(Nonequilibrium) dynamics of diffusion processes with non-conservative drifts.

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The nonequilibrium Fokker-Planck dynamics with a non-conservative drift field, in dimension $N \geq 2$, *can be related with the non-Hermitian quantum mechanics in a real scalar potential V and in a purely imaginary vector potential -iA of real amplitude A*, [1]. Since Fokker-Planck probability density functions may be obtained by means of Feynman's path integrals, the previous observation points towards a general issue of "magnetically affine" propagators, possibly of quantum origin, in real and Euclidean time. In below we shall follow the $N = 3$ "magnetic thread", within which one may keep under a computational control formally and conceptually different implementations of magnetism (or surrogate magnetism) in the dynamics of diffusion processes. We shall focus on interrelations (with due precaution to varied, not evidently compatible, notational conventions) of: (i) the pertinent non-conservatively drifted diffusions, (ii) the classic Brownian motion of charged particles in the (electro)magnetic field, (iii) diffusion processes arising within so-called Euclidean quantum mechanics (which from the outset employs non-Hermitian "magnetic" Hamiltonians), (iv) limitations of the usefulness of the Euclidean map $\exp(-itH_{quant}) \to \exp(-tH_{Eucl})$, regarding the probabilistic significance of inferred (path) integral kernels in the description of diffusion processes.

I. INTRODUCTION.

A. Preliminaries

We are inspired by the observation, [1], that the dynamics of nonequilibrium diffusion processes in dimension $N \geq 2$ shows an affinity with the "non-Hermitian electromagnetic quantum mechanics", actually appearing in its *fully* Euclidean version, c.f. [2, 3, 5, 6] and [8, 9].

To set the ground for further discussion, we bring to the reader's attention the origin of the transformation $-iA \to A$, employed in [1], on the Lagrangian level of description of the appropriate classical system in real and Euclidean times respectively.

Namely, let us consider the classical Lagrangian with the vector field entry (all dimensional constants have been scaled away, to facilitate computations): $\mathcal{L}_{cl}(\vec{q}(t), \dot{\vec{q}}(t)) = \frac{1}{2}\dot{\vec{q}}^2(t) + \dot{\vec{q}}(t) \cdot \vec{F}(\vec{q}(t)) - V(\vec{q}(t))$. The Wick rotation $\tau = it, t \geq 0$ leads us to the complex-valued version of the Lagrangian (remember about the need for an overall sign change of the resultant expression) $\mathcal{L}_{Wick}(\vec{x}(\tau), \dot{\vec{x}}(\tau)) = -\mathcal{L}_{cl}(\vec{q}(t), \dot{\vec{q}}(t)) = \frac{1}{2}\dot{\vec{x}}^2(\tau) - i\dot{\vec{x}}(\tau) \cdot \vec{F}(\vec{x}(\tau)) + V(\vec{x}(\tau))$, with a purely imaginary vector entry $-i\dot{\vec{x}} \cdot \vec{F}(\vec{x})$. The subsequent transformation $\vec{F} \to -i\vec{F}$ gives rise to the real-valued version of the fully Euclidean Lagrangian $\mathcal{L}_{Eucl}(\vec{x}(\tau), \dot{\vec{x}}(\tau)) = \frac{1}{2}\dot{\vec{x}}^2(\tau) - \dot{\vec{x}}(\tau) \cdot \vec{F}(\vec{x}(\tau)) + V(\vec{x}(\tau))$. We indicate that no assumptions were made about the properties (conservative or not) of the vector field $\vec{F}$, which may as well be a pure gradient (e.g. conservative) one.

We realise that for "true" (like e.g. solenoidal) magnetic potentials $\vec{F} \equiv \vec{A}$ this would set a link with the Euclidean variant of the (classical) Maxwell theory, [2, 10, 11]. For future purposes we indicate that the transformation $\vec{A} \to -i\vec{A}$ replaces the L^2-Hermitian operator $H_{quant} = -\frac{1}{2}(\vec{\nabla} - i\vec{A})^2$, [12, 13], by the non-Hermitian one $H_{Eucl} = -\frac{1}{2}(\vec{\nabla} - \vec{A})^2$, [1]. On the level of motion operators (with scaled away physical constants), we ultimately arrive at the fully Euclidean (but non-Hermitian) outcome: $\exp(-itH_{quant}) \to \exp(-\tau H_{quant}) \to \exp(-\tau H_{Eucl})$.

In passing we note an important sign difference, which needs to be accounted for, once we make a comparison with the reasoning of Ref. [2], where H_{Eucl} and its L^2-adjoint H^*_{Eucl} play a decisive role. Namely, given the above Hermitian generator H_{quant}, a transformation $\vec{A} \to +i\vec{A}$ employed in [2], actually produces an operator $H^*_{Eucl} = -\frac{1}{2}(\vec{\nabla} + \vec{A})^2$, which is the adjoint of our H_{Eucl}. This sign difference, while in a comparative vein, can be easily "corrected" by changing the sign of the vector potential in all relevant formulas of [2].

B. Drifted diffusion process.

Our further attention will focus on Markovian diffusion processes driven by non-conservative (generically non-gradient) time-independent drift fields $\vec{F}(\vec{x})$. Let us consider a diffusion process $\vec{X}(t)$, associated with the stochastic differential equation of the Langevin-type, (interpreted in terms of infinitesimal time increments)

$$d\vec{X}(t) = \vec{F}(\vec{X}(t))dt + \sqrt{2\nu}d\vec{W}(t), \tag{1}$$

where a vector field $\vec{F}(\vec{x})$ stands for a forward drift of the process, ν is a diffusion constant (2ν is interpreted as the variance parameter), and $\vec{W}(t)$ is the normalised Wiener noise in R^3, defined by expectation values $\langle W_i \rangle = 0$ and $\langle W_j(s) W_j(t) \rangle = \delta_{ij}\delta(s-t)$ for all $i, j = 1, 2, ...N$.

Accordingly, if an initial probability density function $\rho_0(\vec{x})$ is given, its time evolution $\rho_0(\vec{x}) = \rho(\vec{x}, 0) \to \rho(\vec{x}, t) = [\exp(tL^*)\rho_0](x)$ follows the Fokker-Planck equation [15, 16, 25]:

$$\partial_t \rho = \nu \Delta \rho - \vec{\nabla} \cdot (\vec{F}\rho) = L^* \rho, \tag{2}$$

where the Fokker-Planck operator $L^* = \nu\Delta - \vec{\nabla} \cdot (\vec{F} \cdot)$ is a Hermitian $L^2(R^N)$ adjoint of the diffusion generator $L = \nu\Delta + \vec{F} \cdot \vec{\nabla}$, [15, 16].

We anticipate the existence of a transition probability density function $p(\vec{y}, s, \vec{x}, t)$, $0 \leq s < t \leq T$, ($T \to \infty$ is admissible) for the diffusion process (1), (2): $\rho(\vec{x}, t) = \int p(\vec{y}, s, \vec{x}, t)\rho(\vec{y}, s) d^3y$, [15, 17]. We presume $p(\vec{y}, s, \vec{x}, t)$ to be a (possibly fundamental) solution of the F-P equation, with respect to variables $\vec{x}$ and t, i.e.

$$\partial_t p(\vec{y}, s, \vec{x}, t) = L^*_{\vec{x}} p(\vec{y}, s, \vec{x}, t). \tag{3}$$

We interpret the transition pdf as a (path) integral kernel of the evolution operator $\exp[(t-s)L^*]$, c.f. [1, 25, 26], see also [17]. (Note that in the literature there exist other notational conventions for this pdf.)

In connection with the notion of the diffusion generator L, we indicate that given any continuous and bounded function $f(\vec{x})$, we can introduce a function (actually a conditional expectation value of f, interpreted as an observable, [15])

$$u(\vec{y}, s) = \mathbb{E}[f(\vec{X}_t)|\vec{X}_s = \vec{y}] = \int f(\vec{x}) p(\vec{y}, s, \vec{x}, t) d^N x \tag{4}$$

which solves a final value $u(\vec{y}, s \to t) = f(\vec{y})$ parabolic problem in any a priori prescribed time interval $s \in [0, t]$

$$-\partial_s u(\vec{y}, s) = L_{\vec{y}} u(\vec{y}, s), \tag{5}$$

with the diffusion generator of the form $L = \nu\Delta + \vec{F} \cdot \vec{\nabla}$. Accordingly, [15], we have:

$$-\partial_s p(\vec{y}, s, \vec{x}, t) = L_{\vec{y}} p(\vec{y}, s, \vec{x}, t) \tag{6}$$

as a parabolic companion of Eq. (3) (albeit running in the reverse sense of time). Notice that equations (3) and (6) involve non-Hermitian operators L^* and L, which are $L^2(R^3)$ adjoint to each other. This is a standard feature of Markovian diffusions.

Remark 1: If the Fokker-Plack equation (2) admits an invariant (stationary) probability density $\rho_*(\vec{x})$ as a solution, then the operators L and L^*, which are non-Hermitian in L^2, become Hermitian in function spaces $L^2_{\rho_*}$ and $L^2_{\rho_*^{-1}}$ respectively. Here, $L^2_{\rho_*}$ indicates the Hilbert space , whose scalar product is weighted by ρ_*, according to $(f, g)_{\rho_*} = \int f(\vec{x}) g(\vec{x}) \rho_*(\vec{x}) d^N x$, and analogously for $L^2_{\rho_*^{-1}}$.

C. "Magnetic" affinities.

To conform with the notation of [1, 2, 5, 25, 26], let us set $\nu = 1/2$ (this amounts to rescaling the time label in the Fokker-Planck equation, cf. [5]). By employing the identity $\vec{\nabla} \cdot (\vec{F}\rho) = (\vec{F} \cdot \vec{\nabla})\rho + \rho(\vec{\nabla} \cdot \vec{F})$ we arrive at the following form of the F-P operator L^*:

$$L^* = \frac{1}{2}\Delta - \vec{F} \cdot \vec{\nabla} - (\vec{\nabla} \cdot \vec{F}) = \frac{1}{2}(\vec{\nabla} - \vec{F})^2 - \mathcal{V} = -(H_{Eucl} + \mathcal{V}), \tag{7}$$

where $H_{Eucl} = -\frac{1}{2}(\vec{\nabla} - \vec{F})^2$ and $\mathcal{V}$ has a specific functional form

$$\mathcal{V} = \frac{1}{2}[(\vec{\nabla} \cdot \vec{F}) + \vec{F}^2], \tag{8}$$

which is omnipresent in the discussion of diffusion processes in the classical and quantum realm, [16, 17, 23]. For clarity of discussion, we reproduce an intermediate form taken by L^* in the derivation of (7): $L^* = [\frac{1}{2}\Delta - \vec{F} \cdot \vec{\nabla} - (1/2)(\vec{\nabla} \cdot \vec{F})] + (1/2)\vec{F}^2] - \mathcal{V}$.

The diffusion generator L reads

$$L = \frac{1}{2}\Delta + \vec{F} \cdot \vec{\nabla} = \frac{1}{2}(\vec{\nabla} + \vec{F})^2 - \mathcal{V} = -(H^*_{Eucl} + \mathcal{V}), \tag{9}$$

with $H^*_{Eucl} = -\frac{1}{2}(\vec{\nabla} + \vec{F})^2$ and $\mathcal{V}$ given by Eq. (8). The intermediate form taken by L reads: $L = [\frac{1}{2}\Delta + \vec{F} \cdot \vec{\nabla} + (1/2)(\vec{\nabla} \cdot \vec{F}) + (1/2)\vec{F}^2] - \mathcal{V}$.

We indicate that for a divergenceless drift, $div\vec{F} = \vec{\nabla} \cdot \vec{F} = 0$, the F-P operator simplifies to $L^* = (1/2)\Delta - \vec{F} \cdot \vec{\nabla}$, whose form is fully congruent with that for $L = (1/2)\Delta + \vec{F} \cdot \vec{\nabla}$. Consequently, by merely changing the sign of $\vec{F}$, we can map L^* into L and back.

The above rewriting of L^* and L is highly suggestive, [1, 2, 5], since non-Hermitian operators $H_{Eucl} = -(1/2)(\vec{\nabla} - \vec{F})^2$ and $H^*_{Eucl} = -(1/2)(\vec{\nabla} + \vec{F})^2$, show a Euclidean affinity (normally restricted to magnetic potentials $\vec{F} \equiv \vec{A}$, [1, 2]) with (Hermitian) quantum mechanical magnetic Schrödinger operators, [8, 9].

D. Goals.

We shall be interested in interpreting the transition probability densities of the diffusion processes in question, as integral kernels of the motion operator $\exp(tL^*)$:

$$p(\vec{y}, s, \vec{x}, t) = [e^{L^*(t-s)}](\vec{y}, \vec{x}) = [e^{-(H_{Eucl} + \mathcal{V})(t-s)}](\vec{y}, \vec{x}). \tag{10}$$

It is a classic observation, [25, 26], that Fokker-Planck transition probablity density functions and probability densities, for diffusions with non-conservative drifts, are amenable to Feynman's path integration routines. In case of conservative drifts, the same goal can be achieved by means of a multiplicative (Doob-like) conditioning of the related (positive) Feynman-Kac kernel, [5, 9, 13, 16, 17, 27], provided the existence of stationary pdfs is granted.

In the path integral derivation, we need to define the action functional in terms of a suitable Lagrangian, with the obvious advantage that in the quadratic case the classical action alone normally suffices for the evaluation of the path integral, c.f. [9, 12, 13], see also [34, 35]. By this reason, a substantial part of the discussion in Ref. [1], has been dedicated to linear drift fields, with emphasis on the existence of nonequilibrium steady states $\rho_*(\vec{x})$, and non-vanishing steady currents $\vec{j}_*(\vec{x}, t)$.

Our major aim is to examine the $N = 3$ "(electro)magnetic thread", which comprises conceptually different "(electro)magnetic" implementations of drifted diffusion processes. This amounts to revisiting: (i) the classic theory of the Brownian motion in a magnetic field, [28, 29] and [30–32], (ii) past research on path integral solutions of the Fokker-Planck equation for a system with non-conservative forces, [25, 26], (iii)the Euclidean time dynamics, generated by Hermitian Schrödinger operators in magnetic fields and path integral kernels of related Schrödinger semigroups, [8, 9, 13], (iv) electromagnetic dynamical features deduced within so-called Euclidean quantum mechanics, [2, 3] (see also [6, 23, 24] for a complementary analysis of the Lagrangian variational principle with a classical action).

In particular, we aim at a clear discrimination between the physically motivated impact of magnetism on diffusing charges, and "electromagnetic" analogies appearing in the study of nonequilibrium diffusion processes with non-conservative drifts. The latter, may not necessarily embody the very concept of diffusing charged particles, but nonetheless might satisfactorily mimic (simulate) the electromagnetic-looking behavior of diffusion currents, in terms of a potentially useful "surrogate magnetism".

We shall derive a number of exact transition pdfs for the pertinent spatial diffusion processes. While departing from phase-space derivations of Refs. [28, 29], we may extract formulas which can be reinterpreted solely in the configuration space (via fine tuning and scaling away of various parameters, and temporarily suspending the involved fluctuation-dissipation relationship).

In the course of our discussion, we perform a detailed derivation of the propagator associated with would-be natural Euclidean analogue $H_{Eucl} = -(1/2)(\vec{\nabla} - \vec{F})^2$ of $H_{quant} = -(1/2)(\vec{\nabla} + i\vec{F})^2$, for a solenoidal drift field $\vec{\nabla} \cdot \vec{F} = 0$. This is accomplished by a direct evaluation of the path integral kernel of $\exp(-tH_{Eucl})$.

Our motivation comes from the fact that the integral kernel of the legitimate (Hermitian) Schrödinger semigroup $\exp(-tH_{quant})$ is complex-valued. We have verified that the kernel of $\exp(-tH_{Eucl})$ is real-valued, but shows a number of deficits (detected before in Ref. [5]) which preclude its unrestricted usefulness *as it stands* for the description of diffusion processes.

In passing, we mention that the complex-valued integral kernel of $\exp[-\beta H_{quant}]$, with $\beta \sim 1/kT$, has an interpretation as the unnormalized density matrix (which in turn yields a legitimate partition function of the quantum statistical system) in the classic study of the diamagnetism of free electrons, [33, 34].

II. SIGNATURES OF NONEQUILIBRIUM.

Concerning the nonequilibrium properties of diffusion processes in $N \geq 2$, we restrict our attention to relaxation scenarios, within which $\rho(\vec{x},t)$ asymptotically (as $t \to \infty$) approaches a stationary (steady state) probability density $\rho_*(\vec{x})$. If $D \geq 2$ such pdf may in principle coexist with the non-vanishing steady current $j_*(\vec{x}) \neq 0$, c.f. [1]. To this end we assume form the start, that drift fields $F(\vec{x})$ are non-conservative, i.e. cannot be represented in the pure gradient form.

The diffusion current notion appears through rewriting the Fokker -Planck equation $\partial_t \rho = (1/2)\Delta\rho - \vec{\nabla}\cdot(\vec{F}\cdot\rho)$ as the continuity equation for $\rho(\vec{x},t)$:

$$\begin{aligned} \partial_t \rho = -\vec{\nabla}\cdot\vec{j} = -\vec{\nabla}\cdot(\vec{v}\rho), \\ \vec{v} = \vec{F} - \vec{\nabla}\ln\rho^{1/2}, \end{aligned} \tag{11}$$

where $\vec{v}$ is a current velocity field.

Let us assume that the Fokker-Planck equation (1) admits a stationary pdf $\rho_*(\vec{x})$. In view of $\partial_t \rho_* = 0$, the related asymptotic diffusion current $\vec{j}_* = \rho_* \vec{v}_*$, with $\vec{v_*} = \vec{F} - \vec{\nabla}\ln\rho_*^{1/2}$ needs either to vanish, $\vec{j}_*(\vec{x}) = 0$, or to be divergenceless, $\vec{\nabla}\cdot\vec{j}_* = 0$.

The choice of the drift field in gradient form $\vec{F} = \vec{\nabla}\ln\rho_*^{1/2}$, compare e.g. Ref.[16], would secure a relaxation property $\rho(\vec{x},t) \to \rho_*(\vec{x})$, with no steady current at all, since then $\vec{j}_* = 0$ identically.

By denoting $\rho_* = \exp(-2\phi)$, (that amounts to $\rho_*^{1/2} = \exp(-\phi)$, the notation predominantly used in [15, 16]) we are left with the identity $\partial_t \rho_* = -\vec{\nabla}\cdot[\rho_*(\vec{F}+\vec{\nabla}\phi)] = 0$. The non-vanishing steady current $\vec{j}_* \neq 0$ may coexist with ρ_*, if the drift field $\vec{F}$ is selected as the non-gradient one.

Let us consider non-conservative drifts, which decompose into a sum comprising any (no necessarily related to ρ_*) gradient entry $-\vec{\nabla}\phi$, and the non-gradient one $\vec{A}$ (this notation sets a correspondence with a magnetic vector potential in R^3). We have

$$\begin{aligned} \vec{F} = \vec{A} - \vec{\nabla}\phi \Longrightarrow \\ \mathcal{V} = \frac{1}{2}[(\vec{\nabla}\phi)^2 - \Delta\phi] + \frac{1}{2}[\vec{A}^2 + \vec{\nabla}\cdot\vec{A}] - \vec{A}\cdot\nabla\phi, \end{aligned} \tag{12}$$

Assuming $-\phi = \ln\rho_*^{1/2}$, we realise that the steady diffusion current $\vec{j} = \vec{A}\rho_*$ must be divergenceless. Accordingly,

$$0 = \vec{\nabla}\cdot(\vec{A}\rho_*) = (\vec{A}\cdot\vec{\nabla})\rho_* + \rho_*(\vec{\nabla}\cdot\vec{A}) \Longrightarrow \vec{\nabla}\cdot\vec{A} = 2\vec{A}\cdot\vec{\nabla}\phi. \tag{13}$$

In view of (13), the magnetic contribution to $\mathcal{V}$ in (12) reduces to $\frac{1}{2}\vec{A}^2$. If we additionally assume that $\vec{A}$ is a solenoidal vector field, $\vec{\nabla}\cdot\vec{A} = 0$, we arrive at the orthogonality relation $\vec{A}\cdot\vec{\nabla}\phi = 0$, valid for all $\vec{x} \in R^N$. This correlates $\vec{A}$ with $\vec{\nabla}\phi$.

In connection with (12), we realize that if we skip ϕ (set $\phi \equiv 0$ or equal to any constant), we stay with $\vec{F} = \vec{A}$ and $\mathcal{V} = \frac{1}{2}[\vec{A}^2 + \vec{\nabla}\cdot\vec{A}]$, which reduces to $\frac{1}{2}\vec{A}^2$ for solenoidal and constant vector fields. On the other hand, if we disregard $\vec{A}$ by setting $\vec{A} \equiv \vec{0}$, we are left with $\vec{F} = -\vec{\nabla}\phi$ and $\mathcal{V} = \frac{1}{2}[(\vec{\nabla}\phi)^2 - \Delta\phi]$.

Remark 2: The "magnetic affinity" (7)-(9) acquires a deeper meaning, if the drift field $\vec{A}(\vec{x})$ induces the non-vanishing "magnetic matrix" $\mathbf{B}[\vec{A}] : B_{ij} = \partial_i A_j - \partial_j A_i$, with $1 \leq i,j \leq N$, $\mathbf{B} \neq \mathbf{0}$. This implies $\mathbf{B}[\vec{F}] \neq \mathbf{0}$ as well. We note that for well behaved functions $\phi(\vec{x})$ (with continuous mixed derivatives), $\mathbf{B}[\vec{\nabla}\phi] = \mathbf{0}$. Accordingly, without the $\vec{A}$ contribution in Eq. (12) we are left with a conservative vector field $\vec{F} = -\vec{\nabla}\phi$. Since conservatively drifted diffusions have received an ample coverage in the literature, c.f. [15] and [16], we shall focus our attention on the non-conservative ones.

In $N = 3$ dimensions, the non-conservativeness of the vector field $\vec{F}$ stems from its non-vanishing curl, $curl\vec{F} = \vec{\nabla}\times\vec{F} \neq \vec{0}$. For the conservative (i.e. gradient) drift $\vec{F} \sim -\vec{\nabla}\phi$ we have $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$.

The non-conservative field itself $\vec{F}$ needs not to be purely rotational and may encompass both the gradient and rotational contributions on an equal footing, like in a decomposition (12), c.f. [1, 2, 5]), $\vec{F} = \vec{A} - \vec{\nabla}\phi$, where $\vec{A}$ is considered to be purely rotational. For well behaved ϕ we have $\vec{\nabla}\times(\vec{F}-\vec{A}) = 0$, in affinity with the standard gauge transformation of any electromagnetic potential, which leaves the Lorentz force intact, c.f. [9] and check the Lorentz force appearance in the Brownian motion, [28–32].

Would we have left ϕ aside by setting $\phi = 0$, the resultant drift field would reduce to $\vec{F} = \vec{A}$, with the path-wise dynamics (1), driven by the purely rotational (like solenoidal) vector field, c.f [5, 6].

III. PATH INTEGRATION HINTS.

A. Lagrangian dynamics may look "electromagnetic".

A distinctive research topic in Ref. [1] has been a discussion of general path integral formulas for propagators (fundamental solutions, transition probability densities) of Fokker-Planck equations with non-conservative drifts. That basically refers to quadratic Lagrangians, with semiclassical options for a direct computability of involved path integrals, which incorporate $N \geq 3$ analogues of the familiar $N = 3$ magnetic coupling. See e.g. also [25, 26], and classic derivations of path integral formulas for propagators of "magnetic" Schrödinger equations, analytically continued to "imaginary time", [5, 8, 9, 34].

The path integral context for non-conservatively drifted diffusion processes has been set in Refs. [25, 26], and revived in [1], through the formula "for the propagator associated with the Langevin system" (actually the integral kernel $\exp(tL^*)(y,x)$ of the operator $\exp(tL^*)$):

$$p(\vec{y},0,\vec{x},t) = \exp(L^* t)(\vec{y},\vec{x}) = \int_{\vec{x}(\tau=0)=\vec{y}}^{\vec{x}(\tau=t)=\vec{x}} \mathcal{D}\vec{x}(\tau)\, \exp\left[-\int_0^t d\tau \mathcal{L}(\vec{x}(\tau),\dot{\vec{x}}(\tau))\right], \tag{14}$$

where the τ-dynamics stems from the Euclidean Lagrangian $\mathcal{L}_{Eucl}$ (cf. Section I.A and [1, 25, 26]), hereby denoted $\mathcal{L}$:

$$\mathcal{L}(\vec{x}(\tau),\dot{\vec{x}}(\tau)) = \frac{1}{2}\left[\dot{\vec{x}}(\tau) - \vec{F}(\vec{x}(\tau))\right]^2 + \frac{1}{2}\vec{\nabla}\cdot\vec{F}(\vec{x}(\tau)) = \frac{1}{2}\dot{\vec{x}}^2(\tau) - \dot{\vec{x}}(\tau)\cdot\vec{F}(\vec{x}(\tau)) + \mathcal{V}(\vec{x}(\tau)), \tag{15}$$

with $\mathcal{V}(\vec{x})$ given by Eq. (8).

We recall that the "normal" (e.g. non-Euclidean) classical Lagrangian would have the form $L = T - V$ with $T = \dot{\vec{x}}^2/2$ and $V(\dot{\vec{x}},\vec{x},t) = \mathcal{V} - \dot{\vec{x}}\cdot\vec{F}$. See e.g. the Appendix A in Ref. [1] and [23]. Note that the diffusion-induced Lagrangian (15) actually has the Euclidean form $\mathcal{L} = \mathcal{T} + V$. This has consequences for the derived versions of the second Newton law (e.g. the sign of the derived Lorentz force analogue, in this connection see Remark in below).

Since we have in hands an explicit Lagrangian (15), (while keeping in memory its relevance for the evaluation of path integrals in the quadratic case), we may ask for the dynamical output in terms of the Euler-Lagrange equations, still without specifying detailed properties of the vector field $\vec{F}(\vec{x}(t),t)$, except for tentatively admitting a direct dependence on time. To compress the resulting formulas we pass to the notation $\vec{x} = (x_1,x_2,x_3)$, so that $V(x,\dot{x},t) = \mathcal{V}(x,t) - \sum_j \dot{x}_j F_j(x,t)$, c.f. [23]. The Euler-Lagrange equations for the Lagrangian $\mathcal{L} = \mathcal{T} + \mathcal{V}$, read:

$$\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = 0 \Longrightarrow \frac{\partial V}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{x}_i} + \frac{\partial V}{\partial \dot{x}_i}\right) = 0 \tag{16}$$

for $i = 1,2,3$. Accordingly, we have

$$\ddot{x}_i = \left(\frac{\partial \mathcal{V}}{\partial x_i} + \frac{\partial F_i}{\partial t}\right) + \sum_j B_{ij}\dot{x}_j, \tag{17}$$

where the notation $B = (B_{ij})$ refers to the antisymmetric matrix:

$$B_{ij} = \frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} \tag{18}$$

named in Ref. [1] a magnetic matrix, c.f. our discussion following Eq. (13).

In $N = 3$ dimensions, nonvanishing components of the magnetic matrix, actually define the magnetic vector

$$\vec{\nabla}\times\vec{F} = \vec{B} = (B_1 = B_{32}, B_2 = B_{13}, B_3 = B_{21}) = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1). \tag{19}$$

Denoting $\sum_j B_{ij}\dot{x}_j = F_i^{magn}$, we realize that

$$\vec{F}^{magn} = -\dot{\vec{x}}\times(\vec{\nabla}\times\vec{F}) = -\dot{\vec{x}}\times\vec{B}, \tag{20}$$

as required (up to a sign, which is opposite to that in the "classical" case) from the magnetic part of the Lorentz force.

The electric analogue of this force reads, c.f. Eq. (16) $\vec{F}^{el} = \vec{\nabla}\mathcal{V} + \partial\vec{F}/\partial t$ and is opposite to that valid in the "classical" case.

We note that the decomposition of the derived Euclidean Lorentz force $\vec{F}^{Lorentz} = \vec{F}^{el} + \vec{F}^{magn}$ into a sum of electric and magnetic contributions is not "clean". The "electric" term has explicit $\vec{A}$-dependent contributions, through $-\vec{\nabla}\mathcal{V}$ and $\partial\vec{F}/\partial t$.

In passing we note that the derived "magnetic" force is always orthogonal to the velocity, since $\vec{F}^{magn} \cdot \dot{\vec{x}} = \sum_{ij} B_{ij}\dot{\vec{x}}_i\dot{\vec{x}}_j = 0$.

To have a clear view of the above Euclidean-looking analogue of the standard Lorentz force, let us invoke the text-book wisdom, [36]. Namely, the standard ("classical") Lorentz force expression for a particle of mass m and charge q_c (any sign and size):

$$\ddot{\vec{x}} = \frac{q_c}{m}[\vec{E}(\vec{x},t) + \dot{\vec{x}} \times \vec{B}(\vec{x},t)] \tag{21}$$

derives from the classical Lagrangian (where we temporarily restore relevant dimensional constants)

$$\mathcal{L}_{cl}(\vec{x},\dot{\vec{x}},t) = \frac{1}{2}m\dot{\vec{x}}^2 + q_c\dot{\vec{x}} \cdot \vec{A}(\vec{x},t) - q_c U(\vec{x},t). \tag{22}$$

We note that by passing from $\mathcal{L}_{cl}$ to $(1/m)\mathcal{L}_{cl}$ and absorbing the q_c/m coefficient in the redefined functions $(q/m)\vec{A} = \vec{F}$ and $(q/m)U = \mathcal{V}$, we recover

$$\mathcal{L}_{cl} = \frac{1}{2}\dot{\vec{x}}^2 - [\mathcal{V} - \dot{\vec{x}} \cdot \vec{F}] = T - V \Longrightarrow -\frac{\partial V}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i}\right) = 0 \tag{23}$$

with $V = \mathcal{V} - \dot{\vec{x}} \cdot \vec{F}$. This implies the sign inversion of the inferred "classical" Lorentz force expression

$$\ddot{x}_i = -\left[\left(\frac{\partial \mathcal{V}}{\partial x_i} + \frac{\partial F_i}{\partial t}\right) + \sum_j B_{ij}\dot{x}_j\right], \tag{24}$$

if compared with the "nonclassical" outcome (17) (devoid of dimensional constants).

Accordingly, the two formalisms lead to the opposite (in sign) Lorentz forces, if evaluated for the very same charge. That in view of the sign difference on the right-hand side of the Euler-Lagrange equations (17) and (23).

Remark 3: We point out the gauge invariance of both the electric and magnetic fields, and thence the Lorentz force expressions, while to the contrary, the stochastic differential equation (1), and generators (7), (9), are sensitive to the choice of gauge. Namely, once we have given the drift field $F(\vec{x},t)$, and an (electric) potential $\phi(\vec{x},t)$, we can pass to the gauge transformed pair of functions $\vec{G} = \vec{F} + \vec{\nabla}\eta(\vec{x},t)$ and $\Phi = \phi - \partial_t\eta(\vec{x},t)$, where $\eta(\vec{x},t)$ is any scalar function. These functions determine the very same $\vec{E} = -\vec{\nabla}\Phi - \partial_t\vec{G}$ and $\vec{B} = \vec{\nabla} \times \vec{G}$ as the former pair $\vec{F}$ and ϕ, [36].

B. Lagrangian signatures of stationary pdfs.

Let us consider the action functional (e.g. minus exponent) in Eq. (14), in association with the drift field (12), where $-\phi = \ln \rho_*^{1/2}$. By following arguments of Section II.D of Ref. [1] (while adopted to our notation), we readily infer that the term $\dot{\vec{x}} \cdot \vec{F}$ in the Lagrangian (15) contributes:

$$\int_0^t \dot{\vec{x}} \cdot [-\vec{\nabla}\phi(\vec{x}(\tau)) + \vec{A}(\vec{x}(\tau))]d\tau = -\int_0^t \frac{d}{d\tau}\phi(\vec{x}(\tau))d\tau + \int_0^t \dot{\vec{x}} \cdot \vec{A}(\vec{x}(\tau))d\tau = \phi(\vec{x}(0)) - \phi(\vec{x}(t)) + \int_0^t \dot{\vec{x}} \cdot \vec{A}(\vec{x}(\tau))d\tau \tag{25}$$

to the action functional.

Thence, the related probability density function (path integral kernel of $\exp(tL^*)$) should arise in the form:

$$p(\vec{y},0,\vec{x},t) = e^{\phi(\vec{y})-\phi(\vec{x})}\, k(\vec{y},0,\vec{x},t) \tag{26}$$

where the new function $k(\vec{y},0,\vec{x},t)$ is no longer a transition probability density (does not integrate to one) but an integral kernel (e.g. the propagator) of another motion operator (to be identified in below):

$$k(\vec{y},0,\vec{x},t) = \int_{\vec{x}(\tau=0)=\vec{y}}^{\vec{x}(\tau=t)=\vec{x}} \mathcal{D}\vec{x}(\tau)\, \exp\left[-\int_0^t d\tau \mathcal{L}_{magn}(\vec{x}(\tau),\dot{\vec{x}}(\tau))\right]. \tag{27}$$

By employing the property (13) (the existence condition for steady current, c.f. Section II.D in [1]), we arrive at

$$\mathcal{V} = V + \frac{1}{2}\vec{A}^2, \qquad V(\vec{x}) = \frac{1}{2}[(\vec{\nabla}\phi)^2 - \Delta\phi], \tag{28}$$

where the functional form of V is a consequence of our gradient choice $F = -\nabla\phi$ for the drift field.

In the above,

$$\mathcal{L}_{magn}(\vec{x}(\tau), \dot{\vec{x}}(\tau)) = \frac{1}{2}\dot{\vec{x}}^2(\tau) - \dot{\vec{x}} \cdot \vec{A}(\vec{x}(\tau)) + \mathcal{V}(\vec{x}(\tau)) = \frac{1}{2}\left[\dot{\vec{x}}(\tau) - \vec{A}(\vec{x}(\tau))\right]^2 + V(\vec{x}(\tau)). \tag{29}$$

On the level of operators, the passage from the transition kernel p of (26) to k of (27), amounts to the similarity transformation, discussed in Section II.E of Ref. [1], c.f. also for analogous considerations (pertaining to conservative gradient drifts) in Section 2.4 of Ref. [16]:

$$H_{magn} = e^{\phi} L^* e^{-\phi} = -\frac{1}{2}(\vec{\nabla} - \vec{A})^2 + \mathcal{V}. \tag{30}$$

The outcome can be readily verified by resorting to the operator identity $e^{\phi}\vec{\nabla}e^{-\phi} = \vec{\nabla} - (\vec{\nabla}\phi)$.

Returning back to the formula (26), we readily recognize the term $\exp(-tH_{magn})(\vec{y}, \vec{x}) = k(\vec{y}, 0, \vec{x}, t)$, whose path integral evaluation involves $\mathcal{L}_{magn}$, c.f (27) - (30).

Let us disregard $\vec{A}$ contributions and consider $\vec{F} = -\vec{\nabla}\phi$ in the formulas (14) and (25). We readily arrive at the factorised transition probability density of the conservatively-drifted diffusion process, whose form is analogous to (26). The integral kernel $k(\vec{y}, 0, \vec{x}, t)$ takes the form

$$k_{st}(\vec{y}, 0, \vec{x}, t) = \int_{\vec{x}(\tau=0)=\vec{y}}^{\vec{x}(\tau=t)=\vec{x}} \mathcal{D}\vec{x}(\tau)\, \exp\left[-\int_0^t d\tau \mathcal{L}_{st}(\vec{x}(\tau), \dot{\vec{x}}(\tau))\right], \tag{31}$$

where (c.f. Eq. (15) and note that V replaces $\mathcal{V}$.)

$$\mathcal{L} = \mathcal{L}_{st} + \dot{\vec{x}} \cdot \vec{\nabla}\phi, \qquad \mathcal{L}_{st}(\vec{x}(\tau), \dot{\vec{x}}(\tau)) = \frac{1}{2}\dot{\vec{x}}^2(\tau) + V(\vec{x}(\tau)), \tag{32}$$

and $\mathcal{L}_{st}$ is known to give rise to the standard Feynman-Kac propagator, see e.g [9, 13, 15–17, 27].

On the operator level, one encounters the similarity transformation of the form (30), provided all $\vec{A}$ contributions are neglected. One arrives at

$$H_{st} = e^{\phi} H e^{-\phi} = -\frac{1}{2}\Delta + V. \tag{33}$$

The pertinent transformation is discussed in minute detail in Ref. [15], see also [16].

We indicate that in principle, we can consider a purely conservative drift term $\vec{F} = -\vec{\nabla}\phi$ in the Fokker-Planck equation (1). Then, the auxiliary potential $\mathcal{V}(\vec{x})$, Eq. (8), takes the form (28) with no magnetic contribution. The notation V instead of $\mathcal{V}$ of Eq. (28), distinguishes the gradient drift-induced version of the potential from the general form (8).

If $\rho_* = \exp(-2\phi)$, we have $\vec{F} = \vec{\nabla}\ln\rho_*^{1/2} = -\vec{\nabla}\phi$, [16, 17, 27] and the above potential V can be given another form, [16, 21], $V = (1/2)\Delta\rho_*^{1/2}/\rho_*^{1/2}$, which notoriously reappears in the in the literature on confined diffusion processes in classical and quantum contexts, [2, 5, 6, 16, 21, 23, 24].

Remark 4: We note that the transition probability density (26), with the Feynman-Kac kernel entry (31), can be derived directly from (14), by inserting the Lagrangian $\mathcal{L} = \mathcal{L}_{st} + \dot{\vec{x}} \cdot \vec{\nabla}\phi$ instead of (15). By formally choosing $\phi(\vec{x}) = \vec{x}^2/2$, we can rewrite $\mathcal{L}$ in the form $\mathcal{L} = \frac{1}{2}(\dot{\vec{x}} + \vec{x})^2 - \frac{3}{2}$. Remembering that $\vec{F} = -\vec{\nabla}\phi = -\vec{x}$, we hereby restore the primary form (15) of the Lagrangian $\mathcal{L} = \frac{1}{2}(\dot{\vec{x}} - \frac{1}{2}\vec{F})^2 + \vec{\nabla} \cdot \vec{F}$, while adopted to our special (conservatively drifted) case. For completeness, we note that the Fokker-Planck operator L^* related to the pdf (26), directly stems from (7) and takes the form $L^* = \frac{1}{2}\Delta + \vec{x} \cdot \vec{\nabla} + 3 = -[-\frac{1}{2}[(\Delta + \vec{x})^2 + V]$, with $V = (\vec{x}^2 - 3)/2$.

IV. A DETOUR: BROWNIAN MOTION IN A MAGNETIC FIELD.

A. Frictionless regime.

The original study of Sect. 3 in Ref. [28] refers to a phase-space description of the damped random motion of a charged particle (with charge q_e) in a constant magnetic field and fluctuating electric environment. In contrast with [28], in Ref. [29] the frictional contribution has been skipped. The magnetic field has been presumed to be constant and oriented in the z-direction of the Cartesian reference frame, $\vec{B} = (0, 0, B)$.

The frictionless Langevin-type equation in velocity space has been considered as the randomized version of the second Newton law, with the magnetic part of the Lorentz force subject to random fluctuations, in conformity with standard white-noise statistics assumptions. For clarity of discussion, we reproduce the version employed in Ref. [28]:

$$\frac{d\vec{u}}{dt} = \frac{q_c}{mc}\vec{u} \times \vec{B} + \vec{\mathcal{E}}(t). \tag{34}$$

Here $\mathcal{E}$ *is not* an electric part of the Lorentz form but represents the white noise term, which we traditionally interpret in terms of the statistics of the Wiener process increments, while rewriting Eq. (34) in the form analogous to (1) (up to the proper adjustment of parameters)

$$d\vec{u}(t) = \vec{F}(\vec{u}(t))dt + \sqrt{2q}d\vec{W}(t) \tag{35}$$

with a diffusion constant q and the forward drift given as a vector field in velocity space

$$\vec{F}(\vec{u}(t)) = -\Lambda\vec{u}(t), \tag{36}$$

where

$$\Lambda = \begin{pmatrix} 0 & -\omega_c & 0 \\ \omega_c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{37}$$

and the cyclotron frequency parameter ω_c reads $\omega_c = q_c B/mc$.

In Refs. [28, 29] one can find detailed derivations, which we skip in the present paper. An important outcome is that any appropriate probability density function $\rho(\vec{u}, t)$ is a solution of the Fokker-Planck equation of the form (2):

$$\partial_t \rho = q\Delta_{\vec{u}}\rho - \omega_c \left[\nabla_{\vec{u}} \times \vec{u}\rho\right]_{i=3} = q\Delta_{\vec{u}}\rho - (\vec{\nabla}_{\vec{u}} \cdot \vec{F})\rho = L^*_{\vec{u}}\rho, \tag{38}$$

where

$$\vec{F}(\vec{u}) = \omega_c(u_2, -u_1, 0). \tag{39}$$

The underlying Markovian process in the velocity space is time homogeneous, and has a transition probability density obeying the initial condition $p(\vec{v}_0, 0, \vec{u}, t) \to \delta^3(\vec{u} - \vec{v}_0)$ as $t \to 0$:

$$p(\vec{v}_0, 0, \vec{u}, t) = \left(\frac{1}{4\pi q t}\right)^{3/2} \exp\left(-\frac{(\vec{u} - U(t)\vec{v}_0)^2}{4qt}\right). \tag{40}$$

Here, the rotation matrix $U(t) = \exp(-t\Lambda)$ reads:

$$U(t) = \begin{pmatrix} cos(\omega_c t) & sin(\omega_c t) & 0 \\ -sin(\omega_c t) & cos(\omega_c t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{41}$$

The transition pdf (40) is a solution of the Fokker-Planck equation (38). An extension of $p(\vec{v}_0, 0, \vec{u}, t)$ to the general form of the transition pdf $p(\vec{v}, s, \vec{u}, t)$ is immediate. The time-homogeneity of the process implies that we can replace the t by $(t - s)$ label, in conjunction with the replacement of $\vec{v}_0$ by $\vec{v}$, in the formula (40).

B. Reintroducing friction.

The frictionless diffusion described above, can be deduced from the general phase-space formalism, set for the Brownian motion in a magnetic field [28], as long as we do not insist on the validity of the fluctuation-dissipation relationships, and stay on the velocity space level of the analysis.

We begin from a redefinition of the Langevin-type equation (35) to encompass the simplest damping scenario

$$\frac{d\vec{u}}{dt} = -\beta\vec{u} + \frac{q_c}{mc}\vec{u} \times \vec{B} + \vec{\mathcal{E}}(t), \tag{42}$$

where $\beta > 0$ stands for a friction coefficient. In (42) we assume $\vec{B} = (0, 0, B)$, hence a rewriting (36), (37) is still valid, except for another form of the matrix Λ, which reads:

$$\Lambda = \begin{pmatrix} \beta & -\omega_c & 0 \\ \omega_c & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}. \tag{43}$$

Accordingly (we note that Λ can be decomposed into a sum o commuting matrices):

$$e^{-t\Lambda} = e^{-\beta t}U(t) \tag{44}$$

with $U(t)$ given by Eq. (41).

The corresponding Fokker-Planck equation retains the functional form (35), but $\vec{F}$ presently looks otherwise

$$\vec{F}(\vec{u}) = (\omega_c u_2 - \beta u_1, -\omega_c u_1 - \beta u_2, -\beta u_3), \tag{45}$$

and the previous purely rotational drift $\vec{\nabla} \cdot \vec{F} = 0$ has been modified to yield $\vec{\nabla} \cdot \vec{F} = -3\beta$. The vector field $\vec{F}$, Eq. (36), can be written as a sum of purely rotational and conservative (while in velocity space) contributions, c.f. Eq. (12).

The transition probability density of the frictional diffusion process, conditioned by the initial data u_0 at $t_0 = 0$ reads:

$$p(\vec{v_0}, 0, \vec{u}, t) = \left(2\pi\frac{q}{\beta}(1 - e^{-2\beta t})\right)^{-3/2} \exp\left(-\frac{(\vec{u} - e^{-\beta t}U(t)\vec{v_0})^2}{2\frac{q}{\beta}(1 - e^{-2\beta t})}\right). \tag{46}$$

The process is time-homogeneous, hence the above formula in fact defines $p(\vec{v}, s, \vec{u}, t) = p(t - s, \vec{v}, \vec{u})$.

For any positive value of β, an asymptotic stationary pdf (stationary solution of the Fokker-Planck equation) has the form (we follow the notation of Ref. [16])

$$\rho_*(\vec{u}) = \left(\frac{\beta}{2\pi q}\right)^{3/2} \exp\left(-\frac{\beta\vec{u}^2}{2q}\right). \tag{47}$$

It is a gaussian with mean zero and variance q/β, devoid of any rotational features. The standard fluctuation-dissipation relationship can be retrieved by introducing the notation $D = q/\beta^2$, with $q = k_B T\beta/m$, [28].

C. Disregarding magnetism.

Let us keep intact the friction term in Eq. (42), but completely disregard the magnetic contribution. Setting $\vec{B} = \vec{0}$, we pass to $\vec{F} = -\beta\vec{u}$, and thence arrive at the familiar Ornstein-Uhlenbeck process. Its transition pdf comes out in the canonical form:

$$p(\vec{v_0}, 0, \vec{u}, t) = (2\pi\beta D t)^{-3/2} \exp\left(-\frac{(\vec{u} - e^{-\beta t}\vec{v_0})^2}{2\beta D(1 - e^{-2\beta t})}\right). \tag{48}$$

The OU process relaxes to the asymptotic pdf of the form (47) with $q/\beta^2 = \beta D$.

The corresponding diffusion generator (in velocity space) is

$$L = \beta^2 D\Delta_{\vec{u}} - \beta\vec{u} \cdot \vec{\nabla}_{\vec{u}}, \tag{49}$$

while the Fokker-Planck operator reads

$$L^* = \beta^2 D\Delta_{\vec{u}} + \beta\vec{\nabla}_{\vec{u}} \cdot (\vec{u}\cdot). \tag{50}$$

The process is time homogeneous, hence we can freely replace t by $(t - s)$ and $\vec{v}_0$ by $\vec{v}$, in the formula (48).

V. TRANSFORMATION TO SPATIAL NON-CONSERVATIVE PROCESSES.

All previous derivations were intended to set the grounds for a clean identification of spatial magnetic-looking diffusion processes, which are actually devoid of any obvious phase-space connotations, c.f. [28], while staying in conformity with the general formalism of Ref. [1], specified to $N = 3$.

Note that for a negatively charged particle $q_e = -|q_e|$, we have $\omega_c = -\omega$, with $\omega = |\omega_c|$ and thence $\vec{F}(\vec{u}) = \omega(-u_2, u_1, 0)$. Not incidentally, vector functions of this particular functional form, like e.g. $\vec{A} = (B/2)(-y, x, 0)$, notoriously appear in the discussion of the minimal electromagnetic coupling in quantum theory, c.f. [8, 9], and likewise in the description of non-conservative random dynamical systems, [2, 5, 25].

In view of $\vec{\nabla} \times \vec{A} = (0, 0, B) = \vec{B}$, we may interpret $\vec{A}$ as an exemplary vector potential of the magnetic field $\vec{B}$ oriented in the z-direction. We thus arrive at another (alternative to this described in Refs. [28, 29]) view on the implementation of electromagnetic perturbations of random motion, developed in the Euclidean-looking random motion theory of Refs. [2, 5].

All basic formulas of Section III can be easily transformed from the velocity space diffusion to the closely affine spatial one. Our aim is to establish a direct link with the notation of Ref. [1]. This needs $\vec{u} \to \vec{x}$ replacements, and proper adjustments of various parameters.

1. *No friction.*

We shall rely on notational conventions of [5, 8]. Let us recast Eq. (1) to the form (we set $\nu = 1/2$):

$$d\vec{X}(t) = \vec{A}(\vec{X}(t))dt + d\vec{W}(t), \tag{51}$$

where the former drift $\vec{F}(\vec{x})$ is replaced by $\vec{A}(\vec{x}) = (B/2)(-x_2, x_1, 0)$. Accordingly, the Fokker-Planck equation (2) takes the form

$$\partial_t \rho = \frac{1}{2}\Delta\rho - \vec{\nabla}(\vec{A}\rho), \tag{52}$$

with $\vec{A}(\vec{x}(t)) = +\Lambda\vec{x}(t)$, where

$$\Lambda = \begin{pmatrix} 0 & -B/2 & 0 \\ B/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{53}$$

The resultant transition probability density function retains the functional form (45), except for the obvious replacements $\vec{u} \to \vec{x}$ and $\omega_c \to -B/2$.

Clearly:

$$p(\vec{y}, s, \vec{x}, t) = \left(\frac{1}{2\pi(t-s)}\right)^{3/2} \exp\left[-\frac{(\vec{x} - U(t-s)\vec{y})^2}{2(t-s)}\right], \tag{54}$$

with

$$U(t) = \begin{pmatrix} cos(Bt/2) & -sin(Bt/2) & 0 \\ +sin(Bt/2) & cos(Bt/2) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{55}$$

and $0 \le s < t$, is a fundamental solution of Eq.(51).

The transition pdf (54) solves the pair of equations (3) and (6) with adjoint generators

$$\begin{aligned} L^* &= \frac{1}{2}\Delta - \vec{A}\vec{\nabla} = -\left[-\frac{1}{2}(\vec{\nabla} - \vec{A})^2 + \mathcal{V}\right], \\ L &= \frac{1}{2}\Delta + \vec{A}\vec{\nabla} = -\left[-\frac{1}{2}(\vec{\nabla} + \vec{A})^2 + \mathcal{V}\right], \end{aligned} \tag{56}$$

where in view of $\vec{\nabla} \cdot \vec{A} = 0$, we have $\mathcal{V} = \vec{A}^2 = (B^2/4)(x^2 + y^2)$.

We realize, that the Lagrangian (14) takes here the form (29), but without the V contribution, i.e.

$$\mathcal{L} = \frac{1}{2}\left[\dot{\vec{x}} - \vec{A}(\vec{x}(\tau))\right]^2. \tag{57}$$

2. Frictional case.

The frictional case of Section IV.B gets transformed accordingly. If we additionally set $\beta = 1$, we arrive at

$$p(\vec{y}, s, \vec{x}, t) = \left[\pi(1 - e^{-2(t-s)})\right]^{-3/2} \exp\left(-\frac{(\vec{x} - e^{-(t-s)}U(t-s)\vec{y})^2}{(1 - e^{-2(t-s)})}\right). \tag{58}$$

However, now the drift term in the Fokker-Planck equation (1) is a sum of two contributions:

$$\vec{F} = \vec{A} - \vec{x} = \vec{A} - \vec{\nabla}\phi, \tag{59}$$

where $\vec{A} = (B/2)(-y, x, 0)$, $\vec{x} = (x, y, z)$ and $\phi = \vec{x}^2/2$. We realize that $\vec{A} = (1/2)(\vec{B} \times \vec{x})$, with $\vec{B} = B(0, 0, 1)$, and thus $\vec{\nabla} \cdot \vec{F} = -3$.

The Fokker-Planck operator L^* appears in the functional form of Eq. (7), with the scalar potential (8):

$$\begin{aligned} L^* &= \frac{1}{2}\Delta - \vec{F} \cdot \vec{\nabla} + 3 = -[-\frac{1}{2}(\vec{\nabla} - \vec{F})^2 + \mathcal{V}], \\ \mathcal{V} &= \frac{1}{2}(\vec{F}^2 - 3) = \frac{\vec{A}^2}{2} + \frac{1}{2}(\vec{x}^2 - 3). \end{aligned} \tag{60}$$

It is perhaps amusing to note that $\mathcal{V}$ vanishes on the ellipsoid $\vec{A}^2 + \vec{x}^2 = 3$, while taking negative values in its interior and positive on the outer side.

At this point, we can resort to the discussion of Section II, where the condition (12) validates the coexistence of the steady current with the asymptotic stationary pdf. Indeed, the stationary pdf derives from:

$$\rho_*(\vec{x}) = \pi^{-3/2} \exp(-\vec{x}^2), \tag{61}$$

Since $\vec{A} = (B/2)(-y, x, 0)$, we have $\vec{\nabla} \cdot \vec{A} = 0$, in conjunction with $\phi(\vec{x}) = \vec{x}^2/2$. Accordingly the condition (12) is valid, and the diffusion process (57) admits the divergenceless steady current

$$j_*(\vec{x}) = \vec{A}(\vec{x})\, \rho_*(\vec{x}). \tag{62}$$

The present case actually is an explicit illustration to our discussion of Section III.B, concerning Lagrangian signatures of stationary pdfs, and the existence of steady currents. The corresponding Lagrangian stems from the general formula (15). However, in view of Eq. (25) it can be reduced to the effective form (29): $\mathcal{L}_{magn} = \frac{1}{2}\left[\dot{\vec{x}}(\tau) - \vec{A}(\vec{x}(\tau))\right]^2 + V(\vec{x}(\tau))$. which entails the evaluation of the integral kernel (27). The F-P pdf (26) emerges after accounting for the multiplicative conditioning of this kernel.

3. No magnetism.

By disregarding the magnetic contribution, we arrive at a simplified form of the transition function (58), which actually is the transition pdf of the Ornstein-Uhlenbeck process with the drift $\vec{F} = -\vec{x} = -\vec{\nabla}\phi$, where $\phi = \vec{x}^2/2$, c.f. also [16, 27], (here $0 \leq s < t$):

$$p(\vec{y}, s, \vec{x}, t) = [\pi(t - s)]^{-3/2} \exp\left(-\frac{(\vec{x} - e^{-(t-s)}\vec{y})^2}{(1 - e^{-2(t-s)})}\right). \tag{63}$$

This transition pdf is intimately intertwined with the integral kernel of $\exp(-tH)$, where H is the quantum harmonic oscillator Hamiltonian (with properly tuned parameters), [8, 16].

Namely, we have

$$p(\vec{y}, s, \vec{x}, t) = e^{3(t-s)/2}\, k(\vec{y}, s, \vec{x}, t)\, \frac{\phi_1(\vec{x})}{\phi_1(\vec{y})}, \tag{64}$$

where $\Phi_1(\vec{x}) = \pi^{-3/2} \exp(-\vec{x}^2)$, is the ground state function, while the factor 3/2 in the exponent is the lowest eigenvalue of $H = (1/2)(-\Delta + \vec{x}^2)$. The function $k(\vec{y}, s, \vec{x}, t)$ is the integral kernel of $\exp[-(t - s)H]$, see e.g. Section III.B, where H_{st} differs from H by an additive renormalisation $-3/2$.

The kernel, whose evaluation belongs to the standard path integral inventory (here, in association with the Feynman-Kac formula), has the form, [8, 9, 16]:

$$k(\vec{y},\vec{x},t) = \exp(-3t/2)(\pi[1-\exp(-2t)])^{-3/2} \exp\left[\frac{1}{2}(\vec{x}^2-\vec{y}^2) - \frac{(\vec{x}-e^{-t}\vec{y})^2}{(1-e^{-2t})}\right], \\ = (2\pi \sinh t)^{-3/2} \ \exp\left[-\frac{(\vec{x}^2+\vec{y}^2)\cosh t - 2\vec{x}\cdot\vec{y}}{2\sinh t}\right]. \tag{65}$$

It is instructive to mention that the formal replacement of t by it in Eq. (64), reproduces the familiar propagator $\exp(-itH)(\vec{y},\vec{x})$ of the quantum mechanical harmonic oscillator in R^3, [9] (keeping in memory that we have scaled away all physical constants).

We point out that the kernel (65) of the semigroup $\exp(-tH)$ has the Feynman-Kac path integral representation with the Lagrangian $\mathcal{L} = \frac{1}{2}[\dot{\vec{x}}^2(\tau) + \vec{x}^2(\tau)]$, compare e.g. (32).

Coming back to the OU probability density function (63), we may recall the arguments of Section III.B, while carried out with $\vec{A}$ skipped in all formulas. The corresponding Lagrangian has the form:

$$\mathcal{L} = \frac{1}{2}(\dot{\vec{x}} - \vec{F})^2 + \vec{\nabla}\cdot\vec{F} = \frac{1}{2}\dot{\vec{x}}^2(\tau) + \dot{\vec{x}}\cdot\vec{x} + V(\vec{x}(\tau)), \tag{66}$$

with $V = (1/2)\vec{x}^2 - 3/2$ and $\vec{F} = -\vec{x}$. The Lagrangian (66) rewrites as $\mathcal{L} = (1/2)(\dot{\vec{x}} + \vec{x})^2 - 3$, compare e.g. [12], chap. 6.2.1.17.

The related Fokker-Planck operator derives directly from (60) by skipping the $\vec{A}^2$ term. Accordingly $L^* = \frac{1}{2}\Delta + \vec{x}\cdot\vec{\nabla} + 3 = -[-\frac{1}{2}(\vec{\nabla}+\vec{x})^2 + V]$. We recall that the standard form of $L^* = (1/2)\Delta + \vec{\nabla}(\vec{x}\,\cdot)$ directly follows from Eq. (2).

VI. PROBABILISTIC SIGNIFICANCE OF PATH INTEGRAL KERNELS. OBSTACLES TO BE OBSERVED.

All analytic formulas for transition probability densities of $D = 3$ diffusion processes discussed in Section V, can be obtained (independently form our reasoning based on [28, 29]) by means of standard path integration arguments, [12, 25, 26]. The evaluation often becomes quite tedious and computationally sophisticated, c.f. comments following Eq. (22) in Ref. [25].

Although we have not evaluated explicitly any of these integrals in the present paper, in each subsection of Section V we have reproduced the form of the appropriate Lagrangian, which can be employed in the general path integral formula (15). In this connection see e.g. Appendix D in Ref. [1] for a discussion of $N \geq 2$ path integral procedures for quadratic Lagrangians. The complementary discussion for $N = 1$ can be found in [35].

Since the Euclidean map of Section I.A is quantally motivated, we need to remember about the wealth of available path integral formulas for propagators of various quantum systems, [12]. One should as well be aware of serious jeopardies threatening an uncritical usage of that map. In the context of diffusion processes, all path integral kernels of interest (not only the transition pdfs, c.f. Eqs.(26) and (31) for hints) *must necessarily need to be positive and secure semigroup properties* (akin to the Chapman-Kolmogorov identity for Markovian pdfs).

One should become alert by the fact, that transition probability densities for diffusions with non-conservative drifts, of Sections I to V, appear as (fundamental) solutions of the Fokker-Planck equation with the generator $L^* = -(H_{Eucl} + \mathcal{V})$. Thus, even for the simplest non-conservative drift $\vec{F} = \vec{A}$ with $\vec{\nabla}\cdot\vec{A} = 0$, the operator H_{Eucl} never appears alone as it stands, but is always accompanied by an additive correction (perturbation) $\vec{A}^2/2$.

This raises doubts, [5], about the probabilistic significance (as far as legitimate diffusion processes are concerned) of the integral kernel of the operator $\exp(-tH_{Eucl})$. We point out that the path integral kernel of $exp(-tH_{quant})$, with $t \geq 0$, and $H_{quant} = -\frac{1}{2}(\vec{\nabla} - i\vec{A})^2$ is complex-valued, [5, 12, 13, 34, 35]. The pertinent complex-valuedness can be removed by the supplementary Euclidean map $\vec{A} \to i\vec{A}$, [2], or $\vec{A} \to i\vec{A}$, (to keep the convention of Ref. [**?**]), c.f. the Appendix.

A. Link with Schrödinger semigroups: $\mathcal{L}_{cl} \to \mathcal{L}_{Wick}$.

In the light of our preliminary discussion in Section I.A, concerning the Euclidean map $it \to t, t \geq 0$ (actually the Wick rotation $\tau = it$), it appears useful to recall some features of this formal affinity between quantum mechanical and diffusion-type patterns of dynamical behavior (all physical constants are scaled away) in the simplest case of Hermitian Hamiltonians, when Schrödinger semigroups can be safely introduced, [12, 16].

Given $H = -(1/2)\Delta + V$ with a confining potential V, we execute: $\exp(-iHt)\psi_0 = \psi_t \Longrightarrow \exp(-tH)\Psi_0 = \Psi_t$.

In the absence of external potentials (free case) we have $H = -(1/2)\Delta$ (keep in memory our scaling $\nu \to 1/2$.) One knows that the familiar heat kernel

$$p(\vec{y},\vec{x},t) \equiv k(\vec{y},\vec{x},t) = [\exp(\frac{t}{2}\Delta)](\vec{y},\vec{x}) = (2\pi t)^{-3/2}\, \exp[-(\vec{x}-\vec{y})^2/2t]. \tag{67}$$

has a path integral representation (14) with the Lagrangian (15) reduced to $\mathcal{L} = (1/2)\dot{\vec{x}}^2(\tau)$ (i.e. $\mathcal{L}_{Wick}$ of Section I.A).

This kernel can be formally deduced by performing $t \to it$ in the free Schrödinger propagator

$$K(\vec{y},\vec{x},t) = [\exp(i\frac{t}{2}\Delta)](\vec{y},\vec{x}) = (2\pi it)^{-1/2}\, \exp[+i(\vec{x}-\vec{y})^2/2t], \tag{68}$$

which is associated with $\mathcal{L}_{cl} = (1/2)\dot{\vec{q}}^2(t)$, c.f. Section I.A., via the Feynman path integrand $\exp[i\int_0^t ds\, \mathcal{L}_{cl}(\dot{\vec{q}}(s),\vec{q}(s))]$, [9, 12].

Let us consider $H = (1/2)(-\Delta + \vec{x}^2)$, which differs from H_{st} of Section III.B by the missing additive renormalisation constant $-3/2$, c.f. (63)-(66). The integral kernel $k(\vec{y},\vec{x},t)$ of $\exp(-tH)$ is given by Eq. (65), provided we replace $(t-s)$ by t. The (Euclidean, actually Wick) Lagrangian in the path integral (Feynman-Kac) formula for (65) reads $\mathcal{L}_{Wick} = (1/2)[\dot{\vec{x}}^2(\tau) + \vec{x}^2(\tau)]$.

By formally executing $t \to it$ in the formula (65), one arrives at the quantum harmonic oscillator propagator

$$K(\vec{y},\vec{x},t) = [\exp(-iHt)](\vec{y},\vec{x}) = (2\pi i \sin t)^{-3/2} \exp\left[+i\frac{(\vec{x}^2+\vec{y}^2)\cos t - 2\vec{x}\cdot\vec{y}}{2\sin t}\right], \tag{69}$$

associated with $\mathcal{L}_{cl} = (1/2)[\dot{\vec{q}}^2(t) - \vec{q}^2(t)]$.

Note that we here bypass the problem of giving meaning to square roots of trigonometric functions when they take negative values, e.g. an issue of Maslov corrections (indices), [12].

Eq. (69) should be set against its semigroup version (c.f (65))

$$k(\vec{y},\vec{x},t) = \exp(-tH)(\vec{y},\vec{x}) = (2\pi \sinh t)^{-3/2}\ \exp\left[-\frac{(\vec{x}^2+\vec{y}^2)\cosh t - 2\vec{x}\cdot\vec{y}}{2\sinh t}\right]. \tag{70}$$

We note that the above semigroup kernel can be related to the legitimate transition pdf of the OU process (63), by means of the conditioning formula (64), see also (26).

Remark 5: In relation to the "Euclidean time" label, we may invoke the statistical physics lore of the 50-ies and 60-ties, by passing to an integral kernel of the density operator, that is parameterized by equilibrium values of the temperature. To this end one should set e.g. $t \equiv \hbar\omega/k_BT$ for a harmonic oscillator with a proper frequency ω and remember about evaluating the normalization factor $1/Z_T$, where Z_T stands for a partition function of the system, [9]. In particular, let us mention that the complex-valued integral kernel of $exp(-\beta H_{quant})$ with $\beta \sim 1/k_BT$, where T stands for a temperature, can be interpreted as an unnormalized density matrix in the study the diamagnetism of free electrons, [33, 34].

B. Charged particle in a constant vector potential.

We shall look for possible deficits of the formal map $\mathcal{L}_{cl} \to \mathcal{L}_{Wick} \to \mathcal{L}_{Eucl}$, by invoking a simple example of the Schrödinger propagator for a "free particle with a (constant) vector potential", [12], chap. 6.2.1.3. While adopted to our notation (with scaled away physical constants, presuming the charge $e = +|e| \equiv 1$), we consider the motion operator $\exp(-itH_{quant})$ with $H_{quant} = -(1/2)(\vec{\nabla} - i\vec{A})^2$, where $\vec{A}$ is a constant vector field.

The path integral formula for the integral kernel in question

$$K(\vec{y},\vec{x},t) = \int \mathcal{D}(s) \exp[i\int_0^t ds\, \mathcal{L}_{cl}(\dot{\vec{q}}(s),\vec{q}(s))] = (2\pi it)^{-3/2} \exp[i\frac{(\vec{x}-\vec{y})^2}{2t}) + i\vec{A}\cdot(\vec{x}-\vec{y})] \tag{71}$$

derives from the Lagrangian

$$\mathcal{L}_{cl} = (1/2)\dot{\vec{q}}^2(t) + \dot{\vec{q}}(t)\cdot\vec{A}. \tag{72}$$

The Euclidean map $\mathcal{L}_{cl} \to \mathcal{L}_{Wick}$ leads to the integral kernel of $\exp(-\tau H_{quant})$, deriving from the Lagrangian

$$\mathcal{L}_{Wick} = (1/2)\dot{\vec{x}}^2(\tau) - i\dot{\vec{q}}(\tau)\cdot\vec{A}. \tag{73}$$

The subsequent map $\vec{A} \to -i\vec{A}$ defines na incomplete Lagrangian (15), with the (constant) term $\vec{A}^2/2$ actually missing

$$\mathcal{L}^{inc}_{Eucl}(\vec{x}(\tau), \dot{\vec{x}}(\tau)) = \frac{1}{2}\dot{\vec{x}}^2(\tau) - \dot{\vec{x}}(\tau) \cdot \vec{A}(\vec{x}(\tau)), \tag{74}$$

and $K(\vec{y}, \vec{x}, t)$ gets transformed into

$$k^{inc}(\vec{y}, \vec{x}, t) = (2\pi t)^{-3/2} \exp\left[-\frac{(\vec{x}-\vec{y})^2}{2t} - \vec{A} \cdot (\vec{x}-\vec{y})\right]. \tag{75}$$

The "incompleteness" of both expressions (73) and (74) can in principle be healed. A clean parallel with formulas (14) and (15) may be established by incorporating an additive correction (perturbation)

$$\mathcal{L}_{Eucl} = \mathcal{L}^{inc}_{Eucl} + (1/2)\vec{A}^2, \tag{76}$$

which, via (14) asserts that we recover a legitimate transition pdf

$$p(\vec{y}, \vec{x}, t) = (2\pi t)^{-3/2} \exp\left[-\frac{(\vec{x}-\vec{y})^2}{2t} - \vec{A} \cdot (\vec{x}-\vec{y}) - t\frac{\vec{A}^2}{2}\right] = (2\pi t)^{-3/2} \exp\left\{-\frac{[(\vec{x}-\vec{y}) + t\vec{A}]^2}{2t}\right\} \tag{77}$$

of the fairly standard version of the diffusion process (1) with a *constant* vector drift field $\vec{F} \equiv \vec{A}$.

Indeed, the appearance of the additional term $\vec{A}^2/2$ in (75), amounts to multiplying the kernel (74) by $\exp[-(t/2)\vec{A}^2]$. We point out that the diffusion process pdf (76) is consistently associated with the integral kernel (74) due to: (i) positive-definiteness of the kernel, (ii) a taming factor $\exp(-t\vec{A}^2/2)$ turning k into p according to $p(\vec{y}, \vec{x}, t) = \exp(-t\vec{A}^2) k^{inc}(\vec{y}, \vec{x}, t)$.

C. Solenoidal vector potential.

Presently, we shall discuss relationships (maps) between path integral kernels (propagators) dderiving from Lagrangians (72)-(74), under the assumption the $\vec{A}$ is no longer a constant vector, but a solenoidal vector field (devoid of any explicit time-dependence). Our choice is $\vec{A} = (B/2)(-y, x, 0)$ so that $\vec{\nabla} \times \vec{A} = (0, 0, B)$. We shall assume $B = 2$ at some point in below.

The path integration with the Lagrangian of the form (72) in the action functional, is a text-book classic (we recall about scaled away physical constants, including the pre-definiton of charges which equate ± 1). The propagator reads:

$$\begin{aligned} &\exp(-itH_{quant})(\vec{y}, \vec{x}) = \left(\frac{1}{2\pi i t}\right)^{3/2} \left(\frac{B/2}{\sin(Bt/2)}\right) \cdot \\ &\exp\left\{(i/2)\left[B(-x_1 y_2 + x_2 y_1) + [(y_1 - x_1)^2 + (y_2 - x_2)^2]\frac{B/2}{\tan(Bt/2)} + \frac{(y_3 - x_3)^2}{t}\right]\right\}. \end{aligned} \tag{78}$$

with $H_{quant} = -(1/2)(\vec{\nabla} - i\vec{A})^2$.

This may be directly compared with standard Feynman's path integral formula for the charged particle in a uniform magnetic field, while setting $m = 1$, $\hbar = 1 = c$ and $\omega = B$, [9, 12].

The semigroup kernel, given by the path integral formula stemming from he Lagrangian (73) is as well a literature classic, with the wealth of various evaluation procedures, [8, 9]. It reads:

$$\begin{aligned} &\exp(-tH_{quant})(\vec{y}, \vec{x}) = \frac{B}{4\pi \sinh(Bt/2)} \left(\frac{1}{2\pi t}\right)^{1/2} \cdot \\ &\exp\left\{(iB/2)(-x_1 y_2 + x_2 y_1) - \frac{B}{4}\left[(y_1 - x_1)^2 + (y_2 - x_2)^2\right] \coth(Bt/2) - \frac{(y_3 - x_3)^2}{2t}\right\}. \end{aligned} \tag{79}$$

Note that by setting $t \to it$ in Eq. (79) (this, in conformity with the map $\tau \equiv it$ of Section I.A), we recover the formula (78).

The presence of an imaginary factor i in the first term of the exponent, makes the kernel complex-valued. This precludes its probabilistic significance within the diffusion process framework of sections I-V, were only real-valued and strictly positive kernel functions were admitted.

By *formally* passing to the imaginary magnetic field $B \to -iB$ (strictly speaking $\vec{A} \to -i\vec{A}$), we obtain (see also [5]) the real-valued kernel function

$$\exp(-tH_{Eucl})(\vec{y},\vec{x}) = \frac{B}{4\pi \sin(Bt/2)} \left(\frac{1}{2\pi t}\right)^{1/2} \cdot \\ \exp\left\{-(B/2)(-x_1y_2 + x_2y_1) - \frac{B}{4}\left[(y_1-x_1)^2 + (y_2-x_2)^2\right]\cot(Bt/2) - \frac{(y_3-x_3)^2}{2t}\right\}. \tag{80}$$

However, an obvious problem needs to be raised, [5]. Since trigonometric functions have replaced the hyperbolic ones in the expression (80), the kernel function is real-valued, but remains positive *only* for times $0 < tB/2 \in (2n\pi, 2n\pi + \pi/2)$ with $n = 0, 1, 2, ...$ [5]. This time limitation allows as well to bypass the negative sign problem for $cot(Bt/2)$, which would make the kernel an unbounded function (an exemplary signature of this misbehavior is $\cot(Bt/2) < 0$ for $Bt/2 \in (\pi/2, \pi)$, approaching $-\infty$ for $Bt/2 \to \pi$).

The above sign/unboundedness obstacles have been created by a formal transformation $\vec{A} \to -i\vec{A}$), while executed directly in the path integral kernel expression (79). To keep this formal transformation under control, we provide in the Appendix a detailed evaluation of the path integral kernel $\exp(-tH_{Eucl})(\vec{y},\vec{x})$, by employing the "canonical" methodology, suitable for quadratic Lagrangians, [9].

1. *Generators versus Lagrangians. Resume.*

Taking seriously the path integral viewpoint of Section III, we should try to verify, whether the assumption of the diffusive motion can at be reconciled with the dynamics induced by motion operators $e^{-tH_{Eucl}}$ and $e^{-tH^*_{Eucl}}$, see also Section VI. In case of solenoidal vector field $\vec{A}$, $\vec{\nabla}\cdot\vec{A} = 0$, we have:

$$H_{Eucl} = -\frac{1}{2}(\vec{\nabla} - \vec{A})^2 = -\frac{1}{2}\Delta + \vec{A}\cdot\vec{\nabla} - \frac{1}{2}\vec{A}^2, \\ H^*_{Eucl} = -\frac{1}{2}(\vec{\nabla} + \vec{A})^2 = -\frac{1}{2}\Delta - \vec{A}\cdot\vec{\nabla} - \frac{1}{2}\vec{A}^2, \tag{81}$$

and recall that H_{Eucl} and H^*_{Eucl} are Euclidean analogues of standard Hermitian operators, appropriate (up to dimensional scalings) for the quantum Schrödinger dynamics with the minimal electromagnetic coupling: $H^{\pm}_{quant} = -(1/2)(\vec{\nabla} \pm i\vec{A})^2$.

In passing w note that $H_{Eucl} + \frac{1}{2}\vec{A}^2 = -L^*$, c.f. Eq. (7), where L^* is the Fokker-Planck generator of a legitimate diffusion process with the forward drift $\vec{F} = \vec{A}$. Compare e.g. Sections V.1 and VI.B, in particular Eqs. (56).

To evaluate the propagator of $\exp(-tH_{Eucl})(\vec{y},\vec{x}) = k(\vec{y},0,\vec{x},t)$, with $H_{Eucl} = -\frac{1}{2}(\vec{\nabla}-\vec{A})^2$, we refer to (56), and the Lagrangian (57), but presently without the $\mathcal{V}$ contribution. Accordingly:

$$\mathcal{L}_{Eucl} = \frac{\dot{\vec{x}}^2}{2} - \dot{\vec{x}}\cdot\vec{A}. \tag{82}$$

For clarity of discussion, we refer to Eqs. (21), (22) in Section III, where the classical (non-Euclidean) Lagrangian for a charged particle in the electromagnetic field has been invoked, c.f. also [36].. The charge label q_c has been left undefined. Let us choose $q_c = -|q_c| = -1$, and additionally $m = 1$. Then $\mathcal{L}$ (22) for a particle in a magnetic field B takes the form (we presume $U = 0$)

$$\mathcal{L} = \frac{\dot{\vec{x}}^2}{2} + \dot{\vec{x}}\cdot\vec{A}. \tag{83}$$

We note that by choosing in (22) the opposite (positive) sign of the charge $q_c = |q_c| = 1$, we actually would have changed the sign of the second term in Eq. (83). (The same sign change would result from replacing $\vec{A}$ by $-\vec{A}$).

Accordingly, $\mathcal{L} = \frac{\dot{\vec{x}}^2}{2} - \dot{\vec{x}}\cdot\vec{A}$, (for positive charge) formally (set $m = 1$) coincides with $\mathcal{L}_{Eucl}$,(83) appropriate for the negative charge (c.f. our discussion about sign changes in the Euler-Lagrange equations in Section III).

2. *The (negative) outcome.*

The *resultant kernel function*, whose explicit evaluation has been transferred to the Appendix, ultimately appears in the form "almost identical" with (80), see e.g (A.21)

$$k(\vec{y},0,\vec{x},t) = \frac{1}{2\pi|\sin t|}\left(\frac{1}{2\pi t}\right)^{1/2} \exp\left\{-x_1y_2 + x_2y_1 - \frac{1}{2}\left[(y_1-x_1)^2 + (y_2-x_2)^2\right]\cot t - \frac{(y_3-x_3)^2}{2t}\right\}. \tag{84}$$

The kernel is real-valued, positive, but unbounded (that in view of the sign changes of the $\cot t$ factor in the Gaussian exponent). Its another serious deficit is that the semigroup composition law (akin to the Chapman-Kolmogorow identity) is not valid for all times $t \in R^+$, see e.g (A.27).

We point out that the $1/sin(Bt/2)$ of (80), in (A.21) has been replaced by the $1/|sin(t/2)|$, (we indicate that the pertinent derivation refers to a simplified case of $B = 2$). An explicit presence of the $\cot t$ multiplier in the exponent of the Gaussian function in (A.21), prohibits the validity of the semigroup composition law (see Section A.), unless $\cot t > 0$, i.e. for time intervals $t \in (2n\pi, 2n\pi + \pi/2)$ only.

We point out that contrary to the wealth of evaluation procedures, which are known for the integral kernel of $\exp(-tH_{quant})$, [8, 9], to our knowledge no explicit evaluation of the kernel for $\exp(-tH_{Eucl})$ has been available in the literature. The derivation of (A.21) stems directly from the path integration basic principles, [9, 12]. To the contrary, the expression (80), has been obtained by means of a formal analytic continuation procedure executed directly in the function $\exp(-tH_{quant})(\vec{y},\vec{x})$, (79).

The conclusion is, that the path integration procedure can be completed for $\exp[-tH_{Eucl}](\vec{x},\vec{y})$ as a matter of principle. That is regarded as the action functional evaluation problem for quadratic Lagrangians. However, the outcome (i.e. kernel functions (80) and (A.21)), is incompatible with the presumed Markovian diffusion picture. Clearly, the motion operator $\exp[-tH_{Eucl}]$ cannot be "as it stands" related to any legitimate diffusion process within the ramifications of sections I-V. This defect can be overcome by admitting perturbations by suitable scalar potentials. This was the case in our previous discussion of solutions of the Fokker-Planck equation with the generator $L^* = -(H_{Eucl} + \mathcal{V})$ in Sections V and VI.B.1. where the (indispensable to secure positivity, continuity and boundedness properties) minimal additive perturbation had the form $\vec{A}^2/2$.

Remark 6: We mention that the kernel formula for $\exp(-tH_{Eucl})$, displayed in Ref. [2], sect. 5.2, p 92, is incorrect. In addition to previous outcomes (78)-(80), we have made a direct check (tedious, with the Wolfram Mathematica 12 assistance) to demonstrate that the kernel of $\exp(-tH_{Eucl})$, Eq. (80), with $B = 2$, actually is a solution of the partial differential equation

$$\partial_t k = -H_{Eucl}k = -\frac{1}{2}\Delta k + \vec{A}\cdot\vec{\nabla}k - \frac{1}{2}\vec{A}^2 k = -\frac{1}{2}(\vec{\nabla} - \vec{A})^2 k.$$

Provided, all differentiations are carried out with respect to $\vec{x}$, and the time domain includes only intervals where $sin(t)$ and $cot(t)$ are positive, additionally avoiding the infinitesimal vicinity of troublesome points like e.g. $2n\pi + \pi/2$, $n = 0, 2, ...$ Our check is unquestionable for $0 < t < \pi/2$.

VII. LINKS WITH EUCLIDEAN QUANTUM MECHANICS: DISSIPATIVE COUNTERPART OF QUANTUM DYNAMICS IN R^+.

A. Agreeing the notation.

For clarity reasons, we refer to our previous discussion (21)-(24), see also (34) -(42) and introductory paragraphs of Section V, concerning an impact of physical dimensional constants, in particular that of the charge q_c sign choice. The particular form (39) of the vector field $\vec{F} = \omega_c(u_2, -u_1, 0)$ involves $\omega_c = q_c B/mc$ where q_c stands for the charge label. Choosing the negative charge $q_c = -|q_c|$ we introduce $\omega_c = -\omega, \omega > 0$ and thence $\vec{F} = \omega(-u_2, +u_1, 0)$. After suitable rescaling, we may pass to the discussion of Section V, where $\vec{A} = (B/2)(-y, x, 0)$ has encoded the negative charge sign input, and we have the notation fully agreed with that of sections I to V.

We recall that on physical grounds, [36], the classical Lagrangian for a charge in an electromagnetic field has the dimensional form

$$\mathcal{L}_{cl}^{phys} = \frac{1}{2}\dot{\vec{x}}^2 + \dot{\vec{x}}\cdot(q_c\vec{A}) - q_c U \tag{85}$$

where $\vec{A}$ is a magnetic potential and U is an electric (scalar) potential.

To stay in conformity with the notation of Section I.A, and our subsequent discussion of various Lagrangians, we should keep in mind that $q_c\vec{A} \equiv \vec{F}$. This observation is of relevance, once we pass to dimensionless expressions like $L^* = -(H_{Eucl} + \mathcal{V})$ with $H_{Eucl} = -\frac{1}{2}(\vec{\nabla} - \vec{F})^2$ and $\mathcal{V} = \frac{1}{2}(\vec{F}^2 + \vec{\nabla}\cdot\vec{F})$, c.f. (7)-(9).

We note that the standard dimensional form of the H_{quant}, [36], after rescaling encodes $q_c \equiv \pm|q|$, where $|q| = 1$:

$$H^{phys}_{quant} = \frac{1}{2m}(-i\hbar\vec{\nabla} - q_c\vec{A})^2 = -\frac{\hbar^2}{2m}(\vec{\nabla} - i\frac{q_c}{\hbar}\vec{A})^2 \to H_{quant} = -\frac{1}{2}[\vec{\nabla} - i(\pm\vec{A})]^2, \tag{86}$$

as used throughout the paper. We note that in Eq.(85), the sign choice for $q_c\vec{A}$ needs a parallel sign choice for q_cU. Accordingly, choosing $\vec{F} \equiv \vec{A}$ we actually make an implicit choice of $q_c = +1$ in q_cU as well.

B. Further uses of Euclidean maps.

1. *No magnetism.*

The rationale for Euclidean maps has been outlined in Section I.A. From now on, instead of introducing $\tau = it$ for $t \in R^+$ to map $\mathcal{L}_{cl}$ into $\mathcal{L}_{Wick}$, we shall directly invoke a replacement $t \to -it$. This results in $\exp(-iHt) \to \exp(-Ht)$, in conformity with $it \to t$, actually used in the preamble to Section VI.

Let us consider the formal map $t \to -it$ on the level of (properly rescaled) Schrödinger equation and its complex adjoint (ψ stands for a wave function, while $\overline{\psi}$ for its complex conjugate). Denoting $H = -\frac{1}{2}\Delta + \Omega$ the L^2 Hermitian generator of motion, with virtually any confining potential Ω (this notation is introduced for the scalar potential, to avoid confusion with previously employed $\mathcal{V}$, V and U, purpose-dependent potentials), we adopt a formal recipe:

$$\begin{aligned} i\partial_t\psi = H\psi \ &\to \ \partial_t\theta_* = -H\theta_*, \\ -i\partial_t\overline{\psi} = H\overline{\psi} \ &\to \ \partial_t\theta = H\theta. \end{aligned} \tag{87}$$

C.f. for rationale in Refs. [2]-[7] and [17]-[21]. All dynamical rules are confined to $t \geq 0$.

In terms of the Madelung (polar) decomposition of Schrödinger wave functions (we implicitly presume the L^2 normalization of ψ, so that $|\psi|^2 \equiv \rho$ has an interpretation of the probability density function) we have:

$$\begin{aligned} \psi = \exp(R + iS) &\to \theta_* = \exp(\overline{R} - \overline{S}), \\ \overline{\psi} = exp(R - iS) &\to \theta = \exp(\overline{R} + \overline{S}), \end{aligned} \tag{88}$$

where $R, S, \overline{R}, \overline{S}$ are (appropriate) real-valued functions, while the complex function $\overline{\psi}$ is a conjugate of ψ. The rationale (e.g. a more detailed relationship between R, S and $\overline{R}, \overline{S}$) can be found in Refs. [3, 4].

In view of (88), we have (L^2 normalization being implicit) $\overline{R} = \frac{1}{2}\ln(\theta_*\theta)$, and $\overline{S} = \frac{1}{2}\ln(\theta/\theta_*)$. We hereby anticipate that $\rho \to \overline{\rho} = \theta_*\theta$.

Except for the overline gained by R and S in the mapping to $\overline{R}$ and $\overline{S}$, the only really significant step in (83) is the Euclidean mapping accompanying $R \to \overline{R}$ (and thus $\rho \to \overline{\rho}$):

$$S \to i\overline{S}. \tag{89}$$

Let us check how the map (89) works, if employed *formally* in the coupled system (of local conservation laws, [28, 29]), which derives directly from the Schrödinger equation, [3, 4, 22, 23]. The Hermitian generator choice $H = -\frac{1}{2}\Delta + \Omega$, implies:

$$\begin{aligned} \partial_t R &= -\frac{1}{2}\Delta S - \vec{\nabla}R \cdot \vec{\nabla}S, \\ \partial_t S &= -\frac{1}{2}(\vec{\nabla}S)^2 + \frac{1}{2}\Delta R + \frac{1}{2}(\vec{\nabla}R)^2 - \Omega. \end{aligned} \tag{90}$$

We point out that the first equation (90) is equivalent to the continuity equation $\partial_t\rho = -\vec{\nabla}\cdot(\vec{v}\rho)$, for L-normalised $\rho = \psi\overline{\psi} = |\psi|^2$, with the current velocity $\vec{v} = \vec{\nabla}S$ in the gradient form, [22].

By setting in Eqs. (90) $R \to \overline{R}$, $t \to -it$ and $S \to i\overline{S}$, we arrive at:

$$\begin{aligned} \partial_t\overline{R} &= -\frac{1}{2}\Delta\overline{S} - \vec{\nabla}\overline{R} \cdot \vec{\nabla}\overline{S}, \\ \partial_t\overline{S} &= -\frac{1}{2}(\vec{\nabla}\overline{S})^2 - \frac{1}{2}\Delta\overline{R} - \frac{1}{2}(\vec{\nabla}\overline{R})^2 + \Omega. \end{aligned} \tag{91}$$

The first equation is again a continuity equation $\partial_t\overline{\rho} = -\vec{\nabla}\cdot(\overline{V}\rho)$ for L-normalized $\overline{\rho} = \theta\theta^*$, with $\overline{V} = \vec{\nabla}\overline{S}$ in the gradient form.

We note a conspicuous change of the sign of the potential terms, while passing from (90) to (91), compare e.g. also [3–5, 16, 17, 19]:

$$\begin{aligned} \partial_t S + \frac{1}{2}(\vec{\nabla}S)^2 = Q - \Omega, \\ \partial_t\overline{S} + \frac{1}{2}(\vec{\nabla}\overline{S})^2 = \Omega - \overline{Q}, \\ Q = Q(R) = \frac{1}{2}(\Delta R + (\vec{\nabla}R)^2) \to \overline{Q} = Q(\overline{R}), \end{aligned} \tag{92}$$

In addition to current velocity fields $\vec{v} = \vec{\nabla}S$ and $\vec{V} = \vec{\nabla}\overline{S}$, we introduce so-called osmotic velocities $\vec{u} = \vec{\nabla}R = \vec{\nabla}\ln\rho^{1/2}$, and anlgously $\vec{U}$. By applying the gradient to both sides of the dynamical laws in (92), we get the familiar hydrodynamical evolution equations (to be considered in conjunction with the corresponding continuity equations), which derive respectively from the quantum and dissipative (here we anticipate an implicit diffusive scenario) dynamics, [3, 6]:

$$\begin{aligned} \partial_t\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \vec{\nabla}(Q - \Omega), \\ \partial_t\vec{V} + (\vec{V}\cdot\vec{\nabla})\vec{V} = \vec{\nabla}(\Omega - \overline{Q}). \end{aligned} \tag{93}$$

We point out that in the derivation, the property $\frac{1}{2}\vec{\nabla}\vec{v}^2 = (\vec{v}\cdot\vec{\nabla})\vec{v}$, valid for gradient vector fields, has been employed.

We can give the specific potential Q (de Broglie-Bohm quantum potential-related, [21]) other equivalent forms. Namely, by setting $\vec{u} = \frac{1}{2}\vec{\nabla}\ln\rho$, we have

$$Q = Q(\rho) = \frac{1}{2}(\vec{u}^2 + \vec{\nabla}\cdot\vec{u}) = \frac{1}{2}\frac{\Delta\rho^{1/2}}{\rho^{1/2}}. \tag{94}$$

See e.g. also Ref. [21]. An analogous formula holds true for $\overline{Q} = Q(\overline{\rho})$, with $\vec{u} \to \vec{U} = \vec{\nabla}\overline{R}$.

2. With magnetism.

Let us consider the rescaled Schrödinger equation with the minimal magnetic coupling, [23, 36], paired with its adjoint one (here H_{quant} still is a Hermitian operator)

$$\begin{aligned} i\partial_t\psi = [-\frac{1}{2}(\vec{\nabla} - i\vec{A})^2 + \Omega] = H_{quant}\psi, \\ -i\partial\overline{\psi} = H_{quant}\overline{\psi}. \end{aligned} \tag{95}$$

Employing the Madelung decomposition (88), we arrive at the magnetic extension of the formulas (90), [22, 23]:

$$\begin{aligned} \partial_t R = -\frac{1}{2}\Delta S - \vec{\nabla}R\cdot(\vec{\nabla}S - \vec{A}) + \frac{1}{2}\vec{\nabla}\cdot\vec{A}, \\ \partial_t S = -\frac{1}{2}(\vec{\nabla}S - \vec{A})^2 + Q - \Omega. \end{aligned} \tag{96}$$

with $Q = Q(R)$ defined in (92). We introduce the current velocity in the non-gradient form (c.f. also [22])

$$\vec{v} = \vec{\nabla}S - \vec{A} \tag{97}$$

and take the gradient of the second equation (96). Since

$$\begin{aligned} \frac{1}{2}\nabla\vec{v}^2 = \vec{v}\times(\vec{\nabla}\times\vec{v}) + \vec{v}\cdot\vec{\nabla}\vec{v}, \\ \vec{\nabla}\times(\vec{v} + \vec{A}) = \vec{\nabla}\times\vec{\nabla}S = 0 \to \vec{\nabla}\times\vec{v} = -\vec{\nabla}\times\vec{A}, \end{aligned} \tag{98}$$

after setting $\vec{\nabla}\partial_t S = \partial_t\vec{V} + \partial_t\vec{A}$ we arrive at the local conservation law:

$$\partial_t\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \vec{v}\times(\vec{\nabla}\times\vec{A}) - \partial_t\vec{A} + \vec{\nabla}(Q - \Omega). \tag{99}$$

With identifications $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\partial_t \vec{A} - \vec{\nabla}\Omega$, we make explicit an impact of a minimal electromagnetic coupling on the level of the inferred local conservation law:

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \mathcal{F}_{Lorentz} + \vec{\nabla} Q, \tag{100}$$

with the Lorentz force acting upon the charge $q_c = 1$

$$\begin{aligned} \mathcal{F}_{Lorentz} &= \vec{E} + \vec{v} \times \vec{B}, \\ \vec{E} &= -\partial_t \vec{A} - \vec{\nabla}\Omega, \\ \vec{B} &= \vec{\nabla} \times \vec{A}. \end{aligned} \tag{101}$$

We point out that the case of $\partial_t \vec{A} = 0$, is of particular relevance in connection with the discussion of Sections I to V, set e.g. Eq. (17) against Eq. (24).

3. *Euclidean map in the presence of magnetism.*

Following the pattern of Section VII.B.2 we should execute the maps $t \to -it$ and $S \to iS$. Looking at the continuity equation (first equation (96)), we realise that to be left with a real-valued expression, we must perform a supplementary transformation $\vec{A} \to \pm i\vec{A}$. In turn, the choice of $\vec{A} \to +i\vec{A}$ would make incomplete the expected sign inversion of the right-hand-side of Eq. (99).

Therefore, we opt to employ the $\vec{A} \to -i\vec{A}$ convention of Ref. [1], c.f. also Sections I to III in the present paper. Accordingly, Eqs. (96) are mapped into:

$$\begin{aligned} \partial_t \overline{R} &= -\frac{1}{2}\Delta \overline{S} - \vec{\nabla}\overline{R} \cdot (\vec{\nabla}\overline{S} + \vec{A}) - \frac{1}{2}\vec{\nabla} \cdot \vec{A}, \\ \partial_t \overline{S} &= -\frac{1}{2}(\vec{\nabla}\overline{S} + \vec{A})^2 + \Omega - Q. \end{aligned} \tag{102}$$

Here, the formula (97) takes a Euclidean form $\vec{V} = \vec{\nabla}\overline{S} + \vec{A}$, and we readily recover

$$\partial_t \vec{V} + (\vec{V} \cdot \vec{\nabla})\vec{V} = -[\mathcal{F}_{Lorentz} + \vec{\nabla}\overline{Q}], \tag{103}$$

in conjunction with $\partial_t \overline{\rho} = -\vec{\nabla} \cdot (\vec{V}\overline{\rho})$. Note a conspicuous change of sign of the force term in (103), if compared with (100).

Notice that our mappings actually imply $\vec{v} \to i\vec{V}$, which we may as well insert directly in (99) to arrive at the outcome (102).

Choosing the map $t \to -it$, $S \to iS$ and $\vec{A} \to -i\vec{A}$ in Schrödinger equations (87) has profound consequences. The resultant Euclidean system does not refer to Hermitian generators of motion, but to the non-Hermitian ones:

$$\begin{aligned} -\partial_t \theta_* &= (H_{Eucl} + \Omega)\theta_*, \\ \partial_t \theta &= (H^*_{Eucl} + \Omega)\theta, \end{aligned} \tag{104}$$

where we recall the factorisation ansatz $\overline{\rho} = \theta_* \theta$, and the definition of $\overline{S} = \frac{1}{2}\ln(\theta/\theta_*)$, c.f. (88).

4. *Hasegawa's correspondence.*

We realize that by merely identifying the external potential Ω with the previously employed $\mathcal{V}$, of Eqs. (8) and (12), $\Omega \equiv \mathcal{V}$, we uncover an obvious (albeit at the moment formal) link with the diffusion processes discussed at some length in Sections I to VI. In particular, let us assume that actually the forward drift of the Langevin-type equation (1) equals $\vec{F} = \vec{A}$. Then $\Omega = \mathcal{V} = \frac{1}{2}(\vec{A}^2 + \vec{\nabla} \cdot \vec{A})$. The associated Fokker -Planck operator has the form

$$L^* = -[H_{Eucl} + \mathcal{L}] = -[-\frac{1}{2}(\vec{\nabla} - \vec{A})^2 + \frac{1}{2}(\vec{A}^2 + \vec{\nabla} \cdot \vec{A})]. \tag{105}$$

Clearly, we deal here with the special case of the system (102), provided we identify θ_* with the solution of the Fokker-Planck equation $\overline{\rho}$, while θ is set identically equal 1. This has been noticed in Ref. [6], except for the wrong sign identification preceding

Ω in the analog of our formula (101) (Eq. (9b) in [6]. Because of $L = \frac{1}{2}\Delta + \vec{F} \cdot \vec{\nabla} = -(H^*_{Eucl} + \mathcal{V})$, a particular choice of $\theta = 1$ provides a legitimate solution of the equation $[H^*_{Eucl} + \mathcal{V}]\theta = \partial_t \theta$.

It has been noticed in Ref.[6], that other (e.g. without the $\theta = 1$ restriction) factorizations of the form $\theta_* \theta = \overline{\rho}$ can be introduced, while in the framework of the coupled system (103), with the potential $\Omega = \frac{1}{2}(\vec{A}^2 + \vec{\nabla} \cdot \vec{A})$. The corresponding current velocity in the continuity equation, then acquires the form $\overline{V} = \vec{A} - \frac{1}{2}\vec{\nabla} \ln(\theta/\theta^*)$ and refers to conditioned diffusion processes (see our discussion in below).

C. Schrödinger's boundary data and interpolation problem: Deciphering diffusive dynamics.

Properly selected integral kernels of various motion operators of the form $\exp(-Ht)$, are vitally important in connection with the concept of the Schrödinger interpolating two-gate formula, and more generally in connection with the Schrödinger boundary data problem (pertains to Schrödinger bridges and Bernstein transition densities), [14, 17, 37, 38, 40] see e.g. also [2, 3, 16, 27].

Integral kernels $k(\vec{y}, s, \vec{x}, t)$ of a probabilistic significance need to obey a number of restrictions, among which we list most important for our purposes (see e.g. [2], section 4): strict positivity, joint continuity in all variables $\vec{x}, \vec{y}, s, t$, and semigroup composition property (analogue of the Chapman-Kolmogorov identity) $\int k(\vec{y}, s, \vec{z}, r) k(\vec{z}, r, \vec{x}, t) d^3z = k(\vec{y}, s, \vec{x}, t)$, for $0 \le s < r < t \le T$. (Not disregarding the boundedness of the kernel fucntion.)

Given $k(\vec{y}, s, \vec{x}, t)$ with the above properties, the generated time evolution we consider in the finite time interval $[0, T]$ $0 < s < t < T$. In principle $T > 0$ can be arbitrarily large, and $T \to \infty$ limit may be often kept under control, c.f. [27], leading to well defined conditioned diffusion processes.

Note that the original formulation of the Euclidean quantum mechanics, has been time-symmetric from the outset and defined in the time interval $[-T/2, T/2]$, [3, 4]. Our subsequent discussion refers to the shifted time span $t \in [0, T]$ instead of $t \in [-T/2, T/2]$.

We can formally introduce the (Bernstein, [2, 3, 6, 17, 27]) probability density function with respect to $\vec{x}$ and t, with a priori fixed initial $s = 0, \vec{y}$ and terminal $r = T, \vec{z}$ variables:

$$\rho_B(\vec{x}, t) = \frac{k(\vec{y}, 0, \vec{x}, t)\, k(\vec{x}, t, \vec{z}, T)}{k(\vec{y}, 0, \vec{z}, T)}. \tag{106}$$

In view of the semigroup composition property, there holds $\int \rho_B(\vec{x}, t) d^3x = 1$ for all times $t \in [0, T]$.

Given ρ_B, we may here ask for a (Markovian, possibly conditioned) diffusion process underlying the time evolution of $\rho_B(\vec{x}, t)$, [2–5, 27].

Let us introduce a more general version of Eq. (106), by resorting to functions $\theta(\vec{x}, t)$ and $\theta^*(\vec{y}, s)$, which are obtained from respectively terminal $g(\vec{x})$, and initial $f(\vec{y})$ data for the evolution in the interval $[0, T]$:

$$\begin{aligned} \theta(\vec{x}, t) &= \int k(\vec{x}, t, \vec{z}, T)\, g(\vec{z})\, d^3z \\ \theta^*(\vec{x}, t) &= \int k(\vec{u}, 0, \vec{x}, t)\, f(\vec{u})\, d^3u. \end{aligned} \tag{107}$$

Let us anticipate that (we no longer use the subscript B)

$$\rho(\vec{x}, t) = \theta^*(\vec{x}, t)\, \theta(\vec{x}, t) \tag{108}$$

actually stands for a factorised probability density function. Integrating with respect to $\vec{x}$ and admitting interchanges of involved integrals, we arrive at:

$$\int \rho(\vec{x})\, d^3x = \int d^3u \int d^3z\, f(\vec{u}) k(\vec{u}, 0, \vec{z}, T)\, g(\vec{z}) = \int d^3u \int d^3z\, m(\vec{u}, \vec{z}), \tag{109}$$

where $m(\vec{u}, \vec{z}) = f(\vec{u}) k(\vec{u}, 0, \vec{z}, T)\, g(\vec{z})$, and we impose the (Schrödinger's) boundary data restriction

$$\begin{aligned} \int d^3u\, m(\vec{u}, \vec{z}) &= \rho(\vec{z}, T), \\ \int d^3z\, m(\vec{u}, \vec{z}) &= \rho(\vec{u}, 0). \end{aligned} \tag{110}$$

It is known, that once a suitable integral kernel $k(\vec{y},s,\vec{x},t)$ is selected, then a unique solution of the boundary data problem, in terms of functions $f(\vec{y})$ and $g(\vec{x})$, can be obtained, [3, 4, 17, 37–40].

Accordingly, $\rho = \theta^*\theta$, provides a probability density function, which interpolates between the boundary pdfs in the time interval $[0,T]$. The (forward) transition probability density function of the inferred Markovian diffusion process, is given in the form

$$p(\vec{y},s,\vec{x},t) = k(\vec{y},s,\vec{x},t)\frac{\theta(\vec{x},t)}{\theta(\vec{y},s)} \tag{111}$$

and clearly gives rise to $\rho(\vec{x},t) = \int p(\vec{y},s,\vec{x},t)\rho_B(\vec{y},s)d^3y$.

We can also introduce another (backward) transition probability density function

$$p^*(\vec{y},s,\vec{x},t) = k(\vec{y},s,\vec{x},t)\frac{\theta^*(\vec{y},s)}{\theta^*(\vec{x},t)} \tag{112}$$

which induces a backward in time evolution of $\rho(\vec{x},t)$. Indeed, $\int p^*(\vec{y},s,\vec{x},t)\rho_B(\vec{x},t)d^3x = \rho(\vec{y},s)$, and allows to infer the backward drift $\vec{b}_*$ of the diffusion process.

We point out that standard approaches to the Schrödinger interpolation problem involve contractive semigroups, whose generators are Hermitian, [3, 4, 14, 17]. This is however not a must, since non-Hermitian setting may do a job as well, [2].

Given a transition probability density function $p(\vec{y},s,\vec{x},t)$ of the (Markovian) diffusion process

$$\rho(\vec{x},t) = \int p(\vec{y},s,\vec{x},t)\rho(\vec{y},s)d^3y. \tag{113}$$

The factorised form of $\rho = \theta^*\theta$, Eq. (108), is here admissible as well.

We demand $\rho(\vec{x},t)$ to obey the Fokker-Plack equation in the standard form (2),

$$\partial_t\rho = (1/2)\Delta\rho - \vec{\nabla}(\vec{b}\rho) = -\vec{\nabla}\cdot(\vec{v}\rho). \tag{114}$$

We recall that $\vec{v} = \vec{b} - \frac{1}{2}\vec{\nabla}\ln\rho$, and we have other useful relations: $\vec{b}_* = \vec{b} - \vec{u}$, $\vec{u} = \vec{\nabla}\ln\rho$.

The forward drift $\vec{b}$ derives from the general stochastic (Ito) formula, universally valid for Markovian diffusion processes, [3, 4, 22, 23]

$$D\vec{X}(t) = \vec{b}(\vec{x},t) = \lim_{\Delta t\downarrow 0}\frac{1}{\Delta t}\int(\vec{y}-\vec{x})p(\vec{x},t,\vec{y},t+\Delta t)d^3y. \tag{115}$$

The definition (115) of the forward drift field for a Markovian diffusion process, actually is a special case of the more general (Ito) formula, valid for any smooth function $f(\vec{x},t)$ of the random variable $\vec{X}(t)$, [4, 22, 23], see also [24]:

$$\begin{aligned}\lim_{\Delta s\downarrow 0}\frac{1}{\Delta s}&\left[\int p(\vec{x},t,\vec{y},t+\Delta t)f(\vec{y},t+\Delta t)d^3y - f(\vec{x},t)\right]\\ &= Df(\vec{X}(t),t) = \left[\partial_t + (\vec{b}\cdot\vec{\nabla}) + \frac{1}{2}\Delta\right]f(\vec{x},t)\end{aligned}, \tag{116}$$

where $\vec{X}(t) \equiv \vec{x}$.

The forward drift formally appears through $(D(\vec{X})(t) = \vec{b}(\vec{x},t)$, $\vec{X}(t) = \vec{x}$. The forward acceleration field is introduced accordingly, [3, 4]:

$$D^2\vec{X}(t) = D\vec{b}(\vec{X}(t),t) = \partial_t\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{b} + \frac{1}{2}\Delta\vec{b}. \tag{117}$$

We can as well evaluate the backward drift

$$D_*\vec{X}(t) = \vec{b}_*(\vec{x},t) = \lim_{\Delta t\downarrow 0}\int(\vec{x}-\vec{y})p_*(\vec{y},t-\Delta t,\vec{x},t)d^3x \tag{118}$$

and the backward acceleration formula, [2, 4]

$$D_*^2\vec{X}(t) = D_*\vec{b}_*(\vec{X}(t)) = \partial_t\vec{b}_* + (\vec{b}_*\cdot\vec{\nabla})\vec{b}_* - \frac{1}{2}\Delta\vec{b}_*. \tag{119}$$

1. *Gradient case:* $\vec{V} = \vec{\nabla} S$

In conformity with (88), let us denote

$$\begin{aligned} \vec{b} &= \vec{V} + \vec{U} = \vec{\nabla}(\overline{S} + \overline{R}), \\ \vec{b}_* &= \vec{V} - \vec{U} = \vec{\nabla}(\overline{S} - \overline{R}). \end{aligned} \tag{120}$$

We get, [4]:

$$D^2 \vec{X}(t) = D(\vec{V} + \vec{U}) = \partial_t(\vec{V} + \vec{U}) + (\vec{V} + \vec{U}) \cdot \vec{\nabla}\vec{V} + \frac{1}{2}\Delta(\vec{V} + \vec{U}) + (\vec{V} + \vec{U}) \cdot \vec{\nabla}\vec{U} \tag{121}$$

and, analogously

$$D_*^2 \vec{X}(t) = D_*(\vec{V} - \vec{U}) = \partial_t(\vec{V} - \vec{U}) + (\vec{V} - \vec{U}) \cdot \vec{\nabla}\vec{V} - \frac{1}{2}\Delta(\vec{V} - \vec{U}) + (\vec{V} - \vec{U}) \cdot \vec{\nabla}\vec{U}. \tag{122}$$

The stochastic version of the second Newton law presently reads:

$$(D^2 + D_*^2)\vec{X}(t) = \partial_t \vec{V} + \vec{V} \cdot \vec{\nabla}\vec{V} + \vec{\nabla}\overline{Q}, \tag{123}$$

where $\overline{Q}$ stems from Eq. (94), and we remember that $\vec{U}$ is a gradient field, $\vec{U} = \vec{\nabla} \ln \overline{\rho}$. In view of the second equation (93), we thus get

$$(D^2 + D_*^2)\vec{X}(t) = \vec{\nabla}\Omega, \tag{124}$$

which is a diffusion-induced analog of the second Newton law.

2. *Non-gradient case:* $\vec{V} = \vec{\nabla}\overline{S} + \vec{A}$.

Presently we need to recast the gradient definition (120) of forward and backward drifts, in conformity with the assumption (97), whose Euclidean form is $\vec{V} = \vec{\nabla}\overline{S} + \vec{A}$.

Essentially nothing changes in the reasoning (114)-118). However, presently (119) takes the non-gradient form:

$$\begin{aligned} \vec{b} &= \vec{V} + \vec{U} = \vec{\nabla}(\overline{S} + \overline{R}) + \vec{A}, \\ \vec{b}_* &= \vec{V} - \vec{U} = \vec{\nabla}(\overline{S} - \overline{R}) + \vec{A}. \end{aligned} \tag{125}$$

We notice that the formulas (121) and (122) retain their validity, and likewise the formula (123).

However now, in view of the non-gradient form of the current velocity field, we need to refer to equations (102) and (103). Since (103) rewrites as

$$\partial_t \vec{V} + (\vec{V} \cdot \vec{\nabla})\vec{V} + \vec{\nabla}\overline{Q} = -\mathcal{F}_{Lorentz}, \tag{126}$$

we recover the stochastic second Newton law in the form

$$(D^2 + D_*^2)\vec{X}(t) = -\mathcal{F}_{Lorentz} \tag{127}$$

with the (anticipated) sign inversion of the Lorentz force (c.f. for comparison (100)), which is a signature of the Euclidean version of the second Newton law.

3. *Resume.*

The above discussion effectively sets link with a family of diffusion processes, which stems from the stochastic differential equation (of the infinitesimal form (1)) with the forward drift

$$\vec{F}(\vec{x}, t) = \frac{\vec{\nabla}\theta(\vec{x}, t)}{\theta(\vec{x}, t)} + \vec{A}(\vec{x}(t)). \tag{128}$$

Notice, that by skipping the magnetic input $\vec{A}$, we are left with the well developed framework of conditioned diffusopn processes, [3–5], within which the Hermitian motion generator H_{st}, Eq. (38) might appear as a generator of the Schrödinger semigroup, whose kernel is defined via the Feynman-Kac formula.

With the corresponding transition probability density (111) and (112) in hands we can evaluate the backward drift $\vec{F}_*(\vec{x},t) = -\vec{\nabla}\ln\theta^*(\vec{x},t) + \vec{A}(\vec{x})$. The current velocity (now we turn back to the notation of sections I to V) reads $\vec{v} = \frac{1}{2}\vec{\nabla}\ln\frac{\theta}{\theta_*} + \vec{A}$.

The factorised probability density $\rho = \theta_*\theta$ and the current velocity $\vec{v}$ obey two coupled equations (the Fokker-Planck equation is hereby transformed into a continuity equation):

$$\begin{aligned}\partial_t\rho &= -\vec{\nabla}(\rho\vec{v}),\\ \partial_t\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} &= -[\mathcal{F}_{Lorentz} + \vec{\nabla}Q],\end{aligned} \tag{129}$$

c.f. Eqs. (101) and (103).

We may cross-check the validity of formulas for $\vec{F}$, $\vec{F}^*$ and v, by taking $\vec{F}$ for granted and resorting to the rewriting of the F-P equation as the continuity equation. Namely, we have $\partial_t\rho = -\vec{\nabla}\cdot[(\vec{F} - \vec{\nabla}\ln\rho^{1/2})\rho(\vec{x},t)]$. Inserting $\vec{F} = \vec{\nabla}\ln\theta + \vec{A}$, and $\rho = \theta_*\theta$, we readily arrive at $\partial_t\rho = -\vec{\nabla}(\rho\vec{v})$ with $\vec{v}$ given above.

Analogously, by means of direct substitutions, we may check that $\partial_t\rho = -\vec{\nabla}(\vec{v}\rho)$ can be rewritten as the backward Fokker-Planck equation $-\partial_t\rho = \frac{1}{2}\Delta\rho + \vec{\nabla}\cdot(\vec{F}^*\rho)$, $t \in [0,T]$, [5, 23]. We note the validity of the formula $\vec{F}^* = \vec{F} - \vec{\nabla}\ln\rho$.

This completes the identification of "magnetic perturbations" in the present framework, while allowing to bypass obstacles identified in Section VI, by admitting perturbations of Euclidean generators H_{Eucl} and H^*_{Eucl} by suitable potentials denoted Ω in the above

VIII. OUTLOOK

Although motivated by $d \geq 3$ considerations of Ref. [1], we took a liberty to stay on the level of R^3, where the potential novelties of the non-Hermitian theoretical framework could have been thoroughly tested against the current status of what is named the *Brownian motion in a magnetic field*, [28, 30–32]. An additional impetus came from the need to reconsider some observations of Ref. [5], where the probabilistic status of the integral kernel of $\exp(-tH_{Eucl})$ has been questioned. The probabilistic significance issue for integral kernels generated by non-Hermitian operators has proved to be vital for the understanding of non-conservative diffusion processes, and specifically their links (affinities, analogies, similarities) with electromagnetic perturbations of diffusing particles.

The related pros and cons discussion has been carried out logically. In Section II we gave an outline of the theoretical framework of [1], while restricted to R^3. Section III has been devoted to"magnetic"-looking Lagrangian dynamics, which actually is a Euclidean version of the standard (classic) one, with an emphasis on Lorentz force connotations. Our main motivation was to deduce (most useful in the quadratic case) Lagrangian action functionals, and next complete a standard route towards an evaluation of path integral expressions for involved propagators (actually transition pdfs, with reference to diffusion processes).

In Section IV we have recalled the original appearance of magnetic perturbations in the standard picture of the Brownian motion, [28], followed in Section V by a transcription to spatial non-conservative processes, and paid some attention to a possible meaning of retained magnetic perturbation features (not completely lost in "translation").

In view of the significance of H_{Eucl} in the discussion of Ref. [1], we have performed in Section V a complete evaluation of the integral kernel of $\exp(-tH_{Eucl})$, by starting from the first path integration principles. We have identified a numer of defective properties of this kernel, which strongly limit its probabilistic significance. For probabilistic applications (e.g. construction of transition probability densities of conditioned stochastic processes, [2, 17]) we need strictly positive kernel functions, satisfying the semigroup composition law.

These properties were employed in section VII, where we have proposed a remedy to obstacles induced by the "bare" H_{Eucl} generator of motion. This amounts to an alternative view (if compared with Brownian motion standards, [28]) on the "magnetic" dynamics of non-conservative diffusion processes. Eqs. (126)-(128) succinctly summarize this endeavour, by making explicit the (electro)"magnetic" impact on diffusion currents (strictly speaking, on the deduced current velocity of the diffusion process).

We find worth mentioning our attention paid to the identification of diffusion currents, and specifically to the dynamics of currents and related current velocities.

Appendix A: Evaluation of $\exp[-tH_{Eucl}](\vec{x},\vec{y})$.

1. The Lagrangian.

The integral kernel $k(\vec{y},0,\vec{x},t) = \exp(-tH_{Eucl})(\vec{x},\vec{y})$, where H_{Eucl}, in case of quadratic Lagrangians is known to be proportional to $\exp(-S)$, where S stands for the classical action functional. The normalization factor is given by the van Vleck formula, [1, 12], so that $k(\vec{y},0,\vec{x},t)$ takes the form

$$k(\vec{y},0,\vec{x},t) = \frac{1}{2\pi}^{3/2}\sqrt{\det\left(-\frac{\partial^2 S(\vec{y},0,\vec{x},t)}{\partial_{x_i}\partial_{y_j}}\right)}e^{-S(\vec{y},0,\vec{x},t)}. \tag{A1}$$

Recalling (14) and (27), we specify the Lagrangian $\mathcal{L}_{Eucl} = \frac{1}{2}\dot{\vec{x}}^2 - \dot{\vec{x}}\cdot\vec{A}$. The solenoidal vector field has the form $\vec{A} = (-y,x,0)$, so that $\frac{1}{2}\vec{\nabla}\times\vec{A}) = (0.0.1)$.

From now on, we skip the Euclidean label in $\mathcal{L}_{Eucl}$, since all arguments refer to the Euclidean setting. The Lagrangian, to be employed in the path integral procedure (14), (15), has the form:

$$\mathcal{L}(x,y,\dot{x},\dot{y},\dot{z}) = \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - (\dot{x}y - x\dot{y}) = \mathcal{L}_1(x,y,\dot{x},\dot{y}) + \mathcal{L}_2(\dot{z}), \tag{A2}$$

where

$$\mathcal{L}_1(x,y,\dot{x},\dot{y}) = \frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right) - (\dot{x}y - x\dot{y}) \tag{A3}$$

and

$$\mathcal{L}_2(\dot{z}) = \frac{1}{2}\dot{z}^2. \tag{A4}$$

2. Evaluation of the action functional.

Let S be the Euclidean action

$$S = \int_0^t \mathcal{L}(x,y,\dot{x},\dot{y},\dot{z})ds = \int_0^t \mathcal{L}_1(x,y,\dot{x},\dot{y})ds + \int_0^t \mathcal{L}_2(\dot{z})ds = S_1 + S_2. \tag{A5}$$

The Euler-Lagrange equations are of the form

$$\begin{aligned}
&\frac{\partial\mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \dot{y} - \frac{d}{dt}(\dot{x}-y) = 2\dot{y} - \ddot{x} = 0,\\
&\frac{\partial\mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}} = -\dot{x} - \frac{d}{dt}(\dot{y}+x) = -2\dot{x} - \ddot{y} = 0,\\
&\frac{\partial\mathcal{L}}{\partial z} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{z}} = -\frac{d}{dt}\dot{z} = 0.
\end{aligned} \tag{A6}$$

Let the initial conditions have the form $(x(0),y(0),z(0)) = (x_1,x_2,x_3)$ and $(x(t),y(t),z(t)) = (y_1,y_2,y_3)$. From the last equation of the system (79) we get

$$z(s) = c_1 + c_2 s, \tag{A7}$$

so after taking into account the boundary conditions

$$z(0) = c_1 = x_3, \qquad z(t) = c_1 + c_2 t = y_3, \tag{A8}$$

we obtain

$$z(s) = x_3 + \frac{y_3 - x_3}{t}s. \tag{A9}$$

The action S_2 can be readily calculated:

$$S_2 = \int_0^t \frac{1}{2}\dot{z}^2\, ds = \frac{1}{2}\int_0^t \left(\frac{y_3 - x_3}{t}\right)^2 ds = \frac{(y_3 - x_3)^2}{2t}. \tag{A10}$$

The first two equations of the system of equations (A.8) are slightly more complicated. From the first equation of (A.8) we get $\dot{y} = \ddot{x}/2$, which upon substituting to the second equation, leads to

$$4\dot{x} + \dddot{x} = 0. \tag{A11}$$

The solution of this equation reads

$$x(s) = c_1 + c_2\cos(2s) + c_3\sin(2s). \tag{A12}$$

By substituting the obtained $x(s)$ to $\dot{y} = \ddot{x}/2$ we get

$$\dot{y}(s) = -2c_2\cos(2s) - 2c_3\sin(2s), \qquad y(s) = c_4 - c_2\sin(2s) + c_3\cos(2s). \tag{A13}$$

Given the initial conditions, we obtain the following system of equations, from which $c_i, i = 1, 2, 3, 4$ need to be retrieved:

$$\begin{aligned} x(0) &= c_1 + c_2 = x_1, \\ x(t) &= c_1 + c_2\cos(2t) + c_3\sin(2t) = y_1, \\ y(0) &= c_4 + c_3 = x_2, \\ y(t) &= c_4 - c_2\sin(2t) + c_3\cos(2t) = y_2. \end{aligned} \tag{A14}$$

Its solution reads

$$\begin{aligned} c_1 &= \frac{1}{2}\left(x_1 + y_1 + (y_2 - x_2)\cot t\right), \\ c_2 &= \frac{1}{2}\left(x_1 - y_1 - (y_2 - x_2)\cot t\right), \\ c_3 &= \frac{1}{2}\left(x_2 - y_2 + (y_1 - x_1)\cot t\right), \\ c_4 &= \frac{1}{2}\left(x_2 + y_2 + (y_1 - x_1)\cot t\right). \end{aligned} \tag{A15}$$

After substituting (A.16), (A.17) to solutions (A.14), (A.15), and next performing integrations, we can evaluate S_1:

$$S_1 = \int_0^t \left[\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - (\dot{x}y - x\dot{y})\right] ds = x_1y_2 - x_2y_1 + \frac{1}{2}\left[(y_1 - x_1)^2 + (y_2 - x_2)^2\right]\cot t. \tag{A16}$$

Accordingly, the classical action functional reads

$$S = S_1 + S_2 = x_1y_2 - x_2y_1 + \frac{1}{2}\left[(y_1 - x_1)^2 + (y_2 - x_2)^2\right]\cot t + \frac{(y_3 - x_3)^2}{2t}. \tag{A17}$$

3. Evaluation of the van Vleck (Morette-van Hove) factor.

Let the action functional S be given by Eq. (A.19). Then, the van Vleck matrix reads

$$-\frac{\partial^2 S}{\partial x_n \partial y_m} = \begin{pmatrix} \cot t & 1 & 0 \\ -1 & \cot t & 0 \\ 0 & 0 & \frac{1}{t} \end{pmatrix}, \tag{A18}$$

where $n, m = 1, 2, 3$. The determinant of this matrix reads

$$\det\left(-\frac{\partial^2 S}{\partial x_n \partial y_m}\right) = \frac{1}{t\sin^2 t}.. \tag{A19}$$

Hence, the normalizing factor takes the form (the square root is taken as an arithmetic one in R^+

$$\sqrt{\frac{\det\left(-\frac{\partial^2 S}{\partial x_n \partial y_m}\right)}{(2\pi)^3}} = \frac{1}{2\pi|\sin t|}\left(\frac{1}{2\pi t}\right)^{1/2}. \tag{A20}$$

This gives us the final form of the sought for propagator

$$k(\vec{y},0,\vec{x},t) = \frac{1}{2\pi|\sin t|}\left(\frac{1}{2\pi t}\right)^{1/2} \exp\left\{-x_1y_2 + x_2y_1 - \frac{1}{2}\left[(y_1-x_1)^2 + (y_2-x_2)^2\right]\cot t - \frac{(y_3-x_3)^2}{2t}\right\}. \tag{A21}$$

We note, that the functional form of $\exp[-tH^*_{Eucl}](\vec{x},\vec{y})$, differs from this given in Eq. (A.21), by merely changing a sign of the $x_1y_2 - x_2y_1$ contribution in the exponent. This corresponds to the sign inversion of the involved vector potential $\vec{A}$.

4. Verification of the semigroup composition law.

If we turn back to the diffusion process (1)-(3), the semigroup identity $\exp(sL^*)\exp[(t-s)L^*] = \exp(-tL^*)$, is related to the Chapman-Kolmogorov property of transition pdfs: $p(\vec{z},0,\vec{x},t) = \int p(\vec{z},0,\vec{y},s)p(\vec{y},s,\vec{x},t)d^3y$, with $0 < s < t$. It is by no means obvious, that after passing from $-L^* = H_{Eucl} + \mathcal{V}$ to the "bare" generator H_{Euclid}, the integral kernel of $\exp(-tH_{Eucl})$ inherits the previous composition property: $p_s p_{t-s} = p_t \Rightarrow k_s k_{(t-s)} = k_t$.

Presuming that the kernel function (A.24) is a valid integral kernel of $\exp(-tH_{Eucl})$, we shall explicitly verify the validity of the composition rule $\exp(-sH_{Eucl})\exp[-(t-s)H_{Eucl}] = \exp(-tH_{Eucl})$, directly in terms of integral kernels. This amounts to demanding :

$$k(\vec{z},0,\vec{x},t) = \int k(\vec{z},0,\vec{y},s)k(\vec{y},s,\vec{x},t)d^3y, \tag{A22}$$

where the integration is over R^3.

The right-hand side of the above equation is of the form

$$\begin{aligned}
&\int k(\vec{z},0,\vec{y},s)k(\vec{y},s,\vec{x},t)d^3y = \frac{1}{(2\pi)^3}\frac{1}{\sqrt{s(t-s)}|\sin s\sin(t-s)|} \\
&\int \exp\left\{-x_1y_2 + x_2y_1 - \frac{1}{2}\left[(y_1-x_1)^2 + (y_2-x_2)^2\right]\cot s - \frac{(y_3-x_3)^2}{2s}\right. \\
&\left.-y_1z_2 + y_2z_1 - \frac{1}{2}\left[(z_1-y_1)^2 + (z_2-y_2)^2\right]\cot(t-s) - \frac{(z_3-y_3)^2}{2(t-s)}\right\} dy_1dy_2dy_3.
\end{aligned} \tag{A23}$$

One must be careful, while evaluating the Gaussian integrals. A single y_3 integration is unproblematic, since

$$I_3 = \int dy_3 \exp\left\{-\frac{(y_3-x_3)^2}{2s} - \frac{(z_3-y_3)^2}{2(t-s)}\right\} = \sqrt{\frac{2\pi s(t-s)}{t}} \exp\left[-\frac{(z_3-x_3)^2}{2t}\right]. \tag{A24}$$

The trouble appears, if we pass to the integration over y_2. We are working with Gaussian integrals, hence factors facing y_2^2 in (A.26) should be negative. For instance, if $s \in (\pi/2,\pi)$ and $t-s \in (\pi/2,\pi)$ the integration with respect to y_2, along the whole real line R, is impossible.

In case, when the integration over y_2 produces a finite outcome (e.g. when *cots* and *cot*$(t-s)$ are positive), we obtain

$$\begin{aligned}
I_2 &= \int dy_2 \exp\left\{-x_1y_2 + y_2z_1 - \frac{1}{2}(y_2-x_2)^2\cot s - \frac{1}{2}(z_2-y_2)^2\cot(t-s)\right\} \\
&= \sqrt{\frac{2\pi\sin s\sin(t-s)}{\sin t}} \exp\left\{\frac{1}{2}\left[-z_2^2\cot(t-s) - x_2^2\cot s + \frac{(z_1-x_1+z_2\cot(t-s)+x_2\cot s)^2}{\cot(t-s)+\cot s}\right]\right\}.
\end{aligned} \tag{A25}$$

The integration with respect to y_1 involves the same precautions as in the case of y_2, leading to

$$\begin{aligned}
I_1 &= \int dy_1 \exp\left\{-y_1z_2 + x_2y_1 - \frac{1}{2}(y_1-x_1)^2\cot s - \frac{1}{2}(y_1-z_1)^2\cot(t-s)\right\} \\
&= \sqrt{\frac{2\pi\sin s\sin(t-s)}{\sin t}} \exp\left\{\frac{1}{2}\left[-z_1^2\cot(t-s) - x_1^2\cot s + \frac{(x_2-z_2+z_1\cot(t-s)+x_1\cot s)^2}{\cot(t-s)+\cot s}\right]\right\}.
\end{aligned} \tag{A26}$$

An ultimate outcome, clearly is:

$$\int k(\vec{z},0,\vec{y},s)k(\vec{y},s,\vec{x},t)d^3y = \frac{1}{(2\pi)^3}\frac{1}{\sqrt{s(t-s)}|\sin s\sin(t-s)|}I_1\cdot I_2\cdot I_3 =$$
$$\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{t}}\frac{1}{|\sin t|}\exp\left\{-x_1z_2+x_2z_1-\frac{1}{2}\left[(z_1-x_1)^2+(z_2-x_2)^2\right]\cot t-\frac{(z_3-x_3)^2}{2t}\right\}. \tag{A27}$$

The composition rule surely leads to $k(\vec{z},0,\vec{x},t)$ of the form (A.21), for all $0<s<t<\pi/2$, when a simultaneous positivity of $\sin s$, $\sin t$, $\sin(t-s)$, and $\cot t$ is secured. Notice that the $\sin t$ positivity extends to $0<s<t<\pi$, but for $t\in(\pi/2,\pi)$ the coefficient $\cot t$ in the gaussian exponent becomes negative and explodes to $-\infty$ as $t\to\pi$. This rules out a consistent continuous extension of the $t\in(0,\pi/2)$ time domain, except for a sequence of disjoint time intervals $t\in(2n\pi,2n\pi+\pi/2)$, with $n=0,1,2,\ldots$.

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