, \dots \quad (259)$$

that appear in complex-conjugate pairs as before in Eq. 257

$$\lambda_{(-n)}^{[Reset]} = \frac{4\pi^2}{L^2} n^2 - i\mu \frac{2\pi}{L} n + \gamma = (\lambda_n^{[Reset]})^* \quad (260)$$

The spectral Kemeny time of Eq. 53

$$\begin{aligned} \tau_*^{[Reset]Spectral} &= \sum_{n=1}^{+\infty} \left(\frac{1}{\lambda_n^{[Reset]}} + \frac{1}{\lambda_{-n}^{[Reset]}} \right) = \sum_{n=1}^{+\infty} \left(\frac{1}{\frac{4\pi^2}{L^2} n^2 + \gamma + i\mu \frac{2\pi}{L} n} + \frac{1}{\frac{4\pi^2}{L^2} n^2 + \gamma - i\mu \frac{2\pi}{L} n} \right) \\ &= -\frac{1}{\gamma} + \frac{L}{2\sqrt{\mu^2 + 4\gamma}} \left[\coth \left(L \frac{\sqrt{\mu^2 + 4\gamma} + \mu}{4} \right) + \coth \left(L \frac{\sqrt{\mu^2 + 4\gamma} - \mu}{4} \right) \right] \\ &= -\frac{1}{\gamma} + \frac{L \sinh \left(\frac{L}{2} \sqrt{\mu^2 + 4\gamma} \right)}{2\sqrt{\mu^2 + 4\gamma} \sinh \left(L \frac{\sqrt{\mu^2 + 4\gamma} + \mu}{4} \right) \sinh \left(L \frac{\sqrt{\mu^2 + 4\gamma} - \mu}{4} \right)} \end{aligned} \quad (261)$$

is in agreement with the real-space Kemeny time computed in Eq. H28 of Appendix H.

The asymptotic linear behavior for large L reads

$$\tau_*^{[Reset]Spectral} \underset{L \rightarrow +\infty}{\sim} = \frac{L}{\sqrt{\mu^2 + 4\gamma}} \quad (262)$$

So for any non-vanishing drift $\mu \neq 0$, the presence of resetting $\gamma > 0$ only changes the coefficient of asymptotic linear behavior for large L of Eq. 252.

For vanishing force $\mu = 0$, the Kemeny time of Eq. 261 reduces to

$$\begin{aligned} \mu = 0 : \quad \tau_*^{[Reset]Spectral} &= -\frac{1}{\gamma} + \frac{L \sinh(L\sqrt{\gamma})}{4\sqrt{\gamma} \sinh^2(\frac{L}{2}\sqrt{\gamma})} = -\frac{1}{\gamma} + \frac{L \cosh(\frac{L}{2}\sqrt{\gamma})}{2\sqrt{\gamma} \sinh(\frac{L}{2}\sqrt{\gamma})} \\ &= \frac{1}{\gamma} \left[\frac{\frac{L}{2}\sqrt{\gamma}}{\tanh(\frac{L}{2}\sqrt{\gamma})} - 1 \right] \end{aligned} \quad (263)$$

So the presence of resetting $\gamma > 0$ produces the asymptotic linear behavior

$$\mu = 0 : \quad \tau_*^{[Reset]Spectral} \underset{L \rightarrow +\infty}{\simeq} = \frac{L}{2\sqrt{\gamma}} \quad (264)$$

instead of the diffusive scaling of Eq. 253 for $\gamma = 0$.

As already stressed after Eq. 53, the Kemeny time does not depend on the resetting probability $\Pi(y)$. In the present example, a natural choice is the uniform resetting probability

$$\Pi(y) = \frac{1}{L} \quad \text{for } 0 \leq y \leq L \quad (265)$$

so that the steady state $p_*^{[Reset]}(x)$ with resetting of Eqs 48 49 is also uniform and coincides with the steady state $p_*(x)$ without resetting of Eq. 250

$$p_*^{[Reset]}(x) = p_*(x) = \frac{1}{L} \quad \text{for } 0 \leq x \leq L \quad (266)$$

Then the conclusion is that the convergence towards the uniform distribution is always better in the presence of resetting.

VI. DIRECTED TRAP MODEL ON THE PERIODIC RING OF N SITES

For Markov chains on the periodic ring of N sites, one can write general formula as in the previous section concerning irreversible diffusions on the ring. In order to avoid repetitions, we focus in the present section on the specific example of the Directed Trap model in order to compare with the reflecting boundary conditions considered in section III.

A. Directed Trap model with arbitrary site-dependent trapping times ϑ_x on the periodic Ring of N sites

The model is defined on a ring of L sites with periodic boundary conditions $x + L \equiv x$ with the master equation

$$\partial_t P_t(x) = \frac{P_t(x-1)}{\vartheta_{x-1}} - \frac{P_t(x)}{\vartheta_x} \quad (267)$$

So ϑ_x represents the trapping time already discussed in Eqs 105 106, but the difference with respect to the symmetric trap model discussed in Eq. 104 is that now, when the particle escapes from site x , it jumps towards the right neighbor $(x+1)$.

The steady state $P_*(x)$ at site x is simply proportional to its trapping time ϑ_x

$$P_*(x) = \frac{\vartheta_x}{\sum_{y=1}^N \vartheta_y} \quad (268)$$

as in Eq. 107 for the symmetric trap model.

The Mean-First-Passage-Time $\tau(x+1, x)$ at site $(x+1)$ when starting at x is directly the trapping time ϑ_x

$$\tau(x+1, x) = \vartheta_x \quad (269)$$

As a consequence, the Mean-First-Passage-Time $\tau(x, x_0)$ at x when starting at x_0 can be decomposed in the two regions $x > x_0$ and $x < x_0$ into the following sums that take into account the directed motion along the ring

$$\begin{aligned} \text{for } x_0 \in \{1, \dots, x-1\}: \quad \tau(x, x_0) &= \tau(x, x-1) + \tau(x-1, x-2) + \dots + \tau(x_0+1, x_0) = \sum_{z=x_0}^{x-1} \tau(z+1, z) = \sum_{z=x_0}^{x-1} \vartheta_z \\ \text{for } x_0 \in \{x+1, \dots, N\}: \quad \tau(x, x_0) &= \tau(x, x-1) + \dots + \tau(2, 1) + \tau(1, N) + \tau(N, N-1) + \dots + \tau(x_0+1, x_0) \\ &= \sum_{z=1}^{x-1} \tau(z+1, z) + \sum_{z=x_0}^N \tau(z+1, z) = \sum_{z=1}^{x-1} \vartheta_z + \sum_{z=x_0}^N \vartheta_z \end{aligned} \quad (270)$$

The real-space Kemeny time τ_*^{Space} of Eq. 2 can be thus evaluated for the special case $x_0 = 1$ using Eq. 270

$$\begin{aligned} \tau_*^{Space} &= \sum_{x=1}^N p_*(x) \tau(x, x_0 = 1) = \sum_{x=2}^N p_*(x) \tau(x, x_0 = 1) \\ &= \frac{\sum_{x=2}^N \vartheta_x \sum_{y=1}^{x-1} \vartheta_y}{\sum_{z=1}^N \vartheta_z} = \frac{\sum_{y=1}^{N-1} \sum_{x=y+1}^N \vartheta_y \vartheta_x}{\sum_{z=1}^N \vartheta_z} \end{aligned} \quad (271)$$

that is always smaller than the Kemeny time of Eq. 109 for the symmetric version of the trap model with reflecting boundary conditions involving the same trapping times. In addition, Eq. 271 can be rewritten only in terms of the sum of all the trapping times and of the sum of their squares

$$\tau_*^{Space} = \frac{\left[\sum_{y=1}^N \vartheta_y \right]^2 - \left[\sum_{y=1}^N \vartheta_y^2 \right]}{2 \sum_{z=1}^N \vartheta_z} \quad (272)$$

so that it does not depend on the positions of the trapping times along the ring as expected.

B. Special case of the pure Directed Trap model $\vartheta_x = \frac{1}{2}$ for $x = 1, \dots, N$

When the trapping times take the same value along the ring

$$\vartheta_x = \frac{1}{2} \quad \text{for } x = 1, 2, \dots, N \quad (273)$$

one obtains the Directed version of the symmetric continuous-time random walk on the interval with reflecting boundary conditions of Eq. 76.

1. Real-space Kemeny time

The steady state of Eq. 268 is uniform

$$p_*(x) = \frac{1}{N} \quad (274)$$

and coincides with the steady state of Eq. 79 for the symmetric model. The real-space Kemeny time of Eq. 271

$$\tau_*^{Space} = \frac{1}{2} \frac{\sum_{x=2}^N \sum_{z=1}^{x-1} 1}{\sum_{y=1}^N 1} = \frac{N-1}{4} \quad (275)$$

is linear with respect to the size N , in contrast to the diffusive scaling of Eq. 80 for the symmetric model.

2. Spectral Kemeny time

For the master equation of Eq. 267, the eigenvalue equation reads

$$-\lambda r(x) = 2r(x-1) - 2r(x) \quad (276)$$

On this periodic ring, the eigenvectors are simply the Fourier modes

$$r_n(x) = e^{i2\pi n \frac{x}{N}} \quad \text{with } n = 0, 1, \dots, N-1 \quad (277)$$

with the complex eigenvalues

$$\lambda_n = 2(1 - e^{-i2\pi \frac{n}{N}}) \quad \text{with } n = 0, 1, \dots, N-1 \quad (278)$$

so that the spectral Kemeny time of Eq. 5 that can be computed using Eq. I6

$$\tau_*^{Spectral} = \sum_{n=1}^{N-1} \frac{1}{\lambda_n} = \frac{1}{2} \sum_{n=1}^{N-1} \frac{1}{1 - e^{-i2\pi \frac{n}{N}}} = \frac{N-1}{4} \quad (279)$$

is in agreement with the real-space Kemeny time of Eq. 275.

3. Spectral Kemeny time in the presence of stochastic resetting at rate γ

In the presence of stochastic resetting at rate γ , the spectral Kemeny time of Eq. 279 is modified into Eq. 53 that can be computed using Eq. I7

$$\tau_*^{[Reset]Spectral} = \sum_{n=1}^{N-1} \frac{1}{\gamma + \lambda_n} = \frac{1}{2} \sum_{n=1}^{N-1} \frac{1}{(\frac{\gamma}{2} + 1) - e^{-i2\pi \frac{n}{N}}} = \frac{N}{(\gamma + 2) \left[1 - (\frac{\gamma}{2} + 1)^{-N}\right]} - \frac{1}{\gamma} \quad (280)$$

so that the behavior for large N is linear with a modified prefactor with respect to Eq. 279

$$\tau_*^{[Reset]Spectral} \underset{N \rightarrow +\infty}{\simeq} \frac{N}{\gamma + 2} \quad (281)$$

VII. GENERALIZATIONS OF THE KEMENY TIME FOR ABSORBING MARKOV PROCESSES

In subsection II H, we have considered the Kemeny times of absorbing Markov process conditioned to survive forever. In the present section, we discuss the analogs of the Kemeny time for the initial unconditioned processes.

A. Analogs of the Kemeny times for absorbing Markov processes

For the Markov process that gets absorbed in the dead configuration 0, it is interesting to analyze the analogs of the spectral Kemeny time and of the real-space Kemeny times [34, 35]:

(i) on one hand, one can introduce the analog of the spectral Kemeny time of Eq. 5

$$\tau_*^{Spectral[Abs]} \equiv \sum_{n=1}^N \frac{1}{\lambda_n^{[Abs]}} \quad (282)$$

that involves the sum of the inverse of the N absorbing-eigenvalues $\lambda_n^{[Abs]}$ associated to the propagator of Eq. 39.

(ii) on the other hand, one can introduce the analog of the real-space Kemeny time Eq. 2: however it reduces to the single term $x = 0$ associated to the absorbing steady state $P_*^{[Abs]}(x) = \delta_{x,0}$

$$\sum_x P_*^{[Abs]}(x) \tau(x, x_0) = \tau^{[Abs]}(0, x_0) \quad \text{dependent of } x_0 \quad (283)$$

so that it will depend on the initial point x_0 in contrast to the ergodic case of Eq. 2. Note that this Mean-First-Passage-Time $\tau^{[Abs]}(0, x_0)$ at the absorbing site $x = 0$ when starting at x_0 , represents the Mean-Absorption-Time where the dynamics stops.

To analyze the properties of the two times of Eqs 282 and 283 it is useful to consider the properties of the Green function when the Markov process is absorbing, and to mention the changes with respect to the properties of the Green function of ergodic cases described in section II.

B. Properties of the Green function $G^{[Abs]}(x, x_0)$ for the absorbing Markov chain

From the absorbing propagator of Eq. 39 that converge towards the absorbing state at $x = 0$

$$P_*^{[Abs]}(x) = \delta_{x,0} \quad (284)$$

one can define the absorbing Green function via the analog of Eq. 16

$$\begin{aligned} G^{[Abs]}(x, x_0) &\equiv \int_0^{+\infty} dt \left[P_t^{[Abs]}(x|x_0) - \delta_{x,0} \right] = \int_0^{+\infty} dt \left[\sum_{n=1}^N e^{-t\lambda_n^{[Abs]}} \langle x|r_n \rangle \langle l_n|x_0 \rangle \right] \\ &= \sum_{n=1}^N \frac{\langle x|r_n \rangle \langle l_n|x_0 \rangle}{\lambda_n^{[Abs]}} \end{aligned} \quad (285)$$

The forward and backward equations of Eq. 17 become

$$\begin{aligned} \sum_y w^{[Abs]}(x, y) G^{[Abs]}(y, x_0) &= \delta_{x,0} - \delta_{x,x_0} \\ \sum_{y_0} G^{[Abs]}(x, y_0) w^{[Abs]}(y_0, x_0) &= \delta_{x,0} - \delta_{x,x_0} \end{aligned} \quad (286)$$

while the orthogonality relations of Eq. 18 read

$$\begin{aligned} \text{for any } x_0 : \quad 0 &= \langle l_0 | G^{[Abs]} | x_0 \rangle = \sum_{x=0}^N G^{[Abs]}(x, x_0) \\ \text{for any } x : \quad 0 &= \langle x | G^{[Abs]} | r_0 \rangle = \sum_{x_0=0}^N G^{[Abs]}(x, x_0) \delta_{x_0,0} = G^{[Abs]}(x, 0) \end{aligned} \quad (287)$$

The second relation is obvious from the definition of the first expression Eq. 285 when the starting point is the absorbing point itself $x_0 = 0$ from which one cannot move.

Using $G^{[Abs]}(x, 0) = 0$ and the orthonormalization of Eq. 34, the trace of the Green function of Eq. 285 that can be computed as a sum of the diagonal elements $G(x, x)$ over living configurations $x \neq 0$

$$\text{Tr}(G^{[Abs]}) = \sum_{x=1}^N G^{[Abs]}(x, x) = \sum_{n=1}^N \frac{\sum_x \langle l_n | x \rangle \langle x | r_n \rangle}{\lambda_n^{[Abs]}} = \sum_{n=1}^N \frac{\langle l_n | r_n \rangle}{\lambda_n^{[Abs]}} = \sum_{n=1}^N \frac{1}{\lambda_n^{[Abs]}} = \tau_*^{Spectral[Abs]} \quad (288)$$

corresponds to the sum of the inverses of the eigenvalues introduced in Eq. 282.

C. Link with the averaged time $t_{[0,+\infty]}^{[Abs]}(x, x_0)$ spent in the living configuration x when starting at x_0

The averaged time $t_{[0,T]}^{[Abs]}(x, x_0)$ spent in the living configuration $x \neq 0$ when starting at x_0 remains finite for $T \rightarrow +\infty$ (in contrast to the extensive behavior in T of Eq. 21 for the ergodic dynamics)

$$t_{[0,+\infty]}^{[Abs]}(x, x_0) \equiv \int_0^{+\infty} dt P_t^{[Abs]}(x|x_0) = G^{[Abs]}(x, x_0) \quad \text{for } x \neq 0 \quad (289)$$

and directly corresponds to the Green function $G^{[Abs]}(x, x_0)$. It is only for the absorbing configuration $x = 0$ that one still has the extensive behavior in T of Eq. 23

$$t_{[0,T]}^{[Abs]}(0, x_0) \equiv \int_0^T dt P_t^{[Abs]}(x=0|x_0) \underset{T \rightarrow +\infty}{\simeq} T + G(0, x_0) + o(T^0) \quad (290)$$

D. Link with the Mean-First-Passage-time $\tau^{[Abs]}(0, x_0)$ at the dead configuration $x = 0$ when starting at x_0

The averaged time $t_{[0,T]}^{[Abs]}(0, x_0)$ spent in the dead configuration $x = 0$ is directly given by the difference between the total time T and the Mean-First-Passage-time $\tau^{[Abs]}(0, x_0)$

$$t_{[0,T]}^{[Abs]}(0, x_0) = T - \tau^{[Abs]}(0, x_0) \quad (291)$$

so the comparison with Eq. 290 yields the simple relation

$$\tau^{[Abs]}(0, x_0) = -G^{[Abs]}(0, x_0) \quad (292)$$

Plugging Eq. 292 into the backward Eq. 286 for $G^{[Abs]}(x = 0, y_0)$ yields the standard backward equation of Eq. 27 for the Mean-First-Passage-time $\tau^{[Abs]}(0, x_0)$

$$\sum_{y_0} \tau^{[Abs]}(0, y_0) w^{[Abs]}(y_0, x_0) = -1 \quad (293)$$

while plugging Eq. 292 into the first orthogonality relation of Eq. 287 yields that the Mean-First-Passage-time $\tau^{[Abs]}(0, x_0)$

$$\tau^{[Abs]}(0, x_0) = -G^{[Abs]}(0, x_0) = \sum_{x \neq 0} G^{[Abs]}(x, x_0) = \sum_{x \neq 0} t_{[0,+\infty[}^{[Abs]}(x, x_0) \quad (294)$$

can be rewritten as the sum of the averaged times $t_{[0,+\infty[}^{[Abs]}(x, x_0)$ of Eq. 289 spent in all the living configurations $x \neq 0$ before absorption, as expected.

E. Link with the probability $q(x|x_0)$ to visit x before absorption at 0 when starting at x_0

For the living configuration $x \neq 0$, it is useful to introduce the probability distribution $Q_{t_1}(x|x_0)$ of the first-passage time t_1 at x when starting at $x_0 \neq x$, whose normalization over $t_1 \in]0, +\infty[$

$$\int_0^{+\infty} dt_1 Q_{t_1}(x|x_0) = q(x|x_0) \in [0, 1] \quad (295)$$

represents the probability $q(x|x_0)$ to visit x before absorption at 0 when starting at x_0 .

1. Time-decomposition of the propagator $P_t^{[Abs]}(x|x_0)$ between two living distinct configurations $0 \neq x \neq x_0 \neq 0$

The propagator $P_t^{[Abs]}(x|x_0)$ can be decomposed for two living distinct configurations $0 \neq x \neq x_0 \neq 0$ with respect to the first-passage time t_1 at x via the convolution

$$P_t^{[Abs]}(x|x_0) = \int_0^t dt_1 Q_{t_1}(x|x_0) P_{t-t_1}^{[Abs]}(x|x) \quad \text{for } 0 \neq x \neq x_0 \neq 0 \quad (296)$$

The integration over time $t \in [0, +\infty[$ yields that for two different living configurations $0 \neq x \neq x_0 \neq 0$, the Green function $G^{[Abs]}(x, x_0)$ of Eq. 285

$$G^{[Abs]}(x, x_0) = \int_0^{+\infty} dt P_t^{[Abs]}(x|x_0) = \int_0^{+\infty} dt_1 Q_{t_1}(x|x_0) \int_0^{+\infty} ds P_s^{[Abs]}(x|x) = q(x|x_0) G^{[Abs]}(x, x) \quad (297)$$

reduces to the product of the probability $q(x|x_0)$ of Eq. 295 and of the Green function $G^{[Abs]}(x, x)$ at coinciding points, that will be analyzed in the next subsection.

Plugging Eq. 297 into the backward Eq. 286 for $G^{[Abs]}(x, y_0)$ with $0 \neq x \neq x_0 \neq 0$ yields the backward equation for the probability $q(x|y_0)$ to visit x before 0 when starting at y_0

$$\sum_{y_0} q(x|y_0) w^{[Abs]}(y_0, x_0) = 0 \quad (298)$$

which is indeed a standard method to compute $q(x|y_0)$ with the boundary conditions $q(x|x_0 = x) = 1$ and $q(x|x_0 = 0) = 0$ (see for instance the textbooks [57–59]).

2. Time-decomposition of the propagator $P_t^{[Abs]}(x|x)$ for a living configuration $x \neq 0$

Let us now write the analog of the decomposition of Eq. 296 for $x_0 = x \neq 0$

$$P_t^{[Abs]}(x|x) = e^{tw^{[Abs]}(x,x)} + \int_0^t dt_1 e^{t_1 w^{[Abs]}(x,x)} \sum_{y \neq x} w^{[Abs]}(y,x) \int_0^{t-t_1} dt_2 Q_{t_2}(x|y) P_{t-t_1-t_2}^{[Abs]}(x|x) \quad (299)$$

with the following physical meaning:

(1) the first term $e^{tw^{[Abs]}(x,x)}$ represents the probability to have remained in the configuration x during the time t , where the opposite of the diagonal element of Eq. 7

$$-w^{[Abs]}(x,x) \equiv \sum_{y \neq x} w^{[Abs]}(y,x) > 0 \quad (300)$$

represents the total rate out of the configuration x .

(2) the second term of Eq. 299 is the time-convolution of three functions:

(2a) the first function $\left[e^{t_1 w^{[Abs]}(x,x)} w^{[Abs]}(y,x) \right]$ represents the probability that the first escape from configuration x occurs at time t_1 towards the configuration y .

(2b) the second function $Q_{t_2}(x|y)$ represents the probability distribution of the first-passage-time t_2 at configuration x when starting at configuration y , as introduced in Eq. 295

(2c) the third function $P_{t-t_1-t_2}^{[Abs]}(x|x)$ represents the propagator from x to x in the remaining available time $(t-t_1-t_2)$.

The integration over time $t \in [0, +\infty[$ of Eq. 299 yields that the Green function $G^{[Abs]}(x,x)$ at coinciding points of Eq. 285 satisfies for $x \neq 0$ the consistency equation

$$G^{[Abs]}(x|x) = \frac{1}{[-w^{[Abs]}(x,x)]} \left[1 + \sum_{y \neq x} w^{[Abs]}(y,x) q(x|y) G^{[Abs]}(x|x) \right] \quad (301)$$

So $G^{[Abs]}(x|x)$ can be rewritten in terms of the probabilities $[1 - q(x|y)]$ of never returning to x when starting at configurations $y \neq x$ that are accessible from x via the strictly positive transition rates $w^{[Abs]}(y,x) > 0$

$$G^{[Abs]}(x|x) = \frac{1}{\sum_{y \neq x} w^{[Abs]}(y,x) [1 - q(x|y)]} \quad (302)$$

F. Conclusion for the spectral Kemeny time $\tau_*^{Spectral[Abs]}$ and the Mean-First-Passage-time $\tau^{[Abs]}(0, x_0)$

In summary, the spectral Kemeny time $\tau_*^{Spectral[Abs]}$ of Eq. 288 corresponds to the sum of $G^{[Abs]}(x|x)$ over $x \neq 0$, while the absorbing time $\tau^{[Abs]}(0, x_0)$ of Eq. 294 when starting from x_0 can be rewritten as the sum of Eq. 297 over the living configurations $x \neq 0$ as

$$\tau^{[Abs]}(0, x_0) = \sum_{x \neq 0} G^{[Abs]}(x, x_0) = \sum_{x \neq 0} G^{[Abs]}(x, x) q(x|x_0) \quad (303)$$

Since the probability $q(x|x_0)$ of Eq. 295 is smaller than unity $q(x|x_0) \leq 1$, the general conclusion is that the spectral Kemeny time $\tau_*^{Spectral[Abs]}$ is an upper bound for the absorbing time $\tau^{[Abs]}(0, x_0)$ for any starting configuration x_0 [34, 35]

$$\text{for any } x_0 : \tau^{[Abs]}(0, x_0) \leq \sum_{x \neq 0} G^{[Abs]}(x, x) = \sum_{n=1}^N \frac{1}{\lambda_n^{[Abs]}} \equiv \tau_*^{Spectral[Abs]} \quad (304)$$

This inequality becomes an equality only if the starting point x_0 satisfies $q(x|x_0) = 1$ for any other living configuration $x \neq 0$

$$\tau^{[Abs]}(0, x_0) = \tau_*^{Spectral[Abs]} \equiv \sum_{n=1}^N \frac{1}{\lambda_n^{[Abs]}} \quad \text{only if for any } x \neq 0 : q(x|x_0) = 1 \quad (305)$$

In the following subsections, we focus on such examples in the one-dimensional geometry.

G. Application to Markov chains on the interval $\{0, 1, \dots, N\}$ with absorption at $x = 0$

Let us consider the Markov chains on the interval $\{0, 1, \dots, N\}$ with absorption at $x = 0$

$$\begin{aligned}\partial_t P_t(0) &= w(0, 1)P_t(1) \\ \partial_t P_t(1) &= w(1, 2)P_t(2) - [w(0, 1) + w(2, 1)]P_t(1) \\ \partial_t P_t(x) &= w(x, x-1)P_t(x-1) + w(x, x+1)P_t(x+1) - [w(x-1, x) + w(x+1, x)]P_t(x) \quad \text{for } x = 2, \dots, N-1 \\ \partial_t P_t(N) &= w(N, N-1)P_t(N-1) - w(N-1, N)P_t(N)\end{aligned}\tag{306}$$

1. *Mean-First-Passage-time $\tau^{[Abs]}(0, x_0 = N)$ at the dead configuration $x = 0$ when starting at the other boundary $x_0 = N$*

When the starting point is $x_0 = N$, the particle cannot reach the absorbing configuration 0 without visiting all the other living configurations $x = N-1, \dots, 1$, so the probability $q(x|N)$ of Eq. 295 is unity

$$q(x|N) = 1 \quad \text{for } x = 1, 2, \dots, N-1, N\tag{307}$$

Eq. 305 then yields that the Mean-First-Passage-time $\tau^{[Abs]}(0, x_0 = N)$ when starting at $x_0 = N$ coincides with the spectral Kemeny time

$$\tau^{[Abs]}(0, x_0 = N) = \tau_*^{Spectral[Abs]} \equiv \sum_{n=1}^N \frac{1}{\lambda_n^{[Abs]}}\tag{308}$$

Besides this result concerning the Mean-First-Passage-time $\tau^{[Abs]}(0, x_0 = N)$ on which we focus here as an example of the generalization of the Kemeny time for absorbing Markov chains, let us mention that the whole probability distribution $Q_{t_1}(x=0|x_0=N)$ of the first-passage time t_1 at $x=0$ when starting at $x_0=N$, whose normalization over $t_1 \in]0, +\infty[$ of Eq. 295 is unity, has a very simple Laplace transform in terms of the N eigenvalues $\lambda_n^{[Abs]}$ (see [36] and references therein)

$$\int_0^{+\infty} dt_1 e^{-st_1} Q_{t_1}(x=0|x_0=N) = \sum_{n=1}^N \frac{\lambda_n^{[Abs]}}{s + \lambda_n^{[Abs]}}\tag{309}$$

This means that $Q_{t_1}(x=0|x_0=N)$ is the convolution of the N exponential distributions $\mathcal{E}_n(t) = \lambda_n^{[Abs]} e^{-t\lambda_n^{[Abs]}}$.

Let us now return to Eq. 308. The Mean-First-Passage-Time $\tau^{[Abs]}(0, y_0)$ can be computed as a function of the starting point y_0 via the backward Eq. 293 with the boundary condition $\tau^{[Abs]}(0, 0) = 0$

$$\begin{aligned}-1 &= [\tau^{[Abs]}(0, 2) - \tau^{[Abs]}(0, 1)] w(2, 1) + [0 - \tau^{[Abs]}(0, 1)] w(0, 1) \\ -1 &= [\tau^{[Abs]}(0, x_0 + 1) - \tau^{[Abs]}(0, x_0)] w(x_0 + 1, x_0) + [\tau^{[Abs]}(0, x_0 - 1) - \tau^{[Abs]}(0, x_0)] w(x_0 - 1, x_0) \quad \text{for } x_0 = 2, \dots, N-1 \\ -1 &= [\tau^{[Abs]}(0, N-1) - \tau^{[Abs]}(0, N)] w(N-1, N)\end{aligned}\tag{310}$$

The solution

$$\tau^{[Abs]}(0, x_0) = \sum_{x=1}^{x_0} \frac{1}{w(x-1, x)} \left[1 + \sum_{z=x}^{N-1} \prod_{y=x}^z \frac{w(y+1, y)}{w(y, y+1)} \right]\tag{311}$$

reads for $x_0 = N$

$$\tau^{[Abs]}(0, N) = \sum_{x=1}^N \frac{1}{w(x-1, x)} \left[1 + \sum_{z=x}^{N-1} \prod_{y=x}^z \frac{w(y+1, y)}{w(y, y+1)} \right]\tag{312}$$

So even if the individual eigenvalues $\lambda_n^{[Abs]}$ are not known, Eq. 311 gives an explicit expression for arbitrary transitions rates $w(., .)$ of the spectral Kemeny time $\tau_*^{Spectral[Abs]}$ of Eq. 308.

2. *Simplest example of the continuous-time random walk with rates $w(x \pm 1, x) = 1$ with absorption at 0*

Let us consider the simplest example where all the rates have the same value unity

$$\begin{aligned} w(x+1, x) &= 1 \quad \text{for } x = 1, 2, \dots, N-1 \\ w(x, x+1) &= 1 \quad \text{for } x = 0, 1, 2, \dots, N-1 \end{aligned} \quad (313)$$

Then the Mean-First-Passage-Time of Eq. 312 reduces to

$$\tau^{[Abs]}(0, N) = \sum_{x=1}^N \left[1 + \sum_{z=x}^{N-1} 1 \right] = \sum_{x=1}^N [1 + N - x] = \frac{N^2 + N}{2} \quad (314)$$

Let us now compare with the spectral analysis. For the absorbing Markov chain of Eq. 56 with the rates of Eq. 313, the eigenvalue equation reads for the right eigenvector $r(x)$ associated to the eigenvalue λ

$$\begin{aligned} x = 0 : \quad & -\lambda r(0) = r(1) \\ x = 1 : \quad & -\lambda r(1) = r(2) - 2r(1) \\ x = 2, \dots, N-1 : \quad & -\lambda r(x) = r(x-1) + r(x+1) - 2r(x) \\ x = N : \quad & -\lambda r(N) = r(N-1) - r(N) \end{aligned} \quad (315)$$

Since $\lambda = 0$ is associated to the absorbing steady state $r_0(x) = \delta_{x,0}$, let us now focus on the non-vanishing eigenvalues $\lambda \neq 0$, where the first equation simply allows to compute $r(0)$, so we can now focus on the living configurations $x = 1, \dots, N$. The bulk recurrence for $x = 2, \dots, N-1$ and the boundary condition at $x = N$ are exactly the same as in Eq. 56, so the not-normalized eigenvector $r_k(x)$ associated to the eigenvalue $\lambda(k)$ of Eq. 82 is given by Eq. 83, while the boundary equation for $x = 1$ in Eq. 315

$$0 = [2 - \lambda(k)]r_k(1) - r_k(2) = 2 \cos(k) \cos \left[k \left(N - \frac{1}{2} \right) \right] - \cos \left[k \left(N - \frac{3}{2} \right) \right] = \cos \left[k \left(N + \frac{1}{2} \right) \right] \quad (316)$$

selects the N possible wavevectors k as a function of the size N

$$k_n = \frac{-\frac{\pi}{2} + n\pi}{N + \frac{1}{2}} = \pi \frac{2n-1}{2N+1} \quad \text{with } n = 1, \dots, N \quad (317)$$

with the corresponding eigenvalues of Eq. 82

$$\lambda_n^{[Abs]} = 2[1 - \cos(k)] = 2 \left[1 - \cos \left(\pi \frac{2n-1}{2N+1} \right) \right] \quad \text{with } n = 1, \dots, N \quad (318)$$

The spectral Kemeny time of Eq. 308 that involves these N non-vanishing eigenvalues $\lambda_n^{[Abs]}$

$$\tau_*^{Spectral[Abs]} \equiv \sum_{n=1}^N \frac{1}{\lambda_n^{[Abs]}} = \frac{1}{2} \sum_{n=1}^N \frac{1}{1 - \cos \left(\pi \frac{2n-1}{2N+1} \right)} = \frac{N^2 + N}{2} \quad (319)$$

is in agreement with the Mean-First-Passage-Time $\tau^{[Abs]}(0, N)$ of Eq. 314 as it should.

H. Diffusion processes on the interval $]0, L]$ with absorption at $x = 0$ and reflecting boundary at $x = L$

1. Mean-First-Passage-time $\tau^{[Abs]}(0, x_0 = L)$ at the dead configuration $x = 0$ when starting at the other boundary $x_0 = L$

Let us now consider the analog of Eq. 308 for diffusion processes on the interval $]0, L]$ with absorption at $x = 0$ and reflecting boundary at $x = L$

$$\tau^{[Abs]}(0, x_0 = L) = \tau_*^{Spectral[Abs]} \equiv \sum_{n=1}^{+\infty} \frac{1}{\lambda_n^{[Abs]}} \quad (320)$$

where the Mean-First-Passage-Time $\tau^{[Abs]}(0, x_0 = L)$ at $x = 0$ when starting at $x_0 = L$ was written in Eq. 148

$$\tau^{[Abs]}(0, L) = \int_0^L dy \frac{1}{D(y)e^{-U(y)}} \int_y^L dz e^{-U(z)} \quad (321)$$

in terms of the diffusion coefficient $D(y)$ and in terms of the potential $U(x)$ of Eq. 126 with $x_{ref} = L$

$$\begin{aligned} U(x) &\equiv \int_x^L dy \frac{F(y)}{D(y)} \\ U'(x) &= -\frac{F(x)}{D(x)} \end{aligned} \quad (322)$$

2. Simplest example with diffusion coefficient $D(x) = 1$ and no force $F(x) = 0$

The simplest example corresponds to the diffusion coefficient $D(x) = 1$ and to the vanishing force $F(x) = 0$ leading to the vanishing potential $U(x) = 0$. Then the Mean-First-Passage-Time of Eq. 321 reduces to

$$\tau^{[Abs]}(0, L) = \int_0^L dy \int_y^L dz = \int_0^L dy (L - y) = \frac{L^2}{2} \quad (323)$$

For this diffusion with reflecting boundary condition at L and absorbing boundary condition at 0 , the eigenvalues have already been written in Eq. 220

$$\lambda_n^{[Abs]} = \left(n - \frac{1}{2}\right)^2 \frac{\pi^2}{L^2} \quad \text{for } n = 1, 2, \dots, +\infty \quad (324)$$

The corresponding spectral Kemeny time

$$\tau_*^{Spectral[Abs]} \equiv \sum_{n=1}^{+\infty} \frac{1}{\lambda_n^{[Abs]}} = \frac{L^2}{\pi^2} \sum_{n=1}^{+\infty} \frac{1}{\left(n - \frac{1}{2}\right)^2} = \frac{L^2}{2} \quad (325)$$

is in agreement with the MFTP of Eq. 323.

VIII. CONCLUSIONS

In this paper, we have first described the general properties of the Kemeny time τ_* needed to converge towards the steady state $P_*(x)$ for continuous-time Markov processes. We have then given many illustrative examples where the Kemeny time can be explicitly computed as a function of the system-size, either via its real-space definition where one needs to average the Mean-First-Passage-Time $\tau(x, x_0)$ over the final configuration x drawn with the steady state $P_*(x)$ (which turns out to be independent of the initial configuration x_0), or via its spectral definition where one needs to know all the eigenvalues $\lambda_n \neq 0$ of the opposite generator. We have considered both reversible processes satisfying detailed-balance, where the eigenvalues are real, and irreversible processes characterized by non-vanishing steady currents, where the eigenvalues can be complex. We have also emphasized the specific properties of the Kemeny times for Markov processes with stochastic resetting at constant rate. Finally, for Markov processes with killing or absorption that converge towards a dead configuration, we have analyzed the Kemeny time of the process conditioned to survive forever, while we have also discussed what happens to the analogs of the real-space Kemeny time and of the spectral Kemeny time for the initial unconditioned process.

In the illustrative examples of these various aspects, we have chosen to focus only on the simplest continuous-time Markov processes involving jumps and/or diffusion in the one-dimensional geometry, either on intervals with reflecting boundary conditions, or on rings with periodic boundary conditions. But it is of course very interesting to consider other geometries for the space of configurations:

(i) In the field of Monte-Carlo sampling, in order to accelerate the convergence towards the steady state with respect to the detailed-balance dynamics, the idea of 'lifting' consists in duplicating the configuration space into two copies with stochastic switching between the two copies and in imposing directed flows in each copy: the simplest geometry is thus a ladder, and the real-space Kemeny time has been explicitly computed, either for continuous-time Markov processes [50] or for discrete-time Markov processes [51]. Note that this type of intermittent Markov processes have been also much studied recently with many other motivations under the name "run-and-tumble" processes.

(ii) In dimensions $d > 1$, whenever the eigenvalues in dimension d can be computed from some underlying one-dimensional eigenvalues, they can be used to evaluate the spectral Kemeny time [51].

As a final remark, let us mention that some results of the present paper can be used to obtain the Kemeny times for boundary-driven one-dimensional non-equilibrium Markov dynamics [96].

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Appendix A: Relations with the spectral zeta-function and the spectral partition function

In this Appendix, we describe how the spectral Kemeny time $\tau_*^{Spectral}$ of Eq. 5 is related to the spectral zeta-function and to the spectral partition function.

1. Spectral zeta-function $\zeta^{Spectral}(s)$ and spectral partition function $\mathcal{Z}^{Spectral}(t)$

The spectral Kemeny time $\tau_*^{Spectral}$ of Eq. 5 corresponds to the special case $s = 1$

$$\tau_*^{Spectral} \equiv \sum_{n>0} \frac{1}{\lambda_n} = \zeta^{Spectral}(s=1) \quad (A1)$$

of the spectral zeta-function

$$\zeta^{Spectral}(s) \equiv \sum_{n>0} \frac{1}{\lambda_n^s} \quad (A2)$$

defined for any complex s that makes sense, as in the Riemann zeta-function corresponding to the sum of the integers $\lambda_n = n$ over $n = 1, 2, \dots, +\infty$. Such spectral zeta-functions have been much studied, in particular for Laplacians on compact manifolds and for Schrödinger operators with an infinite discrete spectrum $N = +\infty$ (see the reviews [97–99] and references therein) together with the spectral partition function

$$\mathcal{Z}^{Spectral}(t) \equiv \sum_{n>0} e^{-t\lambda_n} \quad (A3)$$

that is directly related to the spectral zeta-function of Eq. A2 via the Mellin transform

$$\zeta^{Spectral}(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} dt t^{s-1} \mathcal{Z}^{Spectral}(t) \quad (A4)$$

In particular, the spectral Kemeny time of Eq. A1 corresponds to the integral of the spectral partition function over time

$$\tau_*^{Spectral} \equiv \sum_{n>0} \frac{1}{\lambda_n} = \int_0^{+\infty} dt \mathcal{Z}^{Spectral}(t) \quad (A5)$$

2. Spectral zeta-function $\zeta^{Spectral}(s)$ for integer s in terms of Mean-First-Passage-Times

It is interesting to consider the spectral zeta function $\zeta^{Spectral}(s)$ of Eq. A2 when s is an integer $s = 2, 3, \dots$ higher than $s = 1$ corresponding to the Kemeny time $\tau_* = \zeta^{Spectral}(s=1)$ to better understand their physical meanings in terms of Mean-First-Passage-Times. For $s = 2$, Eq. A6 corresponds to the trace of the square G^2 of the Green matrix G and can be evaluated using Eq. 26 and the Kemeny time $\tau_* = \text{Trace}(G)$

$$\begin{aligned} \zeta^{Spectral}(s=2) &= \sum_{n>0} \frac{1}{\lambda_n^2} = \text{Trace}(G^2) = \sum_{x=1}^N \sum_{y=1}^N G(x, y) G(y, x) = \sum_{x=1}^N \sum_{y=1}^N [G(x, x) - P_*(x)\tau(x, y)] [G(y, y) - P_*(y)\tau(y, x)] \\ &= \sum_{x=1}^N \sum_{y=1}^N [G(x, x)G(y, y) - P_*(x)\tau(x, y)G(y, y) - P_*(y)\tau(y, x)G(x, x) + P_*(x)\tau(x, y)P_*(y)\tau(y, x)] \\ &= \sum_{x=1}^N \sum_{y=1}^N P_*(x)P_*(y)\tau(x, y)\tau(y, x) - \tau_*^2 \end{aligned} \quad (A6)$$

So $\zeta^{Spectral}(s=2)$ contains the information on the average of the product of the two MFPT $\tau(x, y)$ and $\tau(y, x)$ when both configurations x and y are drawn with the steady state distribution $P_*(.)$.

This computation can be generalized for any higher integer $s = 3, 4, ..$

$$\begin{aligned} \text{Integer } s : \quad \zeta^{Spectral}(s) &= \sum_{n>0} \frac{1}{\lambda_n^s} = \text{Trace}(G^s) = \sum_{x_1} \sum_{x_2} \dots \sum_{x_s} G(x_s, x_{s-1}) \dots G(x_2, x_1) G(x_1, x_s) \\ &= (-1)^s \left[\sum_{x_1} \sum_{x_2} \dots \sum_{x_s} \left[\prod_{k=1}^s P_*(x_k) \right] \tau(x_s, x_{s-1}) \dots \tau(x_2, x_1) \tau(x_1, x_s) - \tau_*^s \right] \end{aligned} \quad (\text{A7})$$

So $\zeta^{Spectral}(s)$ for integer s contains the information on the average of the product of the s MFPT along a cycle of s configurations $(x_1, .., x_s)$ when these s configurations $(x_1, .., x_s)$ are all drawn with the steady state distribution $P_*(.)$.

3. Spectral partition function $\mathcal{Z}(t)$ for the convergence of the propagator towards the steady state

It is also interesting to mention the physical interpretation of the spectral partition function of Eq. A3 via the computation of the following observable measuring the convergence of the propagator $P_t(.|.)$ towards the steady state $P_*(.)$ for two configurations (x, y)

$$\begin{aligned} \sum_x \sum_y P_*(x) P_*(y) \left[\frac{P_{\frac{t}{2}}(x|y)}{P_*(x)} - 1 \right] \left[\frac{P_{\frac{t}{2}}(y|x)}{P_*(y)} - 1 \right] &= \sum_x \sum_y \left[P_{\frac{t}{2}}(x|y) - P_*(x) \right] \left[P_{\frac{t}{2}}(y|x) - P_*(y) \right] \\ &= \sum_x \sum_y \left[P_{\frac{t}{2}}(x|y) P_{\frac{t}{2}}(y|x) - P_*(x) P_{\frac{t}{2}}(y|x) - P_{\frac{t}{2}}(x|y) P_*(y) + P_*(x) P_*(y) \right] \\ &= \sum_x P_t(x|x) - 1 = \text{Trace}(e^{wt}) - 1 = \sum_{n>0} e^{-t\lambda_n} = \mathcal{Z}(t) \end{aligned} \quad (\text{A8})$$

As a consequence, the Kemeny time $\tau_* = \zeta(s=1)$ of Eq. A4 also corresponds to the integration over time $t \in [0, +\infty[$ of Eq. A8

$$\begin{aligned} \tau_* &= \int_0^{+\infty} dt \mathcal{Z}(t) = \int_0^{+\infty} dt \sum_x \sum_y P_*(x) P_*(y) \left[\frac{P_{\frac{t}{2}}(x|y)}{P_*(x)} - 1 \right] \left[\frac{P_{\frac{t}{2}}(y|x)}{P_*(y)} - 1 \right] \\ &= \int_0^{+\infty} dt \sum_x \sum_y \left[P_{\frac{t}{2}}(x|y) - P_*(x) \right] \left[P_{\frac{t}{2}}(y|x) - P_*(y) \right] \end{aligned} \quad (\text{A9})$$

Appendix B: Specific spectral properties of reversible Markov chains satisfying detailed-balance

In this Appendix, we focus on reversible Markov chains satisfying detailed-balance in order to recall their very specific spectral properties.

1. Markov matrix w satisfying detailed-balance in terms of the potential $U(x)$ and $D(x, y) = \sqrt{w(x, y)w(y, x)}$

The detailed-balance condition means that for any pair of configurations (x, y) , the steady flow $\mathcal{F}(x, y) \equiv w(x, y)P_*(y)$ from y to x is equal to the steady flow $\mathcal{F}(y, x) \equiv w(y, x)P_*(x)$ from x to y

$$\mathcal{F}(x, y) \equiv w(x, y)P_*(y) = w(y, x)P_*(x) \equiv \mathcal{F}(y, x) \quad (\text{B1})$$

so that the steady current corresponding to the difference of the two flows vanishes on all links. It is convenient to write the steady state as the Boltzmann equilibrium in the potential $U(x)$

$$P_*(x) = \frac{e^{-U(x)}}{Z} \quad (\text{B2})$$

with the normalization given by the partition function

$$Z = \sum_x e^{-U(x)} \quad (\text{B3})$$

On a given link (x, y) , since the ratio of the two transition rates is fixed by the detailed-balance condition of Eq. B1

$$\frac{w(x, y)}{w(y, x)} = \frac{P_*(x)}{P_*(y)} = e^{U(y)-U(x)} \quad (\text{B4})$$

it is useful to introduce the symmetric positive function

$$D(x, y) = \sqrt{w(x, y)w(y, x)} = D(y, x) \quad (\text{B5})$$

that can be considered as the diffusion coefficient associated to the link (x, y) , in order to rewrite the transition rates as

$$w(x, y) = D(x, y) \sqrt{\frac{P_*(x)}{P_*(y)}} = D(x, y) e^{-\frac{U(x)-U(y)}{2}} \quad (\text{B6})$$

The two equal flows of Eq. B1 on the link (x, y) correspond to the symmetric function

$$\mathcal{F}(x, y) = \mathcal{F}(y, x) = w(x, y)P_*(y) = D(x, y)\sqrt{P_*(x)P_*(y)} = D(x, y) \frac{e^{-\frac{U(x)+U(y)}{2}}}{Z} \quad (\text{B7})$$

The diagonal element of Eq. 7 reads

$$w(x, x) \equiv - \sum_{y \neq x} w(y, x) = - \sum_{y \neq x} D(x, y) e^{-\frac{U(y)-U(x)}{2}} \quad (\text{B8})$$

The parametrization of the Markov matrix elements of Eqs B6 and B8 allows to rewrite the Markov matrix using bra and ket notations as a sum over links

$$w = - \sum_y \sum_{x > y} D(x, y) e^{-\frac{U(x)+U(y)}{2}} [|x\rangle - |y\rangle] \left[e^{U(x)} \langle x| - e^{U(y)} \langle y| \right] \quad (\text{B9})$$

where it is now obvious on each link contribution that the left and right eigenvectors of Eq. 12 are given by Eq. 13 and by Eq. 14 with Eq. B2.

The form of Eq. B9 that involves the the potential $U(\cdot)$ and the function $D(\cdot, \cdot)$ computed from the initial Markov matrix elements via Eqs B4 B5, can be also rewritten in terms of the steady state of Eq. B2 and the link steady flows of Eq. B7 as

$$w = - \sum_y \sum_{x > y} \mathcal{F}(x, y) [|x\rangle - |y\rangle] \left[\frac{\langle x|}{P_*(x)} - \frac{\langle y|}{P_*(y)} \right] \quad (\text{B10})$$

2. Similarity transformation towards a quantum Hermitian Hamiltonian H

The change of variables involving the steady state $P_*(\cdot)$ of Eq. B2 or the potential $U(\cdot)$

$$P_t(x|x_0) = \sqrt{\frac{P_*(x)}{P_*(x_0)}} \psi_t(x|x_0) = e^{-\frac{U(x)}{2}} \psi_t(x|x_0) e^{\frac{U(x_0)}{2}} \quad (\text{B11})$$

transforms the forward dynamics of Eq. 9 for the propagator $P_t(x, x_0)$ into the following Euclidean Schrödinger equation for the quantum propagator $\psi_t(x|x_0)$

$$-\partial_t \psi_t(x|x_0) = \sum_y H(x, y) \psi_t(y|x_0) \quad (\text{B12})$$

where the matrix elements of the quantum Hermitian Hamiltonian H are obtained from the Markov matrix w via the similarity transformation

$$H(x, y) = -e^{\frac{U(x)}{2}} w(x, y) e^{-\frac{U(y)}{2}} \quad (\text{B13})$$

i.e. at the matrix level using bra and ket notations, Eq. B9 translates

$$\begin{aligned} H &= \sum_y \sum_{x > y} D(x, y) e^{-\frac{U(x)+U(y)}{2}} \left[e^{\frac{U(x)}{2}} |x\rangle - e^{\frac{U(y)}{2}} |y\rangle \right] \left[e^{\frac{U(x)}{2}} \langle x| - e^{\frac{U(y)}{2}} \langle y| \right] \\ &\equiv \sum_y \sum_{x > y} |Q_{x,y}\rangle \langle Q_{x,y}| \end{aligned} \quad (\text{B14})$$

into a sum over links of the not-normalized projectors ($|Q_{x,y}\rangle\langle Q_{x,y}|$), where the elementary ket $|Q_{x,y}\rangle$ localized on the two sites (x, y) can be also rewritten in terms of the transition rates of Eq. B6 or in terms of steady state of Eq. B2 and the link steady flow of Eq. B7

$$\begin{aligned} |Q_{x,y}\rangle &= \sqrt{D(x,y)} e^{-\frac{U(x)+U(y)}{2}} \left[e^{\frac{U(x)}{2}} |x\rangle - e^{\frac{U(y)}{2}} |y\rangle \right] \\ &= \sqrt{w(y,x)} |x\rangle - \sqrt{w(x,y)} |y\rangle \\ &= \sqrt{\mathcal{F}(x,y)} \left[\frac{|x\rangle}{\sqrt{P_*(x)}} - \frac{|y\rangle}{\sqrt{P_*(y)}} \right] \end{aligned} \quad (\text{B15})$$

with the corresponding different expressions for its norm

$$\begin{aligned} \langle Q_{x,y} | Q_{x,y} \rangle &= D(x,y) \left[e^{\frac{U(x)-U(y)}{2}} + e^{\frac{U(y)-U(x)}{2}} \right] \\ &= w(y,x) + w(x,y) \\ &= \mathcal{F}(x,y) \left[\frac{1}{P_*(x)} + \frac{1}{P_*(y)} \right] \end{aligned} \quad (\text{B16})$$

3. Consequences for the spectral properties of the Markov matrix satisfying detailed-balance

The similarity transformation of Eq. B13 yields that the N eigenvalues λ_n of $[-w]$ are the eigenvalues of the hermitian Hamiltonian H . So the $(N-1)$ non-vanishing eigenvalues λ_n for $n \in \{1, \dots, N-1\}$ are real and positive

$$\lambda_n \in]0, +\infty[\quad \text{for } n \in \{1, \dots, N-1\} \quad (\text{B17})$$

The corresponding orthonormalized basis of eigenvectors $|\phi_n\rangle$

$$\begin{aligned} H|\phi_n\rangle &= \lambda_n |\phi_n\rangle \\ \langle \phi_n | \phi_m \rangle &\equiv \delta_{n,m} \end{aligned} \quad (\text{B18})$$

appears in the spectral decomposition of the quantum propagator

$$\psi_t(x|x_0) \equiv \langle x | e^{-tH} | x_0 \rangle = \sum_{n=0}^{N-1} e^{-t\lambda_n} \langle x | \phi_n \rangle \langle \phi_n | x_0 \rangle \quad (\text{B19})$$

that translated via Eq. B11 into the following spectral decomposition for the propagator

$$P_t(x|x_0) = \sqrt{\frac{P_*(x)}{P_*(x_0)}} \psi_t(x|x_0) = \sum_{n=0}^{N-1} e^{-t\lambda_n} \left[\sqrt{P_*(x)} \langle x | \phi_n \rangle \right] \left[\frac{\langle \phi_n | x_0 \rangle}{\sqrt{P_*(x_0)}} \right] \quad (\text{B20})$$

The comparison with Eq. 35 allows to compute the right and left eigenvectors of $[-w]$ in terms of the eigenvectors ϕ_n of the quantum Hamiltonian H

$$\begin{aligned} \langle x | r_n \rangle &= \sqrt{P_*(x)} \langle x | \phi_n \rangle = \frac{e^{-\frac{U(x)}{2}}}{\sqrt{Z}} \langle x | \phi_n \rangle \\ \langle l_n | x_0 \rangle &= \frac{\langle \phi_n | x_0 \rangle}{\sqrt{P_*(x_0)}} = \langle \phi_n | x_0 \rangle \frac{e^{\frac{U(x_0)}{2}}}{\sqrt{Z}} \end{aligned} \quad (\text{B21})$$

In particular for $n = 0$, the quantum ground-state associated to the zero eigenvalue $\lambda = 0$ is the square-root of the steady state $P_*(x)$

$$\phi_0(x) = \sqrt{P_*(x)} = \frac{e^{-\frac{U(x)}{2}}}{\sqrt{Z}} \quad (\text{B22})$$

4. Consequence of detailed-balance for the spectral partition function $\mathcal{Z}(t)$ and for the Kemeny time τ_*

At the level of finite-time propagators, the detailed-balance condition of Eq. B1 translates into the property

$$P_t(x|x_0)P_*(x_0) = P_t(x_0|x)P_*(x) \quad (\text{B23})$$

As a consequence, the observable of Eq. A8 corresponding to the spectral partition function can be rewritten as

$$\begin{aligned} \mathcal{Z}(t) &= \sum_x \sum_{x_0} P_*(x)P_*(x_0) \left[\frac{P_{\frac{t}{2}}(x|x_0)}{P_*(x)} - 1 \right] \left[\frac{P_{\frac{t}{2}}(x_0|x)}{P_*(x_0)} - 1 \right] \\ &= \sum_x \sum_{x_0} P_*(x)P_*(x_0) \left[\frac{P_{\frac{t}{2}}(x|x_0)}{P_*(x)} - 1 \right]^2 \\ &= \sum_{x_0} P_*(x_0) \frac{\left[P_{\frac{t}{2}}(x|x_0) - P_*(x) \right]^2}{P_*(x)} \equiv \sum_{x_0} P_*(x_0) \chi_{x_0}^2(t) \end{aligned} \quad (\text{B24})$$

i.e. it corresponds to the average over the initial position x_0 drawn with the steady state $P_*(x_0)$ of the chi-squared function

$$\chi_{x_0}^2(t) \equiv \frac{\left[P_{\frac{t}{2}}(x|x_0) - P_*(x) \right]^2}{P_*(x)} \quad (\text{B25})$$

that measures the convergence of the propagator $P_{\frac{t}{2}}(x|x_0)$ towards the steady state $P_*(x)$ as a function of the time $\frac{t}{2}$ for a given initial configuration x_0 . Then the Kemeny time $\tau_* = \zeta(s=1)$ of Eq. A4 also corresponds to the integration over time $t \in [0, +\infty[$ of Eq. B24

$$\begin{aligned} \tau_* &= \int_0^{+\infty} dt \mathcal{Z}(t) = \int_0^{+\infty} dt \sum_x \sum_{x_0} P_*(x)P_*(x_0) \left[\frac{P_{\frac{t}{2}}(x|x_0)}{P_*(x)} - 1 \right]^2 \\ &= \int_0^{+\infty} dt \sum_{x_0} P_*(x_0) \frac{\left[P_{\frac{t}{2}}(x|x_0) - P_*(x) \right]^2}{P_*(x)} \end{aligned} \quad (\text{B26})$$

Appendix C: Statistical properties of time-additives observables during $[0, T]$ in terms of the Green function

In this Appendix, we recall how the Green function described in section II allows to analyze the statistical properties of all the time-additives observables of the Markov process.

1. Time-additive observables of trajectories that can be reconstructed from the empirical density

For each Markov trajectory $x(0 \leq t \leq T)$, the empirical density $\rho_{[0,T]}(x, x_0)$ represents the fraction of the time spent at configuration x during the time-window $[0, T]$ when starting at configuration x_0

$$\rho_{[0,T]}(x, x_0) \equiv \frac{1}{T} \int_0^T dt \delta_{x(t), x} \quad (\text{C1})$$

It plays an essential role since it allows to reconstruct any time-additive observable $\mathcal{O}_{[0,T]}(x_0)$ parametrized by some function $O(x)$

$$\mathcal{O}_{[0,T]}(x_0) \equiv \frac{1}{T} \int_0^T dt O(x(t)) = \sum_{x=1}^N O(x) \rho_{[0,T]}(x, x_0) \quad (\text{C2})$$

2. Role of the Green function for the convergence of the averaged values of time-additive observables

The averaged value of the empirical density of Eq. C1 over the trajectories $x(0 \leq t \leq T)$ corresponds to the averaged time $t_{[0,T]}(x, x_0)$ of Eq. 290 divided by T

$$\langle \rho_{[0,T]}(x, x_0) \rangle_{Traj} \equiv \frac{t_{[0,T]}(x, x_0)}{T} \underset{T \rightarrow +\infty}{\simeq} P_*(x) + \frac{G(x, x_0)}{T} + o\left(\frac{1}{T}\right) \quad (C3)$$

This allows to obtain the averaged values of the time-additive observables of Eq. C2

$$\langle \mathcal{O}_{[0,T]}(x_0) \rangle_{Traj} = \sum_{x=1}^N O(x) \langle \rho_{[0,T]}(x, x_0) \rangle_{Traj} \underset{T \rightarrow +\infty}{\simeq} \sum_{x=1}^N O(x) P_*(x) + \frac{1}{T} \left[\sum_{x=1}^N O(x) G(x, x_0) \right] + o\left(\frac{1}{T}\right) \quad (C4)$$

So the Green function $G(x, x_0)$ governs the convergence of order $1/T$ towards the value O_* computed from the steady state $P_*(x)$

$$O_* \equiv \sum_{x=1}^N O(x) P_*(x) \quad (C5)$$

3. Role of the Green function for the large-time variances of time-additive observables

The variance of the time-additive observable $\mathcal{O}_{[0,T]}(x_0)$ of Eq. C2 can be also computed at the leading order $1/T$ in terms of the Green function G (see [50] for more detailed calculations):

$$\langle \mathcal{O}_{[0,T]}^2(x_0) \rangle_{Traj} - \langle \mathcal{O}_{[0,T]}(x_0) \rangle_{Traj}^2 \underset{T \rightarrow +\infty}{\simeq} \frac{2}{T} \sum_x \sum_y O(x) G(x, y) O(y) P_*(y) \quad (C6)$$

For the special case of the function $O(x) = \delta_{x,a}$ that corresponds in Eq. C2 to the empirical density $\rho_{[0,T]}(a, x_0)$ at configuration a , Eq. C6 yields that the variance of the empirical density $\rho_{[0,T]}(a, x_0)$ at configuration a involves the Green function $G(a, a)$ at coinciding points

$$\langle \rho_{[0,T]}^2(a, x_0) \rangle_{Traj} - \langle \rho_{[0,T]}(a, x_0) \rangle_{Traj}^2 \underset{T \rightarrow +\infty}{\simeq} \frac{2G(a, a)P_*(a)}{T} \quad (C7)$$

As a consequence, the Kemeny time τ_* corresponding to the trace of the Green function of Eq. 30 can be recovered if one computes

$$\frac{T}{2} \sum_{x=1}^N \frac{\langle \rho_{[0,T]}^2(x, x_0) \rangle_{Traj} - \langle \rho_{[0,T]}(x, x_0) \rangle_{Traj}^2}{P_*(x)} \underset{T \rightarrow +\infty}{\simeq} \sum_{x=1}^N G(x, x) = \tau_* \quad (C8)$$

4. Discussion

The Green function also allows to analyze the averaged value and the variance of more general time-additive observables than Eq. C2 that involve not only the time spent in the various configurations but also all the jumps between two different configurations that occur during the time-window $[0, T]$, as described in [56]. Beyond the averaged values of Eq. C5 and the variance of Eq. C6, the Green function allows to analyze also the higher cumulants, for instance of order three or four. However if one wishes to study the whole series of higher cumulants, it is better to use the theory of large deviations (see the reviews [100–102] and references therein) that has become the unifying language in the field of non-equilibrium processes (see the reviews with different scopes [103–111], the PhD Theses [112–117] and the Habilitation Thesis [118]). In particular, many works have been devoted recently to the large deviations properties of many interesting time-additive observables for various Markov processes [56, 103, 108–111, 113, 118–160].

Appendix D: Parametrization of one-dimensional diffusion processes with two functions

The Fokker-Planck Eq. 116 for one-dimensional diffusion processes involves two functions: the force $F(x)$ and the diffusion coefficient $D(x)$. In this Appendix, we describe various other choices for the two functions that can be chosen to parametrize diffusion processes.

1. Replacing the Fokker-Planck force $F(x)$ by the Stratonovich force $f^S(x)$ or the Ito force $f^I(x)$

When the diffusion coefficient $D(x)$ depends explicitly on the position x , the corresponding Langevin Stochastic Differential Equation involving a Brownian motion $B(t)$ will depend on the Stratonovich or the Ito interpretations

$$\begin{aligned} dx(t) &= f^S(x(t)) dt + \sqrt{2D(x(t))} dB(t) && [\text{Stratonovich Interpretation}] \\ dx(t) &= f^I(x(t)) dt + \sqrt{2D(x(t))} dB(t) && [\text{Ito Interpretation}] \end{aligned} \quad (\text{D1})$$

where the Stratonovich force $f^S(x)$ and the Ito force $f^I(x)$ have to be computed from the Fokker-Planck force $F(x)$ and the diffusion coefficient $D(x)$ via

$$\begin{aligned} f^S(x) &= F(x) + \frac{D'(x)}{2} \\ f^I(x) &= F(x) + D'(x) \end{aligned} \quad (\text{D2})$$

The mathematical literature is always written in terms of the Ito force $f^I(x)$, so that the adjoint operator of Eq. D3 reads

$$\mathcal{L}_{x_0}^\dagger = [F(x_0) + \partial_{x_0} D(x_0)] \partial_{x_0} = f^I(x_0) \partial_{x_0} + D(x_0) \partial_{x_0}^2 \quad (\text{D3})$$

Note that when the Fokker-Planck force vanishes $F(x) = 0$, the operator \mathcal{L} and its adjoint \mathcal{L}^\dagger coincide

$$\text{when } F(x) = 0 : \mathcal{L}_x = \partial_x D(x) \partial_x = \mathcal{L}_x^\dagger \quad (\text{D4})$$

and the corresponding steady state is uniform $p_*(x) = \text{cst}$. This explains why the Fokker-Planck force $F(x)$ is often more convenient from a physical point of view.

2. Replacing the Fokker-Planck force $F(x)$ by the potential $U(x)$ with derivative $U'(x) = -\frac{F(x)}{D(x)}$

As explained in the main text, it is convenient to replace the Fokker-Planck force $F(x)$ by the potential $U(x)$ of Eq. 126 both in the reversible and in the irreversible cases:

(i) In the reversible diffusion processes satisfying detailed-balance discussed in section IV, it is convenient to further replace the potential $U(x)$ by the normalized steady state $p_*(x) = \frac{e^{-U(x)}}{Z}$ of Eq. 125. Then the diffusion process is characterized by its steady state $p_*(x)$ and its diffusion coefficient $D(x)$, that are the two functions that we have chosen to write the real-space Kemeny time of Eq. 150. Note that in the mathematical literature (see [84] for instance), the non-normalized steady state is called "the speed density $m(x)$ ", while the other function defining the process is called "the scale density $s(x) \propto \frac{1}{D(x)m(x)}$ ".

(ii) In the irreversible diffusion processes on the ring discussed in section V, then the steady state $p_*(x)$ is given by the non-equilibrium expression of Eq. 231, and it is thus more convenient to keep the potential $U(x)$ and the diffusion coefficient $D(x)$ as the two functions defining the non-equilibrium diffusion process.

Appendix E: Eigenvalues for the Diffusion $D(x) = 1$ with the constant force $F(x) = \mu$ on the interval $[0, L]$

In this Appendix, we describe the eigenvalues for the Diffusion $D(x) = 1$ with the constant force $F(x) = \mu$ on the interval $[0, L]$ for various boundary conditions.

1. Similarity transformation towards the free quantum Hamiltonian

For the diffusion coefficient $D(x) = 1$ and the constant force $F(x) = \mu$, the potential $U(x)$ of Eq. 126 is linear

$$U(x) = -\mu x \quad (\text{E1})$$

while the quantum potential of Eq. 131 reduces to the constant

$$V(x) = \frac{F^2(x)}{4} + \frac{F'(x)}{2} = \frac{\mu^2}{4} \quad (\text{E2})$$

The eigenvalue Eq. B19 for the quantum Hamiltonian of Eq. 130 with the potential of Eq. F1 reduces to

$$\lambda\phi(x) = H\phi(x) = -\phi''(x) + \frac{\mu^2}{4}\phi(x) \quad (\text{E3})$$

while the zero-current boundary condition of Eqs 137-139 at $x = L$ reads

$$0 = \frac{\mu}{2}\phi(L) - \phi'(L) \quad (\text{E4})$$

As a consequence, the solution $\phi(x)$ can be written as the linear combination

$$\phi(x) = \phi(L) \left(\cos[k(L-x)] - \frac{\mu}{2k} \sin[k(L-x)] \right) \quad (\text{E5})$$

in terms of the wavevector

$$k \equiv \sqrt{\lambda - \frac{\mu^2}{4}} \quad (\text{E6})$$

associated to the eigenvalue

$$\lambda = \frac{\mu^2}{4} + k^2 \quad (\text{E7})$$

Now one needs to impose the boundary condition at $x = 0$ to determine the possible wavevectors k and the corresponding eigenvalues λ . In the two following subsections, we consider two different boundary conditions.

2. Eigenvalues for the reflecting boundary condition at $x = 0$

Let us impose the zero-current boundary condition of Eqs 137-139 at $x = 0$ to the solution of Eq. E5

$$\begin{aligned} 0 &= \frac{\mu}{2}\phi(0) - \phi'(0) = \frac{\mu}{2}\phi(L) \left(\cos(kL) - \frac{\mu}{2k} \sin(kL) \right) - \phi(L) \left(k \sin(kL) + \frac{\mu}{2} \cos(kL) \right) \\ &= -\frac{\phi(L)}{k} \left(\frac{\mu^2}{4} + k^2 \right) \sin(kL) \end{aligned} \quad (\text{E8})$$

So apart from the vanishing eigenvalue $\lambda = 0$ in Eq. E7 corresponding to $k = i\frac{\mu}{2}$, the other wavevectors should satisfy

$$\sin(kL) = 0 \quad (\text{E9})$$

and thus read

$$k_n = n \frac{\pi}{L} \quad \text{with } n = 1, 2, \dots, +\infty \quad (\text{E10})$$

with the corresponding excited eigenvalues

$$\lambda_n = \frac{\mu^2}{4} + k_n^2 = \frac{\mu^2}{4} + n^2 \frac{\pi^2}{L^2} \quad \text{with } n = 1, 2, \dots, +\infty \quad (\text{E11})$$

that are used to compute the spectral Kemeny time given in Eq. 172 of the main text.

3. Eigenvalues for the absorbing boundary condition at $x = 0$

Let us impose the absorbing boundary condition at $x = 0$ to the solution of Eq. E5

$$0 = \phi(0) = \phi(L) \left(\cos(kL) - \frac{\mu}{2k} \sin(kL) \right) \quad (\text{E12})$$

So the wavevectors $k_n^{[abs]}$ are the solutions of

$$\tan(k_n^{[Abs]}L) = \frac{2}{\mu} k_n^{[Abs]} \quad (\text{E13})$$

and will give the absorbing eigenvalues

$$\lambda_n^{[Abs]} = \frac{\mu^2}{4} + \left[k_n^{[Abs]} \right]^2 \quad (\text{E14})$$

Appendix F: Eigenvalues for the diffusion in the saw-tooth potential on the interval $[0, L]$ of subsection IV G

In this Appendix, we analyze the eigenvalues for the diffusion in the saw-tooth potential $U(x)$ on the interval $[0, L]$ considered in section IV G of the main text.

1. Spectral analysis of the quantum Hamiltonian associated to the saw-tooth potential

The quantum potential of Eq. 131 associated to the diffusion coefficient $D(x) = 1$ and to the force of Eq. 160

$$V(x) = \frac{F^2(x)}{4} + \frac{F'(x)}{2} = \frac{\mu^2}{4} + \mu\delta(x - \rho L) \quad (\text{F1})$$

reduces to the constant $\frac{\mu^2}{4}$ and to a delta impurity of strength μ located at $x = \rho L$. The eigenvalue Eq. B19 for the quantum Hamiltonian of Eq. 130 with the potential of Eq. F1 reads

$$\lambda\phi(x) = H\phi(x) = -\phi''(x) + \left[\frac{\mu^2}{4} + \mu\delta(x - \rho L)\right]\phi(x) \quad (\text{F2})$$

with the zero-current boundary conditions of Eqs 137–139 at $x = 0$ and $x = L$

$$\begin{aligned} 0 &= -\frac{U'(0)}{2}\phi(0) - \phi'(0) = -\frac{\mu}{2}\phi(0) - \phi'(0) \\ 0 &= -\frac{U'(L)}{2}\phi(L) - \phi'(L) = \frac{\mu}{2}\phi(L) - \phi'(L) \end{aligned} \quad (\text{F3})$$

As a consequence, the solution $\phi(x)$ can be written in the two regions as

$$\begin{aligned} \phi(x) &= \phi(0) \left(\cos[kx] - \frac{\mu}{2k} \sin[kx] \right) \quad \text{for } 0 \leq x \leq \rho L \\ \phi(x) &= \phi(L) \left(\cos[k(L-x)] - \frac{\mu}{2k} \sin[k(L-x)] \right) \quad \text{for } \rho L \leq x \leq L \end{aligned} \quad (\text{F4})$$

in terms of the wavevector $k \equiv \sqrt{\lambda - \frac{\mu^2}{4}}$ of Eq. E6 and in terms of the two constants $\phi(0)$ and $\phi(L)$ that can be determined as follows. The delta function $\delta(x - \rho L)$ in Eq. F2 leads to the following continuity condition for $\phi(x)$ at $x = \rho L \pm \epsilon$

$$0 = \phi(\rho L - \epsilon) - \phi(\rho L + \epsilon) = \phi(0) \left(\cos[k\rho L] - \frac{\mu}{2k} \sin[k\rho L] \right) - \phi(L) \left(\cos[k(1-\rho)L] - \frac{\mu}{2k} \sin[k(1-\rho)L] \right) \quad (\text{F5})$$

and to the following discontinuity condition for $\phi'(x)$ at $x = \rho L \pm \epsilon$

$$\begin{aligned} 0 &= \phi'(\rho L - \epsilon) - \phi'(\rho L + \epsilon) + \mu\phi(\rho L) = \phi(0) \left(-k \sin[k\rho L] - \frac{\mu}{2} \cos[k\rho L] \right) - \phi(L) \left(k \sin[k(1-\rho)L] + \frac{\mu}{2} \cos[k(1-\rho)L] \right) \\ &\quad + \mu\phi(0) \left(\cos[k\rho L] - \frac{\mu}{2k} \sin[k\rho L] \right) \\ &= \phi(0) \left(\frac{\mu}{2} \cos[k\rho L] - \left[k + \frac{\mu^2}{2k} \right] \sin[k\rho L] \right) - \phi(L) \left(\frac{\mu}{2} \cos[k(1-\rho)L] + k \sin[k(1-\rho)L] \right) \end{aligned} \quad (\text{F6})$$

Eqs F5 and F6 give two different expressions for the ratio $\frac{\phi(L)}{\phi(0)}$

$$\begin{aligned} \frac{\phi(L)}{\phi(0)} &= \frac{\left(k \cos[k\rho L] - \frac{\mu}{2} \sin[k\rho L] \right)}{\left(k \cos[k(1-\rho)L] - \frac{\mu}{2} \sin[k(1-\rho)L] \right)} \\ \frac{\phi(L)}{\phi(0)} &= \frac{\left(\frac{\mu}{2} \cos[k\rho L] - \left[k + \frac{\mu^2}{2k} \right] \sin[k\rho L] \right)}{\left(\frac{\mu}{2} \cos[k(1-\rho)L] + k \sin[k(1-\rho)L] \right)} \end{aligned} \quad (\text{F7})$$

The consistency between these two expressions

$$\begin{aligned}
0 &= \left(k \cos[k\rho L] - \frac{\mu}{2} \sin[k\rho L] \right) \left(\frac{\mu}{2} \cos[k(1-\rho)L] + k \sin[k(1-\rho)L] \right) \\
&\quad - \left(\frac{\mu}{2} \cos[k\rho L] - \left[k + \frac{\mu^2}{2k} \right] \sin[k\rho L] \right) \left(k \cos[k(1-\rho)L] - \frac{\mu}{2} \sin[k(1-\rho)L] \right) \\
&= \left(\frac{\mu^2}{4} + k^2 \right) \left[\cos[k\rho L] \sin[k(1-\rho)L] + \sin[k\rho L] \cos[k(1-\rho)L] - \frac{\mu}{k} \sin[k\rho L] \sin[k(1-\rho)L] \right]
\end{aligned} \tag{F8}$$

determines the possible values of the wavevector k of Eq. E6, i.e. of the eigenvalues $\lambda = k^2 + \frac{\mu^2}{4}$. So besides the vanishing eigenvalue $\lambda = 0$ corresponding to $k = i\frac{\mu}{2}$, the other possible wavevectors k for the excited states are the solutions of

$$\begin{aligned}
0 &= \cos[k\rho L] \sin[k(1-\rho)L] + \sin[k\rho L] \cos[k(1-\rho)L] - \frac{\mu}{k} \sin[k\rho L] \sin[k(1-\rho)L] \\
&= \sin(kL) - \frac{\mu}{k} \sin[k\rho L] \sin[k(1-\rho)L]
\end{aligned} \tag{F9}$$

Let us first focus on the special case $\rho = \frac{1}{2}$ in the next subsection.

2. Symmetric case $\rho = \frac{1}{2}$: decomposition of the spectrum into the symmetric and antisymmetric sectors

For $\rho = \frac{1}{2}$, Eq. F9 can be factorized into

$$0 = \sin\left(\frac{kL}{2}\right) \left[2 \cos\left(\frac{kL}{2}\right) - \frac{\mu}{k} \sin\left(\frac{kL}{2}\right) \right] \tag{F10}$$

This factorization is a consequence of the mirror symmetry $x \rightarrow L - x$ with respect to the middle-point $x = \frac{L}{2}$ of the Hamiltonian H with the quantum potential of Eq. F1, that allows to decompose the spectrum into its symmetric and antisymmetric sectors as follows.

a. Symmetric sector of the spectrum

When Eq. F10 is satisfied via the vanishing of the first factor

$$\sin\left(\frac{kL}{2}\right) = 0 \tag{F11}$$

then the ratio of Eq. F7 reduces to unity

$$\frac{\phi(L)}{\phi(0)} = 1 \tag{F12}$$

so that the wave-function $\phi(x)$ of Eq. F4 is symmetric with respect to the middle-point $x = \frac{L}{2}$

$$\phi(x) = \phi(L - x) \tag{F13}$$

In this symmetric sector, the possible non-vanishing wavevectors $k_n^{[sym]}$ satisfying Eq. F11 and the corresponding eigenvalues $\lambda_n^{[sym]}$ read

$$\begin{aligned}
k_n^{[sym]} &= \frac{2\pi n}{L} && \text{with } n = 1, 2, \dots, +\infty \\
\lambda_n^{[sym]} &= \frac{\mu^2}{4} + [k_n^{[sym]}]^2 = \frac{\mu^2}{4} + \frac{4\pi^2 n^2}{L^2} && \text{with } n = 1, 2, \dots, +\infty
\end{aligned} \tag{F14}$$

So one can compute the spectral Kemeny time associated to this symmetric sector

$$\tau_*^{Spectral[sym]} \equiv \sum_{n=1}^{+\infty} \frac{1}{\lambda_n^{[sym]}} = \frac{L^2}{4\pi^2} \sum_{n=1}^{+\infty} \frac{1}{n^2 + \left(\frac{\mu L}{4\pi}\right)^2} = \frac{2}{\mu^2} \left[\frac{\frac{\mu L}{4}}{\tanh\left(\frac{\mu L}{4}\right)} - 1 \right] \tag{F15}$$

with its asymptotic linear behavior for large L

$$\tau_*^{Spectral[sym]} \underset{L \rightarrow +\infty}{\simeq} \frac{2}{\mu^2} \left[\left(\frac{\mu L}{4} \right) \text{sgn}(\mu) \right] = \frac{L}{2|\mu|} \quad (\text{F16})$$

Note that the condition of Eq. F11 corresponds to Eq. E9 with the replacement $L \rightarrow \frac{L}{2}$: indeed, the mirror symmetry of Eq. F13 for the eigenvector means that one could consider only the first half $x \in [0, \frac{L}{2}]$ of the system with a reflecting boundary condition at $\frac{L}{2}$. Physically, this means that the symmetric sector of the spectrum describes only the relaxation within each valley, while the relaxation between the two valleys involves the antisymmetric sector of the spectrum described in the next subsection.

b. Antisymmetric sector of the spectrum

When Eq. F10 is satisfied via the vanishing of the second factor

$$2 \cos\left(\frac{kL}{2}\right) - \frac{\mu}{k} \sin\left(\frac{kL}{2}\right) = 0 \quad (\text{F17})$$

the ratio of the second Eq. F7 reduces to

$$\frac{\phi(L)}{\phi(0)} = -1 \quad (\text{F18})$$

so that the wave-function $\phi(x)$ of Eq. F4 is antisymmetric with respect to the middle-point $\frac{L}{2}$

$$\phi(x) = -\phi(L - x) \quad (\text{F19})$$

and in particular vanishes at the middle-point $\phi(\frac{L}{2}) = 0$. In this antisymmetric sector, the possible wavevectors $k_m^{[anti]}$ satisfying Eq. F17 are the solutions of

$$\tan\left(\frac{k_m^{[anti]} L}{2}\right) = \frac{2}{\mu} k_m^{[anti]} \quad (\text{F20})$$

and will give the eigenvalues

$$\lambda_m^{[anti]} = \frac{\mu^2}{4} + [k_m^{[anti]}]^2 \quad (\text{F21})$$

Note that the condition of Eq. F20 corresponds to Eq. E13 with the replacement $L \rightarrow \frac{L}{2}$: indeed, the antisymmetry of Eq. F19 for the eigenvector means that one could consider only the first half $x \in [0, \frac{L}{2}]$ of the system with an absorbing boundary condition at $\frac{L}{2}$.

c. Lowest eigenvalue $\lambda_{m=1}^{[anti]} \equiv \lambda_1$ for $\mu > 0$ where the maximum $U(x = \frac{L}{2}) = \mu \frac{L}{2}$ separates two valleys

It is now interesting to make the link with the main text: for $\mu > 0$ and $\rho = \frac{1}{2}$, where the maximum $U(x = \frac{L}{2}) = \mu \frac{L}{2}$ separates the two valleys with minima at the two boundaries $U(x = 0) = 0 = U(x = L)$, the Kemeny time of Eq. 179 grows exponentially with the system size L . The lowest eigenvalue $\lambda_1 \equiv \lambda_{m=1}^{[anti]}$ in the antisymmetric sector of Eq. F21 is then expected to be exponentially small in L and thus smaller than the constant $\frac{\mu^2}{4}$, so that the corresponding wavevector $k_{m=1}^{[anti]}$ is purely imaginary

$$k_1^{[anti]} = i\kappa$$

$$\kappa \equiv \sqrt{\frac{\mu^2}{4} - \lambda_1} = \frac{\mu}{2} - \frac{\lambda_1}{\mu} + O(\lambda_1^2) \quad (\text{F22})$$

and Eq. F20 becomes

$$\begin{aligned}
0 &= \tanh\left(\frac{\kappa L}{2}\right) - \frac{2\kappa}{\mu} \\
&= \tanh\left(\frac{\mu L}{4} - \frac{L}{2\mu}\lambda_1\right) - 1 + \frac{2}{\mu^2}\lambda_1 + O(\lambda_1^2) \\
&= \left[\tanh\left(\frac{\mu L}{4}\right) - 1\right] + \lambda_1 \frac{2}{\mu^2} \left[1 - \frac{L\mu}{4 \cosh^2\left(\frac{\mu L}{4}\right)}\right] + O(\lambda_1^2)
\end{aligned} \tag{F23}$$

The lowest eigenvalue λ_1 thus reads at leading order

$$\lambda_1 = \frac{\mu^2}{2} \left(\frac{1 - \frac{\sinh\left(\frac{\mu L}{4}\right)}{\cosh\left(\frac{\mu L}{4}\right)}}{1 - \frac{L\mu}{4 \cosh^2\left(\frac{\mu L}{4}\right)}} \right) = \frac{\mu^2}{2} \left(\frac{e^{-\frac{\mu L}{4}}}{\cosh\left(\frac{\mu L}{4}\right) - \frac{L\mu}{4 \cosh\left(\frac{\mu L}{4}\right)}} \right) \underset{L \rightarrow +\infty}{\simeq} \mu^2 e^{-\frac{\mu L}{2}} \tag{F24}$$

i.e. the relaxation time τ_1 associated to this lowest eigenvalue λ_1

$$\tau_1 \equiv \frac{1}{\lambda_1} \underset{L \rightarrow +\infty}{\simeq} \frac{1}{\mu^2} e^{\frac{\mu L}{2}} \tag{F25}$$

has exactly the same behavior for large L as the Kemeny time of Eqs 179 discussed in the main text.

3. Lowest eigenvalue λ_1 for $\mu > 0$ and $\rho \neq (0, 1/2, 1)$ with the maximum $U(x = \rho L)$ between two valleys

Let us now generalize the previous analysis concerning the lowest eigenvalue λ_1 for $\mu > 0$ when $\rho \neq (0, 1/2, 1)$. Plugging the analog of Eq. F22

$$\begin{aligned}
k &= i\kappa \\
\kappa &\equiv \sqrt{\frac{\mu^2}{4} - \lambda_1} = \frac{\mu}{2} - \frac{\lambda_1}{\mu} + O(\lambda_1^2)
\end{aligned} \tag{F26}$$

into Eq. F9 yields

$$\begin{aligned}
0 &= \sinh[\kappa L] - \frac{\mu}{\kappa} \sinh[\kappa \rho L] \sinh[\kappa(1 - \rho)L] \\
&= \sinh\left[\frac{\mu}{2}L\right] - \frac{1}{2} \sinh\left[\frac{\mu}{2}\rho L\right] \sinh\left[\frac{\mu}{2}(1 - \rho)L\right] \\
&\quad - \frac{\lambda_1}{\mu} \left[L \cosh\left[\frac{\mu}{2}L\right] + \frac{4}{\mu} \sinh\left[\frac{\mu}{2}\rho L\right] \sinh\left[\frac{\mu}{2}(1 - \rho)L\right] - 2\rho L \cosh\left[\frac{\mu}{2}\rho L\right] \sinh\left[\frac{\mu}{2}(1 - \rho)L\right] - 2(1 - \rho)L \sinh\left[\frac{\mu}{2}\rho L\right] \cosh\left[\frac{\mu}{2}(1 - \rho)L\right] \right] + O(\lambda_1^2) \\
&= \left(\cosh\left[\frac{\mu}{2}(1 - 2\rho)L\right] - e^{-\frac{\mu}{2}L} \right) - \frac{\lambda_1}{\mu} \left[\left(L + \frac{2}{\mu} \right) \cosh\left[\frac{\mu}{2}L\right] - L \sinh\left[\frac{\mu}{2}L\right] + L(1 - 2\rho) \sinh\left[\frac{\mu}{2}(1 - 2\rho)L\right] - \frac{2}{\mu} \cosh\left[\frac{\mu}{2}(1 - 2\rho)L\right] \right] + O(\lambda_1^2)
\end{aligned} \tag{F27}$$

So the lowest eigenvalue λ_1 reads at leading order for $\mu > 0$ and $\rho \neq (0, 1/2, 1)$

$$\begin{aligned}
\lambda_1 &= \mu \frac{\cosh\left[\frac{\mu}{2}(1 - 2\rho)L\right] - e^{-\frac{\mu}{2}L}}{\frac{2}{\mu} \cosh\left[\frac{\mu}{2}L\right] + L e^{-\frac{\mu}{2}L} + L(1 - 2\rho) \sinh\left[\frac{\mu}{2}(1 - 2\rho)L\right] - \frac{2}{\mu} \cosh\left[\frac{\mu}{2}(1 - 2\rho)L\right]} \\
&\underset{L \rightarrow +\infty}{\simeq} \frac{\mu^2}{2} e^{-L \frac{\mu}{2} [1 - |1 - 2\rho|]}
\end{aligned} \tag{F28}$$

i.e. the relaxation time τ_1 associated to this lowest eigenvalue λ_1

$$\tau_1 \equiv \frac{1}{\lambda_1} \underset{L \rightarrow +\infty}{\simeq} \frac{2}{\mu^2} e^{L \frac{\mu}{2} [1 - |1 - 2\rho|]} = \begin{cases} \frac{2}{\mu^2} e^{L\mu\rho} = \frac{2}{\mu^2} e^{B_1(L)} & \text{for } 0 < \rho < \frac{1}{2} \\ \frac{2}{\mu^2} e^{L\mu(1-\rho)} = \frac{2}{\mu^2} e^{B_2(L)} & \text{for } \frac{1}{2} < \rho < 1 \end{cases} \tag{F29}$$

has exactly the same behavior for large L as the Kemeny time of Eqs 177 and 178 discussed in the main text.

Appendix G: Kemeny times for diffusion processes on infinite intervals $(x_R - x_L) \rightarrow +\infty$

In this Appendix, we focus on diffusion processes on infinite intervals $(x_R - x_L) \rightarrow +\infty$: even if the normalized steady state $P_*(x)$ exists on the infinite interval, the Kemeny time τ_* can diverge as illustrated by the two following simple examples.

1. Divergence of the Kemeny time τ_* for the linear valley $U(x) = \mu x$ with $\mu > 0$ and $D(x) = 1$ on $[0, +\infty[$

Let us consider the limit $L \rightarrow +\infty$ of the case of Eq. 161 with $\rho = 1$ corresponding to the linear potential $U(x) = \mu x$ with $\mu > 0$ and $D(x) = 1$: the normalized steady state exists for $L \rightarrow +\infty$

$$p_*(x) = \mu \frac{e^{-\mu x} \theta(0 \leq x \leq L)}{1 - e^{-\mu L}} \underset{L \rightarrow +\infty}{\simeq} \mu e^{-\mu x} \quad \text{for } x \in [0, +\infty[\quad (\text{G1})$$

but the corresponding Kemeny time computed in Eqs 168 169 diverges

$$\tau_* = \frac{2}{\mu^2} \left[\frac{\frac{\mu L}{2}}{\tanh\left(\frac{\mu L}{2}\right)} - 1 \right] \underset{L \rightarrow +\infty}{\simeq} \frac{L}{|\mu|} \underset{L \rightarrow +\infty}{\simeq} +\infty \quad (\text{G2})$$

Since the quantum potential of Eq. 131 associated to the diffusion coefficient $D(x) = 1$ and to the force $F(x) = -\mu$

$$V(x) = \frac{F^2(x)}{4} + \frac{F'(x)}{2} = \frac{\mu^2}{4} \quad \text{for } x \in]0, +\infty[\quad (\text{G3})$$

is simply constant for $x \in]0, +\infty[$, it is clear that there will be a continuum of excited states with eigenvalues $\lambda \in]\frac{\mu^2}{4}, +\infty[$, as can be also seen from the large L -limit as the exact spectrum for finite L given in Eq. 171.

More generally, the Kemeny time will diverge whenever there is a continuum of eigenvalues

$$\tau_*^{\text{Spectral}[Continuum]} = +\infty \quad (\text{G4})$$

2. Divergence of the Kemeny time τ_* for the quadratic valley $U(x) = \omega \frac{x^2}{2}$ and $D(x) = 1$ on $] -\infty, +\infty[$

The Ornstein-Uhlenbeck process corresponding to the linear restoring force towards the origin $x = 0$ with the parameter $\omega > 0$

$$F(x) = -\omega x \quad (\text{G5})$$

and to the diffusion coefficient $D(x) = 1$ leads to the quadratic potential in Eq. 126

$$U(x) = \omega \frac{x^2}{2} \quad (\text{G6})$$

The corresponding Gaussian steady state

$$p_*(x) = \frac{e^{-U(x)}}{Z} = \sqrt{\frac{\omega}{2\pi}} e^{-\omega \frac{x^2}{2}} \quad (\text{G7})$$

can be plugged into the real-space Kemeny time of Eq. 150 to obtain that it diverges

$$\tau_*^{\text{Space}}(x) = +\infty \quad (\text{G8})$$

Since the quantum potential of Eq. 131 is also quadratic

$$V(x) = \frac{F^2(x)}{4} + \frac{F'(x)}{2} = \frac{\omega^2 x^2}{4} - \frac{\omega}{2} \quad (\text{G9})$$

the quantum Hamiltonian of Eq. 130 corresponds to the quantum harmonic oscillator with its well-known linear spectrum

$$\lambda_n = \left(n + \frac{1}{2}\right) \omega - \frac{\omega}{2} = n\omega \quad \text{with } n = 0, 1, \dots, +\infty \quad (\text{G10})$$

So the spectrum is fully discrete but the spectral Kemeny time nevertheless diverges

$$\tau_*^{Spectral} = \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} = \frac{1}{\omega} \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty \quad (\text{G11})$$

in agreement with the real-space Kemeny time of Eq. G8.

After these two examples where the Kemeny time diverges, it is thus interesting to discuss what conditions will produce a finite Kemeny time τ_* , by considering the real-space Kemeny time and the spectral Kemeny time in the two next subsections respectively.

3. Analysis of the convergence of the real-space Kemeny time τ_*^{Space} at $x_R = +\infty$

For $x_R = +\infty$, the real-space Kemeny time of Eq. 150 reads

$$\tau_*^{Space} = \int_{x_L}^{+\infty} dy \frac{1}{D(y)p_*(y)} \left[\int_{-\infty}^y dz p_*(z) \right] \left[\int_y^{+\infty} dx p_*(x) \right] \quad (\text{G12})$$

so one needs to analyze the convergence of the integral in y when $y \rightarrow +\infty$

Let us assume that the normalized steady state displays the following asymptotic decay for $x \rightarrow +\infty$

$$p^*(x) \underset{x \rightarrow +\infty}{\propto} \frac{e^{-Kx^\sigma}}{x^{1+\nu}} \quad \text{where } \sigma > 0 \text{ or } \sigma = 0; \nu > 0 \quad (\text{G13})$$

The leading behavior of the corresponding cumulative distribution reads

$$\int_y^{+\infty} dz p_*(z) \underset{y \rightarrow +\infty}{\propto} \frac{e^{-Ky^\sigma}}{y^{\sigma+\nu}} \quad (\text{G14})$$

Then the convergence of the Kemeny time of Eq. G12 is determined by the convergence at $(+\infty)$ of the integral

$$\int^{+\infty} dy \frac{\frac{e^{-Ky^\sigma}}{y^{\sigma+\nu}}}{D(y) \frac{e^{-Ky^\sigma}}{y^{1+\nu}}} = \int^{+\infty} dy \frac{y^{1-\sigma}}{D(y)} \quad (\text{G15})$$

i.e. it depends on the behavior of the diffusion coefficient $D(y)$ for $y \rightarrow +\infty$. If the diffusion coefficient behaves asymptotically as the power of exponent η

$$D(y) \underset{y \rightarrow +\infty}{\propto} y^\eta \quad (\text{G16})$$

then the integral of Eq. G15 will converge for

$$\sigma > 2 - \eta \quad (\text{G17})$$

with the following important cases:

(i) When the diffusion coefficient is constant $D(y) = 1$ corresponding to $\eta = 0$, then the real-space Kemeny time is finite $\tau_*^{Space} < +\infty$ only for exponents

$$\sigma > 2 \quad (\text{G18})$$

i.e. only if the asymptotic decay of the steady state of Eq. G13 is more rapid than Gaussian. In particular, the Ornstein-Uhlenbeck considered in the previous subsection corresponding to the Gaussian steady state of exponent $\sigma = 2$ is just on the border where the Kemeny time diverges.

(ii) when the diffusion coefficient is asymptotically linear $D(y) \propto y$ corresponding to $\eta = 1$, then the real-space Kemeny time is finite $\tau_*^{Space} < +\infty$ only for exponents $\sigma > 1$.

(iii) when the diffusion coefficient is asymptotically quadratic $D(y) \propto y^2$ corresponding to $\eta = 2$, then the real-space Kemeny time is finite $\tau_*^{Space} < +\infty$ for any strictly positive exponent $\sigma > 0$.

(iv) if $\sigma = 0$, i.e. if the steady state of Eq. G13 behaves asymptotically as the power-law $\frac{1}{x^{1+\nu}}$ of exponent $\nu > 0$, then real-space Kemeny time is finite $\tau_*^{Space} < +\infty$ only for exponents $\eta > 2$ in the diffusion coefficient of Eq. G16.

4. Analysis of the convergence of the spectral Kemeny time $\tau_*^{Spectral}$ at $n = +\infty$

The convergence of the spectral Kemeny time $\tau_*^{Spectral}$ requires that the spectrum is fully discrete and that the n^{th} eigenvalue λ_n grows more rapidly than linearly for large n .

To make the link with the real-space analysis of the previous subsection, let us now assume that the diffusion coefficient is constant $D(x) = 1$ as considered for Eq. G18 and that the potential $U(x)$ of Eq. 125 defined on $] -\infty, +\infty[$ displays the following asymptotic behaviors

$$U(x) \underset{x \rightarrow \pm\infty}{\simeq} K|x|^\sigma \quad (G19)$$

where the exponent σ was introduced in Eq. G13. The asymptotic behavior of the force of Eq. 126

$$F(x) = -U'(x) \underset{x \rightarrow \pm\infty}{\simeq} -K\sigma \operatorname{sgn}(x)|x|^{\sigma-1} \quad (G20)$$

yields the asymptotic power-law behavior of the quantum potential of Eq. 131

$$V(x) = \frac{F^2(x)}{4} + \frac{F'(x)}{2} \underset{x \rightarrow +\infty}{\simeq} \frac{K^2\sigma^2}{4} x^{2(\sigma-1)} \quad (G21)$$

For large n , the scaling of the eigenvalues λ_n can be obtained via the standard semi-classical formula for high-energy levels

$$\lambda_n \underset{n \rightarrow +\infty}{\propto} n^{q(\sigma)} \quad \text{with} \quad q(\sigma) = \frac{2(\sigma-1)}{\sigma} \quad (G22)$$

So the the spectral Kemeny time $\tau_*^{Spectral}$ will converge for exponents $q(\sigma) > 1$, i.e. for $\sigma > 2$ in agreement with the criterion of Eq. G18 as it should for consistency. In particular, the Ornstein-Uhlenbeck corresponding to $\sigma = 2$ and to the linear behavior $q(\sigma = 2) = 1$ (see Eq. G11) is just on the border where the spectral Kemeny time diverges.

It is also interesting to mention that the case of finite intervals $(x_R - x_L) < +\infty$ can be recovered in the limit of an infinite exponent $\sigma \rightarrow +\infty$ in the potential of Eq. G19, with the corresponding exponent

$$q(\sigma \rightarrow +\infty) = 2 \quad (G23)$$

for the large- n behavior of the eigenvalues of Eq. G22. So the semi-classical formula of Eq. G22 explains the leading quadratic behavior $\lambda_n \propto n^2$ found in all examples on finite intervals with explicit eigenvalues.

5. Example: finite Kemeny time for the diffusion $D(x) = 1$ in the quartic potential $U(x) = \frac{x^4}{2}$ on $] -\infty, +\infty[$

As an example of the condition of Eq. G18 with $D(x) = 1$, let us consider the quartic potential on $] -\infty, +\infty[$ corresponding to the exponent $\sigma = 4$

$$U(x) = \frac{x^4}{2} \quad (G24)$$

The steady state of Eq. 125

$$p_*(x) = \frac{e^{-\frac{x^4}{2}}}{Z} \quad Z = \int_{-\infty}^{+\infty} dx e^{-\frac{x^4}{2}} = 2^{-\frac{3}{4}} \int_{-\infty}^{+\infty} dy y^{\frac{1}{4}-1} e^{-y} = 2^{-\frac{3}{4}} \Gamma\left(\frac{1}{4}\right) \quad (G25)$$

allows to compute the real-space Kemeny time via Eq. 150

$$\tau_* = \int_{-\infty}^{+\infty} dy \frac{1}{p_*(y)} \left[\int_{-\infty}^y dz p_*(z) \right] \left[\int_y^{+\infty} dx p_*(x) \right] = \frac{3\pi\Gamma(\frac{5}{4})}{4\sqrt{2}\Gamma(\frac{3}{4})} \quad (G26)$$

For the quartic potential of Eq. G24 with $D(x) = 1$, the force of Eq. 126 is cubic

$$F(x) = -U'(x) = -2x^3 \quad (G27)$$

so the quantum potential of Eq. 131 is a combination of sextic and quadratic contributions

$$V(x) = \frac{F^2(x)}{4} + \frac{F'(x)}{2} = x^6 - 3x^2 \quad (G28)$$

while the scaling of the eigenvalues λ_n for large n given by the semiclassical formula of Eq. G22 involves the exponent

$$q(\sigma = 4) = \frac{3}{2} \quad (G29)$$

Appendix H: Jump-diffusion and jump-drift processes produced by resetting

In this Appendix, we focus on jump-diffusion and jump-drift processes produced by stochastic resetting (see the review [70] and references therein).

1. Diffusion with diffusion coefficient $D(x)$ and force $F(x)$; resetting with rate $\gamma(x)$ towards $\Pi(y)$

The one-dimensional jump-diffusion dynamics is defined as follows: when at position x , the particle diffuses with the diffusion coefficient $D(x)$ in the force $F(x)$ as in section IV, but with the resetting rate $\gamma(x)$ per unit time, it can also make a non-local jump toward some new position y chosen with the normalized probability distribution $\Pi(y)$.

The propagator $p_t(x|x_0)$ satisfies the Forward dynamics with respect to the final position x

$$\partial_t p_t(x|x_0) = \partial_x [-F(x) + D(x)\partial_x] p_t(x|x_0) + \Pi(x) \int dy \gamma(y) p_t(y|x_0) - \gamma(x) p_t(x|x_0) \quad (\text{H1})$$

and the backward dynamics with respect to the initial position x_0

$$\partial_t p_t(x|x_0) = [F(x_0) + \partial_{x_0} D(x_0)] \partial_{x_0} p_t(x|x_0) + \left[\int dy_0 p_t(x|y_0) \Pi(y_0) - p_t(x|x_0) \right] \gamma(x_0) \quad (\text{H2})$$

2. Steady state $p_*(x)$ with its two types of steady currents

The steady version of the forward Eq. H1 satisfied by the steady state $p_*(x)$

$$\begin{aligned} 0 &= -\frac{d}{dx} \left[F(x) p_*(x) - D(x) \frac{dp_*(x)}{dx} \right] + \Pi(x) \int dy \gamma(y) p_*(y) - \gamma(x) p_*(x) \\ &\equiv -\frac{dj_*(x)}{dx} + \int dy J_*(x, y) \end{aligned} \quad (\text{H3})$$

involves two types of steady currents $j_*(x)$ and $J_*(x, y)$:

(i) the local diffusive current $j_*(x)$ at position x that involves the diffusion coefficient $D(x)$ and the force $F(x)$

$$j_*(x) \equiv F(x) p_*(x) - D(x) \frac{dp_*(x)}{dx} \quad (\text{H4})$$

(ii) the non-local current from y to x produced by the jumps that involve the rates $\gamma(\cdot)$ and the probability distribution $\Pi(\cdot)$

$$J_*(x, y) \equiv \Pi(x) \gamma(y) p_*(y) - \Pi(y) \gamma(x) p_*(x) = -J_*(y, x) \quad (\text{H5})$$

Since the backward Eq. for the MFTP $\tau(x, x_0)$ reads

$$-1 + \frac{\delta_{x, x_0}}{p_*(x)} = [F(x_0) + \partial_{x_0} D(x_0)] \partial_{x_0} \tau(x, x_0) + \left[\int dy_0 \tau(x, y_0) \Pi(y_0) - \tau(x, x_0) \right] \gamma(x_0) \quad (\text{H6})$$

it is convenient to introduce the difference

$$g(x, x_0) \equiv \tau(x, x_0) - \int dy_0 \tau(x, y_0) \Pi(y_0) \quad (\text{H7})$$

that satisfies the differential equation

$$-1 + \frac{\delta_{x, x_0}}{p_*(x)} = \partial_{x_0} [D(x_0) \partial_{x_0} g(x, x_0)] + F(x_0) \partial_{x_0} g(x, x_0) - \gamma(x_0) g(x, x_0) \quad (\text{H8})$$

For arbitrary parameters $[D(x); F(x), \gamma(x)]$, the general solution of this second-order differential equation cannot be written explicitly, so we will focus on two soluble cases in the two following subsections:

- (i) the case where the three parameters $[D(x); F(x), \gamma(x)]$ are space-independent
- (ii) the case without diffusion $D(x) = 0$ while the force $F(x)$ and the resetting rate $\gamma(x)$ are space-dependent.

3. Diffusion $D(x) = 1$ and force $F(x) = \mu$ on the ring $[0, L]$ with resetting at rate $\gamma(x) = \gamma$ towards $\Pi(y)$

For the diffusion with diffusion coefficient $D(x) = 1$ and force $F(x) = \mu$ on the ring $[0, L]$ with resetting at constant rate $\gamma(x) = \gamma$ towards $\Pi(y)$, the spectral Kemeny time has been already computed in Eq. 261 of the main text. The goal of the present section is to give an alternative derivation via the real-space expression of the Kemeny time.

The difference of Eq. H7

$$g(x, x_0) \equiv \tau(x, x_0) - \int_0^L dy_0 \tau(x, y_0) \Pi(y_0) \quad (\text{H9})$$

satisfies the differential equation of Eq. H8

$$-1 + \frac{\delta_{x, x_0}}{p_*^{[Reset]}(x)} = \partial_{x_0}^2 g(x, x_0) + \mu \partial_{x_0} g(x, x_0) - \gamma g(x, x_0) \quad (\text{H10})$$

The general solution of the homogeneous equation is a linear combination of the two exponential solutions $e^{x_0 \kappa_{\pm}}$ where the two coefficients κ_{\pm} are solutions of the quadratic equation

$$0 = \kappa^2 + \mu \kappa - \gamma \quad (\text{H11})$$

and read

$$\kappa_{\pm} = -\frac{\mu}{2} \pm \sqrt{\frac{\mu^2}{4} + \gamma} \equiv -\frac{\mu}{2} \pm \kappa \quad (\text{H12})$$

with the notation

$$\kappa \equiv \sqrt{\frac{\mu^2}{4} + \gamma} \quad (\text{H13})$$

In the two regions $x_0 < x$ and $x_0 > x$, the general solution of the inhomogeneous equation of Eq. H10 reads

$$\begin{aligned} 0 \leq x_0 \leq x : \quad g(x, x_0) &= \frac{1}{\gamma} + e^{-\frac{\mu}{2} x_0} [A_-(x) \cosh(x_0 \kappa) + B_-(x) \sinh(x_0 \kappa)] \\ x \leq x_0 \leq L : \quad g(x, x_0) &= \frac{1}{\gamma} + e^{-\frac{\mu}{2} (x_0 - L)} [A_+(x) \cosh((x_0 - L) \kappa) + B_+(x) \sinh((x_0 - L) \kappa)] \end{aligned} \quad (\text{H14})$$

with their corresponding derivatives with respect to x_0

$$\begin{aligned} 0 \leq x_0 \leq x : \quad \partial_{x_0} g(x, x_0) &= -\frac{\mu}{2} e^{-\frac{\mu}{2} x_0} [A_-(x) \cosh(x_0 \kappa) + B_-(x) \sinh(x_0 \kappa)] \\ &\quad + \kappa e^{-\frac{\mu}{2} x_0} [A_-(x) \sinh(x_0 \kappa) + B_-(x) \cosh(x_0 \kappa)] \\ x \leq x_0 \leq L : \quad \partial_{x_0} g(x, x_0) &= -\frac{\mu}{2} e^{-\frac{\mu}{2} (x_0 - L)} [A_+(x) \cosh((x_0 - L) \kappa) + B_+(x) \sinh((x_0 - L) \kappa)] \\ &\quad + \kappa e^{-\frac{\mu}{2} (x_0 - L)} [A_+(x) \sinh((x_0 - L) \kappa) + B_+(x) \cosh((x_0 - L) \kappa)] \end{aligned} \quad (\text{H15})$$

The four functions $[A_{\pm}(x); B_{\pm}(x)]$ are determined by the following boundary conditions:

(i) The periodicity between $x_0 = 0$ and $x_0 = L$ for the function $g(x, x_0)$ and its derivative $\partial_{x_0} g(x, x_0)$

$$\begin{aligned} 0 &= g(x, x_0 = 0) - g(x, x_0 = L) = A_-(x) - A_+(x) \\ 0 &= \partial_{x_0} g(x, x_0)|_{x_0=0} - \partial_{x_0} g(x, x_0)|_{x_0=L} = \frac{\mu}{2} [A_+(x) - A_-(x)] + \kappa [B_-(x) - B_+(x)] \end{aligned} \quad (\text{H16})$$

yields that the two functions $A_{\pm}(x)$ coincide, as well as the two functions $B_{\pm}(x)$

$$\begin{aligned} A_-(x) &= A_+(x) \equiv A(x) \\ B_-(x) &= B_+(x) \equiv B(x) \end{aligned} \quad (\text{H17})$$

so that from now on we will use the simplified notations $A(x)$ and $B(x)$.

(ii) At $x_0 = x$, the continuity of the function $g(x, x_0)$ and the discontinuity of its derivative $\partial_{x_0}g(x, x_0)$ at $x_0 = x$ required by Eq. H10

$$\begin{aligned}
0 &= [g(x, x_0 = x^+) - g(x, x_0 = x^-)] e^{\frac{\mu}{2}x} \\
&= -A(x) \left[e^{\frac{\mu}{2}L} \cosh((L-x)\kappa) - \cosh(x\kappa) \right] + B(x) \left[e^{\frac{\mu}{2}L} \sinh((L-x)\kappa) + \sinh(x\kappa) \right] \\
&\equiv -A(x)C(x) + B(x)S(x) \\
\frac{e^{\frac{\mu}{2}x}}{p_*^{[Reset]}(x)} &= [\partial_{x_0}g(x, x_0)|_{x_0=x^+} - \partial_{x_0}g(x, x_0)|_{x_0=x^-}] e^{\frac{\mu}{2}x} \\
&= -\frac{\mu}{2}A(x) \left[e^{\frac{\mu}{2}L} \cosh((L-x)\kappa) - \cosh(x\kappa) \right] + \frac{\mu}{2}B(x) \left[e^{\frac{\mu}{2}L} \sinh((L-x)\kappa) + \sinh(x\kappa) \right] \\
&\quad - \kappa A(x) \left[e^{\frac{\mu}{2}L} \sinh((L-x)\kappa) + \sinh(x\kappa) \right] + \kappa B(x) \left[e^{\frac{\mu}{2}L} \cosh((L-x)\kappa) - \cosh(x\kappa) \right] \\
&\equiv -\frac{\mu}{2} [A(x)C(x) - B(x)S(x)] - \kappa [A(x)S(x) - B(x)C(x)]
\end{aligned} \tag{H18}$$

where we have introduced the two functions

$$\begin{aligned}
C(x) &\equiv e^{\frac{\mu}{2}L} \cosh((L-x)\kappa) - \cosh(x\kappa) \\
S(x) &\equiv e^{\frac{\mu}{2}L} \sinh((L-x)\kappa) + \sinh(x\kappa)
\end{aligned} \tag{H19}$$

satisfying

$$\begin{aligned}
C^2(x) - S^2(x) &= e^{\mu L} [\cosh^2((L-x)\kappa) - \sinh^2((L-x)\kappa)] + [\cosh^2(x\kappa) - \sinh^2(x\kappa)] \\
&\quad - 2e^{\frac{\mu}{2}L} [\cosh((L-x)\kappa) \cosh(x\kappa) + \sinh((L-x)\kappa) \sinh(x\kappa)] \\
&= e^{\mu L} + 1 - 2e^{\frac{\mu}{2}L} \cosh(L\kappa) = -2e^{\frac{\mu}{2}L} \left[\cosh(L\kappa) - \cosh\left(L\frac{\mu}{2}\right) \right]
\end{aligned} \tag{H20}$$

The solution of the system H18 for the two functions $A(x)$ and $B(x)$

$$\begin{aligned}
0 &= A(x)C(x) - B(x)S(x) \\
\frac{e^{\frac{\mu}{2}x}}{\kappa p_*^{[Reset]}(x)} &= -A(x)S(x) + B(x)C(x)
\end{aligned} \tag{H21}$$

thus reads

$$\begin{aligned}
A(x) &= -\frac{e^{\frac{\mu}{2}x}S(x)}{p_*^{[Reset]}(x)\kappa[S^2(x) - C^2(x)]} = -\frac{e^{\frac{\mu}{2}x}[\sinh((L-x)\kappa) + e^{-\frac{\mu}{2}L}\sinh(x\kappa)]}{p_*^{[Reset]}(x)2\kappa[\cosh(L\kappa) - \cosh(L\frac{\mu}{2})]} \\
B(x) &= -\frac{e^{\frac{\mu}{2}x}C(x)}{p_*^{[Reset]}(x)\kappa[S^2(x) - C^2(x)]} = -\frac{e^{\frac{\mu}{2}x}[\cosh((L-x)\kappa) - e^{-\frac{\mu}{2}L}\cosh(x\kappa)]}{p_*^{[Reset]}(x)2\kappa[\cosh(L\kappa) - \cosh(L\frac{\mu}{2})]}
\end{aligned} \tag{H22}$$

For $x_0 = x$ with the vanishing MFTP $\tau(x, x_0 = x) = 0$, Eq. H7 reads

$$g(x, x) = -\int_0^L dy_0 \tau(x, y_0) \Pi(y_0) \tag{H23}$$

so that Eq. H7 allows to compute the MFTP $\tau(x, x_0)$ from the function $g(., .)$ via

$$\tau(x, x_0) = g(x, x_0) - g(x, x) \tag{H24}$$

The solution found for $g(., .)$ in Eq. H14 then yields in the two regions

$$\begin{aligned}
x_0 \in [0, x] : \quad \tau(x, x_0) &= g(x, x_0) - g(x, x) \\
&= A(x) \left[e^{-\frac{\mu}{2}x_0} \cosh(x_0\kappa) - e^{-\frac{\mu}{2}x} \cosh(x\kappa) \right] \\
&\quad + B(x) \left[e^{-\frac{\mu}{2}x_0} \sinh(x_0\kappa) - e^{-\frac{\mu}{2}x} \sinh(x\kappa) \right] \\
x_0 \in [x, L] : \quad \tau(x, x_0) &= g(x, x_0) - g(x, x) \\
&= A(x) \left[e^{-\frac{\mu}{2}(x_0-L)} \cosh((L-x_0)\kappa) - e^{-\frac{\mu}{2}(x-L)} \cosh((L-x)\kappa) \right] \\
&\quad - B(x) \left[e^{-\frac{\mu}{2}(x_0-L)} \sinh((L-x_0)\kappa) - e^{-\frac{\mu}{2}(x-L)} \sinh((L-x)\kappa) \right]
\end{aligned} \tag{H25}$$

Let us now compute more explicitly the MFTP $\tau(x, x_0 = 0)$ as a function of the final point x when starting at $x_0 = 0$ using Eq. H22

$$\begin{aligned}
\tau(x, x_0 = 0) &= -A(x) \left[e^{-\frac{\mu}{2}x} \cosh(x\kappa) - 1 \right] - B(x) e^{-\frac{\mu}{2}x} \sinh(x\kappa) \\
&= \frac{[\sinh((L-x)\kappa) + e^{-\frac{\mu}{2}L} \sinh(x\kappa)] [\cosh(x\kappa) - e^{\frac{\mu}{2}x}] + [\cosh((L-x)\kappa) - e^{-\frac{\mu}{2}L} \cosh(x\kappa)] \sinh(x\kappa)}{p_*^{[Reset]}(x) 2\kappa [\cosh(L\kappa) - \cosh(L\frac{\mu}{2})]} \\
&= \frac{\sinh(L\kappa) - e^{\frac{\mu}{2}x} \sinh((L-x)\kappa) - e^{-\frac{\mu}{2}(L-x)} \sinh(x\kappa)}{p_*^{[Reset]}(x) 2\kappa [\cosh(L\kappa) - \cosh(L\frac{\mu}{2})]} \tag{H26}
\end{aligned}$$

in order to compute the real-space Kemeny time

$$\begin{aligned}
\tau_*^{[Reset]Space} &= \int_0^L dx p_*^{[Reset]}(x) \tau(x, x_0 = 0) = \int_0^L dx \frac{\sinh(L\kappa) - e^{\frac{\mu}{2}x} \sinh((L-x)\kappa) - e^{-\frac{\mu}{2}(L-x)} \sinh(x\kappa)}{2\kappa [\cosh(L\kappa) - \cosh(L\frac{\mu}{2})]} \\
&= \frac{L \sinh(L\kappa)}{2\kappa [\cosh(L\kappa) - \cosh(L\frac{\mu}{2})]} - \int_0^L dx \frac{e^{(\frac{\mu}{2}-\kappa)x} [e^{L\kappa} - e^{-\frac{\mu}{2}L}] + e^{(\frac{\mu}{2}+\kappa)x} [e^{-\frac{\mu}{2}L} - e^{-L\kappa}]}{4\kappa [\cosh(L\kappa) - \cosh(L\frac{\mu}{2})]} \\
&= \frac{L \sinh(L\kappa)}{2\kappa [\cosh(L\kappa) - \cosh(L\frac{\mu}{2})]} - \frac{1}{\kappa^2 - \frac{\mu^2}{4}} \tag{H27}
\end{aligned}$$

Using the value $\kappa = \sqrt{\frac{\mu^2}{4} + \gamma}$ of Eq. H13, the final result

$$\begin{aligned}
\tau_*^{[Reset]Space} &= \frac{L \sinh\left(L\sqrt{\frac{\mu^2}{4} + \gamma}\right)}{2\sqrt{\frac{\mu^2}{4} + \gamma} \left[\cosh\left(L\sqrt{\frac{\mu^2}{4} + \gamma}\right) - \cosh\left(L\frac{\mu}{2}\right)\right]} - \frac{1}{\gamma} \\
&= \frac{L \sinh\left(L\sqrt{\frac{\mu^2}{4} + \gamma}\right)}{2\sqrt{\mu^2 + 4\gamma} \sinh\left(L\frac{\sqrt{\mu^2 + 4\gamma} + \mu}{4}\right) \sinh\left(L\frac{\sqrt{\mu^2 + 4\gamma} - \mu}{4}\right)} - \frac{1}{\gamma} \tag{H28}
\end{aligned}$$

is in agreement with the spectral Kemeny time given in Eq. 261 of the main text.

4. Jump-drift process with force $F(x) > 0$ on the ring $[0, L]$ with resetting at rate $\gamma(x)$ towards $\Pi(y) = \delta(y)$

Let us now consider the following jump-drift process without diffusion $D(x) = 0$ with the space-dependent strictly positive force $F(x) > 0$ on the ring periodic $[0, L]$, while the resetting occurs with the space-dependent rate $\gamma(x)$ towards the origin $\Pi(y) = \delta(y)$.

a. Steady state $p_*(x)$

Since there is no diffusion $D(x) = 0$, the local steady current $j_*(x)$ of Eq. H4 is directly proportional to the steady state

$$j_*(x) = F(x)p_*(x) \tag{H29}$$

As a consequence, the steady state Eq. H3 can be rewritten as the following first-order differential equation for the local current $j_*(x)$

$$\frac{dj_*(x)}{dx} + \frac{\gamma(x)}{F(x)} j_*(x) = \delta(x) \int_0^L dy \frac{\gamma(y)}{F(y)} j_*(y) \tag{H30}$$

The current

$$j_*(x) = j_*(0^+) e^{-\int_0^x dy \frac{\gamma(y)}{F(y)}} \quad \text{for } x \in]0, L[\tag{H31}$$

thus decays from its value $j_*(x = 0^+)$ along the ring up to its value at $x = L^-$ that coincides with $x = 0^-$

$$j_*(0^-) = j_*(L^-) = j_*(0^+) e^{-\int_0^L dy \frac{\gamma(y)}{F(y)}} \quad (\text{H32})$$

while the discontinuity at the origin as determined by Eq. H30

$$j_*(0^+) - j_*(0^-) = \int_0^L dy \frac{\gamma(y)}{F(y)} j_*(y) \quad (\text{H33})$$

is satisfied for any $j_*(0^+)$ using Eqs H31 and H32. It is the normalization of the steady state $p_*(x)$ obtained from the current $j_*(x)$ via Eq. H29

$$p_*(x) = \frac{j_*(x)}{F(x)} = \frac{j_*(0^+)}{F(x)} e^{-\int_0^x dy \frac{\gamma(y)}{F(y)}} \quad \text{for } x \in]0, L[\quad (\text{H34})$$

that determines the current $j_*(0^+)$ at the origin

$$1 = \int_0^L dx p_*(x) = \int_0^L dx \frac{j_*(x)}{F(x)} = j_*(0^+) \int_0^L dx \frac{1}{F(x)} e^{-\int_0^x dy \frac{\gamma(y)}{F(y)}} \quad (\text{H35})$$

b. *Mean-First-Passage-Time* $\tau(x, x_0)$

The difference of Eq. H7

$$g(x, x_0) \equiv \tau(x, x_0) - \tau(x, 0) \quad (\text{H36})$$

satisfies the first-order differential equation of Eq. H8

$$-1 + \frac{\delta_{x, x_0}}{p_*(x)} = F(x_0) \partial_{x_0} g(x, x_0) - \gamma(x_0) g(x, x_0) \quad (\text{H37})$$

The change of function

$$g(x, x_0) = h(x, x_0) e^{\int_0^{x_0} dy \frac{\gamma(y)}{F(y)}} \quad (\text{H38})$$

yields that the new function $h(x, x_0)$ satisfies

$$\begin{aligned} \partial_{x_0} h(x, x_0) &= \left[-1 + \frac{\delta_{x, x_0}}{p_*(x)} \right] \frac{1}{F(x_0)} e^{-\int_0^{x_0} dy \frac{\gamma(y)}{F(y)}} = \left[-1 + \frac{\delta_{x, x_0}}{p_*(x)} \right] \frac{p_*(x_0)}{j_*(0^+)} \\ &= \frac{1}{j_*(0^+)} [-p_*(x_0) + \delta_{x, x_0}] \end{aligned} \quad (\text{H39})$$

where we have used the steady state $p_*(x_0)$ of Eq. H34 and the current $j_*(0^+)$ to rewrite the right hand side.

Since $h(x, x_0)$ of Eq. H38 vanishes at $x_0 = 0$ as the function $g(x, x_0)$ of Eq. H36, the integration of Eq. H39 yields in the two regions

$$\begin{aligned} x_0 \in [0, x[: \quad h(x, x_0) &= -\frac{1}{j_*(0^+)} \int_0^{x_0} dy p_*(y) \\ x_0 \in]x, L[: \quad h(x, x_0) &= \frac{1}{j_*(0^+)} \left[-\int_0^{x_0} dy p_*(y) + 1 \right] = \frac{1}{j_*(0^+)} \int_{x_0}^L dy p_*(y) \end{aligned} \quad (\text{H40})$$

where we have used the normalization of the steady state for the second region. Putting everything together, one obtains that the MFTP $\tau(x, x_0)$ can be rewritten in terms of the force $F(x)$ and of the steady state $p_*(x)$ of Eq. H34

$$\begin{aligned} x_0 \in [0, x[: \quad \tau(x, x_0) &= \tau(x, 0) - \frac{1}{j_*(0^+)} e^{\int_0^{x_0} dy \frac{\gamma(y)}{F(y)}} \int_0^{x_0} dy p_*(y) = \tau(x, 0) - \frac{1}{F(x_0) p_*(x_0)} \int_0^{x_0} dy p_*(y) \\ x_0 \in]x, L[: \quad \tau(x, x_0) &= \tau(x, 0) + \frac{1}{j_*(0^+)} e^{\int_0^{x_0} dy \frac{\gamma(y)}{F(y)}} \int_{x_0}^L dy p_*(y) = \tau(x, 0) + \frac{1}{F(x_0) p_*(x_0)} \int_{x_0}^L dy p_*(y) \end{aligned} \quad (\text{H41})$$

In the the present model without diffusion $D(x) = 0$, the particle can only moves along the force $F(x) > 0$ or jump at rate $\gamma(x)$ towards the origin, so the MFTP $\tau(x, x_0)$ vanishes only for $x_0 = x^-$ but not for $x_0 = x^+$

$$\begin{aligned} 0 &= \tau(x, x_0 = x^-) = \tau(x, 0) - \frac{1}{F(x)p_*(x)} \int_0^x dy p_*(y) \\ 0 &\neq \tau(x, x_0 = x^+) = \tau(x, 0) + \frac{1}{F(x)p_*(x)} \int_x^L dy p_*(y) \end{aligned} \quad (\text{H42})$$

So the first line yields the MFTP $\tau(x, 0)$ at position x when starting at the origin $x_0 = 0$

$$\tau(x, 0) = \frac{1}{F(x)p_*(x)} \int_0^x dy p_*(y) \quad (\text{H43})$$

that can be plugged into Eq. H41 to obtain the final solution for the MFTP in the two regions

$$\begin{aligned} x_0 \in [0, x]: \quad \tau(x, x_0) &= \frac{1}{F(x)p_*(x)} \int_0^x dy p_*(y) - \frac{1}{F(x_0)p_*(x_0)} \int_0^{x_0} dy p_*(y) \\ x_0 \in]x, L]: \quad \tau(x, x_0) &= \frac{1}{F(x)p_*(x)} \int_0^x dy p_*(y) + \frac{1}{F(x_0)p_*(x_0)} \int_{x_0}^L dy p_*(y) \end{aligned} \quad (\text{H44})$$

c. Real-space Kemeny time

The real-space Kemeny time of Eq. 2 can be evaluated for the initial position at $x_0 = 0$ using the MFTP $\tau(x, 0)$ of Eq. H43

$$\tau_*^{Space} = \int_0^L dx p_*(x) \tau(x, 0) = \int_0^L dx \frac{1}{F(x)} \int_0^x dy p_*(y) = \int_0^L dy p_*(y) \int_y^L dx \frac{1}{F(x)} \quad (\text{H45})$$

where the steady state is given in Eqs H34 H35.

d. Simplest example with the constant force $F(x) = 1$ and the constant resetting rate $\gamma(x) = \gamma$

The steady state $p_*(x)$ of Eq. H34 then reduces to

$$p_*(x) = \frac{\gamma}{1 - e^{-\gamma L}} e^{-\gamma x} \quad \text{for } x \in]0, L[\quad (\text{H46})$$

and the real-space Kemeny time of Eq. H45 becomes

$$\tau_*^{Space} = \frac{\gamma}{1 - e^{-\gamma L}} \int_0^L dy (L - y) e^{-\gamma y} = \frac{L}{1 - e^{-\gamma L}} - \frac{1}{\gamma} \quad (\text{H47})$$

In the limit $L \rightarrow +\infty$ where one recovers the Sisyphus process on the half-line $x \in [0, +\infty[$ with its exponential steady state (see [95] and references therein)

$$p_*(x) = \gamma e^{-\gamma x} \quad \text{for } x \in]0, +\infty[\quad (\text{H48})$$

the real-space Kemeny time of Eq. H47 diverges

$$\tau_*^{Space} = +\infty \quad (\text{H49})$$

Appendix I: Computation of the spectral Kemeny times involving the roots of unity

In this purely technical Appendix, we describe how to compute the various spectral Kemeny times of the main text that involve the roots of unity.

1. Computation of some spectral Kemeny times involving directly the roots of unity

The characteristic polynomial

$$P(z) = z^N - 1 = \prod_{n=0}^{N-1} (z - \omega_n) \quad (\text{I1})$$

associated to the N roots of unity

$$\omega_n = e^{in\frac{2\pi}{N}} \quad \text{for } n = 0, 1, \dots, N-1 \quad (\text{I2})$$

can be factorized into

$$P(z) = z^N - 1 = (z - 1)Q(z) \quad (\text{I3})$$

where the polynomial $Q(z)$ only involves the $(N-1)$ roots ω_n with $n = 1, \dots, N-1$ that are different from unity $1 = \omega_0$

$$Q(z) = \prod_{n=1}^{N-1} (z - \omega_n) = \frac{z^N - 1}{z - 1} = \sum_{k=0}^{N-1} z^k \quad (\text{I4})$$

Its logarithmic derivative allows to compute the sum of the inverses $\frac{1}{z - \omega_n}$ over $n = 1, \dots, N-1$

$$\sum_{n=1}^{N-1} \frac{1}{z - \omega_n} = \frac{Q'(z)}{Q(z)} = \frac{Nz^{N-1}}{z^N - 1} - \frac{1}{z - 1} = \frac{\sum_{k=1}^{N-1} kz^{k-1}}{\sum_{k=0}^{N-1} z^k} \quad (\text{I5})$$

a. Computation of the spectral Kemeny time of Eq. 279

For $z = 1$, Eq. I5 reduces to

$$\sum_{n=1}^{N-1} \frac{1}{1 - \omega_n} = \frac{\sum_{k=1}^{N-1} k}{\sum_{k=0}^{N-1} 1} = \frac{\frac{N(N-1)}{2}}{N} = \frac{N-1}{2} \quad (\text{I6})$$

that allows to compute the spectral Kemeny time of Eq. 279 of the main text.

b. Computation of the spectral Kemeny time of Eq. 280

For $z \neq 1$, Eq. I5

$$\sum_{n=1}^{N-1} \frac{1}{z - \omega_n} = \frac{N}{z(1 - z^{-N})} - \frac{1}{z - 1} \quad (\text{I7})$$

allows to compute the spectral Kemeny time of Eq. 280 of the main text with $z = \frac{\gamma}{2} + 1$.

2. Computation of some spectral Kemeny times involving the cosine function and the roots of unity

a. Computation of the spectral Kemeny time of Eq. 87

Using ω_n of Eq. I2, one can rewrite

$$\frac{1}{2 - 2\cos(n\frac{2\pi}{N})} = \frac{1}{2 - \omega_n - \frac{1}{\omega_n}} = \frac{1}{(1 - \omega_n)(1 - \frac{1}{\omega_n})} = \frac{-\omega_n}{(1 - \omega_n)^2} = \frac{(1 - \omega_n) - 1}{(1 - \omega_n)^2} = \frac{1}{(1 - \omega_n)} - \frac{1}{(1 - \omega_n)^2} \quad (\text{I8})$$

The derivation of Eq. I5 with respect to z

$$-\sum_{n=1}^{N-1} \frac{1}{(z - \omega_n)^2} = \frac{\sum_{k=2}^{N-1} k(k-1)z^{k-2}}{\sum_{k=0}^{N-1} z^k} - \left(\frac{\sum_{k=1}^{N-1} kz^{k-1}}{\sum_{k=0}^{N-1} z^k} \right)^2 \quad (\text{I9})$$

yields for $z = 1$

$$-\sum_{n=1}^{N-1} \frac{1}{(1 - \omega_n)^2} = \frac{\sum_{k=2}^{N-1} k(k-1)}{\sum_{k=0}^{N-1} 1} - \left(\frac{\sum_{k=1}^{N-1} k}{\sum_{k=0}^{N-1} 1} \right)^2 = \frac{\frac{N(N-1)(N-2)}{3}}{N} - \frac{(N-1)^2}{4} = \frac{(N-1)(N-5)}{12} \quad (\text{I10})$$

So the sum of Eq. I8 over $n = 1, \dots, N-1$ can be obtained from Eqs I6 and I10

$$\sum_{n=1}^{N-1} \frac{1}{2 - 2\cos(n\frac{2\pi}{N})} = \sum_{n=1}^{N-1} \frac{1}{(1 - \omega_n)} - \sum_{n=1}^{N-1} \frac{1}{(1 - \omega_n)^2} = \frac{N-1}{2} + \frac{(N-1)(N-5)}{12} = \frac{N^2-1}{12} \quad (\text{I11})$$

For $N = 2M$, Eq. I11 reads

$$\begin{aligned} \frac{4M^2-1}{12} &= \sum_{n=1}^{2M-1} \frac{1}{2 - 2\cos(n\frac{\pi}{M})} = \sum_{n=1}^{M-1} \frac{1}{2 - 2\cos(n\frac{\pi}{M})} + \frac{1}{2 - 2\cos(\pi)} + \sum_{n=M+1}^{2M-1} \frac{1}{2 - 2\cos(n\frac{\pi}{M})} \\ &= \sum_{n=1}^{M-1} \frac{1}{2 - 2\cos(n\frac{\pi}{M})} + \frac{1}{4} + \sum_{m=1}^{M-1} \frac{1}{2 - 2\cos((2M-m)\frac{\pi}{M})} = 2 \sum_{n=1}^{M-1} \frac{1}{2 - 2\cos(n\frac{\pi}{M})} + \frac{1}{4} \end{aligned} \quad (\text{I12})$$

so that one obtains

$$\sum_{n=1}^{M-1} \frac{1}{2 - 2\cos(n\frac{\pi}{M})} = \frac{1}{2} \left[\frac{4M^2-1}{12} - \frac{1}{4} \right] = \frac{M^2-1}{6} \quad (\text{I13})$$

that allows to compute the spectral Kemeny time of Eq. 87 of the main text.

b. Computation of the spectral Kemeny time of Eq. 101

Using ω_n of Eq. I2, one can rewrite

$$\begin{aligned} \frac{1}{\left(y + \frac{1}{y}\right) - 2\cos\left(n\frac{2\pi}{N}\right)} &= \frac{1}{y + \frac{1}{y} - \omega_n - \frac{1}{\omega_n}} = \frac{1}{(y - \omega_n)(1 - \frac{1}{y\omega_n})} = \frac{-\omega_n}{(y - \omega_n)(\frac{1}{y} - \omega_n)} \\ &= \frac{1}{y^2 - 1} \left[\frac{y^2}{y - \omega_n} - \frac{1}{\frac{1}{y} - \omega_n} \right] \end{aligned} \quad (\text{I14})$$

So the sum of Eq. I14 over $n = 1, \dots, N-1$ can be obtained from Eq. I5 for the two values $z = y$ and $z = \frac{1}{y}$

$$\begin{aligned} \sum_{n=1}^{N-1} \frac{1}{\left(y + \frac{1}{y}\right) - 2\cos\left(n\frac{2\pi}{N}\right)} &= \frac{1}{y^2 - 1} \left[y^2 \sum_{n=1}^{N-1} \frac{1}{y - \omega_n} - \sum_{n=1}^{N-1} \frac{1}{\frac{1}{y} - \omega_n} \right] \\ &= \frac{1}{y^2 - 1} \left[y^2 \frac{Ny^{N-1}}{y^N - 1} - \frac{y^2}{y - 1} - \frac{Ny^{1-N}}{y^{-N} - 1} + \frac{1}{\frac{1}{y} - 1} \right] \\ &= \frac{y}{y^2 - 1} \left[\frac{N(y^N + 1)}{y^N - 1} - \frac{y + 1}{y - 1} \right] \end{aligned} \quad (\text{I15})$$

For $N = 2M$, Eq. I15 reads

$$\begin{aligned}
\frac{y}{y^2 - 1} \left[\frac{2M(y^{2M} + 1)}{y^{2M} - 1} - \frac{y + 1}{y - 1} \right] &= \sum_{n=1}^{2M-1} \frac{1}{\left(y + \frac{1}{y}\right) - 2 \cos\left(n \frac{\pi}{M}\right)} \\
&= \sum_{n=1}^{M-1} \frac{1}{\left(y + \frac{1}{y}\right) - 2 \cos\left(n \frac{\pi}{M}\right)} + \frac{1}{\left(y + \frac{1}{y}\right) - 2 \cos(\pi)} + \sum_{n=M+1}^{2M-1} \frac{1}{\left(y + \frac{1}{y}\right) - 2 \cos\left(n \frac{\pi}{M}\right)} \\
&= \sum_{n=1}^{M-1} \frac{1}{\left(y + \frac{1}{y}\right) - 2 \cos\left(n \frac{\pi}{M}\right)} + \frac{1}{y + \frac{1}{y} + 2} + \sum_{m=1}^{M-1} \frac{1}{\left(y + \frac{1}{y}\right) - 2 \cos\left((2M - m) \frac{\pi}{M}\right)} \\
&= 2 \sum_{n=1}^{M-1} \frac{1}{\left(y + \frac{1}{y}\right) - 2 \cos\left(n \frac{\pi}{M}\right)} + \frac{y}{(y + 1)^2}
\end{aligned} \tag{I16}$$

so that one obtains

$$\begin{aligned}
\sum_{n=1}^{M-1} \frac{1}{\left(y + \frac{1}{y}\right) - 2 \cos\left(n \frac{\pi}{M}\right)} &= \frac{1}{2} \left(\frac{y}{y^2 - 1} \left[\frac{2M(y^{2M} + 1)}{y^{2M} - 1} - \frac{y + 1}{y - 1} \right] - \frac{y}{(y + 1)^2} \right) \\
&= \frac{y}{(y^2 - 1)} \left[\frac{M(y^{2M} + 1)}{y^{2M} - 1} - \frac{(y^2 + 1)}{y^2 - 1} \right]
\end{aligned} \tag{I17}$$

For $y = e^{\frac{\mu}{2}}$ and the label change $M \rightarrow N$, the sum of Eq. I17

$$S_\mu(N) \equiv \sum_{n=1}^{N-1} \frac{1}{2 \cosh\left(\frac{\mu}{2}\right) - 2 \cos\left(n \frac{\pi}{N}\right)} = \frac{1}{2 \sinh\left(\frac{\mu}{2}\right)} \left[\frac{N}{\tanh\left(N \frac{\mu}{2}\right)} - \frac{1}{\tanh\left(\frac{\mu}{2}\right)} \right] \tag{I18}$$

allows to compute the spectral Kemeny time of Eq. 101.

Since the leading behavior for large N of the sum $S_\mu(N)$ of Eq. I18 is linear for $\mu \neq 0$

$$S_\mu(N) \underset{N \rightarrow +\infty}{\simeq} \frac{N}{2 \sinh\left(\frac{|\mu|}{2}\right)} \tag{I19}$$

while it is quadratic for $\mu = 0$ (Eq. I13)

$$S_{\mu=0}(N) = \frac{N^2 - 1}{6} \underset{N \rightarrow +\infty}{\simeq} \frac{N^2}{6} \tag{I20}$$

one can analyze the crossover between these two scaling behaviors for large N via the introduction of the appropriate rescaled variable

$$\hat{N} \equiv N \frac{|\mu|}{2} \tag{I21}$$

that is considered fixed in the double limit $N \rightarrow +\infty$ and $\mu \rightarrow 0$. Then one obtains that the ratio of Eq. I18 and I13

$$\frac{S_\mu(N)}{S_{\mu=0}(N)} \underset{\substack{N \rightarrow +\infty \\ \mu \rightarrow 0 \\ \hat{N} = N \frac{|\mu|}{2} \text{ fixed}}}{\simeq} \frac{3}{\hat{N}^2} \left[\frac{\hat{N}}{\tanh \hat{N}} - 1 \right] \equiv \mathcal{T}(\hat{N}) \tag{I22}$$

involves the same scaling function $\mathcal{T}(\cdot)$ encountered in Eq. 159 of the main text that governs the analog crossover the Kemeny time of Eq. 170 for the corresponding continuous-space diffusion process.

c. Translation to obtain the spectral Kemeny time of Eq. 88

In Eq. 88 of the main text, one needs the sum of Eq. I18 for the values

$$\begin{aligned}
\cosh\left(\frac{\mu}{2}\right) &= 1 + \frac{\gamma}{2} \\
\sinh\left(\frac{\mu}{2}\right) &= \sqrt{\cosh^2\left(\frac{\mu}{2}\right) - 1} = \sqrt{\gamma \left(1 + \frac{\gamma}{4}\right)}
\end{aligned} \tag{I23}$$

leading to

$$\frac{\mu}{2} = 2 \ln \left(\sqrt{\frac{\gamma}{4}} + \sqrt{\frac{\gamma}{4} + 1} \right) \quad (\text{I24})$$

So Eq. I18 becomes the sum

$$\begin{aligned} \Sigma_{\gamma}(N) &\equiv \sum_{n=1}^{N-1} \frac{1}{(\gamma+2) - 2 \cos \left(n \frac{\pi}{N} \right)} = \frac{1}{\sqrt{\gamma(4+\gamma)}} \left[\frac{N}{\tanh \left[2N \ln \left(\sqrt{\frac{\gamma}{4}} + \sqrt{\frac{\gamma}{4} + 1} \right) \right]} - \frac{\gamma+2}{\sqrt{\gamma(4+\gamma)}} \right] \\ &= \frac{1}{\sqrt{\gamma(4+\gamma)}} \left[N \frac{\left(\sqrt{\frac{\gamma}{4}} + \sqrt{\frac{\gamma}{4} + 1} \right)^{4N} - 1}{\left(\sqrt{\frac{\gamma}{4}} + \sqrt{\frac{\gamma}{4} + 1} \right)^{4N} + 1} - \frac{\gamma+2}{\sqrt{\gamma(4+\gamma)}} \right] \end{aligned} \quad (\text{I25})$$

and allows to compute the Kemeny time of Eq. 88 of the main text.

Similarly, the crossover of Eq. I22 translates into

$$\frac{\Sigma_{\gamma}(N)}{\Sigma_{\gamma=0}(N)} \underset{\substack{N \rightarrow +\infty \\ \gamma \rightarrow 0 \\ \hat{N} = \sqrt{\gamma}N \text{ fixed}}}{\simeq} \frac{3}{\hat{N}^2} \left[\frac{\hat{N}}{\tanh \hat{N}} - 1 \right] \equiv \mathcal{T}(\hat{N} = \sqrt{\gamma}N) \quad (\text{I26})$$

that involves the new scaling variable

$$\hat{N} \equiv \sqrt{\gamma}N \quad (\text{I27})$$

and the same scaling function $\mathcal{T}(\cdot)$ encountered in Eq. 159 of the main text that governs the analog crossover the Kemeny time of Eq. 158 for the corresponding continuous-space diffusion process.

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