A q-analog of certain symmetric functions and one of its specializations

Vincent Brugidou

Université de Lille, 59655 Villeneuve d'Ascq cedex, France

Abstract : Let the symmetric functions be defined for the pair of integers (n, r), $n \geq r \geq 1$, by $p_n^{(r)} = \sum m_\lambda$ where m_λ are the monomial symmetric functions, the sum being over the partitions λ of the integer n with length r. We introduce by a generating function, a q-analog of $p_n^{(r)}$ and give some of its properties. This q-analog is related to its the classical form using the q-Stirling numbers. We also start with the same procedure the study of a p, q-analog of $p_n^{(r)}$.

By specialization of this q-analog in the series $\sum_{n=0}^{\infty} q^{\binom{n}{2}} t^n/n!$, we recover in a purely formal way a class of polynomials $J_n^{(r)}$ historically introduced as combinatorial enumerators, in particular of tree inversions. This also results in a new linear recurrence for those polynomials whose triangular table can be constructed, row by row, from the initial conditions $J_r^{(r)} = 1$. The form of this recurrence is also given for the reciprocal polynomials of $J_n^{(r)}$, known to be the sum enumerators of parking functions. Explicit formulas for $J_n^{(r)}$ and their reciprocals are deduced, leading inversely to new representations of these polynomials as forest statistics.

keywords : Symmetric functions, q-analog, q-Stirling numbers, tree inversions, parking functions.
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1 Introduction

This paper is the first in a series whose object is a q-analog defined in a fairly natural way of certain symmetric functions namely the functions defined for each pair of integers (n, r) such that $n \geq r \geq 1$ by

$$p_n^{(r)} = \sum_{|\lambda|=n,\ l(\lambda)=r} m_\lambda. \tag{1.1}$$

In (1.1) the m_λ are the monomial symmetric functions, the sum being over the integer partitions λ of n, of length $l(\lambda) = r$. These functions $p_n^{(r)}$ are introduced with this notation in [11, Example 19 p. 33], whose notations we will follow faithfully. The q-analogs thus defined, that we will note $\left[p_n^{(r)}\right]$, have attractive properties. The article presents the definition and some properties of this q-analog. We also present some applications by specialization in the following formal series sometimes called q-deformation of the exponential series :

$$E_{xp}(t) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{t^n}{n!}. \tag{1.2}$$

This allows new insight into polynomials known as inversion enumerators in rooted forests, or in a reciprocal way as the sum enumerators of parking functions (see Yan's summary article [20] on these notions). New identities are deduced for these polynomials, from which we extract some combinatorial consequences.

The organization of the article is as follows. In Section 2 we set the notations and recall the necessary prerequisites for symmetric functions, integer partitions and q-calculus.

Sections 3 and 4 are devoted to the general study of the $\left[p_n^{(r)}\right]$. In Section 3, starting from a generating function of $p_n^{(r)}$ we give the definition of $\left[p_n^{(r)}\right]$ and its first properties, with the particular case $r = 1$. Section 4 contains Theorem 4.1 which is an important result of the article. It links the q-analog to its classical form through q-Stirling numbers of the second kind. We also give the matrix form and the inverse equations of this theorem through q-Stirling numbers of the first kind. Section 5 begins the generalization to p, q-analog.

The remaining Sections 6 to 9 are dedicated to the applications of $\left[p_n^{(r)}\right]$, obtained by specialization in $E_{xp(t)}$. In Section 6, Theorem 4.1 lead us to polynomials previously introduced in [15] and [19], but in a different way

*E-mail address :*vincent.brugidou@univ-lille.fr

of that used by these authors. More precisely, it is the class of polynomials defined for $n \geq r \geq 1$ that we will denote $J_n^{(r)}$, and which correspond with the notations of [20] to the polynomials $I_{n-r}^{(r,1)}$.

Another important result thus obtained is the linear "positive" recurrence (6.5) of these polynomials. This recurrence makes it possible to determine all the polynomials from the initial conditions $J_r^{(r)} = 1$. As an example the first rows and columns of the table of $J_n^{(r)}$ are given.

In Section 7, we study the reciprocal polynomials of $J_n^{(r)}$, denoted by $\overline{J_n^{(r)}}$, and which are sum enumerators of parking functions. Recurrence (6.5) translates into a linear recurrence between $\overline{J_n^{(r)}}$, that we compare with another linear recurrence relation coming from the application of Goncarev polynomials to parking functions [10].

In Section 8, as an application of Relation (6.5), we obtain explicit formulas for the $J_n^{(r)}$ in Theorem 8.2, which are new results of our work. In Section 9, we give combinatorial interpretations of these explicit formulas by introducing new statistics on forests whose enumerator polynomials are $J_n^{(r)}$ or $\overline{J_n^{(r)}}$. In particular, this gives a q-refinement of the enumeration of functionnal digraph given in [13, page 19].

Complementary articles will follow - one of which is the subject of the preprint [1] - presenting other properties or applications of this q-analog, in particular combinatorial ones.

2 Preliminary

Let us first recall the definitions relating to integer partitions and symmetric functions, refering for more details to [11, Chap. 1] or [16, Chap. 7]. If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ is a partition of the integer n, we set $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_r = n$, $l(\lambda) = r$, and

$$n(\lambda) = \sum_{i \geq 1} (i-1)\, \lambda_i = \sum_{i \geq 1} \binom{\lambda_i'}{2}, \tag{2.1}$$

where $\lambda' = \left(\lambda_1', \lambda_2', ..\right)$ is the conjugate of the partition λ.

If A is a commutative ring, Λ_A is the set of symmetric functions in the variables $X = (x_i)_{i \geq 1}$ with coefficients in A. For our purposes A will be the field of rational functions $\mathbb{Q}(q)$, with eventually some indeterminates adjoined (in particular p, at Section 5). For λ describing the set of partitions, $(m_\lambda), (e_\lambda), (h_\lambda)$ and (p_λ), are the four classical bases of Λ_A. Agreeing that $e_0 = h_0 = 1$ we recall that

$$E(t) = \sum_{n \geq 0} e_n t^n = \prod_{i \geq 1} (1 - x_i t) \ ; \ \ H(t) = \sum_{n \geq 0} h_n t^n = (E(-t))^{-1} = \frac{1}{\prod\limits_{i \geq 1} (1 + x_i t)}. \tag{2.2}$$

For the q-analogs we use, with a few exceptions, the notations of [8], to which we refer for more details. The index q can be omitted if there is no ambiguity. For $(n, k) \in \mathbf{N}^2$, we have

$$[n]_q = 1 + q + q^2 + ... + q^{n-1} \text{ for } n \neq 0 \text{ and } [0]_q = 0,$$

$$[n]_q! = [1]\,[2] \ldots [n] \text{ for } n \neq 0 \text{ and } [0]_q! = 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]!\,[n-k]!} \text{ for } n \geq k \geq 0 \quad \text{and} \begin{bmatrix} n \\ k \end{bmatrix}_q = 0 \text{ otherwise.}$$

The q-derivation of a formal series $F(t)$ is given by

$$D_q F(t) = \frac{F(qt) - F(t)}{(q-1)t} \quad \text{and for } r \geq 1 \ \ D_q^r = D_q \left(D_q^{r-1} F(t) \right) \quad \text{with} \ \ D_q^0 F(t) = F.$$

Specifically

$$D_q^r t^n = [r]! \begin{bmatrix} n \\ r \end{bmatrix} t^{n-r} \text{ for } n \geq r \ \text{ and } \ D_q t^0 = 0. \tag{2.3}$$

We also use as an index instead of q, a rational fraction of q with coefficients in $\mathbb{Q}$. It is necessary to understand in this case what we will give as a definition.

Definition 2.1 *Let $\psi(q) \in \mathbb{Q}(q)$ such that $\psi(1) = 1$. We set for all $n \in \mathbb{N}$*

$$[n]_{\psi(q)} = 1 + \psi(q) + (\psi(q))^2 + ... + (\psi(q))^{n-1} \quad \textit{if } n \geq 1 \quad \textit{and} \quad [0]_{\psi(q)} = 0,$$

then are defined as above $[n]_{\psi(q)}!$, $\left[{n \atop k}\right]_{\psi(q)}$, etc... . In the same way we set

$$D_{\psi(q)} F(t) = \frac{F(\psi(q)t) - F(t)}{(\psi(q) - 1)t}.$$

Finally, some usual notations used in the article are recalled. $D^r F(t)$ is the usual derivation of order r with respect to t of the formal series $F(t)$, $\delta_i^j = \delta_{i,j}$ is the Kronecker symbol which is 1 if $i = j$ and 0 otherwise. If E is a finite set, $|E|$ denotes its cardinality and $\mathbf{2}^E$ the set of parts of E. $\mathbb{N}, \mathbb{Z}$ denote respectively the sets of natural and relative integers and $\mathbb{Q}$ that of rational numbers. If A is a commutative ring and t an indeterminate, $A[t]$ and $A[[t]])$ are the set of polynomials and the set of formal series respectively, in t with coefficients in A. For $n \in \mathbb{N}^* = \mathbb{N} - \{0\}$ we denote $\mathbf{n} = \{1, 2, ...n\}$.

3 Definition of $\left[p_n^{(r)}\right]$

3.1 Some properties of $p_n^{(r)}$

Let us give some properties of $p_n^{(r)}$ defined by (1.1). These properties can be deduced from the results of [11, example 19 p.33], but we give a stand-alone presentation for the reader's convenience.

Proposition 3.1 *Agreeing to set $p_n^{(0)} = \delta_n^0$, we have for all $r \in \mathbf{N}$*

$$\sum_{n \geq r} p_n^{(r)} (-t)^{n-r} = \frac{1}{r!} \frac{D^r E(t)}{E(t)}. \tag{3.1}$$

Proof. Since $D^r t^n = r! \binom{n}{r} t^{n-r}$ it follows by linearity

$$D^r E(t) = r! \sum_{n \geq r} \binom{n}{r} e_n t^{n-r}. \tag{3.2}$$

By performing the following Cauchy product, we have with (3.2) and (2.2)

$$(D^r E(t)) H(-t) = r! t^{-r} \sum_{n \geq r} t^n \sum_{k=r}^{n} \binom{k}{r} e_k h_{n-k} (-1)^{n-k}. \tag{3.3}$$

Definitions of the complete symmetric function and of $p_n^{(r)}$ lead to $h_n = \sum_{r=1}^n \sum_{|\lambda|=n, l(\lambda)=r} m_\lambda = \sum_{r=1}^n p_n^{(r)}$. So, in general

$$e_k h_m = e_k \sum_{l=1}^{m} p_m^{(l)} = \sum_{j=k}^{k+m} \binom{j}{k} p_{k+m}^{(j)}, \tag{3.4}$$

let us explain where the rightmost member of (3.4) comes from.The product of a monomial of e_k and a monomial of h_m gives a monomial of degree $k + m$, whose number j of distinct variables is between k and $k + m$ (for an infinity of variable x_i, or at least if the number of variables is greater than or equal to $k+m$). It is precisely also the form of the monomials of this rightmost member. Moreover in the left handside member of (3.4), the number of identical monomials of degree $k + m$ with j distinct variables corresponds to the number of monomials of e_k of which k distinct variables are taken in these j letters, that is to say $\binom{j}{k}$.

We have by replacing all the products $e_k h_{n-k}$ of (3.3) by (3.4), then inverting the two latest sums :

$$(D^r E(t)) H(-t) = r! \sum_{n \geq r} t^{n-r} \sum_{k=r}^{n} \sum_{j=k}^{n} \binom{k}{r} \binom{j}{k} p_n^{(j)} (-1)^{n-k} = r! \sum_{n \geq r} (-t)^{n-r} \sum_{j=r}^{n} p_n^{(j)} \sum_{k=r}^{j} \binom{j}{k} \binom{k}{r} (-1)^{r-k}. \tag{3.5}$$

In an classical way

$$\binom{j}{k}\binom{k}{r}=\binom{j}{r}\binom{j-r}{k-r},$$

from where it follows

$$\sum_{k=r}^{j}\binom{j}{k}\binom{k}{r}(-1)^{r-k}=\binom{j}{r}\sum_{k=r}^{j}\binom{j-r}{k-r}(-1)^{r-k}=\binom{j}{r}(1-1)^{j-r}=\delta_j^r.$$

Finally by replacing in (3.5) we obtain

$$\left(D^r E(t)\right) H(-t)=r!\sum_{n\geq r}p_n^{(r)}\left(-t\right)^{n-r},$$

(3.1) is deduced from this since $H\left(-t\right)=\left(E(t)\right)^{-1}$. ■

From (3.1) and (3.2) it follows

$$\sum_{n\geq 0}e_n t^n\sum_{n\geq r}p_n^{(r)}(-t)^{n-r}=\sum_{n\geq r}\binom{n}{r}e_n t^{n-r},$$

by equalizing the coefficients, we get the system of linear equations for the unknowns $p_n^{(r)}$

$$\text{for } n\geq r,\quad \sum_{k=r}^{n}(-1)^{k-r}\,e_{n-k}p_k^{(r)}=\binom{n}{r}e_n,$$

whose the following solutions are given by Cramer's rule :

$$p_n^{(r)}=\begin{vmatrix}\binom{r}{r}e_r & 1 & 0 & 0 & . & . & . & 0\\ \binom{r+1}{r}e_{r+1} & e_1 & 1 & 0 & . & . & . & 0\\ \binom{r+2}{r}e_{r+2} & e_2 & e_1 & 1 & 0 & . & . & 0\\ . & . & & . & . & . & . & .\\ . & . & & & . & . & . & .\\ . & . & & & & . & . & 0\\ . & . & & & & & . & 1\\ \binom{n}{r}e_n & e_{n-r} & e_{n-r-1} & . & . & . & e_2 & e_1\end{vmatrix}. \tag{3.6}$$

3.2 Definition of $\left[p_n^{(r)}\right]$

To define $\left[p_n^{(r)}\right]_q$, we will use the following q-analogue of (3.1).

Definition 3.2 *For any pair of integers (n,r) such that $n\geq r\geq 0$, we set*

$$\sum_{n\geq r}\left[p_n^{(r)}\right]_q(-t)^{n-r}=\frac{1}{[r]_q!}\frac{D_q^r E(t)}{E(t)}, \tag{3.1q}$$

which implies in particular $\left[p_n^{(0)}\right]_q=\delta_n^0$.

The subscript q is implied in what follows if there is no ambiguity. It is easy to check that the calculations made in Subsection 3.1 can be transposed for this q-analog of $p_n^{(r)}$, with in particular

$$D_q^r E(t)=[r]!\sum_{n\geq r}\begin{bmatrix}n\\ r\end{bmatrix}e_n t^{n-r}, \tag{3.2q}$$

and the system of linear equations

$$\text{For } n\geq r\quad \sum_{k=r}^{n}(-1)^{k-r}\,e_{n-k}\left[p_k^{(r)}\right]=\begin{bmatrix}n\\ r\end{bmatrix}e_n,$$

of which solutions is the q-analog of (3.6) given by the following proposition :

Proposition 3.3 *For $n \geq r \geq 1$, we have*

$$\left[p_n^{(r)}\right] = \begin{vmatrix} \left[{r \atop r}\right] e_r & 1 & 0 & 0 & . & . & . & 0 \\ \left[{r+1 \atop r}\right] e_{r+1} & e_1 & 1 & 0 & . & . & . & 0 \\ \left[{r+2 \atop r}\right] e_{r+2} & e_2 & e_1 & 1 & 0 & . & . & 0 \\ . & . & & . & . & . & . & . \\ . & . & & & . & . & . & . \\ . & . & & & & . & . & 0 \\ . & . & & & & & . & 1 \\ \left[{n \atop r}\right] e_n & e_{n-r} & e_{n-r-1} & . & . & . & e_2 & e_1 \end{vmatrix}. \tag{3.6q}$$

This determinant can be taken as an alternative definition of $\left[p_n^{(r)}\right]$.

Particular case $r = 1$
Equation (1.1) gives in this case $p_n^{(1)} = m_n = p_n$ thus $\left[p_n^{(1)}\right]$ can be also denoted $[p_n]$. We then

$$\sum_{n\geq 1} [p_n]_q \, (-t)^{n-1} = \frac{D_q E(t)}{E(t)},$$

and the system of linear equations

$$\text{For } n \geq 1 \quad \sum\nolimits_{k=1}^{n} (-1)^{k-1} e_{n-k} \, [p_k] = [n] \, e_n, \tag{3.7}$$

whose solution are

$$[p_n] = \begin{vmatrix} [1]\, e_1 & 1 & 0 & 0 & . & . & 0 \\ [2]\, e_2 & e_1 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & & . & . & 0 \\ . & . & . & & & . & 1 \\ [n]\, e_n & e_{n-1} & e_{n-2} & . & . & e_2 & e_1 \end{vmatrix}. \tag{3.8}$$

By inverting (3.7), we obtain the system of linear equations in the unknwons e_n

$$\text{for } n \geq 1, \quad \sum\nolimits_{k=1}^{n-1} (-1)^{k-1} [p_{n-k}] \, e_k + [n] \, e_n = [p_n] \,,$$

which gives by Cramer's rule

$$[n]! e_n = \begin{vmatrix} [p_1] & [1] & 0 & 0 & . & . & 0 \\ [p_2] & [p_1] & [2] & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & & . & . & . & 0 \\ . & . & & & & . & [n-1] \\ [p_n] & [p_{n-1}] & [p_{n-2}] & . & . & [p_2] & [p_1] \end{vmatrix}. \tag{3.9}$$

We notice that (3.8) and (3.9) are q-analog of the classical case (see the first two equations of [11, Example 8 p. 28]. We will continue in additional articles the study of $[p_n]$ and more generally, of :

$$[p_\lambda] = [p_{\lambda_1}] \, [p_{\lambda_2}] \ldots [p_{\lambda_r}] \,, \tag{3.10}$$

defined for any integer partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$.

4 Relations between $\left[p_n^{(r)}\right]$ and $p_n^{(r)}$

To prove Theorem 4.1 some reminders about the q-analogs of the Stirling numbers must be made. The "classical" Stirling numbers are well known (see for example [4, Chap. 5]). For their q-analogs which are still the subject of research, reference is made to [5, 14]. In [2, 3], Carlitz defined the q-Stirling numbers of the second kind, which we denote $S_q[n,k]$, by the following identity (with our notations) :

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{j=k}^{n} \binom{n}{j} (q-1)^{j-k} S_q[j,k], \tag{4.1}$$

(the index q is omitted if there is no ambiguity). And he got by inversion of (4.1)

$$(1-q)^{n-k} S[n,k] = \sum_{l=k}^{n} (-1)^{l-k} \binom{n}{l} \begin{bmatrix} l \\ k \end{bmatrix}, \tag{4.2}$$

as well as the recurrence relation

$$S[n,k] = S[n-1,k-1] + [k] S[n-1,k], \tag{4.3}$$

which makes it possible to find all the values of $S[n,k]$ for $(n,k) \in \mathbf{N}^2$ by setting $S[n,0] = \delta_n^0$, and $S[n,k] = 0$ if $k > n$. We have $S[n,n] = S[n,1] = 1$ and $S_{q=1}[n,k] = S(n,k)$, where $S(n,k)$ is the classical Stirling numbers of the second kind. (4.3) is the q-analog of the recurrence formula for $S(n,k)$. Note that (4.1) and (4.2) do not have classical correspondent.

Theorem 4.1 *We have for all $n \geq r \geq 1$*

$$\left[p_n^{(r)}\right]_q = \sum_{j=r}^{n} (1-q)^{j-r} S_q[j,r]\, p_n^{(j)}, \tag{4.4}$$

where the $S_q[n,r]$ are the q-Stirling numbers of the second kind of Carlitz.

Proof. The proof begins in a sort of inverse way compared to the proof of Proposition 3.1. Equation (3.1q) gives with $E(t)H(-t) = 1$ and $m = n-r$

$$\sum_{m\geq 0} \left[p_{m+r}^{(r)}\right] (-1)^m t^m = \left(\frac{1}{[r]!} D_q^r E(t)\right) H(-t).$$

Replacing the expression in the right-hand side parenthesis by (3.2q) and as $H(-t) = \sum_{l\geq 0} h_l (-1)^l t^l$, we obtain in equalizing the coefficients

$$\left[p_{m+r}^{(r)}\right] = \begin{bmatrix} r \\ r \end{bmatrix} e_r h_m - \begin{bmatrix} r+1 \\ r \end{bmatrix} e_{r+1} h_{m-1} + \dots + (-1)^{m-1} \begin{bmatrix} r+m-1 \\ r \end{bmatrix} e_{r+m-1} + (-1)^m \begin{bmatrix} r+m \\ r \end{bmatrix} e_{r+m}. \tag{4.5}$$

Using Equation (3.4) for each products $e_{r+k} h_{m-k}$ in (4.5), it follows

$$\left[p_{m+r}^{(r)}\right] = \sum_{k=0}^{m} (-1)^k \begin{bmatrix} r+k \\ r \end{bmatrix} \sum_{j=r+k}^{m+r} \binom{j}{r+k} p_{m+r}^{(j)},$$

then by inverting the sums :

$$\left[p_{m+r}^{(r)}\right] = \sum_{j=r}^{m+r} p_{m+r}^{(j)} \sum_{k=0}^{j-r} (-1)^k \binom{j}{r+k} \begin{bmatrix} r+k \\ r \end{bmatrix}.$$

With the change of index $l = r+k$ and $n = m+r$, we find

$$\left[p_n^{(r)}\right]_q = \sum_{j=r}^{n} p_n^{(j)} \sum_{l=r}^{j} (-1)^{l-r} \binom{j}{l} \begin{bmatrix} l \\ r \end{bmatrix}_q. \tag{4.6}$$

Theorem 4.1 follows from (4.2) . ■

Particular case $r = 1$. We get for all $n \geq 1$,

$$[p_n] = \sum_{j=1}^{n} (1-q)^{j-1} p_n^{(j)}. \tag{4.7}$$

Matrix form and inverse. Since $S[n,k]=0$ for $k<n$, (4.4) can still be written for $n\geq r\geq 1$

$$\left[p_n^{(r)}\right]=\sum_{j=1}^{n}(1-q)^{j-r}\,S[j,r]\,p_n^{(j)}. \tag{4.8}$$

Let the row matrices be defined by $[P_n]=\left(\left[p_n^{(r)}\right]\right)_{r=1}^{n}$ and $P_n=\left(p_n^{(r)}\right)_{r=1}^{n}$. System (4.8) for $n\geq r\geq 1$, can be written in matrix form $[P_n]=P_nA_n$, where A_n is the triangular matrix $(A_{i,j})_{i,j=1}^{n}$ with $A_{i,j}=(1-q)^{i-j}\,S[i,j]$. It is easy to see that we have

$$A_n=U_n\,[S_n]\,U_n^{-1}, \tag{4.9}$$

where $[S_n]$ is the triangular matrix of the q-Stirling numbers of the second kind $[S_n]=(S_q[i,j])_{i,j=1}^{n}$ and U_n is the diagonal matrix

$$U_n=\begin{pmatrix}(1-q)^0 & & & \\ & (1-q)^1 & & \\ & & \dots & \\ & & & (1-q)^{n-1}\end{pmatrix}.$$

The Stirling numbers of the first kind, denoted $s(n,k)$, have as q-analog the q-Stirling numbers of the first kind introduced in [7], which we will denote by $s_q[n,k]$. Let the matrix $[s_n]$ be defined by

$$[s_n]=(s_q[i,j])_{i,j=1}^{n},$$

then we know (see for example [5, p. 96]) that $[S_n]$ and $[s_n]$ are inverses of each other. We deduce with (4.9), that A_n is inversible and that $A_n^{-1}=U_n\,[s_n]\,U_n^{-1}$and $P_n=[P_n]\,A_n^{-1}$ with $A_n^{-1}=(B_{i,j})_{i,j=1}^{n}$and $B_{i,j}=(1-q)^{i-j}\,s[i,j]$. Hence, the corollary equivalent to Theorem 4.1 :

Corollary 4.2 *We have for all $n\geq r\geq 1$,*

$$p_n^{(r)}=\sum_{j=r}^{n}(1-q)^{j-r}\,s_q[j,r]\left[p_n^{(j)}\right]_q,$$

where the $s_q[n,r]$ are the q-Stirling numbers of the first kind.

Remark. It is easy to see that (4.8) generalizes for $n\geq r\geq 0$, to $\left[p_n^{(r)}\right]=\sum_{j=0}^{n}(1-q)^{j-r}\,S[j,r]\,p_n^{(j)}$. One could easily deduce the generalizations of the matrix and inverse forms of this system, with the augmented q-Stirling matrices $\widehat{[S_n]}=(S[i,j])_{i,j=0}^{n}$ and $\widehat{[s_n]}=(s[i,j])_{i,j=0}^{n}$.

5 Extension to p,q-analog

We know that there is a calculus with two parameters, denoted p,q-analog calculus, whose origin dates back at least to 1991, and which is reduced to the q-analog when $p=1$. Let us briefly recall the definitions that generalize those of the q-analog given in Section 2. One can consult for more details [5, 6].

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}=p^{n-1}+p^{n-2}q+\dots+pq^{n-2}+q^{n-1},$$

$$[n]_{p,q}!=[1]_{p,q}\,[2]_{p,q}\dots[n]_{p,q}\quad\text{and}\quad\begin{bmatrix}n\\k\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}!\,[n-k]_{p,q}!},$$

$$D_{p,q}F(t)=\frac{F(pt)-F(qt)}{(p-q)\,t}\quad\text{then }D_{p,q}^{r}F(t)=D_{p,q}\left(D_{p,q}^{r-1}F(t)\right).$$

Specifically

$$D_{p,q}^{r}t^n=[r]_{p,q}!\begin{bmatrix}n\\r\end{bmatrix}_{p,q}t^{n-r}.$$

The definition of the p,q-analog of the $p_n^{(r)}$ follows that of the q-analog.

Definition 5.1 *We set for $n \geq r \geq 0$,*

$$\sum_{n \geq r} \left[p_n^{(r)}\right]_{p,q} (-t)^{n-r} = \frac{1}{[r]_{p,q}!} \frac{D_{p,q}^r E(t)}{E(t)},$$

which implies in particular $\left[p_n^{(0)}\right]_{p,q} = \delta_n^0$.

We verify that the calculations of Section 3 transpose to this p,q-analog, with in particular

$$D_{p,q}^r E(t) = [r]_{p,q}! \sum_{n \geq r} \begin{bmatrix} n \\ r \end{bmatrix}_{p,q} e_n t^{n-r},$$

and

$$\left[p_n^{(r)}\right]_{p,q} = \begin{vmatrix} \begin{bmatrix} r \\ r \end{bmatrix}_{p,q} e_r & 1 & 0 & 0 & . & . & . & 0 \\ \begin{bmatrix} r+1 \\ r \end{bmatrix}_{p,q} e_{r+1} & e_1 & 1 & 0 & . & . & . & 0 \\ \begin{bmatrix} r+2 \\ r \end{bmatrix}_{p,q} e_{r+2} & e_2 & e_1 & 1 & 0 & . & . & 0 \\ . & . & & . & . & . & . & . \\ . & . & & & . & . & . & . \\ . & . & & & & . & . & 0 \\ . & . & & & & & . & 1 \\ \begin{bmatrix} n \\ r \end{bmatrix}_{p,q} e_n & e_{n-r} & e_{n-r-1} & . & . & . & e_2 & e_1 \end{vmatrix}.$$

Particular case $r = 1$: we also denote $\left[p_n^{(1)}\right]_{p,q} = [p_n]_{p,q}$ and we get the following $p, q-$analogs of the classical case (see [11, Example 8]), with inside the determinants the p, q-analogs of the square brackets :

$$[p_n]_{p,q} = \begin{vmatrix} [1]\, e_1 & 1 & 0 & 0 & . & . & 0 \\ [2]\, e_2 & e_1 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & & . & . & 0 \\ . & . & . & & & . & 1 \\ [n]\, e_n & e_{n-1} & e_{n-2} & . & . & e_2 & e_1 \end{vmatrix}, \qquad [n]_{p,q}!\, e_n = \begin{vmatrix} [p_1] & [1] & 0 & 0 & . & . & 0 \\ [p_2] & [p_1] & [2] & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & & . & . & . & 0 \\ . & . & & & & . & [n-1] \\ [p_n] & [p_{n-1}] & [p_{n-2}] & . & . & [p_2] & [p_1] \end{vmatrix}.$$

We leave it to the reader to verify that the proof of (4.6) extends without difficulty to the p, q-analog and gives :

Proposition 5.2 *We have for all $n \geq r \geq 1$,*

$$\left[p_n^{(r)}\right]_{p,q} = \sum_{j=r}^{n} p_n^{(j)} \sum_{l=r}^{j} (-1)^{l-r} \binom{j}{l} \begin{bmatrix} l \\ r \end{bmatrix}_{p,q}.$$

Things get more complicated if we want to extend Theorem 4.1. There are indeed p, q-Stirling numbers of the first and second kind [5, 18]. But as indicated in [5, page 104], there is no p, q-analog of (4.2) which we used to go from (4.6) to (4.4). This apparently prevents getting a p, q-analog of Theorem 4.1.

6 The polynomials $J_n^{(r)}$ and their linear recurrence

We now study the specialization $e_n = q^{\binom{n}{2}}/n!$ which corresponds to the q-deformation of the exponential series defined by (1.2). We begin with two lemmas.

Lemma 6.1 *For the q-deformation of the exponential series defined by* (1.2)*, and $n \geq r+1 \geq 2$, we have*

$$p_n^{(r)} = \frac{(1-q^r)}{r!} q^{\binom{r}{2}} \left[p_{n-r}\right]_{q^r}, \tag{6.1}$$

where for all $n \geq 1$, $[p_n]_{q^r}$ is deduced from (3.8) *with in the determinant $e_n = q^{\binom{n}{2}}/n!$ and $[n] = [n]_{q^r}$ given by Definition* 2.1.

Proof. *Lets start from* (3.6) *with* $e_n = q^{\binom{n}{2}}/n!$. *We factor the first column by* $q^{\binom{r}{2}}/r!$, *then we subtract the first column from the second, which gives*

$$p_n^{(r)} = \frac{q^{\binom{r}{2}}}{r!}\begin{vmatrix} 1 & 0 & 0 & 0 & . & . & . & 0 \\ \dfrac{q^{\binom{r+1}{2}-\binom{r}{2}}}{1!} & \dfrac{1-q^r}{1!} & 1 & 0 & . & . & . & 0 \\ \dfrac{q^{\binom{r+2}{2}-\binom{r}{2}}}{2!} & q^{\binom{2}{2}}\dfrac{1-q^{2r}}{2!} & 1 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \dfrac{q^{\binom{r+j}{2}-\binom{r}{2}}}{j!} & q^{\binom{j}{2}}\dfrac{1-q^{jr}}{j!} & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & 1 \\ \dfrac{q^{\binom{n}{2}-\binom{r}{2}}}{(n-r)!} & q^{\binom{n-r}{2}}\dfrac{1-q^{(n-r)r}}{(n-r)!} & \dfrac{q^{\binom{n-r-1}{2}}}{(n-r-1)!} & . & . & . & \dfrac{q^{\binom{2}{2}}}{2!} & 1 \end{vmatrix}.$$

For $1 \le j \le n-r$, *we have* $1-q^{jr} = (1-q^r)\,[j]_{q^r}$. *We therefore factor the second column by* $(1-q^r)$, *then we expand the determinant with respect to the first row. Hence,*

$$p_n^{(r)} = \frac{q^{\binom{r}{2}}}{r!}\,(1-q^r)\begin{vmatrix} 1 & 1 & 0 & . & . & . & 0 \\ [2]_{q^r}\dfrac{q^{\binom{2}{2}}}{2!} & 1 & 1 & 0 & . & . & 0 \\ . & \dfrac{q^{\binom{2}{2}}}{2!} & . & . & . & . & . \\ . & & . & . & . & . & . \\ [j]_{q^r}\dfrac{q^{\binom{j}{2}}}{j!} & & & . & . & . & 0 \\ . & . & . & . & . & . & 1 \\ [n-r]_{q^r}\dfrac{q^{\binom{n-r}{2}}}{(n-r)!} & \dfrac{q^{\binom{n-r-1}{2}}}{(n-r-1)!} & . & . & . & \dfrac{q^{\binom{2}{2}}}{2!} & 1 \end{vmatrix}.$$

By comparing this determinant to (3.8) *and taking into account Definition 2.1 with* $\psi(q) = q^r$, *we have indeed obtained the second member of* (6.1). ■

Lemma 6.2 *For the formal series* $E_{xp}(t) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} t^n/n!$, *we have*

$$D^r E_{xp}(t) = q^{\binom{r}{2}} E_{xp}(q^r t). \tag{6.2}$$

Proof. On the left handside we have formally

$$D^r E_{xp}(t) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{n!} D^r t^n = \sum_{n=r}^{\infty} \frac{q^{\binom{n}{2}}}{n!}\frac{n!}{(n-r)!} t^{n-r} = \sum_{m=0}^{\infty} q^{\binom{m+r}{2}} \frac{t^m}{m!}.$$

On the right handside we have

$$q^{\binom{r}{2}} E_{xp}(q^r t) = q^{\binom{r}{2}} \sum_{m=0}^{\infty} q^{\binom{m}{2}} \frac{(q^r t)^m}{m!} = \sum_{m=0}^{\infty} q^{\binom{r}{2}+\binom{m}{2}+mr} \frac{t^m}{m!},$$

and we check that

$$\binom{m+r}{2} = \binom{m}{2} + \binom{r}{2} + mr. \tag{6.3}$$

■

It is now possible to introduce very naturally a class of polynomials with positive integer coefficients.

Theorem 6.3 *i) For* $n \ge r \ge 1$ *and* $E_{xp}(t) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} t^n/n!$, *we have*

$$p_n^{(r)} = (1-q)^{n-r} \frac{q^{\binom{r}{2}}}{r!\,(n-r)!} J_n^{(r)}(q), \tag{6.4}$$

where $J_n^{(r)}$ *is a monic polynomial with positive integer coefficients, a constant term equal to* $(n-r)!$ *and whose degree is* $\binom{n-1}{2}-\binom{r-1}{2}$. *Moreover, for all* $r\geq 1$ $J_r^{(r)}=1$.

ii) The polynomials $J_n^{(r)}$ *satisfy when* $n-1\geq r\geq 1$, *the linear recurrence whose coefficients belong to* $\mathbb{N}[q]$ *:*

$$J_n^{(r)}(q)=\sum_{j=1}^{n-r}[r]_q^j\ q^{\binom{j}{2}}\binom{n-r}{j}J_{n-r}^{(j)}(q). \tag{6.5}$$

This recurrence suffices with the initial conditions $J_r^{(r)}=1$ *for* $r\geq 1$, *to calculate all the* $J_n^{(r)}$ *for* $n\geq r+1$.

Proof. Let first prove *i)*

a) For $n=r$. From (3.1) and Lemma 6.2 we have

$$\sum_{n\geq r}p_n^{(r)}(-t)^{n-r}=\frac{1}{r!}\frac{D^rE_{xp}(t)}{E_{xp}(t)}=\frac{q^{\binom{r}{2}}}{r!}\frac{E_{xp}(q^rt)}{E_{xp}(t)}.$$

The constant term of the formal series quotient $E_{xp}(q^rt)/E_{xp}(t)$ is $E_{xp}(q^r0)/Exp(0)=1$. Thefore the constant term of the series on the left handside of the above equation is $p_r^{(r)}=q^{\binom{r}{2}}/r!$. By comparison to (6.4) we get $J_{r,r}=1$ and this polynomial verifies all the assertions of item *i)* of the theorem.

b) For the general case $n\geq r\geq 1$ we reason by induction on n. For $n=r=1$ it is a particular case of a) and it is therefore proved. Let's assume true the theorem for all pairs of integers (l,j) such that $1\leq l\leq n$ and $1\leq j\leq l$. And let's prove it for $n+1$ and all r between 1 and n (since it's already proved for $r=n+1$ by a)). We have with Lemma 6.1

$$p_{n+1}^{(r)}=(1-q^r)\frac{q^{\binom{r}{2}}}{r!}[p_{n+1-r}]_{q^r}, \tag{6.6}$$

and the particular case (4.7) of Theorem 4.1 gives

$$[p_{n+1-r}]_{q^r}=\sum_{j=1}^{n+1-r}(1-q^r)^{j-1}p_{n+1-r}^{(j)}.$$

But for $1\leq r\leq n$ we have $1\leq n+1-r\leq n$, so the induction hypothesis applies to $p_{n+1-r}^{(j)}$, hence

$$[p_{n+1-r}]_{q^r}=\sum_{j=1}^{n+1-r}(1-q^r)^{j-1}(1-q)^{n+1-r-j}\frac{q^{\binom{j}{2}}}{j!\,(n+1-r-j)!}J_{n+1-r}^{(j)},$$

by replacing this expression in (6.6) we get after some easy transformations

$$p_{n+1}^{(r)}=(1-q)^{n+1-r}\frac{q^{\binom{r}{2}}}{r!\,(n+1-r)!}\sum_{j=1}^{n+1-r}[r]^j\binom{n+1-r}{j}q^{\binom{j}{2}}J_{n+1-r}^{(j)}.$$

Let

$$J_{n+1}^{(r)}(q)=\sum_{j=1}^{n+1-r}[r]^j\binom{n+1-r}{j}q^{\binom{j}{2}}J_{n+1-r}^{(j)}, \tag{6.7}$$

then we have verified (6.4) for $n+1$. Moreover, since (6.7) links $J_{n+1}^{(r)}$ to polynomials which have positive integer coefficients by the induction hypothesis with coefficients which are themselves polynomials in q with positive integer, it is the same for $J_{n+1}^{(r)}$.

On the right hand side of (6.7) the degree of each term of the sum is by the induction hypothesis

$$d^\circ\left([r]^j\binom{n+1-r}{j}q^{\binom{j}{2}}J_{n+1-r}^{(j)}\right)=(r-1)j+\binom{j}{2}+\binom{n-r}{2}-\binom{j-1}{2}=\binom{n-r}{2}+rj-1.$$

It is clearly maximum for $j=n+1-r$ and is then $\binom{n}{2}-\binom{r-1}{2}$ (use (6.3) with $m=n-r$). The coefficient of the monomial of maximum degree $n+1-r$, is $J_{n+1-r}^{(n+1-r)}$ which is equal to 1 by a). $J_{n+1}^{(r)}$ is therefore a monic polynomial.

When $j=1$ the term of the sum is $[r]\,(n+1-r)\,J_{n+1-r}^{(1)}(q)$. This polynomial has an order equal to zero and it is the only term of the sum having this property. The order of $J_{n+1}^{(r)}$ is therefore zero and its constant term is by the induction hypothesis $(n+1-r)\,.\,(n-r)!=(n+1-r)!$. Which ends the proof of *i)*.

Proof of item *ii)* Equation (6.5) is none other than the equation (6.7) defining $J_{n+1}^{(r)}$ in the proof of *i)*. (6.5) is therefore proved for all n and r such that $n-1 \geq r \geq 1$. Arrange the polynomials in a triangular table as shown below (we can also set $J_n^{(r)} = 0$ if $n < r$ by convention). We see that linear horizontal recurrence (6.5) makes it possible to determine row by row all the polynomials such as $n > r$. ■

$n \backslash r$	1	2	3	4	5
1	1				
2	1	1			
3	$2+q$	$1+q$	1		
4	$6+6q+3q^2+q^3$	$2+3q+2q^2+q^3$	$1+q+q^2$	1	
5	$24+36q+30q^2+20q^3$ $+10q^4+4q^5+q^6$	$6+12q+12q^2+10q^3$ $+6q^4+3q^5+q^6$	$2+3q+4q^2+$ $3q^3+2q^4+q^5$	$1+q+q^2+q^3$	1

Table 1 : Polynomials $J_n^{(r)}$ for $5 \geq n \geq r \geq 1$.

Exemple of application of Recurrence (6.5) **for** $n = 6, r = 2$:

$$J_6^{(2)} = 4\,(1+q)\,J_4^{(1)} + 6\,(1+q)^2\,qJ_4^{(2)} + 4\,(1+q)^3\,q^3 J_4^{[3]} + (1+q)^4\,q^6,$$

hence : $$J_6^{(2)} = 24 + 60q + 78q^2 + 80q^3 + 68q^4 + 52q^5 + 35q^6 + 20q^7 + 10q^8 + 4q^9 + q^{10}.$$

Case $r = 1$: (6.5) gives for all $n \geq 1$, $\quad J_{n+1}^{(1)}(q) = \sum\limits_{j=1}^{n} \binom{n}{j} q^{\binom{j}{2}} J_n^{(j)}(q).$

Recurrence (6.5) can be generalized in the following form, by setting for all $n \geq 0$, $J_n^{(0)} = \delta_{n,0}$ and $[0]_q^0 = 1$.

Corollary 6.4 *We have for all pair of integers (n, r) such that $n \geq r \geq 0$*

$$J_n^{(r)}(q) = \sum_{j=0}^{n-r} [r]_q^j \; q^{\binom{j}{2}} \binom{n-r}{j} J_{n-r}^{(j)}(q). \tag{6.8}$$

Proof. If $n > r \geq 1$, (6.5) implies (6.8) since the added term contains $J_{n-r}^{(0)}$ which is zero.

If $n = r \geq 1$, the left side of (6.8) is 1 and the right side is $[0]^0\, q^0 \binom{0}{0} J_0^{(0)} = 1$.

If $n \geq r = 0$, the left side of (6.8) is $J_n^{(0)} = \delta_{n,0}$ and on the right side, the sum is reduced to the first term $[0]^0\, q^0 \binom{0}{0} J_n^{(0)} = \delta_{n,0}$. ■

Now let's make the connection with the polynomials introduced previously of which we briefly remind the genesis. We refer to [20] for more details. In [12], Mallow and Riordan defined the inversion polynomial enumerator for rooted trees with n nodes, that they denoted J_n. Then Stanley [15] and Yan [19] successively generalized the definition of inversion polynomial enumerator to sequences of "colored" rooted forests, which correspond to classical parking functions in the terminology of [20, Section 1.4.4]. These polynomials are characterized by a pair of parameters (a, b) and denoted $I_m^{(a,b)}$. It has been proven combinatorially that [19, Corollary 5.1] :

$$\sum_{m \geq 0} (q-1)^m \, I_m^{(a,b)} \frac{t^m}{m!} = \frac{\sum\limits_{m \geq 0} q^{am+b\binom{m}{2}} \dfrac{t^m}{m!}}{\sum\limits_{m \geq 0} q^{b\binom{m}{2}} \dfrac{t^m}{m!}}. \tag{6.9}$$

Note that these polynomials have other combinatorial representations and are still the subject of recent research (see for example [9, 17]).

Corollary 6.5 *We have for all $n \geq r \geq 1$,*

$$J_n^{(r)} = I_{n-r}^{(r,1)} \Leftrightarrow I_m^{(r,1)} = J_{m+r}^{(r)}. \tag{6.10}$$

Proof. For $a = r, b = 1$, (6.9) becomes

$$\sum_{m \geq 0} (q-1)^m \, I_m^{(r,1)} \frac{t^m}{(m)!} = \frac{\sum\limits_{m \geq 0} q^{rm+\binom{m}{2}} \dfrac{t^m}{(m)!}}{\sum\limits_{m \geq 0} q^{\binom{m}{2}} \dfrac{t^n}{n!}}. \tag{6.11}$$

The denominator of this fraction is $E_{xp}(t)$. For the numerator we have by Lemma 2, $D^r E_{xp}(t) = q^{\binom{r}{2}} E_{xp}(q^r t) = q^{\binom{r}{2}} \sum_{m\geq 0} q^{rm+\binom{m}{2}} t^m/m!$. So by multiplying (6.11) by $q^{\binom{r}{2}}/r!$ and with $m = n-r$,

$$\sum_{n\geq r} (q-1)^{n-r} q^{\binom{r}{2}} I_{n-r}^{(r,1)} \frac{t^{n-r}}{r!\,(n-r)!} = \frac{1}{r!} \frac{D^r E_{xp}(t)}{E_{xp}(t)}. \tag{6.12}$$

By identifying the coefficients of (6.12) and those of (3.1) we therefore obtain for $E_{xp}(t)$

$$p_n^{(r)} = (1-q)^{n-r} \frac{q^{\binom{r}{2}}}{r!\,(n-r)!} I_{n-r}^{(r,1)},$$

that is to say by comparison with (6.4), $J_n^{(r)} = I_{n-r}^{(r,1)}$. ■

Case $r = 1$ corresponds to the case of a tree [12]. So we have $J_n^{(1)} = J_n$, which gives another notation for $J_n^{(1)}$.

We see that our study clearly differs from that of the previously cited publications. In these, polynomials $I_m^{(a,b)}$ are introduced combinatorially and Equation (6.9) is also deduced combinatorially. On the contrary we defined formally the polynomials $J_n^{(r)}$ by recursive application of Theorem 4.1 on the q-analog of the symmetric functions $p_n^{(r)}$. Recurrence (6.5), inherent in the proof of Theorem 6.3, will lead in return to a new combinatorial representation of the $J_n^{(r)}$ in Section 9. Note, however, that our approach only applies to the class of polynomials given by (6.10).

7 Reciprocal polynomials

Definition 7.1 *The polynomials $\overline{J_n^{(r)}}$ are defined by*

$$\overline{J_n^{(r)}}(q) = q^{\binom{n-1}{2}-\binom{r-1}{2}} J_n^{(r)}(1/q). \tag{7.1}$$

These are the reciprocal polynomials of $J_n^{(r)}$

Since $\overline{\overline{J_n^{(r)}}} = J_n^{(r)}$, (7.1) is equivalent to

$$J_n^{(r)}(q) = q^{\binom{n-1}{2}-\binom{r-1}{2}} \overline{J_n^{(r)}}(1/q). \tag{7.1bis}$$

It is well known that these reciprocal polynomials are equal to the sum enumerator of the corresponding parking functions which definition we recall. We will refer for more details to [20] from which we almost adopt the notations for the parking functions of our case, i.e when parameters $(a,b) = (r,1)$. Let $m = n-r$ and let $PK_m(r,1)$ designate the set of parking functions associated with the sequence $r, r+1, r+2, ..., r+m-1$. $PK_m(r,1)$ is the set of sequences of integers $\mathbf{a} = (a_1, a_2, ..., a_m)$, such that :

$$a_{(i)} < r+i-1 \quad \text{for} \quad 1 \leq i \leq m,$$

$\left(a_{(1)}, a_{(2)}, ..., a_{(m)}\right)$ being the sequences of a_i ordered in a non-decreasing way.

We therefore have with $|\mathbf{a}| = a_1 + a_2 + ... + a_m$

$$\overline{J_{m+r}^{(r)}}(q) = \sum_{\mathbf{a}\in PK_m(r,1)} q^{|\mathbf{a}|} = S_m(q;r),$$

where the second member is the sum enumerator written $S_m(q;r)$ (instead of $S_m(q;r,r+1,r+2,...,r+m-1)$ in [20]). We verify that this relation is still true if $m = 0$, on the condition that we set $PK_0(r,1) = \emptyset$ and $\sum_{\mathbf{a}\in\emptyset} q^{|\mathbf{a}|} = 1$.

With (7.1*bis*), (6.5) gives :

Corollary 7.2 *The reciprocal polynomials* $\overline{J_n^{(r)}}$ *satisfy when* $n-1 \geq r \geq 1$*, the linear recurrence of which coefficients are elements of* $\mathbb{N}[q]$

$$\overline{J_n^{(r)}}(q) = \sum_{j=1}^{n-r} [r]_q^j \; q^{r(n-r-j)} \binom{n-r}{j} \overline{J_{n-r}^{(j)}}(q) \tag{7.2}$$

We could alternatively use this recurrence to find the table of $\overline{J_n^{(r)}}$ with $\overline{J_r^{(r)}} = 1$.

In [10] Kung and Yan presented an alternative study of parking functions from Goncarov polynomials. It is interesting to compare our linear recurrence (7.2) with a linear recurrence resulting from this study. This is Equation (6.2) in [10] (or (1.27) in [20]), which in our case and with our notations is written :

$$1 = \sum_{k=0}^{m} \binom{m}{k} q^{(r+k)(m-k)} \left(1-q\right)^k \overline{J_{k+r}^{(r)}},$$

which is equivalent with $m+r=n$ and $l=k+r$, to

$$\left(1-q\right)^{n-r} \overline{J_n^{(r)}}(q) = 1 - \sum_{l=r}^{n-1} \binom{n-r}{l-r} q^{l(n-l)} \left(1-q\right)^{l-r} \overline{J_l^{(r)}}(q)\,. \tag{7.3}$$

To see the difference, let's take the example of $n=6, r=2$:

$$\overline{J_6^{(2)}} = 4\left(1+q\right) q^6 \overline{J_4^{(1)}} + 6\left(1+q\right)^2 q^4 \overline{J_4^{(2)}} + 4\left(1+q\right)^3 q^2 \overline{J_4^{(3)}} + \left(1+q\right)^4 \overline{J_4^{(4)}}, \qquad \text{(with 7.2)}$$

$$\left(1-q\right)^4 \overline{J_6^{(2)}} = 1 - q^8 \overline{J_2^{(2)}} - 4q^9\left(1-q\right) \overline{J_3^{(2)}} - 6q^8 \left(1-q\right)^2 \overline{J_4^{(2)}} - 4q^5\left(1-q\right)^3 \overline{J_5^{(2)}}, \qquad \text{(with 7.3)}$$

Let call Table 2, the table - not shown here - deduced from Table 1 by replacing the polynomials $J_n^{(r)}$ by their reciprocals $\overline{J_n^{(r)}}$. The two equations above show that :

* On the one hand, our recurrence relation (7.2) is horizontal using here row 4 of Table 2, and the coefficients belong to $\mathbb{N}[q]$.

* On the other hand, the recurrence relation (7.3) is vertical using here column 2 of Table 2, and the coefficients belong to $\mathbb{Z}[q]$.

8 An explicit formula for the $J_n^{(r)}$

From the recurrence formulas of Section 6, we will prove the following theorem which is declined in two versions corresponding to the two versions (6.5) and (6.8) of the recurrence relation. We first generalize the notations used for integer partitions.

Notation 8.1 *Let* $u = (u_1, u_2, ..., u_k)$ *be a sequence of strictly positive integers, we set*

$$|u| = u_1 + u_2 + ... + u_k \quad \textit{and} \quad n(u') = \binom{u_1}{2} + \binom{u_2}{2} + ... + \binom{u_k}{2}.$$

Note that u' is not defined in this case, only $n(u')$ has a meaning given by the above formula. More generally if $\mathcal{U}$ is the set of infinite sequences of integers $u = (u_1, u_2, ...)$ with $u_i \geq 0$, such that only a finite number of the term u_i are non-zero, we can extend the above notations to all $u \in \mathcal{U}$ by

$$|u| = \sum_{i \geq 1} u_i \quad \text{and} \quad n(u') = \sum_{i \geq 1} \binom{u_i}{2}.$$

Theorem 8.2 *a) For any pair of integers* (n, r) *satisfying* $n-1 \geq r \geq 1$ *we have the explicit formula*

$$J_n^{(r)}(q) = \sum [r]_q^{u_1} [u_1]_q^{u_2} ... [u_{k-1}]_q^{u_k} q^{n(u')} \binom{n-r}{u_1, u_2, ..., u_k}, \tag{8.1}$$

where the sum is over the k-multiplets of strictly positive integers $u = (u_1, u_2, ..., u_k)$ *with* $k \geq 1$ *and* $|u| = n - r$, *with* $n(u') = \sum\limits_{i=1}^{k} \binom{u_i}{2}$ *and where* $\binom{n-r}{u_1, u_2, ..., u_k}$ *is the multinomial coefficient.*

b) For any pair of integers (n, r) *satisfying* $n \geq r \geq 0$ *we have :*

$$J_n^{(r)}(q) = (n-r)! \sum [r]_q^{u_1} \, q^{n(u')} \prod_{i \geq 1} \frac{[u_i]_q^{u_{i+1}}}{u_i!}, \tag{8.2}$$

where the sum is over the sequences $u = (u_i)_{i \geq 1} \in \mathcal{U}$ *satisfying* $|u| = \sum_{i \geq 1} u_i = n - r$ *and with* $n(u') = \sum_{i \geq 1} \binom{u_i}{2}$.

As we will see below, the sum in (8.2) contains only a finite number of non-zero terms. Note that if b) is apparently more complicated than a), it is in fact easier to prove.

Proof. We first show the equivalence of a) and b), when $n - 1 \geq r \geq 1$.

In order for the general term of the sum in (8.2) to be non-zero, it is necessary - taking into account $[0]^n = \delta_n^0$ - that the finite number (say k) of non-zero terms of the sequence u, occupy the first k places in this sequence. In other words, the summation can be limited to these sequences, which we call commencing sequences, and we denote by (8.2*bis*) the summation deduced from (8.2) and coresponding to these commencing sequences. The application, which associates to a multiplet of the sum of (8.1) $(u_1, u_2, ..., u_k)$ with $u_1 + u_2 + ... + u_k = n - r$ and $k \geq 1$, the commencing sequence $u = (u_1, u_2, ..., u_k, 0, 0, ...)$ with $|u| = n - r$, is clearly a bijection between the two summation sets of (8.1) and (8.2*bis*). Moreover, we check that the value of the terms in the respective sums, thus put in 1-1 corespondence are equal. The three sums (8.1), (8.2) and (8.2*bis*) are therefore equal.

Let now prove (8.2*bis*) for all $n \geq r \geq 0$.

1) If $n = r$, the condition $|u| = \sum_{i \geq 1} u_i = n - r = 0$ implies $u_i = 0$ for all $i \geq 1$. The sum in (8.2*bis*) therefore reduces to $0! \, [r]^0 \, q^0 \, [0]^0 \, .../0!.. = 1$ which is indeed equal to $J_r^{(r)}$ for all $r \geq 0$.

2) If $r = 0$ and $n > 0$, it is necessary for a commencing sequence u verifying $|u| = n - r > 0$, that its first term u_1 be non-zero. Therefore, the term indexed by u in the sum of (8.2*bis*) is zero, since its contains the factor $[0]^{u_1} = 0$. Hence, the sum in (8.2*bis*) is zero, which is indeed $J_n^{(0)}$ for $n > 0$.

3) Let now prove by induction on n the general case $n \geq r \geq 0$.

When $n = r = 0$, it has already been checked by 1). Assume that relation (8.2*bis*) is true for any pair of integers (l, j) such that $0 \leq l \leq n - 1$, $0 \leq j \leq l$ and show that it is true for $l = n$, $n > 0$ and $0 \leq j = r \leq n$.

If $r = 0$, it is the case 2) already proved since $n > 0$.

If $r = n$, it is the case 1) already proved.

So suppose $1 \leq r \leq n - 1$. The set of commencing sequences verifying $|u| = \sum_{i \geq 0} u_i = n - r > 0$ can be decomposed according to the value of u_1 into :

* $u_1 = 1$, from where $\sum_{i \geq 2} u_i = n - r - 1$,

and in a general way for $1 \leq j \leq n - r$:

* $u_1 = j$, from where $\sum_{i \geq 2} u_i = n - r - j$.

Equation (8.2*bis*) can therefore be written

$$J_n^{(r)}(q) = \sum_{j=1}^{n-r} [r]^j \, q^{\binom{j}{2}} \binom{n-r}{j} \left[(n-r-j)! \sum_{u_2 + u_3 + ... = n-r-j} [j]^{u_2} \, q^{\binom{u_2}{2} + \binom{u_3}{2} + ...} \prod_{i \geq 2} \frac{[u_i]^{u_{i+1}}}{u_i!} \right]. \tag{8.3}$$

Let $v_i = u_{i+1}$ for $i \geq 1$, then $v = (v_i)_{i \geq 1}$ is a sequence satisfying $|v| = (n - r) - j$ and the square bracket in (8.3) can be rewritten

$$((n-r)-j)! \sum [j]_q^{v_1} \, q^{n(v')} \prod_{i \geq 1} \frac{[v_i]_q^{v_{i+1}}}{v_i!},$$

where the sum is over sequences ν of integers such that $|v| = n - r - j$. But this sum is $J_{n-r}^{(j)}$ according to the induction hypothesis.The second member of (8.3) is therefore

$$\sum_{j=1}^{n-r} [r]_q^j \, q^{\binom{j}{2}} \binom{n-r}{j} J_{n-r}^{(j)}(q,$$

which is indeed equal to $J_n^{(r)}$, according to (6.5). ■

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Case $q=1$. We obtain with (8.1)

$$J_n^{(r)}(1)=\sum_{\substack{u_1+u_2+...+u_k=n-r\\ u_i\geq 1,\ k\geq 1}} r^{u_1}u_1^{u_2}...u_{k-1}^{u_k}\binom{n-r}{u_1,u_2,...,u_k}=rn^{n-r-1}. \tag{8.4}$$

It follows from the generalization, made in [19] to interpret polynomials $I_m^{(a,b)}$ as inversion enumerators, that $J_n^{(r)}(1)$ is equal to the number of forests on n vertices (including the roots), comprising r rooted trees whith specified roots. The set of these forests is denoted by $\mathcal{F}_m(r,1)$ in [20] with here $m=n-r$. Note that according to [16, Proposition 5.3.2], the number of these forests is $J_n^{(r)}(1)=rn^{n-r-1}$, which can also easily be verified by induction.

Equation (8.4) can be linked to a formula given by Katz, whose description and reference can be found in [13, page 19]. This formula gives the number of connected directed graph or functionnal digraph, made up of rooted trees whose roots determine a directed cycle. If the cycle has length r and if the number of nodes including the roots is n, then this number is with our notations [13, Formula page 19] :

$$D(n,r)=(r-1)!\sum_{\substack{u_1+u_2+...+u_k=n-r\\ u_i\geq 1,\ k\geq 1}} r^{u_1}u_1^{u_2}...u_{k-1}^{u_k}\binom{n-r}{u_1,u_2,...,u_k}. \tag{8.5}$$

It is easy to deduce (8.5) from (8.4) by multiplying the number $J_n^{(r)}(1)$ of forests with r rooted trees by the number of cycles $(r-1)!$, that can be formed with the roots. On the other hand, Katz's proof of (8.5) consists to enumerate the digraphs according to the numbers u_i of nodes located at distance i from the cycle. This suggests that conversely, Equation (8.1) could be interpreted, putting aside the factor $(r-1)!$, as a q-refinement of the Katz's enumeration. It is this statistic that is the subject of Section 9.

9 Level statistics on forests

We denote $\mathcal{F}_{n,R}$ the set of rooted forests whose vertices are $V=\mathbf{n}=\{1,2,...,n\}$, and whose roots are a specified subset $R\subseteq\mathbf{n}$ with $|R|=r$. $\mathcal{F}_{n,R}$ is equipotent to $\mathcal{F}_{n-r}(r,1)$ of [20]. We assume until further notice that $n-1\geq r\geq 1$, therefore $R\subset\mathbf{n}$. Let $F\in\mathcal{F}_{n,R}$, k will designate the height of F, i.e. the greatest height of its trees. For i an integer between 0 and k, let $V_i=\{v\in\mathbf{n}\,;\text{ distance of } v \text{ from the root}=i\}$ and $u_i=|V_i|$; in particular $V_0=R$ and $u_0=r$. The map which to $v\in\mathbf{n}$ associates V_i such that $v\in V_i$ is denoted L_F, $L_F(v)$ is the level (or generation) of v.

We necessarily have $u_0+u_1+....u_k=n$. According to the previous section, we have with these notations and $r\leq n-1$ which implies $k\geq 1$

$$|\mathcal{F}_{n,R}|=J_n^{(r)}(1)=\sum_{\substack{u_0+u_1+...+u_k=n\\ u_0=r,\ u_i\geq 1}} u_0^{u_1}u_1^{u_2}...u_k^{u_{k+1}}\binom{n-r}{u_1,u_2,...,u_k}.$$

Level statistics can be described quite generally with the following definitions.

Definition 9.1 *Let $\mathbf{2^n}$ be the set of subsets of $\mathbf{n}$. A ranking ρ on $\mathbf{2}^n$ is the data for each $P\in\mathbf{2}^n$ of a bijection from P into $\mathbf{p}=\{1,...,p\}$ where $p=|P|$, bijection denoted $\rho(P)$.*

Exemples of ranking on $\mathbf{2^n}$: the increasing ranking ρ_+ is the one for which $\rho_+(P)$ is the increasing bijection for all $P\in\mathbf{2^n}$; the decreasing ranking ρ_- is the one for which $\rho_-(P)$ is the decreasing bijection for all $P\in\mathbf{2^n}$. Any combination of ρ_+ and ρ_- is also a ranking on $\mathbf{2^n}$, for example that which is equal to ρ_+ if $|P|$ is even and to ρ_- otherwise.

Definition 9.2 *Let ρ be a ranking on $\mathbf{2^n}$ and $\mathcal{F}_{\mathbf{n},R}$ the set of forests defined above, n and R being given. The weight associated with ρ of a vertex v of $F\in\mathcal{F}_{n,R}$ is $w_\rho(v)=\rho(L_F(v))(v)$*

Note that this weight depends only on ρ and the generation of v. We note $p(v)$ the (only) parent of the vertex $v\in\mathbf{n}-R$.

Example of a weighted forest with $n=13$, $r=3$, $k=4$, $R=\{7,11,2\}$ and ρ_+. The forest $F\in\mathcal{F}_{13,R}$ is represented in Figure 1. For each of its vertices v labeled in black, we have indicated in red the weight associated with ρ_+. The cardinal of each of its level is also indicated on the right.

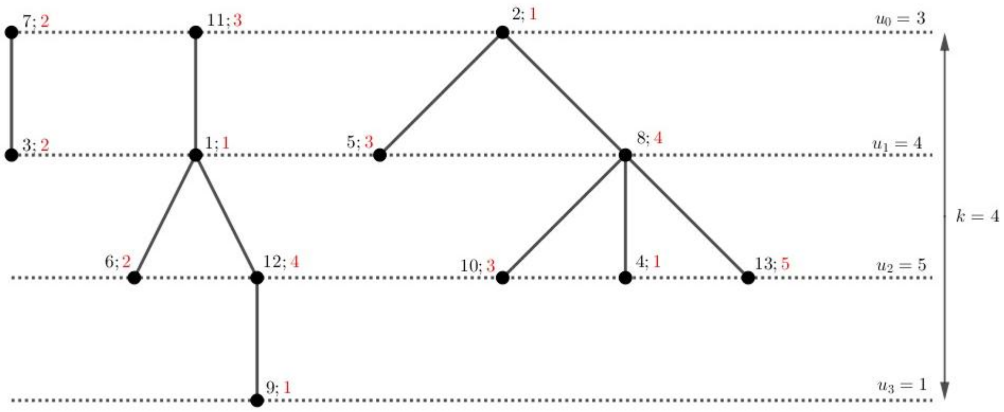

FIGURE 1 – A weighted forest $F \in \mathcal{F}_{13,R}$ with $R = \{7, 11, 2\}$. The label of each vertex is in black, its weight is in red. $u_i = |V_i|$ for $0 \leq i \leq 3$.

Definition 9.3 *Let ρ a ranking on $\mathbf{2}^n$, for all $F \in \mathcal{F}_{n,R}$ (n and $R \subset \mathbf{n}$ given), the level statistic l_ρ associated with ρ is defined by*

$$l_\rho(F) = n(u') + \sum\nolimits_{v \in \mathbf{V}-R} (w_\rho(p(v)) - 1),$$

where $n(u') = \binom{u_1}{2} + \binom{u_2}{2} + ... + \binom{u_k}{2}$ and u_i is the number of vertices at distance i from its roots .

For the forest F represented in Fig. 1, we have

$$l_{\rho_+}(F) = \binom{4}{2} + \binom{5}{2} + \binom{1}{2} + (4-1) + 2(1-1) + 3(4-1) + (2-1) + (3-1) + 2(1-1) = 31.$$

Theorem 9.4 *Let $n \in \mathbb{N}^*$ and $R \subset \mathbf{n}$, $|R| = r$, then for any ranking ρ on $\mathbf{2^n}$, we have*

$$J_n^{(r)}(q) = \sum_{F \in \mathcal{F}_{n,R}} q^{l_\rho(F)}.$$

Proof. For each height k, between 1 and $n-r$, and each k-multiplet $u = (u_1, u_2, ..., u_k)$ with $u_i \geq 1$ and $|u| = u_1 + u_2 + ... + u_k = n - r$, we have $\binom{n-r}{u_1, u_2, ... u_k}$ possibilities of placing the $n-r$ non-root vertices on the k levels, i.e. of choosing the k generations $V_1, V_2, ...V_k$. So we have

$$\sum_{F \in \mathcal{F}_{n,R}} q^{l_\rho(F)} = \sum_{\substack{k \geq 1, u_i \geq 1 \\ u_1 + u_2 + ... + u_k = n-r}} \binom{n-r}{u_1, u_2, ..., u_k} q^{n(u')} \sum_{F \in F(V_0, V_1, ..., V_k)} q^{\sum_{v \in \mathbf{n}-R}(w_\rho(p(v))-1)}. \tag{9.1}$$

Here $F(V_0, V_1, ..., V_k)$ is the subset of forests of $\mathcal{F}_{n,R}$, with a specified height k, and whose generations $V_0 = R, V_1, V_2, ...V_k$ are all specified. This subset of forests is clearly in bijection with the Cartesian product $\prod_{i=0}^{k-1} \mathcal{G}_i$ where, for $i = 0$ to $k-1$, $\mathcal{G}_i$ is the set of maps from V_{i+1} to V_i ; the map $g_i \in \mathcal{G}_i$ associated with a forest of $F(V_0, V_1, ..., V_k)$, being defined by $g_i(v) = p(v)$. We then have

$$\sum_{v \in \mathbf{n}-R} (w_\rho(p(v)) - 1) = \sum_{i=0}^{k-1} \sum_{v \in V_{i+1}} (w_\rho(g_i(v)) - 1),$$

hence,

$$\sum_{F \in F(V_0, V_1, ..., V_k)} q^{\sum\limits_{v \in \mathbf{n}-R} (w_\rho(p(v))-1)} = \prod_{i=0}^{k-1} \Theta_i, \tag{9.2}$$

with

$$\Theta_i = \sum_{g_i \in \mathcal{G}_i} q^{\sum_{v \in V_{i+1}} w_\rho(g_i(v))-1}.$$

Let us calculate Θ_i for i between 0 and $k-1$. By definition of ρ, $w_\rho = \rho(V_i)$ is a bijection from V_i to $\mathbf{u}_i$ and $\rho(V_{i+1})$ is a bijection from $V_{i+1}$to $\mathbf{u}_{i+1}$. So, for each $g_i \in \mathcal{G}_i$, $h_i = \Phi(g_i) = \rho(V_i) \circ g_i \circ (\rho(V_{i+1}))^{-1}$ is a map from $\mathbf{u}_{i+1}$ to $\mathbf{u}_i$. By construction Φ is a bijection from $\mathcal{G}_i$ to $\mathbf{u}_i^{\mathbf{u}_{i+1}}$, so we can make the bijective change of index $h_i = \Phi(g_i)$. We can therefore write

$$\Theta_i = \sum\nolimits_{h_i \in \mathbf{u}_i^{\mathbf{u}_{i+1}}} \prod_{j=1}^{u_{i+1}} q^{h_i(j)-1} = \left(1 + q + ... + q^{u_i - 1}\right)^{u_{i+1}},$$

the second equality above, comes from classical combinatorial results (see for example Th.A p. 127 in [4] with $M = \mathbf{u}_{i+1}$, $N = \mathbf{u}_i$, $u(x,y) = q^{y-1}$ and $\mathcal{R} = M \times N$). By transferring these expressions into (9.1) we obtain

$$\sum_{F \in \mathcal{F}_{n,R}} q^{l_\rho(F)} = \sum_{\substack{k \geq 1, u_i \geq 1 \\ u_1 + u_2 + ... + u_k = n - r}} \binom{n-r}{u_1, u_2, ..., u_k} q^{n(u')} \prod_{i=0}^{k-1} [u_i]_q^{u_{i+1}},$$

which with (8.1) ends the proof. ■

Remark 9.5 *We can include the case $R = V = \mathbf{n}$ in the previous theorem. $\mathcal{F}_{n,V}$ reduces to the empty graph E_n (n vertices without edges) and it suffices to set for all ranking ρ, $l_\rho(E_n) = 0$. We verify that $J_n^{(n)} = 1 = \sum_{F \in \mathcal{F}_{n,V}} q^{l_\rho(F)} = q^{l_\rho(E_n)}$.*

Case of reciprocal polynomials. By replacing the formulas (7.1*bis*) in the formulas of Theorem 8.2 we can obtain explicit formulas for the $\overline{J_n^{(r)}}$. For example we get from (8.1) :

Corollary 9.6 *For any couple of integers (n, r) satisfying $n - 1 \geq r \geq 1$ we have*

$$\overline{J_n^{(r)}}(q) = \sum [r]_q^{u_1} [u_1]_q^{u_2} ... [u_{k-1}]_q^{u_k} q^{\sigma(u) + r(n-r-u_1)} \binom{n-r}{u_1, u_2, ..., u_k}, \tag{9.3}$$

where the sum is over the k-multiplets of strictly positive integers $u = (u_1, u_2, ..., u_k)$ with $k \geq 1$, $|u| = n - r$, and $\sigma(u) = \sum_{1 \leq i,j \leq k,\ j \geq i+2} u_i u_j$

It is possible to define level statistics from the formula (9.3) which can still be written

$$\overline{J_n^{(r)}}(q) = \sum [u_0]_q^{u_1} [u_1]_q^{u_2} ... [u_{k-1}]_q^{u_k} q^{\sigma(\widehat{u})} \binom{n-r}{u_1, u_2, ..., u_k}, \tag{9.4}$$

where the sum is over the multiplets $\widehat{u} = (u_0 = r,\ u_1, u_2, ..., u_k)$ such that $|\widehat{u}| = u_0 + u_1 + ... + u_k = n$ and $\sigma(\widehat{u}) = \sum_{0 \leq i,j \leq k,\ j \geq i+2} u_i u_j$.

We let the reader verify that for any ranking ρ on $\mathbf{2^n}$, $\overline{l_\rho}$ defined below is a statistic for $\overline{J_n^{(r)}}(q)$

$$\overline{l_\rho}(F) = \sigma(\widehat{u}) + \sum_{v \in \mathbf{n} - R} (w_\rho(p(v)) - 1).$$

Let us point out in conclusion that the polynomials $J_n^{(r)}$ and the formulas exposed in this article - as well as their reciprocals - are susceptible to other combinatorial developments which will be presented in future articles.

Moreover in the recent article [1] the author introduces a family of polynomials denoted J_λ, indexed by the set of integer partitions and with coefficients in $\mathbb{Z}$. These polynomials generalize the polynomials J_n in another direction than the polynomials $J_n^{(r)}$. It is conjectured and partially proved in [1] that these polynomials J_λ have, like J_n, strictly positive and log-concave coefficients.

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