PROOF OF THE GINZBURG-KAZHDAN CONJECTURE

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1. Introduction

The goal of this paper is to prove the following theorem, which in particular confirms a conjecture of Ginzburg and Kazhdan [16, Conjecture 1.3.6]:

Theorem 1.1. The variety $\overline{T^*(G/N)}$ has conical symplectic singularities.

Here, G denotes a connected semisimple group over $\mathbb{C}$, $N = [B, B]$ denotes the unipotent radical of some Borel B, and $\overline{T^*(G/N)}$ is the affinization of the quasi-affine variety $T^*(G/N)$. We also prove in Theorem 6.3 that $\overline{T^*(G/N)}$ has symplectic singularities if G is reductive, although the $\mathbb{G}_m$-action we construct is not conical if G is not semisimple.

Theorem 1.1 implies that $\overline{T^*(G/N)}$ conjecturally admits a *symplectic dual* in the sense of Braden-Licata-Proudfoot-Webster. We also prove the following theorem, which determines properties of this conjectural dual:

Theorem 1.2. The variety $\overline{T^*(G/N)}$ is $\mathbb{Q}$-factorial and has terminal singularities. Moreover, if G is simply connected then the ring of functions of $\overline{T^*(G/N)}$ is a unique factorization domain.

We refer to [20], [34], and [16] for some motivation for Theorem 1.1 and for studying $\overline{T^*(G/N)}$ more generally, and to [8], [6], [21], [4], [5], [3] for some motivation for symplectic duality and the relevance of $\overline{T^*(G/N)}$ to the mirror symmetry program.

1.1. **Outline of Proof.** In [20], it is shown that $\overline{T^*(G/N)}$ has symplectic singularities when $G = \mathrm{SL}_n$. We briefly review the approach of [20] so as to compare and contrast with the approach taken in the general case here. When $G = \mathrm{SL}_n$, the variety $\overline{T^*(G/N)}$ admits a description as a hyperkähler reduction of a certain vector space obtained from a quiver [9]. This quiver description gives a stratification of $\overline{T^*(G/N)}$ by hyperkähler varieties of even (complex) codimension, and this stratification in particular includes a dense open smooth subset denoted in [9] as Q^{hks}. A key insight in [20] is that complement of Q^{hks} in $\overline{T^*(G/N)}$ in fact has codimension *four*, which is proved by showing all other strata have codimension at least four. Results of [33], [11] then show that, to prove that $\overline{T^*(G/N)}$ has symplectic singularities, it suffices to show that $\overline{T^*(G/N)}$ is normal and that the smooth locus of $\overline{T^*(G/N)}$ admits a symplectic form, see Lemma 6.4. These facts are proved in [20] using the above stratification and the Marsden-Weinstein theorem.

A quiver type description of $\overline{T^*(G/N)}$ is not expected to exist for general G, even when $G = \mathrm{SO}_{2n}$ [8]. In proving Theorem 1.1, we circumvent this issue by constructing a smooth open subscheme $Q \subseteq \overline{T^*(G/N)}$ that we expect can be identified with Q^{hks} when $G = \mathrm{SL}_n$. We show that the complement of Q in $\overline{T^*(G/N)}$ has codimension at least four (Theorem 6.2), and argue directly that $\overline{T^*(G/N)}$ is normal and that the regular locus of $\overline{T^*(G/N)}$ admits a symplectic form. Using this, we show that $\overline{T^*(G/N)}$ has symplectic singularities in Section 6.2.

To show that the complement of Q has codimension at least four, we analyze the right T-action on $\overline{T^*(G/N)}$ and use the fact that the ring of functions on $\overline{T^*(G/N)}$ is a unique factorization domain when G is simply connected, see Corollary 3.5. In this case, the fact that the singular

Mathematics Subject Classification 2020: 14B05, 22E57, 22E10 locus of $\overline{G/N}$ has codimension at least four allows us to show that a set strictly smaller than Q (in general) has codimension at least four, see Proposition 5.16. This of course gives the fact that the complement of Q has codimension at least four for simply connected G, which we use to derive the corresponding fact for general G in Section 6.1. The fact that $\overline{T^*(G/N)}$ has terminal singularities then follows immediately from a result of Namikawa [29].

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2. Recollections on The Affine Closure of The Basic Affine Space

We now collect the facts we will use regarding $\overline{G/N}$ and $\overline{T^*(G/N)}$ and set the notation which will be used in what follows. None of the material in Section 2 is original. More details, references, and proofs can be found in works such as [2], [17], [7], [16], and [24].

2.1. **Affine Closures of Basic Affine Space and Its Cotangent Bundle.** Hereafter, in every section except[1] Section 5, we let $G_{\mathbb{Z}}$ denote some split reductive group over $\mathbb{Z}$ with choice of maximal torus $T_{\mathbb{Z}}$, and let $G := G_k$ and $T := T_k$ denote the respective base changes to $k := \mathbb{C}$. Denote by $X^\bullet(T)$ denote the lattice of characters for T, and let $X_\bullet(T)$ denote the lattice of cocharacters. Choose some Borel subgroup $B \supseteq T$, and let N denote the unipotent radical of B. Let $\mathfrak{g}$ and $\mathfrak{t}$ denote the Lie algebras of G and T respectively, and let $\mathfrak{g}^*$ and $\mathfrak{t}^*$ denote their respective dual Lie algebras. We will occasionally abuse notation by denoting $\mathfrak{t}(\mathbb{Q}) := \mathrm{Lie}(T_{\mathbb{Q}})(\mathbb{Q})$ and $\mathfrak{t}^*(\mathbb{Q}) := \mathrm{Lie}(T_{\mathbb{Q}})^*(\mathbb{Q})$. With this notation, we have isomorphisms

$$X^\bullet(T) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathfrak{t}^*(\mathbb{Q}) \tag{1}$$

and

$$X_\bullet(T) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathfrak{t}(\mathbb{Q}) \tag{2}$$

both induced by the differential.

By the algebraic Peter-Weyl theorem the ring of functions on G/N

$$A := \Gamma(G/N, \mathcal{O}_{G/N})$$

admits a grading by $X^\bullet(T)$ such that $\mathbb{C} \xrightarrow{\sim} A_0$ and $A_\lambda \neq 0$ only if λ is dominant. Moreover, it is standard (see, for example, [27, Proposition 14.26]) that the variety G/N is quasi-affine; we also reprove this fact below as an instance of a more general claim in Proposition 5.5. Therefore G/N is an open subscheme of its *affine closure* $\overline{G/N} := \mathrm{Spec}(A)$.

The projection map $\pi : T^*(G/N) \to G/N$ is affine, and so the variety $T^*(G/N)$ is also quasi-affine. In particular, we may identify it as an open subscheme of its affinization $\overline{T^*(G/N)} = \mathrm{Spec}(R)$, where

$$R := \Gamma(T^*(G/N), \mathcal{O}_{T^*(G/N)})$$

denotes the ring of global functions on $T^*(G/N)$. The map π induces a canonical map

$$\overline{\pi} : \overline{T^*(G/N)} \to \overline{G/N}$$

[1] In Section 5, we will restrict to the case where $G_{\mathbb{Z}}$ is simply connected.

induced by the inclusion $A \subseteq R$. The action of $G \times T$ on G/N gives a corresponding $G \times T$ action on $\overline{T^*(G/N)}$. We refer to the action of the subgroup $1 \times T$ as 'the' T-action on $\overline{T^*(G/N)}$ and, for any $\lambda \in X^\bullet(T)$, we let R_λ denote the graded summand of R induced from this T-action.

2.2. **Moment Map Notation.** From the $G \times T$-action on G/N, we obtain a moment map $T^*(G/N) \to \mathfrak{g}^* \times \mathfrak{t}^*$ which lifts to a map $T^*(G/N) \to \mathfrak{g}^* \times_{\mathfrak{t}^*/\!\!/W} \mathfrak{t}^*$. Since $\mathfrak{g}^* \times_{\mathfrak{t}^*/\!\!/W} \mathfrak{t}^*$ is affine, we obtain an induced map

$$\overline{\mu} : \overline{T^*(G/N)} \to \mathfrak{g}^* \times_{\mathfrak{t}^*/\!\!/W} \mathfrak{t}^*$$

by the universal property of affinization.

2.3. **Algebraic Gelfand-Graev Action on $\overline{T^*(G/N)}$.** We recall a weaker form of one of the main theorems of [16], see also [17, Section 5.5]:

Theorem 2.1. [16, Section 1.3] There is a canonical $G, T \rtimes W$ action on $\overline{T^*(G/N)}$ lifting the $G \times T$ action such that the map $\overline{\mu}$ is W-equivariant.

2.4. **Finite Generation of Functions on $T^*(G/N)$.** We now summarize some results of [17, Section 3.6] which will be used below. For any $w \in W$, let $\overline{\pi}_w$ denote the composite map $\overline{\pi} \circ w : \overline{T^*(G/N)} \to \overline{G/N}$.

Lemma 2.2. The ring R is finitely generated, and in particular Noetherian. Moreover, the morphism

$$\overline{T^*(G/N)} \xrightarrow{\overline{\mu} \times \times_w \overline{\pi}_w} \mathfrak{g}^* \times_{\mathfrak{t}^*/\!\!/W} \mathfrak{t}^* \times \bigtimes_{w \in W} \overline{G/N}$$

is a $G \times (T \rtimes W)$-equivariant closed embedding, and, for any dominant $\lambda \in X^\bullet(T)$ and any $w \in W$, the restricted multiplication map

$$M_{w,\lambda} : \mathrm{Sym}(\mathfrak{g}) \otimes_{Z\mathfrak{g}} \mathrm{Sym}(\mathfrak{t}) \otimes_k w(A_\lambda) \to R_{w\lambda}$$

is surjective, where $w(A_\lambda)$ denotes the image of A_λ in R under w.

3. Ring of Functions on $T^*(G/N)$

In this section, we study the variety $\overline{T^*(G/N)}$ and its ring of functions R. We first show that the variety $\overline{T^*(G/N)}$ is $\mathbb{Q}$-factorial, and moreover that R is a UFD when G is simply connected, in Section 3.1. We then construct a $\mathbb{G}_m$-action on $\overline{T^*(G/N)}$ in Section 3.2 and record some of its basic properties.

3.1. **Factoriality and $\mathbb{Q}$-Factoriality of Affine Closure of $T^*(G/N)$.** Recall that a normal variety is said to be $\mathbb{Q}$-*factorial* if the cokernel of the map $\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X)$ is torsion. We now show:

Proposition 3.1. The variety $\overline{T^*(G/N)}$ is $\mathbb{Q}$-factorial and, in particular, normal.

Notice that $\overline{G/N}$ and $\overline{T^*(G/N)}$ are normal since the ring of functions on any smooth variety is normal, see, for example, [32, Lemma 28.7.9]. To prove the remainder of Proposition 3.1, we first show the following:

Lemma 3.2. The class group and the Picard group of $\overline{G/N}$ are finite. Moreover, if G is simply connected, both groups are trivial.

Proof. For any normal variety, the Picard group injects into the class group, so it suffices to show the class group of $\overline{G/N}$ is finite and, when G is simply connected, is trivial. The class group of $\overline{G/N}$ agrees with the class group of G/N since its complement has codimension 2 by the stratification (4), see, for example [18, Proposition II.6.5(b)]. Since G/N is smooth, its class group and Picard group agree. However, it is known (see, for example, [27, Theorem 18.32]) that we have an exact sequence

$$X^\bullet(N) \to \mathrm{Pic}(G/N) \to \mathrm{Pic}(G) \to \mathrm{Pic}(N)$$

where $X^\bullet(N) := \mathrm{Hom}_{\mathrm{AlgGp}}(N, \mathbb{G}_m)$. Notice that $X^\bullet(N) = 0$ and $\mathrm{Pic}(N) = 0$ as N is unipotent by [27, Corollary 14.18] and [27, Proposition 14.32], respectively. Therefore $\mathrm{Pic}(G/N) \xrightarrow{\sim} \mathrm{Pic}(G)$. Now our claim follows from the fact that $\mathrm{Pic}(G)$ is finite [27, Corollary 18.23] and the fact that if G is simply connected then $\mathrm{Pic}(G)$ is trivial [27, Corollary 18.24]. □

From Lemma 3.2, we derive the following result, which in particular completes the proof of Proposition 3.1:

Proposition 3.3. The class group of $\overline{T^*(G/N)}$ is isomorphic to the class group of $\overline{G/N}$. In particular, the class group of $\overline{T^*(G/N)}$ is finite and, if G is simply connected, is trivial.

We show this after showing the following lemma:

Lemma 3.4. Assume Y is an integral quasi-affine scheme and let B denote its ring of functions so that $j : Y \to \mathrm{Spec}(B)$ is an open embedding. If B is Noetherian, then the complement of Y in $\mathrm{Spec}(B)$ has codimension at least two.

Proof. Choose some minimal prime $\mathfrak{p}$ in the complement of Y, and let Z denote the integral scheme $\mathrm{Spec}(B/\mathfrak{p})$. Letting $U := \mathrm{Spec}(B) \setminus Z$ and $X := \mathrm{Spec}(B)$, we have a containment of open subschemes

$$X \supseteq U \supseteq Y \tag{3}$$

which induces a map $j^\# : H^0(U; \mathcal{O}_U) \to H^0(Y; \mathcal{O}_Y) = B$. This map is surjective by the induced map of functions given by (3) and injective since Y, and thus $\mathrm{Spec}(B)$, are integral, and thus we see that $j^\#$ is an isomorphism. From this and (3), it follows that the restriction map

$$\mathcal{O}_X(X) \to \mathcal{O}_X(U)$$

is an isomorphism. Note that U is quasi-compact, as it is an open subset of $\mathrm{Spec}(B)$ for B Noetherian. However, for quasi-compact U, the main result of [28] gives that (3) is not an isomorphism if $\mathfrak{p}$ is a divisor, so the height of $\mathfrak{p}$ must be at least two. □

Proof of Proposition 3.3. Since the complement of $T^*(G/N)$ has codimension at least two in $\overline{T^*(G/N)}$ by Lemma 3.4, it suffices to show the analogous claim for $T^*(G/N)$. Since $T^*(G/N)$ is smooth, by the Auslander-Buchsbaum theorem its Picard group and class group agree. Therefore by Lemma 3.2 it is enough to show that the map

$$\pi^* : \mathrm{Pic}(G/N) \to \mathrm{Pic}(T^*(G/N))$$

of abelian groups is an isomorphism. This map is injective, as any line bundle $\mathcal{L}$ for which $\pi^*(\mathcal{L})$ is trivial has the property that $\mathcal{L} \cong z^*(\pi^*(\mathcal{L}))$ is also trivial, where z denotes the zero section. The surjectivity follows from [10, Corollaire IV.21.4.11, Erratum], as $\pi : T^*(G/N) \to G/N$ is a faithfully flat morphism to a normal variety whose its fibers are vector spaces. □

Since the class group of $\mathrm{Spec}(R) = \overline{T^*(G/N)}$ is trivial when G is simply connected, we immediately obtain:

Corollary 3.5. The ring R is a unique factorization domain if G is simply connected.

3.2. **Grading on Functions on $T^*(G/N)$.** We can define a $\mathbb{G}_m$-action on $\overline{T^*(G/N)}$ defined as follows. Let $2\rho^\vee$ denote the product of the positive coroots in T. This defines a map $p : \mathbb{G}_m \to T$ which we denote $\alpha \mapsto h_\alpha$. Define a $\mathbb{G}_m$-action on $G \times \mathfrak{b}$ via

$$(g, \xi)\alpha := (gh_\alpha, \alpha^2 \mathrm{Ad}_{h_\alpha^{-1}}(\xi)).$$

Since, for any $u \in N$

$$(guh_\alpha, \alpha^2 \mathrm{Ad}_{h_\alpha^{-1}} \mathrm{Ad}_{u^{-1}}(\xi)) = (gh_\alpha(h_\alpha^{-1}uh_\alpha), \alpha^2 \mathrm{Ad}_{h_\alpha^{-1}u^{-1}h_\alpha h_\alpha^{-1}}(\xi)) = (gh_\alpha u_0, \alpha^2 \mathrm{Ad}_{u_0^{-1}} \mathrm{Ad}_{h_\alpha^{-1}}(\xi))$$

where $u_0 := h_\alpha^{-1}uh_\alpha$, we see that this gives an action of $\mathbb{G}_m$ on $T^*(G/N) \cong G \overset{N}{\times} \mathfrak{b}$ and in particular equips R with a grading $R = \oplus_i R^i$, where we use superscripts for the grading to disambiguate from the $X^\bullet(T)$-grading on R of Section 2.1. For a nonzero homogeneous element $r \in R$, with respect to this grading, we let $c(r)$ denote the unique integer with $r \in R^{c(r)}$, and refer to $c(r)$ as the *c-grading* of r. We use this term since, when G is semisimple, this grading is conical, as stated in the last point of the following proposition:

Proposition 3.6. The above grading on R has the following properties:

(1) The maps $\overline{\pi}$ and $\overline{\mu}$ are $\mathbb{G}_m$-equivariant, where $\overline{G/N}$ is equipped with a $\mathbb{G}_m$-action via restricting the T-action via p and $\mathbb{G}_m$ acts on $\mathfrak{g}^* \times_{\mathfrak{t}^*/\!\!/W} \mathfrak{t}^*$ via $\alpha(\xi, \nu) = (\alpha^2\xi, \alpha^2\nu)$.
(2) For any nonzero $r \in R_\lambda$ of usual grading h_r, r is homogeneous with respect to the c-grading and moreover $c(r) = 2h_r + \langle \lambda, 2\rho^\vee \rangle$.
(3) The grading is W-equivariant–that is, $c(r) = c(wr)$ for any homogeneous nonzero $r \in R$ and $w \in W$.
(4) If G is semisimple, then the c-grading on R is *conical*–that is, R^i is nonzero only if $i \geq 0$ and $R^0 = k$.

Proof. The first claim immediately follows from the fact that π and μ are $\mathbb{G}_m$-equivariant. By Lemma 2.2 and (1), we see that (2) holds for any λ lying in the closure of the Weyl chamber $\overline{C} = 1\overline{C}$ determined by our choices of B and N. Using this as the base case, we now proceed by induction on the length of $w \in W$. For any $\lambda \in w\overline{C}$, we may choose some simple reflection s such that $sw < w$. Let $r \in R_\lambda$. By induction we see that $s(r)$ and $rs(r)$ are both homogeneous with respect to the c-grading, and thus r is as well since R is an integral domain. Letting $h_{s(r)}$ and h_r denote the respective usual gradings we then obtain

$$c(r) + 2h_{s(r)} + \langle s(\lambda), 2\rho^\vee \rangle = c(r) + c(s(r)) = c(rs(r)) = 2(h_{s(r)} + h_r) + \langle \lambda + s(\lambda), 2\rho^\vee \rangle$$

where both the first and last step use the inductive hypothesis. We therefore see

$$c(r) = 2h_r + \langle \lambda, 2\rho^\vee \rangle$$

which shows (2).

To prove (3), it suffices to show the claim on some generating set. By Lemma 2.2, we may choose this generating set whose elements are a basis of $\mathfrak{g} \oplus \mathfrak{t}$ as well as the elements $w(a)$ for all $a \in A_\lambda$ for λ dominant and $w \in W$. The former case follows from (1) and the W-equivariance of $\overline{\mu}$, so we may assume $r = w(a)$ for some $a \in A_\lambda$. However, for such r, it is known [17, Remark 3.2.2(1)], building on the analogous claim for differential operators [24, Proposition 2.9], [2], that the usual grading of $w(a)$ is $\langle \lambda - w(\lambda), \rho^\vee \rangle$. In particular, we see

$$c(w(a)) = 2\langle \lambda - w(\lambda), \rho^\vee \rangle + \langle w(\lambda), 2\rho^\vee \rangle = \langle \lambda, 2\rho^\vee \rangle$$

is independent of w. Furthermore, when G is semisimple, we have that $\langle \lambda, 2\rho^\vee \rangle > 0$ for all nonzero dominant λ. Therefore for such G each element in the set

$$\{w(A_\lambda) : w \in W\} \cup \mathfrak{g} \oplus \mathfrak{t}$$

has positive c-grading, where λ varies over the dominant nonzero weights. Since this set generates R by Lemma 2.2, we obtain (4). $\square$

Corollary 3.7. The Poisson bracket $\{\cdot,\cdot\}$ on the algebra R lowers c-grading by 2–that is, if $x \in R^i$ and $y \in R^j$, then $\{x, y\} \in R^{i+j-2}$.

Proof. Fixing x, y as above, by Proposition 3.6(2) we may assume that there exists $\lambda, \lambda' \in X^\bullet(T)$ and $h, h' \in \mathbb{Z}^{\geq 0}$ such that $i = \langle \lambda, 2\rho^\vee \rangle + 2h$, $j = \langle \lambda', 2\rho^\vee \rangle + 2h'$, and, with respect to the $\mathbb{G}_m$-action given by fiber dilation, the grading on x, respectively y, is h, respectively h'. The Poisson bracket on R preserves the T-grading and lowers the (usual) degree of a vector field by one (which can be checked locally on $T^*(G/N)$). Therefore we see that $\{x, y\} \in R_{\lambda+\lambda'}$ and its grading from the $\mathbb{G}_m$-action given by fiber dilation is $h + h' - 1$, and so its c-grading is

$$2(h + h' - 1) + \langle \lambda + \lambda', 2\rho^\vee \rangle = i + j - 2$$

using Proposition 3.6(2), as desired. □

Remark 3.8. This $\mathbb{G}_m$-action is also considered when $G = \mathrm{SL}_n$ in [20, Section 5]. In particular, it is shown that this grading is compatible with a natural $\mathbb{G}_m$-action given by identifying $\overline{T^*(\mathrm{SL}_n/N)}$ with a hyperkähler reduction of a certain vector space [9]–see [20, Proposition 5.5] for the precise statement.

4. Preliminary Results in Algebraic Geometry

4.1. **GIT Quotients of Integral Varieties.** We now record two properties of GIT quotients of integral affine varieties with $\mathbb{G}_m$-actions which will be used below.

Lemma 4.1. Assume S is a graded integral finitely generated k-algebra such that there is a positively graded or negatively graded homogeneous element x. Then $\dim(\mathrm{Spec}(S) /\!\!/ \mathbb{G}_m) \leq \dim(\mathrm{Spec}(S)) - 1$, where we equip $\mathrm{Spec}(S)$ with the $\mathbb{G}_m$-action corresponding to the grading on S.

Proof. The fact that S is an integral domain implies that the multiplication map $S_0 \otimes_k k[x] \to S$ is injective, since we may check if an element of S is nonzero by checking if each projection onto each homogeneous summand of S is nonzero. We thus see that the morphism

$$\mathrm{Spec}(S) \to \mathrm{Spec}(S_0 \otimes_k k[x]) \cong \mathrm{Spec}(S_0) \times \mathbb{A}^1 \cong \mathrm{Spec}(S) /\!\!/ \mathbb{G}_m \times \mathbb{A}^1$$

is dominant. Therefore we see that

$$\dim(\mathrm{Spec}(S) /\!\!/ \mathbb{G}_m) + 1 = \dim(\mathrm{Spec}(S) /\!\!/ \mathbb{G}_m) + \dim(\mathbb{A}^1) = \dim(\mathrm{Spec}(S) /\!\!/ \mathbb{G}_m \times \mathbb{A}^1) \leq \dim(\mathrm{Spec}(S))$$

where the second equality uses the fact that S is finitely generated and therefore in particular Noetherian. □

Recall that, for any affine variety $\mathrm{Spec}(S)$ with an action of $\mathbb{G}_m$, this action is determined by a $\mathbb{Z}$-grading on the ring S. Moreover, the fixed point subscheme $\mathrm{Spec}(S)^{\mathbb{G}_m}$ is a closed subscheme cut out by the ideal I generated by the S_i for $i \neq 0$. We have an induced map

$$\mathrm{Spec}(S/I) = \mathrm{Spec}(S)^{\mathbb{G}_m} \to \mathrm{Spec}(S) /\!\!/ \mathbb{G}_m = \mathrm{Spec}(S_0)$$

by composing the inclusion and the quotient map. Moreover, the induced map of rings $S_0 \to S/I$ is surjective since $f = \sum_i f_i \in S$ is congruent to f_0 in S/I. This proves the following observation:

Proposition 4.2. For any affine variety $\mathrm{Spec}(S)$ with an action of $\mathbb{G}_m$, the induced map $\mathrm{Spec}(S)^{\mathbb{G}_m} \to \mathrm{Spec}(S) /\!\!/ \mathbb{G}_m$ is a closed embedding.

4.2. **Fiber of Projection Over Smooth Locus.** Next, we compute the fiber of $\overline{\pi}$ over its smooth locus $\mathcal{S}$. Notice that the complement of $T^*(G/N)$ in $T^*(\mathcal{S})$ has codimension ≥ 2 by the stratification (4). From this, we see that $\mathcal{O}(T^*(\mathcal{S})) \xrightarrow{\sim} \mathcal{O}(T^*(G/N))$, and therefore we obtain a canonical map $\sigma : T^*(\mathcal{S}) \to \overline{T^*(G/N)} \times_{\overline{G/N}} \mathcal{S}$.

Proposition 4.3. The map σ is an isomorphism.

Proof. For any variety Y, we let $\mathcal{T}_Y$ denote its tangent sheaf. Notice that we have a canonical map

$$\overline{T^*(G/N)} \to \mathrm{Spec}(\mathrm{Sym}^{\bullet}_{\overline{G/N}}(\mathcal{T}_{\overline{G/N}}))$$

over $\overline{G/N}$ given by the universal property of affinization and the fact that $\mathrm{Spec}(\mathrm{Sym}^{\bullet}_{\overline{G/N}}(\mathcal{T}_{\overline{G/N}}))$ is affine. Therefore, we obtain a canonical map ϕ given by the composite

$$\overline{T^*(G/N)} \times_{\overline{G/N}} \mathcal{S} \to \mathrm{Spec}(\mathrm{Sym}^{\bullet}_{\overline{G/N}}(\mathcal{T}_{\overline{G/N}})) \times_{\overline{G/N}} \mathcal{S} \cong T^*(\mathcal{S})$$

where this isomorphism is given by the fact that $\mathcal{S}$ is smooth, for which the composite

$$T^*(\mathcal{S}) \xrightarrow{\sigma} \overline{T^*(G/N)} \times_{\overline{G/N}} \mathcal{S} \xrightarrow{\phi} T^*(\mathcal{S})$$

is the identity. Since this map is a section and ϕ is separated we see that σ is a closed embedding of equidimensional integral schemes, and therefore is an isomorphism. $\square$

5. The Singular Locus for Simply Connected G

In this subsection, we study the singular locus of $\overline{T^*(G/N)}$ in the case where G is simply connected. For such G we also introduce the subset Q and study its basic properties. To avoid excessive repetition, *in all of Section 5 we assume that G is simply connected.*

5.1. **Stratification of Affine Closure of G/N.** For any subset of simple coroots or subset of simple roots θ, we let P_θ denote the standard parabolic subgroup containing B and labeled by θ. The goal of Section 5.1 is to prove the following well known proposition whose proof we are unable to find in the literature:

Proposition 5.1. The variety G/N is quasi-affine and its affine closure admits a stratification

$$\overline{G/N} = \bigcup_{\theta} G/[P_\theta, P_\theta] \tag{4}$$

into the orbits of the action of $G \times T$, where θ varies over all subsets of simple roots. Moreover, the smooth locus of $\overline{G/N}$ contains all locally closed subschemes $G/[P_\theta, P_\theta]$ where $|\theta| = 1$.

The first sentence of Proposition 5.1, whose proof heavily uses ideas of [23], also holds by similar methods in a more general context where $N = [B, B]$ is replaced by the commutator of an arbitrary parabolic subgroup, see Proposition 5.5. We use this generalization to prove the second sentence of Proposition 5.1.

5.1.1. *Commutator of Parabolic.* Choose a subset I of the set of simple coroots $\Delta^\vee$ and let J denote its complement. For each simple coroot c, we let $\omega_c \in X^\bullet(T)$ denote the fundamental weight dual to c and choose a nonzero highest weight vector $\vec{v}_c$ in the irreducible representation V_c of highest weight ω_c. (Each $\vec{v}_i$ is of course unique up to nonzero scalar multiple and our constructions will be independent of our choices of $\vec{v}_i$.) If we let L_j denote the line spanned by $\vec{v}_j$ for every $j \in J$, then $P_{\Delta^\vee \setminus \{j\}}$ is stabilizer of L_j, and moreover $P_I = \cap_{j \in J} P_{\Delta^\vee \setminus \{j\}}$, both of which can be seen from for example the fact that any subgroup of G containing our choice of Borel subgroup is a standard parabolic subgroup [19, Theorem 29.3(a)] by computing the simple root vectors in the Lie algebra of the group. (In particular, $P_\emptyset = B$.) When I contains a single element i we will also let $P_i := P_{\{i\}}$

and, when there is no danger of confusion, we will also denote P_i by P_α where α is the simple root associated to i.

We also define the G-representation $V_J := \oplus_{j\in J} V_j$, and let $E_J := \mathrm{span}_{V_J}\{\vec{v}_j : j \in J\}$ which is a subspace of V_J which is closed under the action of P_I and has dimension $|J|$. We naturally obtain a representation

$$\rho_I : P_I \to \mathrm{Aut}(E_J) = \mathbb{G}_m^{|J|} \tag{5}$$

of P_I whose restriction along the map

$$\alpha_J^\vee := \prod_{\alpha^\vee \in J} \alpha^\vee : \mathbb{G}_m^{|J|} \to P_I \tag{6}$$

is an isomorphism. Therefore $\alpha_J^\vee$ is a closed embedding, and moreover if we denote the image of $\alpha_J^\vee$ by $\mathbb{G}_m^J$ we see that by for example [27, Proposition 2.34] we have a semidirect product decomposition

$$P_I \xleftarrow{\sim} Q_I \rtimes \mathbb{G}_m^J \tag{7}$$

where Q_I denotes the kernel of the map of (5), or equivalently the intersection of stabilizers of the $\vec{v}_j$ for $j \in J$. Moreover, since $\mathbb{G}_m^{|J|}$ is abelian, Q_I contains $[P_I, P_I]$ and, since $[P_I, P_I]$ and Q_I are affine (as they are closed subgroup schemes of G), the inclusion map $[P_I, P_I] \hookrightarrow Q_I$ is a closed embedding.

Proposition 5.2. The closed embedding $[P_I, P_I] \hookrightarrow Q_I$ is an isomorphism.

Proof. By (7) we have an isomorphism of varieties $P_I/\mathbb{G}_m^J \xleftarrow{\sim} Q_I$, so Q_I is connected. A theorem of Cartier gives any affine algebraic group in characteristic zero is smooth (see for example [27, Theorem 3.23]) and so any connected affine algebraic group over our characteristic zero field is in particular irreducible; it thus suffices to show that this closed embedding induces an equality of the Lie algebras of $[P_I, P_I]$ and Q_I. Moreover, both $[P_I, P_I]$ and Q_I are normalized by P_I so their Lie algebras in particular admit T-representations. However, the Lie algebra of $[P_I, P_I]$ contains all of $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and each $h_\beta := [e_\beta, f_\beta]$ and each $f_\beta = -2[h_\beta, f_\beta]$ for every simple coroot $\beta^\vee \in I$ where $e_\beta \in \mathfrak{n}_\beta$ and $f_\beta \in \mathfrak{n}_\beta^-$ yield an $\mathfrak{sl}_2$-triple. Therefore the tangent spaces of both groups of Proposition 5.2 are equal, as desired. ☐

We can use similar methods to the proof of Proposition 5.2 to obtain an explicit description of the G-stabilizer of any vector $\vec{v} \in E_J$. By the semidirect product decomposition of (7), to compute the stabilizer of any vector it suffices to compute the stabilizer of $\vec{v}_{\tilde{J}} := \sum_{j\in\tilde{J}} \vec{v}_j$ for any fixed subset $\tilde{J} \subseteq J$. We describe this in terms of $\tilde{I} := \Delta^\vee \setminus \tilde{J}$:

Corollary 5.3. The stabilizer $S_{\vec{v}_{\tilde{J}}}$ of $\vec{v}_{\tilde{J}}$ is $Q_{\tilde{I}}$.

Proof. Letting $S_{\vec{v}_j}$ denote the stabilizer of $\vec{v}_j$, we have

$$S_{\vec{v}_{\tilde{J}}} = \cap_{j\in\tilde{J}} S_{\vec{v}_j}$$

since $V_{\tilde{J}} = \oplus_{j\in\tilde{J}} V_j$. Since the stabilizer of a vector is contained in the stabilizer of the line it spans, the stabilizer of $\vec{v}_j$ in G is $Q_{\Delta^\vee\setminus\{j\}}$ for any $j \in \Delta^\vee$; in other words, $S_{\vec{v}_j} = Q_{\Delta^\vee\setminus\{j\}}$. Therefore

$$\cap_{j\in\tilde{J}} S_{\vec{v}_j} = \cap_{j\in\tilde{J}} Q_{\Delta^\vee\setminus\{j\}} \subseteq \cap_{j\in\tilde{J}} P_{\Delta^\vee\setminus\{j\}} = P_I$$

and so in particular we see that

$$P_I \cap \bigcap_{j\in\tilde{J}} S_{\vec{v}_{\tilde{j}}} = \cap_{j\in\tilde{J}} S_{\vec{v}_j}.$$

However, the kernel $Q_{\tilde{I}}$ of the representation $\rho_{\tilde{I}}$ is evidently the intersection

$$\cap_{j\in\tilde{J}} (S_{\vec{v}_{\tilde{j}}} \cap P_I) = P_I \cap \bigcap_{j\in\tilde{J}} S_{\vec{v}_{\tilde{j}}}$$

so our claim follows from combining these above equalities. ☐

Remark 5.4. We thank the referee for suggesting the above proof of Corollary 5.3, which is more simple than the proof originally given by the author.

5.1.2. *Stratification of Affine Closure of* $G/[P_I, P_I]$. From Proposition 5.2, we can now show the following claim, which when $I = \emptyset$ recovers the first sentence of Proposition 5.1:

Proposition 5.5. The variety $G/[P_I, P_I]$ is quasi-affine. Moreover, we have a $G \times T$-equivariant stratification of its affine closure

$$\overline{G/[P_I, P_I]} = \bigcup_{K \supseteq I} G/[P_K, P_K] \tag{8}$$

such that $G/[P_{K'}, P_{K'}]$ is in the closure of $G/[P_K, P_K]$ if and only if $K \subseteq K'$.

Proof. Notice that the representation morphism $\rho : G \times V_J \to V_J$ induces a map $\tilde{\rho} : G \overset{P_I}{\times} V_J \to V_J$ and so we obtain an induced map

$$\tilde{\rho}|_{E_J} : G \overset{P_I}{\times} E_J \to V_J$$

since $E_J \subseteq V_J$ is P_I-stable. We first claim that $\tilde{\rho}|_{E_J}$ is proper. To see this, notice that, if ϕ denotes the isomorphism $G \overset{P_I}{\times} V_J \xrightarrow{\sim} G/P_I \times V_J$ given by $\phi(g, v) := (gP, gv)$, we have $\text{proj}_{V_J}\phi = \tilde{\rho}$. This shows that $\tilde{\rho}$ is proper, and, since $\tilde{\rho}|_{E_J}$ is the restriction to the closed subvariety $G \overset{P_I}{\times} E_J$, we see $\tilde{\rho}|_{E_J}$ is proper as well. In particular, the image X_I of $\tilde{\rho}|_{E_J}$ is closed. Therefore, X_I is affine and every closed point of X_I can be written as $g\vec{v}$ for some closed point $\vec{v} \in E_J(k)$.

Now, notice the P_I-orbits of E_J are equivalently given by the $\mathbb{G}_m^J$-orbits on E_J. Since we have a $\mathbb{G}_m^J$-equivariant isomorphism $E_J \cong \prod_{j \in J} L_j$ we may explicitly compute the $\mathbb{G}_m^J$-orbits on this space and we see that the $\mathbb{G}_m^J$-orbits on E_J are precisely of the form

$$\mathbb{O}_{K'} := \prod_{j \in K'} \{0\} \times \prod_{j \in J \setminus K'} L_j \setminus \{0\}$$

for some $K' \subseteq J$, such that $\mathbb{O}_{K'}$ lies in the closure of $\mathbb{O}_{K''}$ if and only if $K' \supseteq K''$. Now, using (7), we see that $G\mathbb{O}_{K'}$ is equivalently the G-orbit of $\sum_{j \in J \setminus K'} \vec{v}_j$ and therefore $G\mathbb{O}_{K'} \cong G/Q_{I \cup K'}$ by Corollary 5.3. We therefore have $G\mathbb{O}_{K'} = G/[P_{I \cup K'}, P_{I \cup K'}]$ by Proposition 5.2, which gives the stratification

$$X_I = \bigcup_{K \supseteq I} G/[P_K, P_K] \tag{9}$$

with the closure relations as in Proposition 5.5.

The variety X_I is normal by [23, Proposition 1]. We claim that the complement of G/Q_I in X_I is a codimension two subset. Indeed, this follows from the fact that for any coroot $j \in J$ the Lie algebra of any $Q_{I \cup \{j\}}$ contains a Levi factor SL_2^j of P_j and so, comparing the T-weight spaces of the Lie algebras, we see that $\dim(Q_{I \cup \{j\}}) \geq \dim(Q_I) + 2$ and so $\dim(G/Q_{I \cup \{j\}}) \leq \dim(G/Q_I) - 2$. Therefore since X_I is a normal affine variety (it is a closed subscheme of E), we have that X_I is the affine closure of G/Q_I and so our stratification (9) gives our claim. ☐

5.1.3. *The Symplectic Vector Bundle.* Recall that in [22, Section 2.1], the authors choose an SL_2-triple associated to a fixed simple root α and construct a certain rank two symplectic vector bundle $f_\alpha : V_\alpha \to G/Q_\alpha$ such that V_α admits an action of $G \times T$ for which the map f_α is equivariant for this action and such that the complement of the zero section of V_α can be identified with G/N. In particular, the variety $T^*(V_\alpha)$ has an endomorphism given by the *symplectic Fourier transform*

$$S_\alpha : T^*(V_\alpha) \to T^*(V_\alpha)$$

see for example [17, Appendix B] where, in the notation of [17], we specialize to $\hbar = 0$. This notation is justified as the composite endomorphism (read left to right)

$$R \xleftarrow{\sim} \mathcal{O}(T^*(V_\alpha)) \xrightarrow{\sim} \mathcal{O}(T^*(V_\alpha)) \xrightarrow{\sim} R$$

induced by the restriction maps (which are equivalences since the complement of $T^*(G/N)$ in $T^*(V_\alpha)$ has codimension two) and pulling back by S_α is, by definition, the automorphism s_α of R given by the Gelfand-Graev action, see [17].

Corollary 5.6. The vector bundle V_α is quasi-affine.

Proof. This is a standard argument (see, for example, the proof of [13, Lemma 3.21]) but since the argument is short we repeat it for the convenience of the reader. The morphism f_α is affine as V_α is a vector bundle, and therefore quasi-affine. Since the terminal map from $G/[P_\alpha, P_\alpha]$ is quasi-affine by Proposition 5.5, our claim follows from the fact that composition of quasi-affine morphisms is quasi-affine [32, Lemma 29.13.4]. □

Corollary 5.7. In the stratification (4), we can naturally identify the open subscheme $G/N \cup G/[P_\alpha, P_\alpha]$ with V_α.

Proof. Recall that the complement of $G/N \subseteq V_\alpha$ has codimension two, since at the reduced level it can be identified with the scheme theoretic image of the zero section $G/Q_\alpha \to V_\alpha$ and V_α is a rank two vector bundle. Therefore since V_α is smooth and in particular normal we therefore see that the restriction map gives an equivalence $\mathcal{O}(V_\alpha) \xrightarrow{\sim} A$. Since V_α is quasi-affine by Corollary 5.6, the affinization map for V_α is an open embedding, and so we have an open embedding $V_\alpha \subseteq \overline{G/N}$ given by the composite (read left to right)

$$V_\alpha \subseteq \mathrm{Spec}(\mathcal{O}(V_\alpha)) \xleftarrow{\sim} \overline{G/N}$$

whose restriction to the open G-orbit G/N the open embedding $G/N \subseteq \overline{G/N}$. Therefore the claim follows by comparing the stratification of $V_\alpha = G/N \cup G/[P_\alpha, P_\alpha]$ into the orbits of the action of $G \times T$ to the stratification of Proposition 5.5 with $I = \emptyset$. □

Since V_α is smooth, we in particular see the following, which completes the proof of Proposition 5.1:

Corollary 5.8. The smooth locus of $\overline{G/N}$ contains G/N and $G/[P_\alpha, P_\alpha]$ for any simple root α.

5.2. **Irreducible Elements of Functions on $T^*(G/N)$.** In this section, we use the Gelfand-Graev action to compute some irreducible elements of R:

Lemma 5.9. For all fundamental weights ω_i, $w \in W$, and nonzero $z \in A_{\omega_i}$, the element $w(z) \in R$ is irreducible.

Proof. It suffices to show this in the case when $w = 1$ since any $w \in W$ gives a ring automorphism which in particular preserves irreducibility. Now, if $z = ab$ for some $a, b \in R$, then by the $\mathbb{Z}^{\geq 0}$-grading given by the usual $\mathbb{G}_m$-action on the cotangent bundle, we see that $a, b \in A$. However, in A, we may identify the $X^\bullet(T)$-grading with a $(\mathbb{Z}^{\geq 0})^r$ grading using the fundamental weights. Then the irreducibility of z then follows as $A_0 = k$, which implies that any nonzero element of A_v for $v \in (\mathbb{Z}^{\geq 0})^r$ of length one must be irreducible. □

5.3. **Torus Stabilizers and Projections.** For any global function f on some scheme Y, we let $D(f)$ denote the complement of the vanishing locus of f, and, for any subset of global functions F, we let $D(F) := \cup_{f \in F} D(f)$. We now prove the following proposition, which informally says that for any $\mathfrak{p} \in D(R_\lambda)$, there exists some $w \in W$ such that $\overline{\pi}_w(\mathfrak{p})$ lies in an open locus of $\overline{G/N}$ determined by root hyperplanes which do not contain λ.

Proposition 5.10. Assume $\lambda \in X^\bullet(T)$ and $w \in W$ such that $w\lambda$ lies in the closure of the dominant Weyl chamber. Write $w\lambda = \sum_i n_i\omega_i$ with $n_i \in \mathbb{Z}^{\geq 0}$, and let $S_{w\lambda}$ denote the subset of fundamental weights ω_i for which $n_i \neq 0$. Then $\overline{\pi}_w(D(R_\lambda))$ maps into the open subscheme $\cap_{\omega_i \in S_{w\lambda}} D(A_{\omega_i})$. In particular, if λ is regular then $\overline{\pi}_w(D(R_\lambda))$ maps into G/N.

Proof of Proposition 5.10. By Lemma 2.2, we see that the multiplication map

$$\mathrm{Sym}(\mathfrak{g}) \otimes_{Z\mathfrak{g}} \mathrm{Sym}(\mathfrak{t}) \otimes A_{w\lambda} \to R_{w\lambda}$$

is surjective. As A is generated by the union of the A_ω where ω varies over the fundamental weights, we therefore in particular obtain the multiplication map

$$\mathrm{Sym}(\mathfrak{g}) \otimes_{Z\mathfrak{g}} \mathrm{Sym}(\mathfrak{t}) \otimes \bigotimes_i A_{\omega_i}^{\otimes n_i} \to R_{w\lambda} \tag{10}$$

is surjective.

Now fix some $\mathfrak{p} \in \overline{T^*(G/N)}$ such that $R_\lambda \not\subseteq \mathfrak{p}$, so that $R_{w\lambda} \not\subseteq w\mathfrak{p}$. Let I denote the ideal of R generated by $R_{w\lambda}$ and, for a fixed $\omega_i \in S_{w\lambda}$, let J_i denote the ideal of R generated by A_{ω_i}. The fact that the multiplication map of (10) is surjective implies that $R_{w\lambda} \subseteq J_i$ and so $I \subseteq J_i$. In particular, we see that $A_{\omega_i} \not\subseteq w\mathfrak{p}$. Thus $\overline{\pi}_w(\mathfrak{p}) = A \cap w\mathfrak{p}$ does not contain all of A_{ω_i} for any $\omega_i \in S_{w\lambda}$, as desired. □

From this, we derive the following:

Corollary 5.11. Assume $\lambda_1, ..., \lambda_q \in X^\bullet(T)$ span the vector space $\mathfrak{t}^*(\mathbb{Q})$. Then there exists some $w \in W$ such that $\overline{\pi}_w$ maps the set $\cap_i D(R_{\lambda_i})$ into G/N. In particular T acts with no stabilizer on any point in $\cap_i D(R_{\lambda_i})$.

We show Corollary 5.11 after proving the following Lemma:

Lemma 5.12. Assume $L \subseteq \mathbb{Q}^d$ is some full rank lattice and choose some basis $S := \{\vec{v}_1, ..., \vec{v}_d\} \subseteq L$ of $\mathbb{Q}^d$. Denote by C the $\mathbb{R}^{>0}$-span of S, i.e.

$$C := \{\sum_{i=1}^{d} \alpha_i \vec{v}_i : \alpha_i \in \mathbb{R}^{>0}\}$$

and assume $\mathcal{Z}$ is some closed subset of $\mathbb{R}^d$ which does not contain C and which is closed under scaling by any positive real number. Then the $\mathbb{Z}^{>0}$-span of S contains a point of L not in $\mathcal{Z}$.

Proof. Since C is open, $C \cap \mathcal{Z}^c$ is open, and so in particular $C \cap \mathcal{Z}^c$ contains an open ball of some positive radius since by assumption it is nonempty. Since $C \cap \mathcal{Z}^c$ is closed under the scaling of any positive real number, for any $r \in \mathbb{R}^{>0}$, $C \cap \mathcal{Z}^c$ contains an open ball of radius r. However, for any full rank lattice, there exists some M such that all points in $\mathbb{R}^d$ are distance at most M from a point on that lattice. Choosing $r > M$ we see that there is an element of $x \in L \cap \mathcal{Z}^c$ in the $\mathbb{R}^{>0}$-span of S. As S is a basis and x in particular lies in $\mathbb{Q}^d$, we see that x lies in the $\mathbb{Q}^{>0}$-span of S. There exists some positive integer N such that Nx therefore is a $\mathbb{Z}^{>0}$-linear combination of the $\vec{v}_i$, as desired. □

Proof of Corollary 5.11. Assume $\mathfrak{p} \in \cap_i D(R_{\lambda_i})$. Let $L := X^\bullet(T)$ and choose some subset S of the λ_i such that S is a basis of $X^\bullet(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. If $\mathcal{Z}$ denotes the union of root hyperplanes, we may apply Lemma 5.12 to show there is some $\lambda \in X^\bullet(T)$ which lies in the interior of some Weyl chamber and which is a $\mathbb{Z}^{>0}$-linear combination of the elements of S. Since $\mathfrak{p} \in D(R_{\lambda_i})$ for every i, we see $\mathfrak{p} \in D(R_\lambda)$. Choose the (unique) $w \in W$ which takes λ to the dominant Weyl chamber. Then since $w\lambda$ is regular, by Proposition 5.10 we see that $\overline{\pi}(w\mathfrak{p})$ maps to G/N. In particular, $\overline{\pi}(w\mathfrak{p})$ has no T-stabilizer, and thus neither does $\mathfrak{p}$ itself, since $\overline{\pi}$ is T-equivariant. □

From this we may also derive a similar result for the locus of points whose T-stabilizer has dimension one:

Corollary 5.13. Assume that the T-stabilizer of some point $\mathfrak{p} \in \overline{T^*(G/N)}$ has dimension one. Then there exists some $w \in W$ and some simple root α such that $\overline{\pi}(w\mathfrak{p}) \in G/[P_\alpha, P_\alpha]$. In particular, the T-stabilizer of $\mathfrak{p}$ is $w\mathbb{G}_m^\alpha w^{-1}$.

Proof. For such a point $\mathfrak{p}$, we define the set $\mathcal{D} := \{\lambda \in X^\bullet(T) : \mathfrak{p} \in D(R_\lambda)\}$. Using (1), we may view $\mathcal{D}$ as a subset of $\mathfrak{t}^*(\mathbb{Q})$ and let $V_\mathbb{Q}$ denote the span of $\mathcal{D}$ in $\mathfrak{t}^*(\mathbb{Q})$.

We claim that the dimension of $V_\mathbb{Q}$ is at least $\dim(\mathfrak{t}^*) - 1$. To see this, assume the dimension of $V_\mathbb{Q}$ was less than $\dim(\mathfrak{t}^*) - 1$. In this case, there would be two linearly independent elements $\nu_1, \nu_2 \in \mathfrak{t}(\mathbb{Q})$ such that $\nu_1(s) = 0 = \nu_2(s)$ for any $s \in \mathcal{D}$. Here, we use the finite dimensionality of $\mathfrak{t}(\mathbb{Q})$ to canonically identify it with the vector space dual of $\mathfrak{t}^*(\mathbb{Q})$. Now using the isomorphism (2) we see that we may replace ν_1 and ν_2 by a positive integer multiple if necessary and additionally assume that both ν_1 and ν_2 are in $X_\bullet(T)$. By the definition of $\mathcal{D}$, we have that the image of each map $\nu_i : \mathbb{G}_m \to T$ stabilizes $\mathfrak{p}$. However, the fact that ν_1 and ν_2 form a linearly independent set implies that the subgroup generated by the images of these ν_i has dimension two, violating our assumption on the dimension of the stabilizer of $\mathfrak{p}$.

Let $V_\mathbb{R}$ denote the $\mathbb{R}$-span of the elements of $\mathcal{D}$, let $\mathcal{Z}'$ denote the union of every intersection of two distinct root hyperplanes, and let $\mathcal{Z} := V_\mathbb{R} \cap \mathcal{Z}'$. We also choose a $\mathbb{Q}$-basis S of $V_\mathbb{Q} \subseteq V_\mathbb{R}$ contained in $\mathcal{D} \subseteq V_\mathbb{Q}$ and let L denote the lattice generated by S. Since the dimension of $V_\mathbb{Q}$ is at least $\dim(\mathfrak{t}^*) - 1$, we see that $\mathcal{Z}$ does not contain the $\mathbb{R}^{\geq 0}$-span of S, and so we may apply Lemma 5.12 to see that the $\mathbb{Z}^{\geq 0}$-span of the elements of S contains some $\lambda \in X^\bullet(T)$ such that λ lies on at most one root hyperplane. Since the $\mathbb{Z}^{\geq 0}$-span of S is contained in $\mathcal{D}$ as $\mathcal{D}$ is its own $\mathbb{Z}^{\geq 0}$-span, we see that $\mathfrak{p} \in D(R_\lambda)$.

Choose some $w \in W$ such that $w\lambda$ lies on a root hyperplane cut out by at most one simple coroot. By Proposition 5.10, we see that if $w\lambda$ is contained in no root hyperplanes then $\overline{\pi}(w\mathfrak{p})$ projects into G/N and so in particular the T-action on $\mathfrak{p}$ is free. Thus $w\lambda$ is contained in exactly one root hyperplane cut out by the vanishing of a simple coroot $\alpha^\vee$. Therefore by Proposition 5.10 $\overline{\pi}(w\mathfrak{p}) \in G/[P_\alpha, P_\alpha]$. The latter claim follows from the fact that $\overline{\pi}$ is compatible with the T-action, which shows that the T-stabilizer of $w\mathfrak{p}$ is a closed subgroup scheme of $\mathbb{G}_m^\alpha$. ☐

Corollary 5.14. If the right T-stabilizer of some point $\mathfrak{p} \in \overline{T^*(G/N)}$ has dimension zero or one, then $\mathfrak{p}$ is a smooth point.

Proof. If the T-stabilizer of $\mathfrak{p} \in \overline{T^*(G/N)}$ has dimension zero, then we in particular see that the set of $w\omega_i$ for which $\mathfrak{p} \in D(R_{w\omega_i})$ spans the rational points of $\mathfrak{t}^*$, since otherwise they would be contained in some hyperplane cut out by some $\alpha \in \mathfrak{t}(\mathbb{Q})$ and thus fixed by some subtorus. Therefore by Corollary 5.11 some element of the W-orbit of $\mathfrak{p}$ projects to G/N under $\overline{\pi}$. If the T-stabilizer of $\mathfrak{p}$ has dimension one, by Corollary 5.13 some point in its W-orbit projects to some point of $G/[P_\alpha, P_\alpha]$ under $\overline{\pi}$. In either case, $\pi_w(\mathfrak{p})$ lies in $G/N \cup G/[P_\alpha, P_\alpha]$ for some w. Since $G/N \cup G/[P_\alpha, P_\alpha] \subseteq \mathcal{S}$ by our simply connectedness assumption, we see that $\mathfrak{p}$ is smooth by Proposition 4.3. ☐

5.4. **Codimension of Singular Locus of Affine Closure of $T^*(G/N)$.** We now use the results of Section 5.3 to prove the following, which is a key ingredient in our proof of Theorem 1.1:

Theorem 5.15. The singular locus of $\overline{T^*(G/N)}$ has codimension at least four.

By Corollary 5.14, Theorem 5.15 follows directly from the following proposition:

Proposition 5.16. The locus of points of $\overline{T^*(G/N)}$ whose T-stabilizer has dimension ≥ 2 has codimension at least four.

With the exception of Remark 5.18, the proof of Proposition 5.16 will occupy the remainder of Section 5.4. We first construct a stratification of $\overline{T^*(G/N)}$ by T-invariant locally closed subschemes which will also be used in Section 5.5. Let F denote the set of fundamental weights. For any $(w, \omega) \in W \times F$, set

$$A_{w,\omega} := w(A_\omega) \subseteq R.$$

Notice that $A_\omega = A_{1,\omega}$ for any $\omega \in F$. By Lemma 2.2, the sets

$$\mathscr{S}_S := \cap_{(w,\omega)\in S} V(A_{w,\omega}) \cap \bigcap_{(w,\omega)\notin S} D(A_{w,\omega})$$

give a stratification of $\overline{T^*(G/N)}$ by locally closed T-invariant subschemes, where S varies over subsets of $W \times F$. Moreover, for a fixed S, any two closed points in $\mathscr{S}_S$ have the same (right) T-stabilizer T_S. Since there are finitely many such $\mathscr{S}_S$, Proposition 5.16 follows from the following proposition:

Proposition 5.17. Assume that $S \subseteq W \times F$ such that T_S has dimension at least two. Then $\mathscr{S}_S$ has codimension at least four in $\overline{T^*(G/N)}$.

Proof. Fix a subset $S \subseteq W \times F$ such that T_S has dimension at least two. Letting $S^0 := R$, we recursively construct k-algebras S^1, S^2, S^3, S^4 such that for every $i \in \{0, 1, 2, 3\}$,

$$\dim(\mathrm{Spec}(S^{i+1})) \leq \dim(\mathrm{Spec}(S^i)) - 1 \tag{11}$$

and such that by construction $\mathscr{S}_S$ is a locally closed subscheme of $\mathrm{Spec}(S^4)$. The existence of this construction automatically implies that $\mathscr{S}_S$ has codimension at least four.

Choose some fundamental weight ω such that $(1, \omega) \in S$. Such an ω exists since, if not,

$$\mathscr{S}_S \subseteq \cap_\omega D(A_{1,\omega}) = \cap_\omega D(A_\omega) \subseteq \cap_\omega D(R_\omega)$$

where the intersections are taken over all fundamental weights, and therefore any point of $\mathscr{S}_S$ has no T-stabilizer by, for example, Corollary 5.11. Choose some nonzero $a_1 \in A_\omega$ and let $S^1 = R/a_1$. Note that since

$$\mathrm{Spec}(S^1) \supseteq V(A_\omega) = V(A_{(1,\omega)})$$

and $(1, \omega) \in S$, $\mathscr{S}_S$ is a locally closed subscheme of $\mathrm{Spec}(S^1)$. Moreover, since R is a unique factorization domain and a_1 is an irreducible element of R by Lemma 5.9, S^1 is an integral domain. Since a_1 is a nonzero element in an integral domain, we also see that (11) holds when $i = 0$. We also have that, since a_1 is homogeneous with respect to the $X^\bullet(T)$-grading, the ring S^1 admits a grading by $X^\bullet(T)$. The grading on S^1 has the property that

$$S^1_\lambda \neq 0 \text{ for any } \lambda \in X^\bullet(T) \tag{12}$$

by Lemma 2.2 and the unique factorization of R given by Corollary 3.5.

As T_S has dimension at least two, we may choose two elements $\gamma_1, \gamma_2 \in X_\bullet(T_S)$ which are linearly independent in $\mathrm{Lie}(T_S)(\mathbb{Q})$, and we denote by T_1, respectively T_2 the rank one subtori generated by γ_1, respectively γ_2. Define $S^2 := S^1_{\gamma_1=0}$ so that, by definition, S^2 is the subring of S^1 generated by those S^1_λ such that $\langle\gamma_1, \lambda\rangle = 0$. Then since (12) holds, we may apply Lemma 4.1 to see that (11) holds when $i = 1$. Moreover, since $\mathscr{S}_S$ is a locally closed subscheme of $\mathrm{Spec}(S^1)^{T_1}$ and the quotient map identifies $\mathrm{Spec}(S^1)^{T_1}$ as a closed subscheme of $\mathrm{Spec}(S^2) = \mathrm{Spec}(S^1) /\!\!/ T_1$ by Proposition 4.2, we see that we may view $\mathscr{S}_S$ as a locally closed subscheme of $\mathrm{Spec}(S^2)$. Similarly, define $S^3 = S^2_{\gamma_2=0}$. Since (12) holds for λ which satisfy

$$\langle\gamma_1, \lambda\rangle = 0 \neq \langle\gamma_2, \lambda\rangle$$

we see that we may similarly apply Lemma 4.1 to see that (11) holds when $i = 2$, and, exactly as above, we may view $\mathscr{S}_S$ as a locally closed subscheme of $\mathrm{Spec}(S^3) = \mathrm{Spec}(S^2) /\!\!/ T_2$. Moreover, S^3 is an integral domain as it is a subring of the integral domain S^1.

Finally, choose some $a_2 \in A_\omega$ such that $\{a_1, a_2\}$ is linearly independent and choose some nonzero $b \in A_{w_0,w_0(-\omega)}$. If we set $S^4 := S^3/(a_2 b)$ then, since $\omega \in S$, $\mathscr{S}_S$ (viewed as a locally closed subscheme of $\mathrm{Spec}(S^3)$ via the quotient map as above) is contained $\mathrm{Spec}(S^4)$. We now claim that $a_2 b$ is not zero in S^3. To see this, notice that if $a_2 b = 0$ in $S^3 \subseteq S^1$ then there exists some $f \in R$ such that $a_1 f = a_2 b$ in R. However, since a_1, a_2, b are irreducible in R by Lemma 5.9, this would violate the fact that R is a unique factorization domain, i.e. Corollary 3.5. Since a_2b is a nonzero element in the integral domain S^3, any irreducible component of $\mathrm{Spec}(S^4)$ has codimension 1 in $\mathrm{Spec}(S^3)$. $\square$

Remark 5.18. The reader who is only interested in the proof of the main theorems when G is simply connected may proceed directly to Section 6.2, replacing the usage of Theorem 6.2 in final sentence of the proof of Theorem 6.3 with the usage of Theorem 5.15.

5.5. **Free Locus Has Codimension Four.** Using the notation introduced in Section 5.4, let Z_i denote the closed subscheme $V(A_{\omega_i}) \cap V(A_{w_0,w_0(-\omega_i)})$, and let U_i denote its open complement. In this section, we study properties of the open subscheme $Q := \cup_w w(\cap_{i=1}^r U_i)$. We first give a more explicit description for Q, which can be compared to [14, Proposition 5.1.4]:

Proposition 5.19. We have an equality

$$Q = \cup_w \overline{\pi}_w^{-1}(G/N). \tag{13}$$

Moreover, Q is the set of points of $\overline{T^*(G/N)}$ for which the (right) T action is free and is the set of points of $\overline{T^*(G/N)}$ for which the T-stabilizer has dimension zero. In particular, any point of $\overline{T^*(G/N)}$ whose T-stabilizer has dimension zero has trivial stabilizer.

Proof. To show $\subseteq$ in (13), by W-equivariance it suffices to show $\cap_{i=1}^r U_i \subseteq \cup_w \overline{\pi}_w^{-1}(G/N)$. Choose some homogeneous function $f_i \in A_{\omega_i} \cup A_{w_0,w_0(-\omega_i)}$ and let $\lambda_i \in \{\pm\omega_i\}$ denote the degree of f_i. It further suffices to show that

$$\cap_{i=1}^r D(f_i) \subseteq \cup_w \overline{\pi}_w^{-1}(G/N)$$

which follows from Corollary 5.11. Conversely, by W-equivariance we may show the containment $\overline{\pi}^{-1}(G/N) \subseteq \cup_w w(\cap_{i=1}^r U_i)$, but this follows from the fact that $\overline{\pi}^{-1}(G/N) = \cap_{i=1}^r D(A_{\omega_i}) \subseteq \cap_{i=1}^r U_i$. By the T-equivariance of $\overline{\pi}$, we see that any point in $\overline{T^*(G/N)}$ which maps to G/N under $\overline{\pi}$ must itself have trivial (right) T-stabilizer. Now assume that the right T-action on $\overline{T^*(G/N)}$ for some point $\mathfrak{p}$ has dimension zero. Then for any $\gamma \in X_\bullet(T)$ there exists some $\lambda \in X^\bullet(T)$ and $f \in R_\lambda$ such that $\langle \lambda, \gamma \rangle \neq 0$ and f does not vanish at $\mathfrak{p}$, as otherwise it would be stabilized by the one parameter subgroup $\gamma : \mathbb{G}_m \hookrightarrow T$. In particular, if we let S denote the set of all $\lambda \in X^\bullet(T)$ such that R_λ contains a function which does not vanish at $\mathfrak{p}$, then we see that the elements of S span the real points of $\mathfrak{t}^*$. Thus we see that $\mathfrak{p} \in \overline{\pi}_w^{-1}(G/N)$ for some $w \in W$ by Corollary 5.11 and, therefore by the above must also have trivial T-stabilizer. $\square$

From this, we can derive the codimension result on the complement of Q (for G simply connected) stated in the introduction, whose proof occupies the remainder of this entire subsection:

Corollary 5.20. The complement of Q has codimension at least four.

For this remainder of this section, we fix a simple root α, choose an SL_2-triple for α and use the notation of Section 5.1.3. We also let $p_\alpha : T^*(V_\alpha) \to V_\alpha$ denote the projection map and set $p_{\alpha,s} := p_\alpha \circ S_\alpha$ where $S_\alpha : T^*(V_\alpha) \to T^*(V_\alpha)$ is the symplectic Fourier transform. We also define $Y_\alpha := G/Q_\alpha$ and view Y_α as a closed subscheme of V_α via the zero section map.

Lemma 5.21. The scheme theoretic intersection

$$Z_\alpha := p_\alpha^{-1}(Y_\alpha) \cap p_{\alpha,s}^{-1}(Y_\alpha)$$

has codimension four in $T^*(V_\alpha)$.

Proof. First, the symplectic Fourier transform on V_α is, by construction, an automorphism over Y_α. Therefore the map

$$(p_\alpha, p_{\alpha,s}) : T^*(V_\alpha) \to V_\alpha \times V_\alpha$$

factors through the closed subscheme $V_\alpha \times_{Y_\alpha} V_\alpha$. We wish to compute the dimension of the closed subscheme

$$Z_\alpha = T^*(V_\alpha) \times_{V_\alpha \times_{Y_\alpha} V_\alpha} Y_\alpha$$

of $T^*(V_\alpha)$ where we regard $T^*(V_\alpha)$ (and thus Z_α) as a scheme over Y_α in the natural way. It suffices to do this on an open cover of V_α.

Let C denote the open B-orbit of G/P_α, and let $\tilde{C}$ denote its inverse image under the quotient map $G/Q_\alpha \to G/P_\alpha$. Defining $U_0 := f_\alpha^{-1}(\tilde{C})$, the open subset $U := p_\alpha^{-1}(U_0)$ gives a nonempty open subscheme of $T^*(V_\alpha)$. Moreover, since f_α and p_α are G-equivariant and the action of G on G/Q_α is transitive, we see that U and its $G(k)$-translates cover $T^*(V_\alpha)$. It therefore suffices to show $U \cap Z_\alpha$ has codimension four in U by $G(k)$-equivariance.

The construction of V_α [22, Section 2.1] gives a trivialization $U_0 = f_\alpha^{-1}(\tilde{C}) \cong \tilde{C} \times \mathbb{A}^2$ such that the formula

$$(c, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}), (c, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}) \mapsto (c, x_1 y_2 - y_1 x_2)$$

gives the symplectic form

$$(\tilde{C} \times \mathbb{A}^2) \times_{\tilde{C}} (\tilde{C} \times \mathbb{A}^2) \to \tilde{C} \times \mathbb{G}_a$$

on V_α restricted to this open subset. In particular, we obtain isomorphisms

$$U \cong T^*(\tilde{C} \times \mathbb{A}^2) \cong T^*(\tilde{C}) \times T^*(\mathbb{A}^2) \cong T^*(\tilde{C}) \times \mathbb{A}^2 \times \mathbb{A}^{2,*} \xrightarrow{\sim} T^*(\tilde{C}) \times \mathbb{A}^2 \times \mathbb{A}^2 \tag{14}$$

where the final isomorphism is induced by the symplectic form. Under the composite identification obtained from reading (14) left to right one can directly follow the construction of the symplectic Fourier transform as in, for example [17, Appendix B], to see that it is given by the automorphism

$$(z, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}) \mapsto (z, \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}, \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix})$$

of $T^*(\tilde{C}) \times \mathbb{A}^2 \times \mathbb{A}^2$. Therefore, using the above trivialization, we may identify $p_\alpha|_U$ with the map

$$(z, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}) \mapsto (\overline{z}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix})$$

where $\overline{z}$ is the image of p under the map $T^*(C) \to C$. We may similarly identify the map $p_{\alpha,s}|_U$ with

$$(z, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}) \mapsto (\overline{z}, \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix})$$

again using the identification induced by the composite of the isomorphisms of (14). Therefore, via the isomorphisms of (14), we can identify

$$Z_\alpha \cap U \cong T^*(\tilde{C}) \times \{0\} \times \{0\}$$

which evidently has codimension four in U. □

Proof of Corollary 5.20. We temporarily denote by Z_Q the complement of Q in $\overline{T^*(G/N)}$. Since we can identify Q with the locus where T acts with dimension zero stabilizers Proposition 5.19, we can write $Z_Q = Z_Q^1 \cup Z_Q^{\geq 2}$ where Z_Q^1 denotes the open subscheme of Z_Q consisting of those points whose right T-stabilizer has dimension one and $Z_Q^{\geq 2}$ is the closed subscheme consisting of those points whose right T-stabilizer has dimension at least two. By Proposition 5.16, the codimension of $Z_Q^{\geq 2}$ is at least four, so it suffices to show that Z_Q^1 has codimension at least four.

By Corollary 5.13, we may write Z_Q^1 as the union $Z_Q^1 = \cup_{w,\alpha} Z_{Q,w,\alpha}^1$, where w varies over W, α varies over the simple roots, and $Z_{Q,w,\alpha}^1$ denotes the subset of points of Z_Q^1 such that $\overline{\pi}_w(Z_Q^1) \in$

G/Q_α. In particular we have $w(Z^1_{Q,w,\alpha}) \subseteq \overline{\pi}^{-1}(V_\alpha) \times_{V_\alpha} G/Q_\alpha$ by Corollary 5.7. Notice, however, that

$$\overline{\pi}^{-1}(V_\alpha) = \overline{T^*(G/N)} \times_{\overline{G/N}} V_\alpha \cong (\overline{T^*(G/N)} \times_{\overline{G/N}} \mathcal{S}) \times_{\mathcal{S}} V_\alpha \xleftarrow{\sim} T^*(\mathcal{S}) \times_{\mathcal{S}} V_\alpha \cong T^*(V_\alpha)$$

using the isomorphism given by Proposition 4.3 and the containment $V_\alpha \subseteq \mathcal{S}$ given by Corollary 5.8. Therefore $w(Z^1_{Q,w,\alpha}) \subseteq T^*(V_\alpha) \times_{V_\alpha} Y_\alpha$. We also have that $s_\alpha(w(Z^1_{Q,w,\alpha})) \subseteq T^*(V_\alpha) \times_{V_\alpha} Y_\alpha$ since if there was a point which was not contained in this closed subscheme, using the stratification $V_\alpha = G/N \cup Y_\alpha$ of Corollary 5.7, we would see that T acts freely on this point by Proposition 5.19. Therefore $w(Z_{Q,w,\alpha}) \subseteq p_\alpha^{-1}(Y_\alpha) \cap p_{\alpha,s}^{-1}(Y_\alpha)$ and so $w(Z_{Q,w,\alpha})$ has codimension at least four by Lemma 5.21. Therefore $Z_{Q,w,\alpha} = w^{-1}(w(Z_{Q,w,\alpha}))$ has codimension at least four as well. ☐

6. Proofs of Main Theorems

In this section, we record some consequences of our above computations. First, we extend results on Q above to an arbitrary reductive group in Section 6.1. We then show that one can derive the symplectic singularities of $\overline{T^*(G/N)}$ for all G from this in Section 6.2 and then finish the proof of Theorem 1.2 in Section 6.3.

6.1. **Free Locus from Simply Connected Case.** The group G can be written as a quotient

$$(G^{\mathrm{sc}} \times \tilde{T})/Z \xrightarrow{\sim} G \tag{15}$$

for G^{sc} some simply connected semisimple group, $\tilde{T}$ some torus and Z some closed finite central subgroup scheme of $\tilde{G} := G^{\mathrm{sc}} \times \tilde{T}$. Let $\tilde{N}$ denote the unipotent radical of some Borel which projects into B under the quotient map. The map $\tilde{G}/\tilde{N} \to G/N$ is a finite étale cover. From this, we see:

Lemma 6.1. The natural map induces an isomorphism

$$q : T^*(\tilde{G}/\tilde{N})/Z \xrightarrow{\sim} T^*(G/N)$$

so that in particular if $\tilde{R}$ denotes the ring of functions on $T^*(\tilde{G}/\tilde{N})$ we have $R \xrightarrow{\sim} \tilde{R}^Z$. In other words, the map $\overline{q} : \overline{T^*(\tilde{G}/\tilde{N})} \to \overline{T^*(G/N)}$ induces an isomorphism $\overline{q} : \overline{T^*(\tilde{G}/\tilde{N})} /\!\!/ Z \xrightarrow{\sim} \overline{T^*(G/N)}$.

As above, let Q denote the locus of points of $\overline{T^*(G/N)}$ on which T acts freely.

Theorem 6.2. The locus Q is smooth. Moreover, the complement of Q has codimension at least four.

Proof. We see that $\overline{q}^{-1}(Q)$ is precisely the locus on which the maximal torus in $\tilde{G}$ acts with finite stabilizer group (since $\overline{q}$ is a finite map). By Proposition 5.19 (which also evidently applies in the case where G is the product of a simple simply connected group with some torus) we see that $\overline{q}^{-1}(Q)$ is also the locus on which $\tilde{T}$ acts freely. In particular, the map $\overline{q}^{-1}(Q) \to Q$ induces an isomorphism $\overline{q}^{-1}(Q)/Z \xrightarrow{\sim} Q$. This shows that Q is smooth. Moreover, the fact that $\overline{q}$ is a surjection also shows that the map

$$\overline{q}|_{\overline{T^*(\tilde{G}/\tilde{N})} \setminus \overline{q}^{-1}(Q)} : \overline{T^*(\tilde{G}/\tilde{N})} \setminus \overline{q}^{-1}(Q) \to \overline{T^*(G/N)} \setminus Q$$

is dominant. In particular, we see that the fact that the complement of $\overline{q}^{-1}(Q)$ has codimension at least four (Corollary 5.20) implies that the complement of Q has codimension at least 4. ☐

6.2. **Symplectic Singularities of Normal Varieties Via Codimension.** In this section, we prove Theorem 1.1. When G is semisimple, we have shown that $\overline{T^*(G/N)}$ admits a conical $\mathbb{G}_m$-action compatible with the Poisson bracket in Section 3.2. Therefore, it remains to show the following, which we prove for arbitrary reductive G:

Theorem 6.3. The variety $\overline{T^*(G/N)}$ has symplectic singularities.

We recall the following standard lemma on extendability of differential forms as applied to the theory of symplectic singularities:

Lemma 6.4. Assume X is some irreducible variety.

(1) Assume Z is some closed subscheme of X^{reg} whose codimension is larger than 1. Then if $\omega \in \Omega^2(X^{\mathrm{reg}} \setminus Z)$ is some symplectic form, then ω extends to a symplectic form on X^{reg}.
(2) If in addition X is normal and $\mathrm{codim}_X(X \setminus X^{\mathrm{reg}}) \geq 4$, X has symplectic singularities.

Proof. A standard Hartog's lemma argument gives that ω extends to a symplectic form on X^{reg}, see, for example, [1, Section 4, Remarque (3)]. Now, any normal irreducible variety with a nondegenerate 2-form on the regular locus has symplectic singularities if and only if, for any resolution $p : Y \to X$ of singularities, the induced 2-form $p^*(\omega)$ extends to a 2-form on Y. However, since the codimension of the singular locus is larger than 3, this extension follows directly from the main theorem of [11]. ☐

Proof of Theorem 6.3. We check the hypotheses of Lemma 6.4. The fact that $\overline{T^*(G/N)}$ is normal follows from Proposition 3.1. The variety $T^*(G/N)$ is an open subscheme of $\overline{T^*(G/N)}$ whose complement has codimension at least two by Lemma 3.4, so the symplectic form on $T^*(G/N)$ necessarily extends to a symplectic form on the smooth locus of $\overline{T^*(G/N)}$. Finally, the fact that $\overline{T^*(G/N)}$ has a singular locus of codimension ≥ 4 follows from Theorem 6.2. ☐

6.3. **Consequences of Main Theorem.** We have seen that $\overline{T^*(G/N)}$ has symplectic singularities in Theorem 6.3 and that its singular locus has codimension at least four in Theorem 6.2. Recall that all singular symplectic varieties whose singular locus has codimension at least four are terminal by the main result of [29]. Therefore we immediately obtain the following result which, combined with Proposition 3.1, completes the proof of Theorem 1.2:

Corollary 6.5. The variety $\overline{T^*(G/N)}$ has terminal singularities.

We also claim that, if G is semisimple and not a product of copies of SL_2, that $\overline{T^*(G/N)}$ is singular:

Proposition 6.6. When G is semisimple and not a product of copies of SL_2, the cone point $0 \in \overline{T^*(G/N)}$ is singular.

Proof. Notice that the ideal cutting out the image of the closed embedding given by the zero section

$$\overline{z} : \overline{G/N} \hookrightarrow \overline{T^*(G/N)}$$

is homogeneous for both the torus action and the usual $\mathbb{G}_m$-action induced by scaling fibers on $T^*(G/N)$. Therefore, by Proposition 3.6(2), this ideal is also homogeneous for the conical $\mathbb{G}_m$-action. In particular, the cone point is contained in this closed subscheme.

It is well known that $\overline{G/N}$ is singular when G is semisimple and not a product of copies of SL_2–for example, this follows from the fact that the ring of differential operators on any smooth affine variety is generated by derivations [26, Corollary 15.6] but the results of [24] show that this is not the case for such G. Since the singular locus of a scheme is closed and the singular locus of a scheme with a group action is closed under the action of that group, we see that the singular locus of $\overline{G/N}$ contains the cone point for such G since any nonempty closed $\mathbb{G}_m$-invariant subscheme of $\overline{G/N}$ contains the cone point. Therefore $\dim(T_0(\overline{G/N})) > d := \dim(G/N)$, and, since $\overline{z} \circ \overline{\pi} = \mathrm{id}$, $\overline{\pi}$ induces a surjective map

$$T_0(\overline{T^*(G/N)}) \to T_0(\overline{G/N}) \tag{16}$$

on tangent spaces.

Note also that, by for example Proposition 4.3, the generic fiber of $\overline{\pi}$ has dimension exactly d. Therefore, by upper semicontinuity of the fiber dimension, we see that the fiber F of $\overline{\pi}$ at the cone point of $\overline{T^*(G/N)}$ has dimension at least d. In particular, $T_0(F)$ has dimension at least d and lies in the kernel of the map (16). Therefore by rank-nullity the dimension of $T_0(\overline{T^*(G/N)})$ is larger than $d+d = \dim(\overline{T^*(G/N)})$, and so the cone point is singular. $\square$

On the other hand, the codimension of the singular locus of $\overline{T^*(G/N)}$ is at least four by Theorem 6.2. Thus the $\mathbb{Q}$-factoriality of $\overline{T^*(G/N)}$ in Proposition 3.1 allows us to use [12, Corollary 1.3] to show the following, which generalizes a remark of [16, Section 1.3] for $G = \mathrm{SL}_3$ to all types:

Corollary 6.7. If G is semisimple and not a product of copies of SL_2 then the variety $\overline{T^*(G/N)}$ does not admit a symplectic resolution.

The fact that $\overline{T^*(G/N)}$ admits conical symplectic singularities for semisimple G implies that is a natural object in the study of symplectic duality. The following result determines properties of the conjectural symplectic dual to $\overline{T^*(G/N)}$ and should be compared to the expectations of [8, Section 8]:

Corollary 6.8. There are no nontrivial flat Poisson deformations of $\overline{T^*(G/N)}$.

Proof. It suffices to show that the vector space $HP^2(\overline{T^*(G/N)})$ is zero, see, for example, [15], [31], [30]. In fact, $HP^2(\mathcal{Y})$ vanishes for any normal affine variety $\mathcal{Y}$ with terminal symplectic singularities and finite class group. We give the proof of this well known result here for completeness.

Since $\mathcal{Y}$ has terminal symplectic singularities, we have that $HP^2(\mathcal{Y}) \cong H^2(Y, \mathbb{C})$, where Y denotes the smooth locus of $\mathcal{Y}$ [31]. Now [25, Lemma 4.4.6] shows that the first Chern class gives an isomorphism $\mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} H^2(Y, \mathbb{Q})$. Therefore, since $\mathrm{Pic}(Y)$ is finite (since it is a subgroup of the class group of $\mathcal{Y}$) we see that $H^2(Y, \mathbb{C}) \cong H^2(Y, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong 0$ as desired. $\square$

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