A coloring of the plane without monochromatic right triangles

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February 24, 2023

Abstract

We give a full, correct proof of the following result, earlier claimed in [1]. If the Continuum Hypothesis holds then there is a coloring of the plane with countably many colors, with no monocolored right triangle.

One[1] of Paul Erdős's favorite topics consisted the applications of the Axiom of Choice to construct paradoxical sets and colorings of the Euclidean spaces. Among a large number of other questions, he raised the following: is there a coloring of the plane with countably many colors, with no monocolored right angled triangles. In [1] this was shown to be equivalent to the Continuum Hypothesis.

Recently, the senior author observed that the proof of the positive direction in [1] is incomplete. Here we give a full, correct proof.

We notice that later Schmerl in [2] gave another, more general proof, which, however, is less elementary.

Notation. Definitions. We use the notation and terminology of axiomatic set theory. The ordinals are von Neumann ordinals, ω_1 is the least uncountable ordinal.

Theorem. (CH) *There is $f : \mathbb{R}^2 \to \omega$ with no monochromatic right angles.*

*Partially supported by Hungarian National Research Grant OTKA K 131842

[1] *2010 Mathematics Subject Classification.* Primary 03E50 Secondary 51M04, 05D10. Key words and phrases: Continuum Hypothesis, Ramsey theory of Euclidean spaces

Proof. Using CH, we decompose $\mathbb{R}^2$ and the set $\mathcal{E}$ of planar lines and circles into the increasing, continuous sequence of sets as $\mathbb{R}^2 = \bigcup\{H_\alpha : \alpha < \omega_1\}$ and $\mathcal{E} = \bigcup\{\mathcal{E}_\alpha : \alpha < \omega_1\}$ such that $H_0 = \mathcal{E}_0 = \emptyset$ and
(1) if $x \neq y \in H_\alpha$, then their connecting line and Thales circle are in $\mathcal{E}_\alpha$,
(2) if $x, y, z \in H_\alpha$ are not collinear, then their circuit is in $\mathcal{E}_\alpha$,
(3) if $e_0 \neq e_1 \in \mathcal{E}_\alpha$, then $e_0 \cap e_1 \subseteq H_\alpha$,
(4) if $x \in C \in \mathcal{E}_\alpha$, C is a circle, then the antipodal point of x in C is in H_α,
(5) if $L \in \mathcal{E}_\alpha$ is a line, $x \in H_\alpha \cap L$, then the line L' perpendicular to L with $x \in L'$ also contained in $\mathcal{E}_\alpha$.

This can be done by the usual Skolem-type closing arguments.

We are going to construct the coloring $f : \mathbb{R}^2 \to \omega$ and the function $\varphi : \mathcal{E} \to [\omega]^\omega$ by transfinite recursion on $\alpha < \omega_1$ for $H_{\alpha+1} - H_\alpha$ and for $\mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$, satisfying the following:
(6) $f|(H_{\alpha+1} - H_\alpha)$ is injective,
(7) if $x \in H_{\alpha+1} - H_\alpha$, $e \in \mathcal{E}_\alpha$, $x \in e$, then $f(x) \in \varphi(e)$,
(8) if $C \in \mathcal{E}$ is a circle, $i \in \varphi(C)$, $x, y \in C$, $f(x) = f(y) = i$, then x, y are not antipodal,
(9) if $C \in \mathcal{E}$ is a circle, $i \notin \varphi(C)$, then $|f^{-1}(i) \cap C| \leq 2$,
(10) if $L \in \mathcal{E}$ is a line, $i \notin \varphi(L)$, then $|f^{-1}(i) \cap L| \leq 1$,
(11) if L, L' are perpendicular lines, $\{x\} = L \cap L'$, then $f(x) \notin \varphi(L) \cap \varphi(L')$.

Claim 1. *There is no right triangle monocolored by f.*

Proof. Assume that x, y, z form a right triangle with the right angle at y and $i = f(x) = f(y) = f(z)$. Let C be the circle around x, y, z. If $i \notin \varphi(C)$, then we obtain a contradiction with (9). If $i \in \varphi(C)$, then we get a contradiction with (8), as x, z are antipodal. □

Claim 2. *If $x \in H_{\alpha+1} - H_\alpha$, then there is at most one $e \in \mathcal{E}_\alpha$ such that $x \in e$.*

Proof. By (3). □

We add the following condition:
(12) if $x \in H_{\alpha+1} - H_\alpha$, L is a line with $x \in L \in \mathcal{E}_\alpha$, L' is the line perpendicular to L at x, $\{y\} = L' \cap H_\alpha$, then $f(x) \neq f(y)$.

Notice that $L' \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$ by (3) and (5), and $|L' \cap H_\alpha| \leq 1$ by (1).

Assume that $f|H_\alpha$ and $\varphi|\mathcal{E}_\alpha$ are already constructed, we have to define $f|(H_{\alpha+1} - H_\alpha)$ and $\varphi|(\mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha)$.

Enumerate $H_{\alpha+1} - H_\alpha$ as $\{x_j : j < \omega\}$. By recursion on j define $f(x_j)$ so that

$$f(x_j) \geq \max\{f(x_0), \ldots, f(x_{j-1})\} + 2,$$

$f(x_j)$ satisfies (7), and is different from the (possible) color disqualified by (12). Clearly (6) is satisfied, and also $\omega - f[H_{\alpha+1} - H_\alpha]$ is infinite.

Next we define φ on the circuits in $\mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$. If C is such a circuit, set $A = C \cap H_\alpha$. By (2), we have $|A| \leq 2$.

Case 1. $|A| \leq 1$ or f takes distinct values on the two elements of A.

In this case, set $\varphi(C) = \omega - f[A]$.

Case 2. $|A| = 2$ and f assumes the same value on the elements of A.

In this case, set $\varphi(C) = \omega$.

Finally, we define φ on the lines in $\mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$. Let $L \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$ be a line. Set $B = L \cap H_\alpha$. Notice that by (1), $|B| \leq 1$. If B is nonempty, let y be its unique point.

Case 1. $B = \emptyset$ or $f(y) \notin f[L \cap (H_{\alpha+1} - H_\alpha)]$.

Then let $\varphi(L) = \omega - f[L \cap H_{\alpha+1}]$.

Case 2. $f(y) \in f[L \cap (H_{\alpha+1} - H_\alpha)]$.

Then set $\varphi(L) = (\omega - f[L \cap (H_{\alpha+1} - H_\alpha]) \cup \{f(y)\}$.

Claim 3. $\varphi(e) \in [\omega]^\omega$ $(e \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha)$.

Proof. If $C \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$ is a circuit, then this is obvious in Case 2 and in Case 1 $\varphi(C)$ is ω minus at most 2 elements.

If $L \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$ is a line, the statement follows as $f[H_{\alpha+1} - H_\alpha]$ is a coinfinite set. □

Claim 4. *φ satisfies* (8).

Proof. Assume that $C \in \mathcal{E}_{\alpha+1}$ is a circle, $x, y \in C \cap H_{\alpha+1}$ are antipodal and $f(x) = f(y) = i$.

Case 1. $C \in \mathcal{E}_\alpha$.

In this case one of x, y, say x must be in H_α by (6). Then, by (4), y is also in H_α, and we are finished by induction.

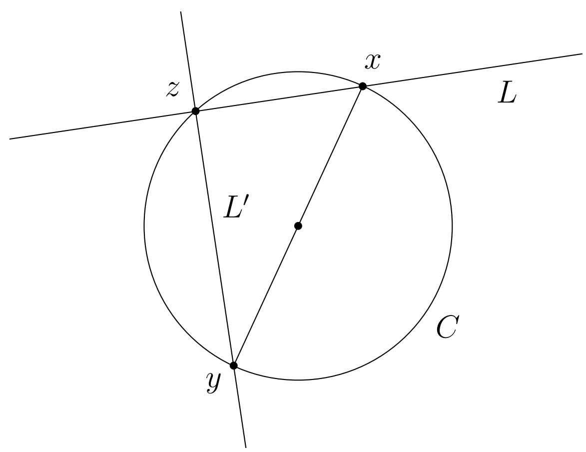
Case 2. $C \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$.

If $x, y \in H_{\alpha+1} - H_\alpha$, then $f(x) \neq f(y)$ by (6).

If $x, y \in H_\alpha$, then $C \in \mathcal{E}_\alpha$ by (1), a contradiction again.

Assume finally, that $x \in H_\alpha$, $y \in H_{\alpha+1} - H_\alpha$. If, in the definition of $\varphi(C)$, Case 1 applies, then $i \notin \varphi(C)$. We can therefore assume that Case 2 holds,

$H_\alpha \cap C = \{x, z\}$ and $f(y) = f(z) = i$. Let L be the connecting line of x and z, L' the connecting line of z and y.



Then $L \in \mathcal{E}_\alpha$ by (1) and then $L' \in \mathcal{E}_\alpha$ by (5). Further, by (10), $i \in \varphi(L) \cap \varphi(L')$, which, with $f(z) = i$, contradicts (11). $\square$

Claim 5. (9) *holds.*

Proof. Assume that $C \in \mathcal{E}_{\alpha+1}$ and $i \notin \varphi(C)$. If $C \in \mathcal{E}_\alpha$, then $C \cap f^{-1}(i)$ does not contain element from $H_{\alpha+1} - H_\alpha$ by (7). If $C \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$ and Case 2 holds in the definition of $\varphi(C)$, then there is nothing to prove. Assume finally, that $C \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$, and Case 1 holds. Then, if $i \notin \varphi(C)$, then $C \cap f^{-1}(i)$ has at most one element in H_α, at most one in $H_{\alpha+1} - H_\alpha$ by (6), that is at most 2. $\square$

Claim 6. (10) *holds.*

Proof. Assume that $L \in \mathcal{E}_{\alpha+1}$, $i \notin \varphi(L)$. If $L \in \mathcal{E}_\alpha$, then $L \cap f^{-1}(i)$ does not increase in $H_{\alpha+1} - H_\alpha$. If $L \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$ and $B = L \cap H_\alpha$ is empty, then $|L \cap f^{-1}(i)| \leq 1$ by (6). Otherwise, B is a singleton by (1), let y be its unique element. If $f(y) \notin f[L \cap (H_{\alpha+1} - H_\alpha)]$, then again $|L \cap f^{-1}(i)| \leq 1$, otherwise we are in Case 2 of the definition of $\varphi(L)$ where we specifically added $f(y)$ to $\varphi(C)$. $\square$

Claim 7. (11) *holds.*

Proof. Assume that $L, L' \in \mathcal{E}_{\alpha+1}$ are perpendicular, $L \cap L' = \{x\}$, $i = f(x) \in \varphi(L) \cap \varphi(L')$.

Case 1. $L, L' \in \mathcal{E}_\alpha$.

In this case $x \in H_\alpha$ by (3), so the configuration in (11) already appers in H_α, $\mathcal{E}_\alpha$.

Case 2. $L \in \mathcal{E}_\alpha$, $L' \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$.

As $i \in \varphi(L')$, by the definition of the latter there is $y \in L' \cap H_\alpha$, $f(y) = i$. This is exactly what is ruled out at the coloring of x by (12).

Case 3. $L, L' \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$.

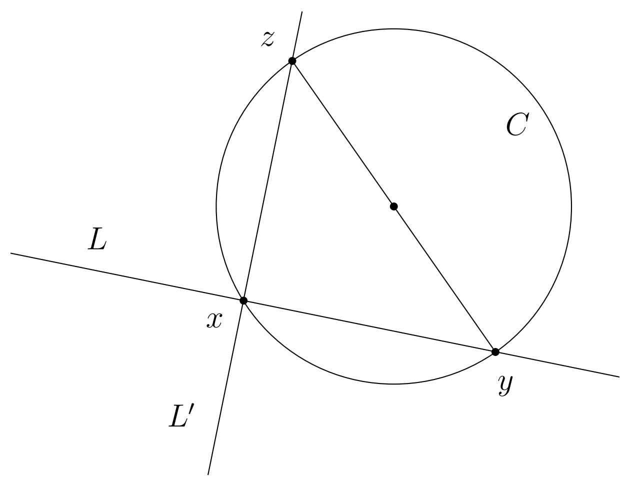
Subcase 3.1. $x \in H_\alpha$.

As $i = f(x) \in \varphi(L)$, by the definition of $\varphi(L)$ there is $y \in L \cap (H_{\alpha+1} - H_\alpha)$ with $f(y) = i$. Likewise, there is $z \in L' \cap (H_{\alpha+1} - H_\alpha)$ with $f(z) = i$. Now y, z are distinct elements of $H_{\alpha+1} - H_\alpha$ and $f(y) = f(z)$, contradicting (6).

Subcase 3.2. $x \in H_{\alpha+1} - H_\alpha$.

As $i = f(x) \in \varphi(L)$, there is $y \in L \cap H_\alpha$, $f(y) = i$. Similarly, there is $z \in L' \cap H_\alpha$, $f(z) = i$.

Let C be the circuit containing x, y, z.



As L, L' are perpendicular, y and z are antipodal in C. As in C there are 3 points of color i, $i \in \varphi(C)$. But this contradicts to the antipodality of y and z. $\square$

The proof of the theorem is concluded. $\square$

References

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