# THE VIRTUAL CACTUS GROUP AND LITTELMANN PATHS

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Abstract. We define a *virtual cactus group* and show that the cactus group action on Littelmann paths is compatible with the virtualization map defined by Pan–Scrimshaw [PS18]. Our *virtual cactus group* generalizes the group with the same name defined for the symplectic Lie algebra in [ATFT22].

## 1. Introduction

Let $\mathfrak{g}$ be a finite dimensional, complex, semisimple Lie algebra. Let $D$ be the Dynkin diagram associated to the root system of $\mathfrak{g}$, $R$ its root system, $\Delta = \{\alpha_i : i \in D\} \subset R$ the set of simple roots, $W = W(R)$ its Weyl group, generated by the simple reflections $\{r_i : i \in D\}$, and $w_0 \in W$ the longest element of the Weyl group. For a connected sub-diagram $J \subseteq D$, of $D$, denote by $\theta_J : J \to J$ the unique Dynkin diagram automorphism that satisfies $\alpha_{\theta_J(j)} = -w_0^J \alpha_j$, for any node $j \in J$, where $w_0^J$ is the longest element of the parabolic subgroup $W^J \subseteq W$ (the Weyl group for $\mathfrak{g}$ restricted to $J$) [BB05]. This leads to the following definition by Halacheva.

**Definition 1.** [Hal20] *The cactus group $J_D$ is the group with generators $s_J$, one for each connected subdiagram $J$ of $D$, and relations given as follows:*

1. $s_J^2 = 1$;
2. $s_I s_J = s_J s_I$ *for $I, J \subseteq D$ connected subsets if the union $J \cup I$ is disconnected;*
3. $s_I s_J = s_{\theta_I(J)} s_I$ *if $J \subset I$.*

Definition 1 is a generalization of the original definition of the cactus group defined by Henriques–Kamnitzer in [HJK04], which was denoted by $J_n$ and which corresponds to the cactus group associated to the Dynkin diagram of type $A_{n-1}$.

1.1. **Main results and aim of the paper.** In this paper we will be concerned with pairs of Dynkin diagrams $(X, Y)$ related by *folding*, that is, there is an injection of sets of nodes $X \hookrightarrow Y$ which induces an injection of the corresponding Lie algebras $\mathfrak{g}_X \hookrightarrow \mathfrak{g}_Y$ as described in [BS17]. The main result and aim of this paper is the "virtualization" of the cactus group $J_X$, as defined by Halacheva in [Hal20], and of its action on $\mathfrak{g}_X$-crystals, transferring certain results obtained for the case $C_n \hookrightarrow A_{2n-1}$ in [ATFT22] to the more general setup described above. This is carried out in Theorem 2 and Theorem 4. It consists in defining a group monomorphism $J_X \hookrightarrow J_Y$ compatible with the action of $J_X$ and $J_Y$ on $\mathfrak{g}_X$, respectively $\mathfrak{g}_Y$-crystals. Moreover, by using

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the virtualization map on Littelmann paths described by Pan–Scrimshaw in [PS18], instead of the Baker virtualization map used in [ATFT22] for Kashiwara–Nakashima tableaux, we obtain a simple rule to compute the partial Schützenberger–Lusztig involutions of Littelmann paths in $\mathfrak{g}_X$-crystals in terms of partial Schützenberger–Lusztig involutions of Littelmann paths in $\mathfrak{g}_Y$-crystals. This is carried out in Theorem 4.

## 2. Acknowledgements

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## 3. The cactus group and crystals

Let $\Lambda$ be the integral weight lattice and $\Lambda^+ \subset \Lambda$ be the *dominant weights*. Recall that irreducible finite-dimensional representations of $\mathfrak{g}$ are in one-to-one correspondence with the set of highest weights $\Lambda^+$. We now recall the definition of a semi-normal crystal as in [BS17].

**Definition 2.** *A semi-normal $\mathfrak{g}$-crystal consists of a non-empty set $B$ together with maps*

$$\begin{aligned} \mathrm{wt} :&B \longrightarrow \Lambda \\ e_i, f_i :&B \longrightarrow B \sqcup \{0\}, i \in D \end{aligned}$$

*such that for all $b, b' \in B$:*

- $b' = e_i(b)$ *if and only if* $b = f_i(b')$*,*
- *if* $f_i(b) \neq 0$ *then* $\mathsf{wt}(f_i(b)) = \mathsf{wt}(b) - \alpha_i$*;*
  *if* $e_i(b) \neq 0$*, then* $\mathsf{wt}(e_i(b)) = \mathsf{wt}(b) + \alpha_i$*, and*
- $\varphi_i(b) - \varepsilon_i(b) = \langle \mathsf{wt}(b), \alpha_i^\vee \rangle$*,*

*where*

$$\begin{aligned} \varepsilon_i(b) &= \max\{a \in \mathbb{Z}_{\geq 0} : e_i^a(b) \neq 0\} \text{ and} \\ \varphi_i(b) &= \max\{a \in \mathbb{Z}_{\geq 0} : f_i^a(b) \neq 0\}. \end{aligned}$$

*To each such crystal $B$ is associated a crystal graph, a coloured directed graph with vertex set $B$ and edges coloured by elements $i \in D$, where if $f_i(b) = b'$ there is an arrow $b \xrightarrow{i} b'$. We say that a crystal is irreducible if its corresponding crystal graph is connected and finite.*

The finite irreducible semi-normal $\mathfrak{g}$-crystals are labeled by the dominant weights $\Lambda^+$. Given a highest weight $\lambda \in \Lambda^+$, the corresponding irreducible crystal is usually denoted by $B(\lambda)$. It encodes important information about the corresponding irreducible finite dimensional representation of $\mathfrak{g}$, $V(\lambda)$. For instance, $\dim(\mathrm{V}(\lambda))$ equals the cardinality of $B$, and, in the weight decomposition $V(\lambda) = \bigoplus_{\mu \leq \lambda} V(\lambda)_\mu$, $\dim(V(\lambda)_\mu)$ equals the cardinality of the set of $b \in B(\lambda)$ such that $\mathrm{wt}(b) = \mu$. Moreover, for a subinterval $J \subset D$, the crystal corresponding to the Levi restriction of $V(\lambda)$ corresponds to the $\mathfrak{g}_J$-crystal $B(\lambda)_J$ with crystal graph obtained from the graph for $B(\lambda)$ by deleting edges with labels $i \notin J$. In this paper, we will only deal with crystals whose crystal graphs decompose into connected components, each of which is isomorphic to crystals of the form $B(\lambda)$. These are also known in the literature as *normal* crystals.

3.0.1. *Schützenberger–Lusztig involutions.* There is an elegant internal action of the cactus group $J_\mathfrak{g}$ on crystals via partial Schützenberger–Lusztig involutions, which are generalizations of Schützenberger–Lusztig involutions originally studied by Berenstein–Kirillov and generalized by Halacheva. For a subinterval $J \subset D$, the partial Schützenberger–Lusztig involution is defined as follows on $B(\lambda)$. Let $v \in B(\lambda)_J$ be a *highest weight* element, and let $v_{w_0^J} \in B(\lambda)_J$ be a *lowest weight* element. In particular $\mathrm{wt}(v_{w_0^J}) = w_0^J(\mathrm{wt}(v))$ Let $b = f_{i_r} \cdots f_{i_1}(v)$ for $i_j \in J, j \in [1, r]$. Then the partial Schützenberger–Lusztig involution is the unique involution $\xi_J : B(\lambda) \to B(\lambda)$ which satisfies for each $j \in J$:

$$\begin{aligned} \xi_J(e_j(b)) &= f_{\theta_J(j)}(\xi_J(b)) \\ \xi_J(f_j(b)) &= e_{\theta_J(j)}(\xi_J(b)) \text{ and} \\ \mathrm{wt}(\xi_J(b)) &= w_0^J(\mathrm{wt}(b)). \end{aligned}$$

In fact, $\xi_J(b) = e_{\theta_J(i_r)} \cdots e_{\theta_J(i_1)}(v)$. If $J = D$, $\xi_J$ is known as the Schützenberger–Lusztig involution, and denoted simply by $\xi$. Each partial Schützenberger–Lusztig involution acts as the corresponding Schützenberger–Lusztig involution applied to each connected component of the Levi-branched crystal $B(\lambda)_J$. If our normal crystal $B$ is not connected, partial Schützenberger–Lusztig involutions are defined in the same way as above, on each connected component.

**Theorem 1** (Halacheva, [Hal20])**.** *Let $B$ be a normal $\mathfrak{g}$-crystal. The cactus group $J_\mathfrak{g}$ acts on $B$ via partial Schützenberger–Lusztig involutions, that is, for $J \subset D$ a subinterval, the assignment $s_J \mapsto \xi_J$ induces a group action.*

## 4. The virtual cactus group

Let $X \hookrightarrow Y$ be an embedding of a twisted Dynkin diagram $X$ into a simply-laced Dynkin diagram $Y$ given by folding. More precisely, there is a Dynkin diagram automorphism $\mathrm{aut} : Y \to Y$ of $Y$ such that there is an edge-preserving bijection $\sigma : X \to Y/\mathrm{aut}$. The injection of Dynkin diagrams is reflected on the Lie algebras as follows. Let $\mathfrak{g}_X$, respectively $\mathfrak{g}_Y$ be the complex simple Lie algebras with Dynkin diagram $X$, respectively $Y$. Then the Dynkin diagram automorphism aut induces a Lie algebra automorphism $\mathrm{aut} : \mathfrak{g}_Y \to \mathfrak{g}_Y$. The set of fixed points under this automorphism has the structure of a Lie algebra isomorphic to $\mathfrak{g}_X$ [Kac90]. This induces an injection $\mathfrak{g}_X \hookrightarrow \mathfrak{g}_Y$. Below we list all such pairs, together with the values of $\theta_X$ and $\theta_Y$. We use the numbering of the vertices given by [BS17].

| **X** | **Y** | $\theta_X$ | $\theta_Y$ |
|---|---|---|---|
| $C_n$ | $A_{2n-1}$ | Id | $\theta_Y(i) = 2n - i$ |
| $B_{2n-1}$ | $D_{2n}$ | Id | Id |
| $B_{2n}$ | $D_{2n+1}$ | Id | $\theta_Y(i) = \begin{cases} i & \text{if } i < 2n \\ 2n, 2n+1 & \text{if } i = 2n+1, 2n \text{ resp.} \end{cases}$ |
| $G_2$ | $D_4$ | Id | Id |
| $F_4$ | $E_6$ | Id | $\theta_Y(i) = \begin{cases} 6,1 & \text{if } i = 1,6 \text{ resp.} \\ 5,3 & \text{if } i = 3,5 \text{ resp.} \\ i & \text{otherwise} \end{cases}$ |

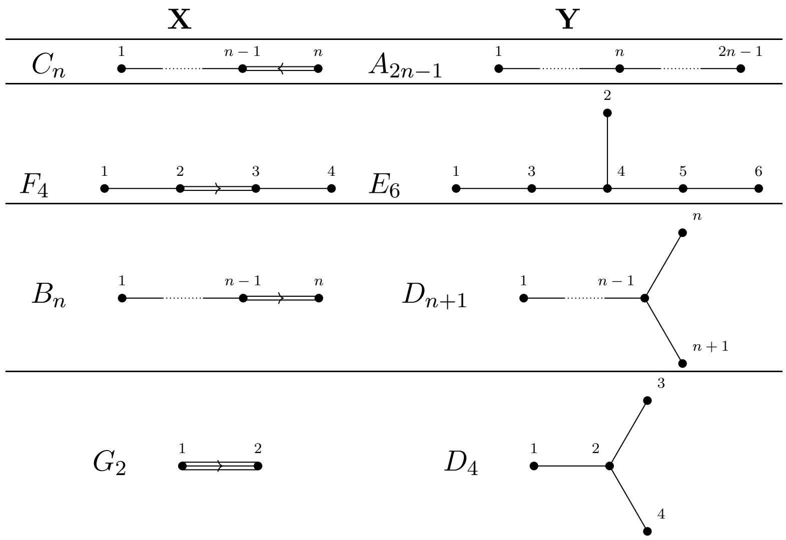
We have $\mathrm{aut} = \theta_Y$, except for the cases where $Y = D_{2n}$, where

$$\mathrm{aut}(i) = \begin{cases} i & i < 2n-1 \\ 2n & i = 2n-1 \\ 2n-1 & i = 2n. \end{cases}$$

We proceed to define a group monomorphism $J_X \hookrightarrow J_Y$. Its image will be isomorphic to what we call the virtual cactus group, generalizing the concept of the virtual symplectic cactus group defined in [ATFT22] for $X = C_n$ and $Y = A_{2n-1}$. We start by stating the following lemma, which immediately follows from the description in the previous section. We will abuse notation and consider the coset $\sigma(I) \in Y/\mathrm{aut}$, as a subset of $Y$, for $I \subset X$. Each non-simply laced Dynkin diagram we consider has what we will call in this note a *branching point* $x_0 \in X$, described in the table below.

| **X** | $x_0$ |
|---|---|
| $C_n$ | $n$ |
| $F_4$ | $2$ |
| $B_n$ | $n-1$ |
| $G_2$ | $2$ |

For the comfort of the reader we include the corresponding Dynkin diagrams as well below.




We now consider the following elements:

$$\tilde{s}_I = \prod s^Y_{\tilde{I}}$$

where $s^Y_{\tilde{I}}$ are the generators of the cactus group $J_Y$ and the product is taken over the connected components $\tilde{I}$ of $\sigma(I)$. Our aim for the rest of this section is to prove the following result.

**Theorem 2.** *The map defined by*

$$\begin{aligned} \Phi : J_X &\to J_Y \\ s_I &\mapsto \tilde{s}_I \end{aligned}$$

*is a monomorphism of groups.*

**Lemma 1.** *Let $I, J \subset X$ such that $J \subset I$. Then*

$$\tilde{s}_I \tilde{s}_J = \tilde{s}_{\theta_I(J)} \tilde{s}_I$$

*Proof.* First assume that $\theta_Y = \mathrm{Id}$. This means $Y = D_{2n}$ for some $n \geq 2$. If $I = X$ then $\sigma(I) = Y$, therefore the statement of Lemma 1 follows from $\theta_Y = \mathrm{Id}$ and the defining Relation 3 for the cactus group $J_Y$. If $I \subset X$ does not contain the branching point $x_0$ then $\sigma|_I : I \to \tilde{I} = \sigma(I)$ is an isomorphism, hence the statement follows trivially. If $I$ is not $X$ but contains the branching point, then either $I$ is of type A, $\sigma(I) = \tilde{I}$ is of type A and $\sigma|_I : I \to \tilde{I}$ is an isomorphism, which implies the claim as in the previous case, or $I$ is of type $G_2$, in which case the claim also follows easily since $J$ is forced to consist of just one vertex.

Assume next that $\theta_Y = \mathrm{aut}$. If $I \subset X$ contains the branching point $x_0$, then $\theta_I = \mathrm{Id}_I$ and $\sigma(I) = \tilde{I}$ is connected. Let us then assume first that $x_0 \in I$. Now, if

$x_0 \in J$ also, then $\sigma(J) = \tilde{J}$ is connected and $\theta_{\tilde{I}}(\tilde{J}) = \tilde{J}$. Now, if $J \subset I$ does not contain a branching point but $I$ does, then either

- $\sigma(J) = \tilde{J}_1 \sqcup \tilde{J}_2$ has two isomorphic connected components, in which case $\theta_{\tilde{I}}(\tilde{J}_1) = \tilde{J}_2$ and $\theta_{\tilde{I}}(\tilde{J}_2)) = \tilde{J}_1$, or
- $\sigma(J) = \tilde{J}$ is connected and isomorphic to $J$, in which case $\theta_{\tilde{I}}(\tilde{J}) = \tilde{J}$.

We conclude then that if $x_0 \in I$ and $\sigma(J) = \tilde{J}$ is connected, then

$$\tilde{s}_I \tilde{s}_J = s^Y_{\tilde{I}} s^Y_{\tilde{J}} = s^Y_{\theta_{\tilde{I}}(\tilde{J})} s^Y_{\tilde{I}} = s^Y_{\tilde{J}} s^Y_{\tilde{I}} = \tilde{s}_J \tilde{s}_I = \tilde{s}_{\theta_I(J)} \tilde{s}_I,$$

as desired. Now, if $x_0 \in I$ and $\sigma_J = \tilde{J}_1 \sqcup \tilde{J}_2$, then we still have $\theta_I = \mathrm{Id}$, so $\theta_I(J) = J$. We have in this case

$$\tilde{s}_I \tilde{s}_J = s^Y_{\tilde{I}} s^Y_{\tilde{J}_1} s^Y_{\tilde{J}_2} = s^Y_{\theta_{\tilde{I}}(\tilde{J}_1)} s^Y_{\tilde{I}} s^Y_{\tilde{J}_2} = s^Y_{\theta_{\tilde{I}}(\tilde{J}_1)} s^Y_{\tilde{I}(\tilde{J}_2)} s^Y_{\tilde{I}} = \tilde{s}_J \tilde{s}_I = \tilde{s}_{\theta_I(J)} \tilde{s}_I.$$

This concludes the proof in the case $x_0 \in I$.

Now let us assume that $x_0 \notin I$. We have two cases: The case where $\sigma(I)$ is connected is trivial because since $\theta_Y = \mathrm{aut}$, we conclude that necessarily $\theta_{\sigma(I)} = \mathrm{aut}\,|_{\sigma(I)} = \mathrm{Id}_{\sigma(I)}$, also $\sigma(J) \subset \sigma(I)$ is connected for each $J \subset I$, and $\tilde{s}_J = s^Y_{\sigma(J)}$ for each $J \subset I$. It remains to consider the case where $\sigma(I)$ has two connected components $\sigma(I) = \tilde{I}_1 \sqcup \tilde{I}_2$. It follows that for each $J \subset I$ we have a decomposition into connected components $\sigma(J) = \tilde{J}_1 \sqcup \tilde{J}_2$, where $\tilde{J}_i \subset \tilde{I}_i, i = 1, 2$. The following identity holds by case-by-case analysis:

$$\sigma(\theta_I(J)) = \theta_{\tilde{I}_1}(\tilde{J}_1) \sqcup \theta_{\tilde{I}_2}(\tilde{J}_2). \tag{1}$$

Therefore we have in this case:

$$\begin{aligned}
\tilde{s}_I \tilde{s}_J &= s^Y_{\tilde{I}_1} s^Y_{\tilde{I}_2} s^Y_{\tilde{J}_1} s^Y_{\tilde{J}_2} \\
&= s^Y_{\tilde{I}_1} s^Y_{\tilde{J}_1} s^Y_{\tilde{I}_2} s^Y_{\tilde{J}_2} \\
&= s^Y_{\theta_{\tilde{I}_1}(\tilde{J}_1)} s^Y_{\tilde{I}_1} s^Y_{\theta_{\tilde{I}_2}(\tilde{J}_2)} s^Y_{\tilde{I}_2} \\
&= s^Y_{\theta_{\tilde{I}_1}(\tilde{J}_1)} s^Y_{\theta_{\tilde{I}_2}(\tilde{J}_2)} s^Y_{\tilde{I}_1} s^Y_{\tilde{I}_2} \\
&= \tilde{s}_{\theta_I(J)} \tilde{s}_I,
\end{aligned}$$

where the last equality follows from (1). This concludes the proof in the cases where $\theta_Y = \mathrm{aut}$ and therefore the whole proof.

□

**Definition 3.** *The virtual cactus group $J^v_X$ is defined by generators $s_{\sigma(I)}$, for each $I \subset X$ connected subdiagram, and by the relations:*

1. $s^2_{\sigma(I)} = 1$;
2. $s_{\sigma(I)} s_{\sigma(J)} = s_{\sigma(J)} s_{\sigma(I)}$ *if the union $J \cup I$ is disconnected;*

*3.* $s_{\sigma(I)}s_{\sigma(J)} = s_{\sigma(\theta_I(J))}s_{\sigma(I)}$ *if* $J \subset I$.

It is clear from the definition that the virtual cactus group $J_X^v$ is isomorphic to the cactus group $J_X$.

*Proof of Theorem 2.* To show that $\Phi$ is a group morphism, we need to show three relations:

(1) $\tilde{s}_I^2 = Id$,
(2) $\tilde{s}_I\tilde{s}_J = \tilde{s}_J\tilde{s}_I$,
(3) $\tilde{s}_I\tilde{s}_J = \tilde{s}_{\theta_I(J)}\tilde{s}_I$.

Note that the third relation has already been established in Lemma 1. To prove (1), note that since the connected components of $\sigma(I)$ are disjoint, the commutation relation *2.* in Definition 1 implies

$$\tilde{s}_I^2 = \prod {s_{\tilde{I}}^Y}^2 = Id$$

To show the second relation, let $I, J \subset X$ be two disjoint, connected intervals. Then necessarily $\sigma(I)$ and $\sigma(J)$ are mutually disjoint. We have then

$$\tilde{s}_I\tilde{s}_J = \prod s_{\tilde{I}}^Y \prod s_{\tilde{J}}^Y = \prod s_{\tilde{J}}^Y \prod s_{\tilde{I}}^Y$$

where the third equality follows from relation *2.* for $J_Y$. Note that the image $\Phi(J_X)$ is a group isomorphic to the virtual cactus group $\tilde{J}_X$ via the isomorphism $\tilde{s}_I \mapsto s_{\sigma(I)}$. Note that this map is well defined because $\sigma(I) = \sigma(J) \iff I = J$. $\square$

## 5. Virtualization of the action of the cactus group on crystals of Littelmann paths

In this section we will borrow most of our notation from [PS18] for practical purposes as well as for the comfort of the reader. Let $\lambda \in \Lambda^+$. We consider $\mathcal{P}(\lambda)$ to be the Littelmann path model for $\lambda$ with paths $\pi : [0, 1] \to \Lambda_{\mathbb{R}}$ of the form

$$\pi(t) = \sum_{i \in D} H_{i,\pi}(t)\Lambda_i,$$

where $H_{i,\pi}(t) = \langle t, \alpha_i^\vee \rangle$ and where $\Lambda_i \in \Lambda^+$ are the fundamental weights for $i \in D$. The set $\mathcal{P}(\lambda)$ has the structure of a crystal isomorphic to $B(\lambda)$ with weight map $\mathrm{wt}(\pi) = \pi(1)$. We refer the reader to [PS18] for the definition of the crystal structure using the notation we use in this section. The original and standard reference of the topic is the paper [Lit95] by Littelmann.

Recall that in this paper we consider embeddings $X \hookrightarrow Y$ given by folding. Let $\Lambda_X$ and $\Lambda_Y$ be the corresponding integral weight lattices. The bijection $\sigma : X \to Y/\mathrm{aut}$ induces a map

$$\Psi : \Lambda_X \to \Lambda_Y$$

given by the assignment

$$\Lambda_i^X \mapsto \sum_{j \in \sigma(i)} \gamma_i (\Lambda^Y)_j,$$

where $\gamma_i$ is given by Table 5.1 in [BS17] and where $\Lambda_i^X$ and $\Lambda_j^Y$ denote the fundamental weights in $\Lambda_X$, respectively $\Lambda_Y$.

**Definition 4.** *Let $\tilde{B}$ be a normal $\mathfrak{g}_Y$-crystal, and a subset $V \subset \tilde{B}$. The virtual root operators of type $X$ are, for $i \in X$:*

$$e_i^v = \prod_{j \in \sigma(i)} \tilde{e}_j^{\gamma_i} \tag{2}$$

$$f_i^v = \prod_{j \in \sigma(i)} \tilde{f}_j^{\gamma_i}, \tag{3}$$

*where $\tilde{e}_i, \tilde{f}_i, i \in Y$ are the root operators for the $\mathfrak{g}_Y$-crystal $\tilde{B}$.*
*A virtual crystal is a pair $(V, \tilde{B})$ such that $V$ has a $\mathfrak{g}_X$-crystal structure defined by*

$$e_i := e_i^v f_i := f_i^v \tag{4}$$

$$\varepsilon_i := \gamma_i^{-1} \tilde{\varepsilon}_j \varphi_i := \gamma_i^{-1} \tilde{\varphi}_j, \tag{5}$$

*where $\tilde{\varepsilon}_j, \tilde{\varphi}_j j \in Y$ denote the maps given by*

$$\tilde{\varepsilon}_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : \tilde{e}_i^a(b) \neq 0\} \text{ and}$$
$$\tilde{\varphi}_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : \tilde{f}_i^a(b) \neq 0\}.$$

*If $\mathfrak{g}_X$-crystal $B$ is crystal isomorphic to a virtual crystal $V \subset \tilde{B}$ via an isomorphism $\phi : B \to V$, then the isomorphism $\phi$ is called a virtualization map.*

For $\lambda \in \Lambda_X^+$, the weight $\psi(\lambda) \in \lambda_Y$, is dominant, that is, $\psi(\lambda) \in \Lambda_Y^+$. Given $\pi \in \mathcal{P}(\lambda)$, consider the path $\Psi(\pi) : [0, 1] \to \Lambda_Y$ defined by

$$\Psi(\pi)(t) = \sum_{i \in D} H_{i,\pi}(t) \psi(\Lambda_i) \tag{6}$$

One of the main results in [PS18] is the following theorem.

**Theorem 3** (Pan–Scrimshaw, [PS18])**.** *The assignment $\pi \mapsto \Psi(\pi)$ induces a virtualization map*

$$\mathcal{P}(\lambda) \to \mathcal{P}(\psi(\lambda))$$
$$\pi \mapsto \Psi(\pi).$$

The principal aim of this section is to describe the action of the cactus group in terms of the virtualization map of Pan–Scrimshaw. For this, given a connected subdiagram $I \subset X$, let

$$\tilde{\xi}_{\sigma(I)} := \prod \xi_{\tilde{I}}^Y$$

where $\xi_{\tilde{I}}^Y$ are the partial Schützenberger–Lusztig involutions in $\mathcal{P}(\psi(\lambda))$ and the product is taken over the connected components $\tilde{I}$ of $\sigma(I)$. Our next aim is to prove the following result, which generalizes [ATFT22, Theorem 5, Theorem 6, Section 9.5].

**Theorem 4.** *Let $\lambda \in \Lambda_X^+$ and $\mathcal{P}(\lambda)$ the corresponding Littelmann path model. Then the following diagram commutes*

$$\begin{tikzcd} \mathcal{P}(\lambda) \arrow[r, "\Psi"] \arrow[d, "\xi_I^X"'] & \mathcal{P}(\psi(\lambda)) \arrow[d, "\tilde{\xi}_{\sigma(I)}"] \\ \mathcal{P}(\lambda) \arrow[r, "\Psi"'] & \mathcal{P}(\psi(\lambda)) \end{tikzcd}.$$

*Moreover, the left inverse $\Psi^{-1}$ can be explicitly computed on $\tilde{\xi}^Y_{\sigma(I)}(\Psi(\mathcal{P}(\lambda)))$.*

*Proof.* First note that since the Littelmann path model $\mathcal{P}(\psi(\lambda))$ is stable under the root operators $\tilde{e}_i, \tilde{f}_i$, it is also stable under the action of the operators $\tilde{\xi}^Y_{\sigma(I)}$ for $I \subset X$ connected. Therefore, all paths in $\tilde{\xi}^Y_{\sigma(I)}(\Psi(\mathcal{P}(\lambda)))$ must be of the form (6), so the left inverse $\Psi^{-1}$ can be explicitly computed on $\tilde{\xi}^Y_{\sigma(I)}(\Psi(\mathcal{P}(\lambda)))$, simply by writing out the corresponding path in this form. We now proceed to show that the diagram commutes. Let $\pi_\nu \in \mathcal{P}(\lambda)_I$ be a highest weight path of weight $\mathrm{wt}(\pi_\nu) = \pi_\nu(1) = \nu$ and $\pi = f_{i_r} \cdots f_{i_1} \pi_\nu$ for $i_j \in I, j \in [1, r]$. Recall that

$$\xi_I^X(\pi) = e_{\theta_I(i_r)} \cdots e_{\theta_I(i_1)} \pi_\nu.$$

Therefore by Theorem 3 we have

$$\Psi(\xi_I^X(b)) = e^v_{\theta_I(i_r)} \cdots e^v_{\theta_I(i_1)} \Psi(\pi_\nu).$$

Now, by Definition 4 and Theorem 3 we have

$$\begin{aligned} \tilde{\xi}_{\sigma(I)}(\Psi(b)) &= \prod \xi^Y_{\tilde{I}}(\Psi(\pi)) \\ &= \prod \xi^Y_{\tilde{I}}\Big(\prod_{j \in \sigma(i_r)} \tilde{f}_j^{\gamma_{i_r}} \cdots \prod_{j \in \sigma(i_1)} \tilde{f}_j^{\gamma_{i_1}}(\Psi(\pi_\nu))\Big) \end{aligned}$$

where the product is taken over the connected components $\tilde{I}$ of $\sigma(I)$. To continue our computations we consider three cases separately:

(1) The subdiagram $\sigma(I) = \tilde{I} \subset Y$ is connected. Then $\theta_I = \mathrm{Id}$, we have $\gamma_{i_j} = 1$ if and only if $\sigma(i_j) = \left\{\tilde{i}_j^1, \tilde{i}_j^2\right\}$ or $\sigma(i_j) = \left\{\tilde{i}_j^1, \tilde{i}_j^2, \tilde{i}_j^3\right\}$ and $\gamma_{i_j} = 2, 3$ if and only if $\sigma(i_j) = \left\{\tilde{i}_j\right\}$. In case $\gamma_{i_j} = 1$ we have $\theta_{\tilde{I}}(\tilde{i}_j^1) = \tilde{i}_j^2$ and $\theta_{\tilde{I}}(\tilde{i}_j^2) = \tilde{i}_j^1$. Moreover, the root operators $\tilde{e}_{\tilde{i}_j^1}$ and $\tilde{e}_{\tilde{i}_j^2}$ commute. In case $\gamma_{i_j} = 2, 3$ we have $\theta_{\tilde{I}}(\tilde{i}_j) = \tilde{i}_j$. All together this implies:

$$\begin{aligned}\tilde{\xi}_{\sigma(I)}(\Psi(b)) &= \xi^Y_{\tilde{I}}(f^v_{i_r}\cdots f^v_{i_1}(\Psi(\pi_\nu)))\\ &= e^v_{\theta_I(i_r)}\cdots e^v_{\theta_I(i_1)}\Psi(\pi_\nu)\\ &= \Psi(\xi^X_I(b)).\end{aligned}$$

(2) The subdiagram $\sigma(I) \subset Y$ is disconnected. Assume $\theta_Y = \text{aut}$. In this case we must have $|\sigma(I)| = 2|I|$, that is, $\sigma(I) = \tilde{I}_1 \sqcup \tilde{I}_2$ is a disconnected union. In particular all root operators $\tilde{e}_s$, $\tilde{f}_t$ with $s,t \in \tilde{I}_1$ commute with the operators $\tilde{e}_u$, $\tilde{f}_v$, with $u,v \in \tilde{I}_2$. Moreover $\gamma_{i_j} = 1$ for all $j \in [1,r]$. Altogether, this implies:

$$\begin{aligned}\tilde{\xi}_{\sigma(I)}(\Psi(b)) &= \xi^Y_{\tilde{I}_1}\xi^Y_{\tilde{I}_2}(f^v_{i_r}\cdots f^v_{i_1}(\Psi(\pi_\nu)))\\ &= \xi^Y_{\tilde{I}_1}\xi^Y_{\tilde{I}_2}(\tilde{f}_{i^1_r}\tilde{f}_{i^2_r}\cdots \tilde{f}_{i^1_1}\tilde{f}_{i^2_1}(\Psi(\pi_\nu)))\\ &= \xi^Y_{\tilde{I}_1}\xi^Y_{\tilde{I}_2}(\tilde{f}_{i^2_r}\cdots \tilde{f}_{i^2_1}\tilde{f}_{i^1_r}\cdots \tilde{f}_{i^1_1}(\Psi(\pi_\nu)))\\ &= \xi^Y_{\tilde{I}_1}(\tilde{e}_{\theta_{\tilde{I}_2}(i^2_r)}\cdots \tilde{e}_{\theta_{\tilde{I}_2}(i^2_1)}\tilde{f}_{i^1_r}\cdots \tilde{f}_{i^1_1}(\Psi(\pi_\nu)))\\ &= \xi^Y_{\tilde{I}_1}(\tilde{f}_{i^1_r}\cdots \tilde{f}_{i^1_1}\tilde{e}_{\theta_{\tilde{I}_2}(i^2_r)}\cdots \tilde{e}_{\theta_{\tilde{I}_2}(i^2_1)}(\Psi(\pi_\nu)))\\ &= \tilde{e}_{\theta_{\tilde{I}_1}(i^1_r)}\cdots \tilde{e}_{\theta_{\tilde{I}}(i^1_1)}\tilde{e}_{\theta_{\tilde{I}_2}(i^2_r)}\cdots \tilde{e}_{\theta_{\tilde{I}_2}(i^2_1)}(\Psi(\pi_\nu))\\ &= \tilde{e}_{\theta_{\tilde{I}_1}(i^1_r)}\tilde{e}_{\theta_{\tilde{I}_2}(i^2_r)}\cdots \tilde{e}_{\theta_{\tilde{I}_1}(i^1_1)}\tilde{e}_{\theta_{\tilde{I}_2}(i^2_1)}(\Psi(\pi_\nu)).\end{aligned}$$

The case $\theta_Y = \text{Id}$ is very similar. □

**Corollary 1.** *The virtual cactus group $J^v_X$ acts on $\mathcal{P}(\psi(\lambda))$ and preserves the image $\Psi(\mathcal{P}(\lambda))$ of $\Psi$.*

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