

Vizing's conjecture holds ^{*}

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Abstract

In 1964 Vizing proved that starting from any k -edge-coloring of a graph G one can reach, using only Kempe swaps, a $(\Delta + 1)$ -edge-coloring of G where Δ is the maximum degree of G . One year later he conjectured that one can also reach a Δ -edge-coloring of G if there exists one. Bonamy *et. al* proved that the conjecture is true for the case of triangle-free graphs. In this paper we prove the conjecture for all graphs.

1 Introduction

In 1964 Vizing proved that the chromatic index of a graph G (*i.e.* the minimum number of colors needed to properly color the edges of G), denoted by $\chi'(G)$, is at most $\Delta(G) + 1$ colors, where $\Delta(G)$ is the maximum degree of G .

Theorem 1. *Any simple graph G satisfy $\chi'(G) \leq \Delta(G) + 1$.*

The proof heavily relies on the use of *Kempe changes*. Kempe changes were introduced by Kempe in his unsuccessful attempt to prove the 4-color theorem, but it turns out that this concept became one of the most fruitful tool in graph coloring. Throughout this paper, we only consider proper edge-colorings, and so we will only write colorings to denote proper edge-colorings. Given a graph G and a coloring β , a Kempe chains C is a maximal bichromatic component (Kempe chains were invented in the context of vertex-coloring, but the principle remains the same for edge-coloring). Applying a *Kempe swap* (or Kempe change) on C consists in switching the two colors in this component. Since C is maximal, the coloring obtained after the swap is guaranteed to be a proper coloring, and if C is not the unique maximal bichromatic component containing these two colors, the coloring obtained after the swap is a coloring different from β , as the partition of the edges is different.

The Kempe swaps induce an equivalence relation on the set of colorings of a graph G ; two colorings β and β' are equivalent if one can find a sequence of Kempe swaps to transform β into β' . In 1964, Vizing actually proved a stronger statement, he proved that any k -coloring of a graph G (with $k > \Delta(G)$) is equivalent to a $(\Delta(G) + 1)$ -coloring of G .

Theorem 2. *Let G be a graph and β a k -coloring of G (with $k > \Delta(G)$). Then there exists a $(\Delta(G) + 1)$ -coloring β' equivalent to β .*

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Note that some graphs only need Δ colors to be properly colored. One year later, Vizing proved that this result is generalizable to multigraphs, and ask the following question:

Question 3. For any (multi)graph G and for any k -coloring β of G , is there always an optimal coloring equivalent to β ?

Note that both in Theorem 1 and in Question 3 we do not have the choice in the target coloring. If we can choose a specific target optimal coloring, then the question can be reformulated as a reconfiguration question.

Question 4. For any (multi)graph G and for any k -coloring β , is any optimal coloring always equivalent to β ?

If true, Question 4 would imply the following conjecture, as it suffices to take the optimal target coloring as an intermediate between the two non-optimal colorings.

Conjecture 5. *Any two non-optimal colorings are equivalent.*

Mohar proved the weaker case where we have one additional color [Moh06].

Theorem 6 ([Moh06]). *All $(\chi'(G) + 2)$ -colorings are equivalent.*

When considering the stronger case, McDonald & al. proved that the conjecture is true for graphs with maximum degree 3 [MMS12], Asratian and Casselgren proved that it is true for graphs with maximum degree 4 [AC16], and Bonamy & al. proved that the conjecture is true for triangle-free graphs. In this paper, we prove that the conjecture is true for all graphs.

Theorem 7. *Let G be a graph, all its $(\chi'(G) + 1)$ -edge colorings are Kempe-equivalent.*

Theorem 7 is a direct consequence of the following Lemma which is the main result of this paper.

Lemma 8. *Let G be a graph, any $(\chi'(G) + 1)$ -coloring of G is equivalent to any $\chi'(G)$ -coloring of G .*

2 General setting of the proof

The proof inherits the technical setup of [BDK⁺21], in this section, we introduce this setting, and give the general outline of the proof of the main result.

2.1 Reduction to $\chi'(G)$ -regular graphs

The general setting of the proof follows that of [BDK⁺21] which itself follows that of [MMS12] and of [AC16]. We first show that we can reduce the problem to the class of regular graphs. Indeed, given a graph G and a coloring β , we can build a graph G' s.t. :

- G' is $\chi'(G)$ -regular,
- the coloring β can be completed into a coloring β' of G' , and
- if a coloring γ' is equivalent to β' in G' , then the restriction of γ' to G is equivalent to β .

To build G' we step-by-step build a sequence $G_1, \dots, G_t = G'$ where at each step, $\delta(G_{i+1}) = \delta(G_i) + 1$ (where $\delta(G_i)$ is the minimum degree of G_i). For any G_i , the graph G_{i+1} is build as follows:

- take two copies of G_i , and
- add a matching between a vertex and its copy if this vertex has minimum degree in G_i .

It is clear that G' is $\chi'(G)$ -regular, and to extend a coloring of G_i to a coloring of G_{i+1} , it suffices to copy the coloring for each copy of G_i , and since the edges of the matching connect two vertices of minimum degree, there will always be an available colors for those edges. Moreover, a Kempe swap in G_i has a natural generalization in G_{i+1} : if a swap in G_i corresponds to more than one swap in G_{i+1} , it suffices to apply the swap on all the corresponding components in G_{i+1} . From now on, we will only consider χ' -regular graphs.

Note that colorings in regular graphs are easier to handle as the two following properties are verified:

- for any $(\Delta(G))$ -coloring of a $\chi'(G)$ -regular graph G , every vertex v is incident to exactly one edge of each color, and each color class is a perfect matching, and
- for any $(\Delta(G) + 1)$ -coloring α of a $\chi'(G)$ -regular graph G , every vertex v is incident to all but one color, we call this color the *missing color* at v , and denote it by $m_\alpha(v)$ (we often drop the α when the coloring is clear from the context).

From now on, in the rest of the paper, we only consider χ' -regular graphs.

2.2 The good the bad and the ugly

The general approach to Theorem 7 is an induction on the chromatic index. Given a graph G , a $\Delta(G)$ -coloring α and a $(\Delta(G) + 1)$ -coloring β , our goal is to find a sequence of Kempe swaps to transform β into α . To do so, we consider a color class in α , say the edges colored 1. These edges induce a perfect matching M in G , thus, if we can find a coloring β' equivalent to β s.t. for any edge e , $\beta'(e) = 1 \Leftrightarrow \alpha(e) = 1$, then we can proceed by induction on $G' = G \setminus M$, noting that $\chi'(G') = \chi'(G) - 1$, and that the restrictions of α and β' to G' only use $\Delta(G) - 1$, and $\Delta(G)$ colors respectively.

So, given a $(\Delta(G) + 1)$ -coloring β of G , we can partition the edges of G into three sets, an edge e is called:

- *good*, if $e \in M$ and $\beta(e) = 1$,
- *bad*, if $e \in M$ and $\beta(e) \neq 1$, and
- *ugly*, if $e \notin M$ and $\beta(e) = 1$.

A vertex missing the color 1 is called a *free vertex*. Toward contradiction, we assume that β is not equivalent to α , and we consider a $(\Delta(G) + 1)$ -coloring β' equivalent to β which minimizes the number of ugly edges among the colorings equivalent to β that minimize the number of bad edges, we call *minimal* such a coloring. Thus, if we can find a coloring β'' equivalent to β' where the number of bad is strictly lower than in β' , or with the same number of bad edges, and strictly fewer ugly edges, we get a contradiction.

2.3 Fan-like tools

In his proof of 64, Vizing introduce a technical tool to apply Kempe swaps on an edge-coloring in very controlled way: *Vizing's fans*. To define them, we first need to define an auxiliary digraph. Given a graph G , a $(\Delta(G) + 1)$ -coloring β of G and a vertex v , the directed graph D_v is defined as follows:

- the vertex set of D_v is the set of edges incident with v , and
- we put an arc between two vertices vv_1 and vv_2 of D_v , if the edge vv_2 is colored with the missing color at v_1 .

The *fan around v starting at the edge e* , denoted by $X_v(e)$, is the maximal sequence of vertices of D_v reachable from the edge e . It is sometime more convenient to speak about the color of the starting edge of a fan, if c is a color, $X_v(c)$ denotes the fan around v starting at the edge colored c incident with v . Note that since the graph G is $\chi'(G)$ -regular, each vertex misses exactly one color, and thus, in the digraph D_v , each vertex has outdegree at most 1. Hence a fan \mathcal{X} is well-defined and we only have three possibilities for the fan \mathcal{X} :

- \mathcal{X} is a path,
- \mathcal{X} is a cycle, or
- \mathcal{X} is a comet (*i.e.* a path with an additional arc between the sink and an internal vertex of the path).

If $X = (vv_1, \dots, vv_k)$ is a fan, v is called the central vertex of the fan, and vv_1 and vv_k are respectively called the first and the last edge of the fan (similarly, v_1 and v_k are the first and last vertex of X respectively).

Given a $(\Delta(G) + 1)$ -coloring β of G , and fan $\mathcal{X} = (vv_1, \dots, vv_k)$ which is a cycle around a vertex v , where each vertex v_i misses the color i (and so each edge vv_i is colored $(i - 1)$), we can define the coloring $\beta' = X^{-1}(\beta)$ as follows:

- for any edge vv_i not in \mathcal{X} , $\beta'(vv_i) = \beta(vv_i)$, and
- for any edge vv_i in \mathcal{X} , $\beta'(vv_i) = i$ and $m(v_i) = i - 1$

The coloring $X^{-1}(\beta)$ is called the *invert* of \mathcal{X} , and we say that X is *invertible* if \mathcal{X} and $X^{-1}(\beta)$ are equivalent. In this paper, we prove that in any coloring, any cycle is invertible.

Lemma 9. *In any $(\chi'(G) + 1)$ -coloring of a $\chi'(G)$ -regular graph G , any cycle is invertible.*

We prove Lemma 9 in Section 2.4, and prove here Theorem 7. The proof of Theorem 7 is derived from the proof of Theorem 1.6 in [BDK⁺21]. We first need the following results from [BDK⁺21] and [AC16] which we restate here (in a slightly different way).

Observation 10 ([BDK⁺21]). *In a minimal coloring, every bad edge is adjacent to an ugly edge.*

Lemma 11 ([BDK⁺21]). *In a minimal coloring, any ugly edge uv is such that the fans $X_v(uv)$ and $X_u(uv)$ are cycles.*

Lemma 12 ([AC16]). *In a minimal coloring both ends of an ugly edge are adjacent to a free vertex.*

We first show that in a minimal coloring, there always exists a bad edge adjacent to an ugly edge and incident with a free vertex.

Lemma 13. *In a minimal coloring, there exists a bad edge adjacent to an ugly edge and incident with a free vertex.*

Proof. Let β be a minimal coloring, if there is no bad edge in β , then all the edges of M are colored 1 in β as desired. So there exists a bad edge e in β , and by Observation 10, e is adjacent to an ugly edge e' . By Lemma 12, there exists a free vertex u adjacent to an end of e' . As u is a free vertex, u is incident with a bad edge, we denote by v the neighbor of u such that the edge uv is bad. If v is a free vertex, then we swap the single edge uv to obtain a coloring with fewer bad edges, so v is not free, and thus uv is adjacent to an ugly edge; this concludes the proof. \square

We are now ready to prove Theorem 7, but we first need some terminology and notations. Given a coloring α , for any pair of colors a, b , we denote by $K(a, b)$ the graph induced in G by the edges colored a and b . The Kempe chain involving these two colors and containing the element $x \in V(G) \cup E(G)$ is denoted by $K_x^\alpha(a, b)$ (we often drop the α when the coloring is clear from the context). It is important to note that if a, b, c and d are 4 different colors, then swapping a component of $K(a, b)$ before or after swapping a component of $K(c, d)$ does not change the coloring obtained after the two swaps.

Note also that in an edge-coloring, any Kempe chain $K(a, b)$ is a connected bipartite subgraph of maximum degree 2, hence it is either a path, or an even cycle. To distinguish the notions of fans that can be paths or cycles, when a Kempe component C of $K(a, b)$ is a path (respectively an even cycle) we say that C is a (a, b) -bichromatic path (respectively a (a, b) -bichromatic cycle). If u is a vertex missing the color a , then $K_u(a, b)$ is a (a, b) -bichromatic path whose ends are u and another vertex missing either a or b .

Proof of Theorem 7. Let β be a minimal coloring. By Lemma 13, there exists a bad edge uv such that u is free and v is incident with an ugly edge vw . By Lemma 11, the fans $X_v(vw)$ and $X_w(vw)$ are both cycles. The vertex v does not belong to $X_v(vw)$, otherwise, by Lemma 9 we invert $X_v(vw)$ and obtain a coloring with strictly fewer bad edges. Hence, the vertex w is missing a color c' different from $c = \beta(uv)$ (otherwise, $X_v(vw)$ is a cycle of size 2 containing u). We now consider the component $C = K_w(c, c')$, note that since w is missing the color c' , this component is a (c, c') -bichromatic path. If the component C does not contain the vertex v , then we swap it to obtain a coloring where w is missing the color of the edge uv and we are done. Thus, C contains v and we have to distinguish whether v is between u and w in C or u is between w and v .

Case 13.1 (u is between w and v in C).

In this case, by Lemma 9 we can invert $X_v(vw)$ to obtain a coloring where the component $K_w(c, c')$ is now a (c, c') -bichromatic cycle that we swap. In the coloring obtained after the swap, $X_v(uv)$ is a cycle, and so by Lemma 9 we can invert it to obtain a coloring with strictly fewer bad edges; a contradiction.

Case 13.2 (v is between w and u in C).

In this case, we consider the cycle $X_w(vw)$. If it does not contain the vertex u , we invert it by Lemma 9 and obtain a coloring where u and v are free, so it suffices to swap the edge uv to obtain a coloring with strictly fewer bad edges. Hence the vertex u belongs to $X_w(vw)$. After inverting this cycle, we obtain a minimal coloring where uv is still bad, v is free, and uw is ugly (the edge vw is not ugly anymore in this coloring). By Lemma 11, the fan $X_u(uw)$ is a cycle. The situation is now similar to the previous case: we invert the cycle $X_u(uw)$ to obtain a coloring where the component $K_w(c, c')$ is a (c, c') -bichromatic cycle. After swapping this cycle we obtain a minimal coloring where $X_u(uv)$ is a cycle. After inverting this cycle, we obtain a coloring with one fewer bad edge; a contradiction. \square

2.4 General outline and notations

The proof is an induction on the size of the cycles. Towards contradiction, assume that there exist non-invertible cycles. A *minimum* cycle \mathcal{V} is a non-invertible cycle of minimum size (*i.e.* in any coloring, any smaller cycle is invertible).

A cycle of size 2 is clearly invertible as it only consists of a single Kempe chain composed of exactly two edges: to invert the cycle, it suffices to apply a Kempe swap on this component; so the size of a minimum cycle is at least 3.

We now need some more notations. For any fan $\mathcal{V} = (vv_1, \dots, vv_k)$, $V(\mathcal{V})$ denotes the set of vertices $\{v_1, \dots, v_k\}$, and $E(\mathcal{V})$ denotes the set of edges $\{vv_1, \dots, vv_k\}$. We denote by $\beta(\mathcal{V})$ the set of colors involved in \mathcal{V} (*i.e.* $\beta(\mathcal{V}) = \beta(E(\mathcal{V})) \cup m(V(\mathcal{V})) \cup m(v)$); if \mathcal{V} involves the color c , $M(X, c)$ denotes the vertex of $V(\mathcal{V})$ missing the color c . There is a natural order induced by a fan on its vertices (respectively on its edges), and if $i < j$ we say that the vertex v_i (respectively the edge vv_i) is before the vertex v_j (respectively the edge vv_j). For two vertices v_i and v_j of \mathcal{V} we define the *subfan* $\mathcal{V}_{[v_i, v_j]}$ as the subsequence $(vv_i, vv_{i+1}, \dots, vv_j)$. We often write $\mathcal{V}_{\geq v_i}$, $\mathcal{V}_{> v_i}$, $\mathcal{V}_{\leq v_i}$ and $\mathcal{V}_{< v_i}$ to respectively denote the subfans (v_i, \dots, v_k) , (v_{i+1}, \dots, v_k) , (v_1, \dots, v_i) , and (v_1, \dots, v_{i-1}) .

If the fan \mathcal{V} is a cycle in a coloring β means applying a sequence of Kempe swaps to obtain the coloring $X^{-1}(\beta)$. If \mathcal{V} is a fan which is a path, inverting \mathcal{V} means applying a sequence of single-edge Kempe swaps on the edges of \mathcal{V} such that the ends of the first edge of \mathcal{V} are missing the same color $\beta(vv_1)$. Note that we often only partially invert paths, *i.e.* we apply a sequence of single-edge Kempe swaps on the edges of the paths until we reach a coloring with a specific missing color at the central vertex.

A cycle $\mathcal{V} = (vv_1, \dots, vv_k)$ is called *saturated* if for any $i, v_i \in K_v(m(v), m(v_i))$. Lemma 2.3 of [BDK⁺21], which we restate here, guarantees that if a cycle is not invertible, then it is saturated.

Lemma 14 ([BDK⁺21]). *Let \mathcal{V} be a cycle, if \mathcal{V} is not saturated, then \mathcal{V} is invertible.*

This directly implies the same result for any minimum cycle.

Lemma 15. *Any minimum cycle is saturated.*

Let $X \subseteq E(G) \cup V(G)$, β a coloring and β' a coloring obtained from β by swapping a component C . The component is called X -stable if :

- for any $v \in X$, $m^\beta(v) = m^{\beta'}(v)$, and
- for any $e \in X$, $\beta(e) = \beta'(e)$.

In this case, the coloring β' is called X -identical to β .

If $S = (C_1, \dots, C_k)$ is a sequence of swaps to transform a coloring β into a coloring β' where each C_j is a Kempe component. The sequence S^{-1} is defined as the sequence of swaps (C_k, \dots, C_1) . Such a sequence is called X -stable if each C_j is X -stable.

Observation 16. *Let $X \subseteq V(G) \cup E(G)$, and S a sequence of swaps that is X -stable. Then the sequence S^{-1} is also X -stable.*

If a sequence S is X -stable, then the coloring obtained after apply S to β is called X -equivalent to β . Note that the notion of X -equivalence is stronger than the notion of X -identity. Since two colorings β and β' may be X -identical but not X -equivalent if there exists a coloring β'' in the sequence between β and β' that is not X -identical to β . We first have the following observation that we will often use in this paper.

Observation 17. Let \mathcal{X} be a subfan in a coloring β_0 , v be a vertex which is not in $V(\mathcal{X})$, and $S = (C_1, \dots, C_k)$ be a sequence of trivial swaps of edges incident with v , $(\beta_1, \dots, \beta_k)$ be the colorings obtained after each swap of S . If for any $i \in \{0, \dots, k\}$, $m^{\beta_i}(v) \notin \beta_0(\mathcal{X})$, then the sequence S is (\mathcal{X}) -stable.

Proof. Otherwise, assume that S is not \mathcal{X} -stable. Since the vertex v is not in $V(\mathcal{X})$, then no edge of \mathcal{X} has been changed during the sequence of swap. Thus the missing color of a vertex of \mathcal{X} has been changed during the sequence of swaps, we denote by x the first such vertex. Let s_i be the swap that change the color of the edge vx , it means that in the coloring β_{i-1} the vertices v and x are missing the same color, so $m^{\beta_{i-1}} \in \beta_0(\mathcal{X})$; a contradiction. \square

The following observation gives a relation between X -equivalence and $(G \setminus X)$ -identity between colorings.

Observation 18. Let β be a coloring, $X \subseteq V(G) \cup E(G)$, β_1 a coloring X -equivalent to β , and β_2 a coloring $(G \setminus X)$ -identical to β_1 . Then, there exists a coloring β_3 equivalent to β_2 that is X -identical to β_2 and $(G \setminus X)$ -identical to β .

Proof. Let S be the sequence of swaps that transforms β into β_1 . Since β_1 is X -equivalent to β , the sequence S is X -stable and thus $E(S) \cap E(X) = V(S) \cap V(X) = \emptyset$. Since β_2 is $(G \setminus X)$ -identical to β_1 , it is S -identical to β_1 . So applying S^{-1} to β_2 is well-defined and gives a coloring β_3 S -identical to β . We first prove that β_3 is $(G \setminus X)$ -identical to β . The coloring β_1 is $(G \setminus S)$ -identical to the coloring β by definition of S , and the coloring β_2 is $(G \setminus X)$ -identical to β_1 , so the coloring β_2 is $(G \setminus (X \cup S))$ -identical to β . Again by definition of S^{-1} the coloring β_3 is $(G \setminus S)$ -identical to β_2 , so it is $(G \setminus (S \cup X))$ -identical to β . Since the coloring β_3 is also S -identical to β , in total, it is $(G \setminus X)$ -identical to β .

We now prove that β_3 is X -identical to β_2 . Since $E(S) \cap E(X) = V(S) \cap V(X) = \emptyset$, we have that $E(X) \subseteq E(G) \setminus E(S)$ and $V(X) \subseteq V(G) \setminus V(S)$. Moreover, the coloring β_3 is $(G \setminus S)$ -identical to β_2 by definition of S , so the coloring β_3 is X -identical to β_2 as desired. \square

If \mathcal{X} is a fan, when two colorings are $(V(\mathcal{X}) \cup E(\mathcal{X}))$ -identical (respectively $(V(\mathcal{X}) \cup E(\mathcal{X}))$ -equivalent), we simply write that the two colorings are \mathcal{X} -identical (respectively \mathcal{X} -equivalent). Similarly, if two colorings are $((V(G) \cup E(G)) \setminus X)$ -identical (respectively $((V(G) \cup E(G)) \setminus X)$ -equivalent), we simply write that the two colorings are $(G \setminus X)$ -identical (respectively $(G \setminus X)$ -equivalent).

Remark that if \mathcal{V} is a cycle in a coloring β , then the coloring $\mathcal{V}^{-1}(\beta)$ is $(G \setminus \mathcal{V})$ -identical to β . So from the previous observation we have the following corollary.

Corollary 19. Let \mathcal{V} be a cycle in a coloring β . If there exists a coloring β' \mathcal{V} -equivalent to β where \mathcal{V} is invertible, then \mathcal{V} is invertible in β .

Proof. Let $\beta'' = \mathcal{V}^{-1}(\beta')$. The coloring β' is \mathcal{V} -equivalent to β and β'' is $(G \setminus \mathcal{V})$ -identical to β' . So by Observation 18 there exists a coloring β_3 that is \mathcal{V} -identical to β'' and $(G \setminus \mathcal{V})$ -identical to β . So the coloring β_3 is $(G \setminus \mathcal{V})$ -identical to $\mathcal{V}^{-1}(\beta)$.

Moreover, the coloring β'' is \mathcal{V} -identical to $\mathcal{V}^{-1}(\beta)$, so the coloring β_3 is also \mathcal{V} -identical to $\mathcal{V}^{-1}(\beta)$. Therefore we have $\beta_3 = \mathcal{V}^{-1}(\beta)$ as desired. \square

From the previous corollary, we have the following observation.

Observation 20. Let \mathcal{V} be a minimum cycle in coloring β , and β' a coloring \mathcal{V} -equivalent to β . Then in the coloring β' , the sequence \mathcal{V} is also a minimum cycle such that for any $e \in E(\mathcal{V})$, $\beta(e) = \beta'(e)$, and for any $v \in V(\mathcal{V})$, $m^\beta(v) = m^{\beta'}(v)$.

We often simply say that the cycle \mathcal{V} is the same minimum cycle in the coloring β' .

A cycle $\mathcal{V} = (vv_1, \dots, vv_k)$ is called *tight* if for every i $v_i \in K_{v_{i-1}}(m(v_i), m(v_{i-1}))$. A simple observation is that any minimum cycle \mathcal{V} is tight.

Observation 21. *Let $\mathcal{V} = (vv_1, \dots, vv_k)$ be a minimum cycle in a coloring β . Then the cycle \mathcal{V} is tight.*

Proof. Assume that \mathcal{V} is not tight, so there exists i such that $v_i \notin K_{v_{i-1}}(m(v_i), m(v_{i-1}))$. Without loss of generality, we assume that $i = 2$ and that each v_j is missing the color j . Note that this means that $\beta(vv_2) = 1$, $\beta(vv_3) = 2$ and $\beta(vv_1) = k$.

We swap the component $C_{1,2} = K_{v_1}(1, 2)$ to obtain a coloring β' that is $(\mathcal{V} \setminus \{v_1\})$ -equivalent to the coloring β . Thus each v_j is missing the color j except v_1 which is now missing the color 2. So now the fan $\mathcal{V}' = X_v(k)$ is equal to $(vv_1, vv_3, \dots, vv_k)$, and thus is a cycle strictly smaller than \mathcal{V} . Since \mathcal{V} is minimum, this cycle is invertible, and we denote by β'' the coloring obtained after its inversion.

The coloring β'' is $(G \setminus \mathcal{V}')$ -identical to the coloring β' , so in particular it is $C_{1,2}$ -identical to the coloring β' . Moreover, the coloring β'' is $(\mathcal{V} \setminus \{vv_1, vv_2, v_2\})$ -identical to the coloring $\mathcal{V}^{-1}(\beta)$, and we have $\beta''(vv_1) = 2$, $\beta''(vv_2) = 1$, and $m^{\beta''}(v_2) = 2$.

So now in this coloring the component $K_{v_1}(1, 2)$ is exactly $C_{1,2} \cup \{vv_1, vv_2\}$, and we swap back this component to obtain a coloring β''' . The coloring β''' is now $C_{1,2}$ -identical to β , and thus it is $(G \setminus \mathcal{V})$ -identical to β . Moreover, it is $(\mathcal{V} \setminus \{vv_1, vv_2, v_2\})$ -identical to β'' , so it is $(\mathcal{V} \setminus \{vv_1, vv_2, v_2\})$ -identical to $\mathcal{V}^{-1}(\beta)$. Finally, we have $\beta'''(vv_1) = 1 = m^\beta(v_1)$, $\beta'''(vv_2) = 2 = m^\beta(v_2)$, and $m^{\beta'''}(v_2) = 1 = \beta(vv_2)$, so the coloring β''' is \mathcal{V} -identical to $\mathcal{V}^{-1}(\beta)$. Since it is also $(G \setminus \mathcal{V})$ -identical to β , we have $\beta''' = \mathcal{V}^{-1}(\beta)$ as desired. \square

The proof of Lemma 9, is a consequence of the two following Lemmas.

Lemma 22. *Let \mathcal{V} be a minimum cycle. For any color c different from $m(v)$, the fan $X_v(c)$ is a cycle.*

Lemma 23. *Let \mathcal{X} and \mathcal{Y} be two cycles around a vertex v . For any pair of vertices (z, z') in $(\mathcal{V} \cup \mathcal{X} \cup \mathcal{Y})^2$, the fan $\mathcal{Z} = X_z(c_{z'})$ is a cycle containing z' .*

We prove Lemma 22 in section 2.5, and Lemma 23 in section 5, and prove here Lemma 9.

Proof of Lemma 9. To prove the Lemma, we prove that the graph G only consists of an even clique where each vertex misses a different color. This is a contradiction since in any $(\Delta(G) + 1)$ -coloring of an even clique, for any color c , the number of vertices missing the color c is always even. By Lemma 22, all the fans around v are cycles, so each neighbor of v misses a different color. Moreover, by Lemma 23, there is an edge between each pair of neighbors of v , so $G = N[v] = K_{\Delta(G)+1}$. By construction, G is $\Delta(G)$ -colorable, so G is an even clique and each vertex misses a different color, this concludes the proof. \square

2.5 Only cycles around v : a proof of lemma 22

In this section, we prove Lemma 22. If \mathcal{X} and \mathcal{X}' are two fans, then \mathcal{X} and \mathcal{X}' are called *entangled* if for any $c \in \beta(\mathcal{X}) \cap \beta(\mathcal{X}')$, $M(X, c) = M(X', c)$. To prove Lemma 22 we need the two following lemmas.

Lemma 24. *Let \mathcal{V} be a minimum cycle in a coloring β and let u and u' be two vertices of \mathcal{V} . Then fan $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_l)$ is a cycle entangled with \mathcal{V} .*

Lemma 25. *Let \mathcal{V} be a minimum cycle in a coloring β , u and u' be two vertices of \mathcal{V} , and $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_l)$. Then for any $j \leq l$, the fan $X_v(\beta(uu_j))$ is a cycle.*

Note that by Lemma 24, we can directly conclude that $N[v]$ is a clique. Moreover, we directly have the following corollary.

Corollary 26. *Let $\mathcal{V} = (vv_1, \dots, vv_k)$ be a minimum cycle in a coloring β . Then for any $j \leq k$, the fan $X_u(m(v))$ is a cycle entangled with \mathcal{V} .*

Proof. Let $j \leq k$ and $\mathcal{U} = X_{v_j}(m(v_{j-1})) = X_{v_j}(\beta(vv_j))$. Then the first edge of \mathcal{U} is vv_j , and since v is missing the color $m(v)$, the second edge of \mathcal{U} is colored $m(v)$. By Lemma 24, \mathcal{U} is a cycle entangled with \mathcal{V} , so since $X_u(m(v)) = \mathcal{U}$, the fan $X_u(m(v))$ is a cycle entangled with \mathcal{V} as desired. \square

We prove Lemma 24 in Section 3, Lemma 25 in Section 4, and prove here Lemma 22.

Proof of Lemma 22. Assume that there exists a fan $\mathcal{W} = (vw_1, \dots, vw_t)$ around v which does not induce a cycle, we first prove that \mathcal{W} is not a path.

Claim 27. *The fan \mathcal{W} cannot induce a path.*

Proof. Without loss of generality, we assume that the vertex v is missing the color 1. Assume that \mathcal{W} induces a path, so $m(v) = m(w_t) = 1$. Let $v' \in \mathcal{V}$, by Corollary 26, we have that $\mathcal{U} = X_{v'}(1)$ is a cycle containing v in β . If we apply a single-edge Kempe swap on vw_t , then we obtain a coloring where $m(w_t) = m(v) = \beta(vw_t)$; we denote by β' this coloring, and without loss of generality, we assume that $\beta(vw_t) = 2$. Again, by Corollary 26, we also have that $\mathcal{U}' = X_{v'}(2)$ is a cycle containing v in the coloring β' , so $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, let $v'w''$ be the first edge they have in common, and let $w = M(\mathcal{U}, \beta(v'w''))$ and $w' = M(\mathcal{U}', \beta(v'w''))$. We now have to distinguish whether $v \in \{w, w'\}$ or not.

Case 27.1 ($v \notin \{w, w'\}$).

In this case, $m_\beta(w) = m_{\beta'}(w) = m_\beta(w') = m_{\beta'}(w')$; we denote by c this color. By Lemma 25, $X_v(c)$ is a cycle containing w in β , and $X_v(c)$ is a cycle containing w' in β' , so $w = w'$; a contradiction.

Case 27.2 ($v \in \{w, w'\}$).

The case $v = w$ and $v = w'$ being symmetrical, we can assume that $v = w$. In this case, in the coloring β' , w' is missing the color c_v , but by Lemma 25 $X_v(1)$ is a cycle containing w or \mathcal{V} is invertible, however, in the coloring β' , $X_v(1)$ induces a path which is a single edge; a contradiction. \square

Thus the fan \mathcal{W} is not a path. Now assume that \mathcal{W} is a comet, then there exists w and w' in \mathcal{W} which are missing the same color c . At least one of them is not in $K_v(1, c)$, the two cases being symmetrical, we can assume without loss of generality that w is not in $K_v(1, c)$. So if we swap the component $K_w(1, c)$, we obtain a coloring where the fan $X_v(\beta(vw_1))$ is a path; a contradiction, so \mathcal{W} is a cycle. \square

3 Fans around \mathcal{V} : a proof of Lemma 24

In this section, we prove Lemma 24 which will be often used in the proof of Lemma 25.

Proof. We first prove that the fan \mathcal{U} cannot induce a path.

Claim 28. *The fan \mathcal{U} cannot induce a path.*

Proof. Otherwise, assume that the fan \mathcal{U} is a path, without loss of generality, we can assume that \mathcal{U} is of minimal length (if \mathcal{U} is not minimal, since it is a path, it contains a strictly smaller path). Thus \mathcal{U} contains only one edge colored with a color in $\beta(\mathcal{V}) \setminus \{c_v\}$: its first edge. We now need to distinguish whether $j' = j - 1$ or not (i.e. whether $u = v_j$ and $u' = v_{j'}$ are consecutive or not in \mathcal{V}).

Case 28.1 ($j' = j - 1$).

In this case, $\mathcal{U} = X_u(uv)$, and the edge colored c_v incident with u is just after uv in \mathcal{U} . As \mathcal{U} is a path, we can invert it until we reach a coloring where $m(u) = m(v) = c_v$. Since \mathcal{U} is minimal, no edge incident with a vertex of \mathcal{V} different from u has been recolored during the inversion. In the coloring obtained after the inversion, the fan $(vv_{j+1}, \dots, vv_j = vu)$ is a path that we can invert until we reach a coloring where $m(v) = m(v_{j+1}) = j$, we denote by β' this coloring. Since \mathcal{V} was tight in the coloring β , in the coloring β' we have $C = K_{v_{j-1}}^{\beta'}(j, j-1) = K_{v_{j-1}}^{\beta}(j, j-1) \cup \{vv_{j-1}\} \setminus \{vv_{j+1}, vv_j = vu\}$, so we swap this component to obtain a coloring where $m(v) = m(u) = j - 1$, then we swap the edge uv and obtain a coloring where (uu_{l-1}, \dots, uu_0) is a path that we invert. In the coloring obtained after the inversion, we have that the component $K_{v_{j-1}}(j, j-1)$ is exactly $C \cup \{vv_j\}$, if we swap this component back we obtain exactly $\mathcal{V}^{-1}(\beta)$.

Case 28.2 ($j' \neq j - 1$).

In this case, since \mathcal{U} is a path, we can invert it until we reach a coloring β' where $m(u) = c_{u'} = j'$. Note that, similarly to the previous case, this inversion has not changed the colors of the edges incident with the vertices of \mathcal{V} , except those incident with u . We now consider the component $K_v(j', c_v)$ (which can have changed during the inversion of \mathcal{U} as we swapped an edge colored j'), and we need to distinguish whether or not the vertices u' and u belong to this component; clearly these vertices does not both belong to this component.

Subcase 28.2.1 ($u' \notin K_v(j', c_u)$).

In this case, we swap the component $C = K_{u'}(j', c_v)$ to obtain a coloring where $(vv_{j+1}, \dots, vv_{j'})$ is a path that we invert until we reach a coloring where $m(v) = m(v_{j+1}) = c_u$, we denote by β' this coloring. As \mathcal{V} was tight in β , we have that $C_j = K_{v_{j-1}}^{\beta'}(j, j-1) = K_{v_{j-1}}^{\beta}(j, j-1) \setminus \{vv_{j+1}, vv_j = vu\}$, so we swap this component to obtain a coloring where $(vv_{j'+1}, \dots, vv_{j-1})$ is a path that we invert until we reach a coloring where $m(v) = m(v_{j'+1}) = j'$. In the coloring obtained after the inversion, the component $K_{u'}(j', c_v)$ is exactly $C \cup \{vu'\}$, thus we swap it back. Note that as $|\{c_{u'}, c_v, j, j-1\}| = 4$, we can swap back C before C_j . In the coloring obtained after swapping back the component, we have that the fan (uu_{l-1}, \dots, uu_0) is a path that we invert. In the coloring obtained after the inversion, the component $K_{v_{j-1}}(j, j-1)$ is exactly $C \cup \{vv_{j-1}, vv_j = vu\}$, thus we swap back this component and obtain exactly $\mathcal{V}^{-1}(\beta)$.

So u' belongs to the component $K_v(j', c_u)$.

Subcase 28.2.2 ($u \notin K_v(c_{u'}, c_u)$).

In this case, we swap the component $C = K_u(j', c_v)$, note that, from the previous case, neither v nor u' belong to this component. In the coloring obtained after the swap, the fan (vv_{j+1}, \dots, vv_j) is a path that we invert until we reach a coloring where $m(v) = m(v_{j+1}) =$

c_u ; we denote by β' this coloring. As \mathcal{V} was tight in β , we have that $C_j = K_{v_{j-1}}^{\beta'}(j, j-1) = K_{v_{j-1}}^\beta(j, j-1) \cup \{vv_{j-1}\} \setminus \{vv_{j+1}, vv_j = vu\}$, so we swap this component to obtain a coloring where $m(v) = m(u) = j-1$, then we swap the edge uv to obtain a coloring where $K_u(c_{u'}, c_v)$ is exactly C . Hence we swap back this component, and in the coloring obtained after the swap, the fan (uu_{l-1}, \dots, uu_0) is a path that we invert until we reach a coloring where $m(u) = j$. In this coloring, the component $K_{v_{j-1}}(j, j-1)$ is exactly $C_j \cup \{vv_{j-1}, vv_j = vu\}$, thus we swap back this component to obtain exactly $\mathcal{V}^{-1}(\beta)$. \square

Before proving that the fan \mathcal{U} is not a comet, we prove that \mathcal{U} and \mathcal{V} are entangled.

Claim 29. *The fans \mathcal{U} and \mathcal{V} are entangled.*

Proof. Assume that \mathcal{U} and \mathcal{V} are not entangled, then there exist $w = v_s \in \mathcal{V}$ and $w' = u_{s'} \in \mathcal{U}$ distinct from w with $m(w) = m(w') = c$. If $m(w) = m(v) = c_v$, then, since \mathcal{V} is saturated, $w \in K_v(c_v, c)$, so we swap $K_{w'}(c_v, c)$ to obtain a coloring where \mathcal{V} is still a cycle of the same size, but where $X_u(c_{u'})$ is a path, by the previous claim, this is a contradiction.

So $m(w) \neq m(v)$, and therefore, we successively swap the components $K_{w'}(t, t+1)$ with $t \in (s, \dots, j)$. Note that this sequence of swaps has not changed the colors of the edges incident with a vertex of \mathcal{V} ; it can though have changed the colors of the edges of \mathcal{U} . However, it is guaranteed that in the coloring obtained after the swaps, there exists a color $c' \in \beta(\mathcal{V})$ such that $X_u(c')$ is a path, which is a contradiction by the previous claim. \square

We now prove that \mathcal{U} is not a comet.

Claim 30. *The fan \mathcal{U} is not a comet.*

Proof. Assume that \mathcal{U} is a comet, then there exist w and w' in \mathcal{U} with $m(w) = m(w') = c$ and where w' is after w in the sequence. By the previous claim, as \mathcal{U} and \mathcal{V} are entangled, we have that $c \notin \beta(\mathcal{V})$. We now consider the component $C_v = K_v(c, c_v)$. If w' is not in C_v , then we swap $C_{w'} = K_{w'}(c, c_v)$ to obtain a coloring where w' belongs to the fan $X_u(c_{u'})$ with $m(w') = m(v)$; this contradicts the fact that $X_u(c_{u'})$ and \mathcal{V} are entangled. Note that if u is in C' , and $m(v) \in \beta(\mathcal{U})$, after swapping C' the sequence $X_u(c_{u'})$ now starts at the edge colored c in β , but this does not change the reasoning. So the vertex w' belongs to C , and thus the vertex w does not belong to C_v , so we can swap $C_w = K_w(c, c_v)$ to obtain a coloring where the sequence $X_u(m(u'))$ contains w which is missing the color $m(v)$, a contradiction. Note that if $u \in C_w$, then after swapping C_w , we obtain a coloring where w' comes before w in the fan $X_u(m(u'))$. Similarly to the previous case, this does not change the reasoning. \square

From the previous claims, the fan \mathcal{U} is a cycle entangled with \mathcal{V} as desired. \square

4 Cycles around v starting with u : a proof of Lemma 25

In this section we prove Lemma 25. To prove the lemma we actually prove a stronger statement, but we need first some definitions.

Definition 31. *Let $i \geq 0$, we define the property $P_{weak}(i)$ as the following: For any minimum cycle \mathcal{V} in a coloring β , for any pair of vertices u and u' of \mathcal{V} , let $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_l)$. If $\beta(uu_{l-i}) \neq m(v)$, then $X_v(\beta(uu_{l-i}))$ is not a path.*

Definition 32. Let $i \geq 0$, we define the property $P(i)$ as follows:

For any minimum cycle \mathcal{V} in a coloring β , for any pair of vertices u and u' of \mathcal{V} , let $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_i)$. If $\beta(uu_{i-1}) \neq m(v)$, then the fan $X_v(\beta(uu_{i-1}))$ is a saturated cycle containing u_{i-1} ,

Lemma 25 is a direct consequence of the following lemma.

Lemma 33. The property $P(i)$ is true for all i .

The proof of the lemma is an induction on i . However, before starting to prove the lemma, we need to introduce the notion of (\mathcal{V}, u) -independent fan for a vertex u of a cycle \mathcal{V} .

4.1 (\mathcal{V}, u) -independent fans

Let \mathcal{V} be a minimum cycle in a coloring β , and u a vertex of \mathcal{V} . A (\mathcal{V}, u) -independent subfan \mathcal{X} is a subfan around v such that $\beta(\mathcal{V}) \cap \beta(\mathcal{X}) = \{\beta(u)\}$. We naturally define a (\mathcal{V}, u) -independent path (respectively a (\mathcal{V}, u) -independent cycle) as a (\mathcal{V}, u) -independent subfan that is also a path (respectively a cycle). If v is a vertex not in \mathcal{X} missing a color c , we say that \mathcal{X} avoids v if the last vertex of \mathcal{X} is also missing the color c .

We first prove the following.

Lemma 34. Let \mathcal{V} be a minimum cycle in a coloring β , u a vertex of \mathcal{V} , $\mathcal{Y} = (uy_1, \dots, uy_r)$ a (\mathcal{V}, u) -independent subfan avoiding v and x the extremity of $K_{y_s}(m(u), m(v))$ which is not y_r . Then the fan $X_v(\beta(uy_1))$ is a path containing x which is missing the color $m(v)$.

We decompose the proof into five separate lemmas.

Proof of Lemma 34. Without loss of generality, we assume that the vertices v and u are respectively missing the colors 1 and 2, and that $\beta(uy_1) = 4$. Since the fan \mathcal{V} is a minimum cycle in the coloring β , it is saturated by Lemma 15, so $u \in K_v(1, 2)$ and thus $y_r \notin K_v(1, 2)$. We now swap the component $C_{1,2} = K_{y_r}(1, 2)$ to obtain a coloring \mathcal{V} -equivalent to β , where \mathcal{Y} is now a (\mathcal{V}, u) -independent path. By Lemma 39, the fan $X_v(4)$ is a comet containing the other extremity of $K_{y_r}(1, 2)$ which is x . In this coloring, the vertex x is missing the color 2, therefore in the coloring β , the fan $X_v(4)$ is a path containing x which is missing the color 1 as desired. \square

Lemma 35. Let $\mathcal{X} = (vv_1, \dots, vv_k)$ be a path of length at least 3 in a coloring β , $u = v_i$ for some $i \in [3, k]$, $u'' = v_{i-1}$, $u' = v_1$, and C a $(\beta(vu), m(u))$ -bichromatic path between u'' and u' that does not contain v . Then β is equivalent to a coloring β' such that:

- β' is $(G \setminus (C \cup \mathcal{X}))$ -identical to β ,
- β' is $(\mathcal{X}_{\geq u})$ -identical to β ,
- for any edge $j \in [2, i-1]$, $m^{\beta'}(v_j) = \beta(vv_j)$,
- $m^{\beta'}(u') = \beta(vu)$,
- for any edge $j \in [1, i-2]$, $\beta'(vv_j) = m^\beta(v_j)$,
- $\beta'(vu'') = \beta(vu')$,
- for any edge $e \in C$:

- if $\beta(e) = \beta(vu)$, then $\beta'(e) = m^\beta(u)$, and
- if $\beta(e) = m^\beta(u)$, then $\beta'(e) = \beta(vu)$.

Proof. Without loss of generality, we assume that the vertices v is missing the color 1, that the edge vu' is colored 2, and that the edge vu is colored 3. Note that this means that $m(u'') = 3$. In the coloring β , the fan \mathcal{X} is a path, so we invert this path, and denote by β_2 the coloring obtained after the inversion. The coloring β_2 is $(G \setminus \mathcal{X})$ -identical to the coloring β so C is still a $(2, 3)$ -bichromatic path between u' and u'' that does not contain v . Moreover, for any edge $j \in [1, i]$, $\beta_2(vv_j) = m^\beta(v_j)$ and $m^{\beta_2}(v_j) = \beta(vv_j)$. So the coloring β_2 is $(\mathcal{X}_{[v_2, v_{i-2}]} \cup \{u'', vu'\})$ -identical to β' . The vertex u' is now missing the color 2, and the edge vu'' is now colored 3. Moreover, the vertex v is now missing the color 2, so $K_v(2, 3) = C \cup \{vu''\}$. We now swap this component and denote by β_3 the coloring obtained after the swap.

The coloring β_3 is $(G \setminus (C \cup \mathcal{X}))$ -identical to the coloring β , so it is $(G \setminus (C \cup \mathcal{X}))$ -identical to β' . Moreover, for any edge $e \in C$:

- if $\beta(e) = 2$, then $\beta_3(e) = 3$, and
- if $\beta(e) = 3$, then $\beta_3(e) = 2$.

So the coloring β_3 is also C -identical to β' ; thus it is $(G \setminus \mathcal{X})$ -identical to β' .

The coloring β_3 is $(\mathcal{X}_{[v_2, v_{i-2}]} \cup \{u'', vu'\})$ -identical to β_2 , so it is $(\mathcal{X}_{[v_2, v_{i-2}]} \cup \{u'', vu'\})$ -identical to β' . In the coloring β_3 , the edge vu'' is now colored 2, and the vertex u' is now missing the color 3. So the coloring β_3 is also $(\{vu'', u'\})$ -identical to β' , and thus it is $\mathcal{X}_{<u}$ -identical to β' . In total, the coloring β_3 is $(G \setminus \mathcal{X}_{\geq u})$ -identical to the coloring β' .

Finally, the coloring β_3 is $\mathcal{X}_{\geq u}$ -identical to the coloring β_2 and the vertices v and u are both missing the color 3. So in the coloring β_3 the fan $X_v(1)$ is now a path. We invert this path and denote by β_4 the coloring obtained after the inversion. The coloring β_4 is $\mathcal{X}_{\geq u}$ -identical to the coloring β , so it is $\mathcal{X}_{\geq u}$ -identical to the coloring β' . Moreover, the coloring β_4 is also $(G \setminus \mathcal{X}_{\geq u})$ -identical to the coloring β_3 , so it is $(G \setminus \mathcal{X}_{\geq u})$ -identical to the coloring β' . In total the coloring β_4 is identical to the coloring β' as desired. \square

Lemma 36. Let $\mathcal{V} = (vv_1, \dots, vv_k)$ a cycle of length at least 3 in a coloring β , $u = v_i$, $u' = v_{i+1}$ and $u'' = v_{i-1}$ three consecutive vertices of \mathcal{V} , $\mathcal{Y} = (uy_1, \dots, uy_l)$ a (\mathcal{V}, u) -independent path, $\beta_{\mathcal{Y}} = \mathcal{Y}^{-1}(\beta)$, C a $(\beta(vu), m(u'))$ -bichromatic path in the coloring $\beta_{\mathcal{Y}}$ between u'' and u' that does not contain v nor u , $X = E(C) \cup E(\mathcal{V}) \cup (V(\mathcal{V}) \cup \{v\} \setminus \{u\})$, and $\beta'_{\mathcal{Y}}$ a coloring X -equivalent to $\beta_{\mathcal{Y}}$. If there exists a coloring β' equivalent to $\beta'_{\mathcal{Y}}$ such that:

- β' is $(G \setminus X)$ -identical to $\beta'_{\mathcal{Y}}$,
- β' is $(\mathcal{V} \setminus \{u', vu'', u, vu\})$ -identical to $\mathcal{V}^{-1}(\beta)$,
- $\beta'(vu'') = \beta'_{\mathcal{Y}}(vu')$,
- $\beta'(vu) = m^{\beta_{\mathcal{Y}}}(u'')$, and
- $m^{\beta'}(u') = \beta'_{\mathcal{Y}}(vu)$,
- for any edge $e \in C$:
 - if $\beta'_{\mathcal{Y}}(e) = \beta(vu)$, then $\beta'(e) = m(u')$, and
 - if $\beta'_{\mathcal{Y}}(e) = m(u')$, then $\beta'(e) = \beta(vu)$.

Then the cycle \mathcal{V} is invertible.

Proof. Let $\gamma = \mathcal{V}^{-1}(\beta)$. Without loss of generality, we assume that the vertex v and u are respectively missing the colors 1 and 2 in the coloring β , and that $\beta(vu) = 3$. This means that $\beta'(vu'') = \beta'_y(vu') = \beta(vu') = m^\beta(u) = 2$ and $m^{\beta'}(u') = \beta'_y(vu) = \beta(vu) = m^\beta(u'') = 3$. By definition the coloring β_y is $(G \setminus (\mathcal{Y} \cup \{u\}))$ -identical to β . Since \mathcal{Y} is a (\mathcal{V}, u) -independent path, we have $E(\mathcal{Y}) \cap E(\mathcal{V}) = \emptyset$, and $V(\mathcal{Y}) \cap V(\mathcal{V}) = \emptyset$. So, in particular $\beta_y(vu) = \beta(vu) = 3$. The coloring β' is $(\{vu\})$ -identical to β'_y , so $\beta'(vu) = 3$.

Since the coloring β' is $(G \setminus X)$ -identical to β'_y and β'_y is X -equivalent to β_y , by Observation 18, there exists a coloring β'' which is X -identical to β' and $(G \setminus X)$ -identical to β_y .

The coloring β'' is $(G \setminus X)$ -identical to β_y , so it is $(G \setminus (X \cup \mathcal{Y} \cup \{u\}))$ -identical to β . This means that β'' is $(G \setminus (\mathcal{V} \cup \mathcal{Y} \cup C))$ -identical to β , and thus it is $(G \setminus (\mathcal{V} \cup \mathcal{Y} \cup C))$ -identical to γ . Moreover, β'' is X -identical to β' , and β' is $(\mathcal{V} \setminus (\{u', vu'', u, vu\}))$ -identical to γ , so β'' is $(\mathcal{V} \setminus (\{u', vu'', u, vu\}))$ -identical to γ . In total, the coloring β'' is $(G \setminus (C \cup \mathcal{Y} \cup \{u', vu'', u, vu\}))$ -identical to γ .

In the coloring β_y , the fan $X_u(2)$ is now a path, and we have $E(X_u(2)) = E(\mathcal{Y})$ and $V(X_u(2)) = V(\mathcal{Y})$. So in any coloring $(\mathcal{Y} \cup \{u\})$ -identical to β_y , the fan $X_u(2)$ is a path. The β'' is $(G \setminus X)$ -identical to β_y , $E(X) \cap E(\mathcal{Y}) = \emptyset$ and $V(X) \cap (V(\mathcal{Y}) \cup \{u\}) = \emptyset$, so β'' is $(\mathcal{Y} \cup \{u\})$ -identical to β_y , and thus $X_u^{\beta''}(2)$ is a path that we invert. Let β_3 be the coloring obtained after the inversion.

By definition of \mathcal{Y} , the coloring β_3 is $(\mathcal{Y} \cup \{u\})$ -identical to the coloring β . So it is \mathcal{Y} -identical to the coloring γ , and u is now missing the color 2. The coloring β_3 is also $(G \setminus (\mathcal{Y} \cup \{u\}))$ -identical to β'' , so it is $(G \setminus (C \cup \{u', vu'', u, vu\}))$ -identical to γ , and we have $\beta_3(vu'') = \beta''(vu'') = 2$, $\beta_3(vu) = \beta''(vu) = 3$ and $m^{\beta_3}(u') = m^{\beta''}(u') = 3$. Note that the coloring β_3 is also C -identical to the coloring β'' .

The path C is a $(2, 3)$ -bichromatic path between u'' and u' and does not contain v nor u , so, in the coloring β_3 , we have $K_{u'}(2, 3) = C \cup \{vu'', vu\}$. We now swap this component and denote by β_f the coloring obtained after the swap. The coloring β_f is $(G \setminus (C \cup \{u', vu'', u, vu\}))$ -identical to the coloring β_3 , so it is $(G \setminus (C \cup \{u', vu'', u, vu\}))$ -identical to γ . Moreover, since β_3 is C -identical to β'' , for any edge $e \in C$:

- if $\beta''(e) = \beta_3(e) = 2$, then $\beta_f(e) = 3$, and
- if $\beta''(e) = \beta_3(e) = 3$, then $\beta_f(e) = 2$.

So the coloring β_f is C -identical to the coloring β_y , and thus it is C -identical to the coloring γ . Finally, we have:

- $m^{\beta_f}(u) = 3 = \beta(vu) = m^\gamma(u)$,
- $\beta_f(vu) = 2 = m^\beta(u) = \gamma(vu)$,
- $m^{\beta_f}(u') = 2 = \beta(vu') = m^\gamma(u')$, and
- $\beta_f(vu'') = 3 = m^\beta(u'') = \gamma(vu'')$.

Finally we have that β_f is $(C \cup \{u', vu'', u, vu\})$ -identical to γ , so it is identical to γ , and \mathcal{V} is invertible as desired. \square

Lemma 37. Let $\mathcal{V} = (vv_1, \dots, vv_i)$ a minimum cycle in a coloring β , $u = v_i$, $u' = v_1$ and $u'' = v_{i-1}$ three consecutive vertices of \mathcal{V} , and $\mathcal{Y} = (uy_1, \dots, uy_l)$ a (\mathcal{V}, u) -independent path, $C = K_{u''}(m(u), m(u'')) \setminus \{vu, vu'\}$, and $X = C \cup E(\mathcal{V}) \cup (V(\mathcal{V}) \cup \{v\} \setminus \{u\})$. In any coloring β'_y that is X -equivalent to the coloring $\beta_y = \mathcal{Y}^{-1}(\beta)$, the fan $X_v(m^\beta(u))$ is not a path.

Proof. Without loss of generality, we assume that the vertices v and u are respectively missing the colors 1 and 2, and that $\beta(vu) = 3$. This means that $\beta(vu') = m^\beta(vu) = 2$, $m^\beta(u'') = \beta(vu) = 3$, and $m^{\beta_y}(u) = 4$. Assume that $\mathcal{X} = X_v^{\beta_y}(2)$ is a path. The vertex v is still missing the color 1 in the coloring β_y and thus it is still missing 1 in β'_y . The coloring β_y is $(\mathcal{V} \setminus \{u\})$ -identical to the coloring β and so is the coloring β'_y . So $\{u', u''\} \subseteq V(\mathcal{X})$ and $\beta'_y(vu) = \beta(vu)$, so $u \in V(\mathcal{X})$, and thus the size of \mathcal{X} is at least 3. Note that this means that $V(\mathcal{V}) = V(\mathcal{X}_{\leq u})$.

The cycle \mathcal{V} is a minimum cycle in β , so by Observation 21, it is tight, and in particular, $u \in K_{u''}(2, 3)$. So the C is a $(2, 3)$ -bichromatic path between u'' and u' that does not contain u nor v . Since \mathcal{Y} is a (\mathcal{V}, u) -independent path, the coloring β_y is C -identical to β . The coloring β'_y is C -equivalent to β_y so C is still the same bichromatic path in the coloring β'_y .

Since \mathcal{X} is a path of path of length at least 3, by Lemma 35 there exists a coloring β' such that:

- β' is $(G \setminus (C \cup \mathcal{X}))$ -identical to β'_y ,
- β' is $(\mathcal{X}_{\geq u})$ -identical to β'_y ,
- for any edge $j \in [2, i-1]$, $m^{\beta'}(v_j) = \beta'_y(vv_j)$,
- $m^{\beta'}(u') = \beta'_y(vu) = 3$,
- for any edge $j \in [1, i-2]$, $\beta'(vv_j) = m^{\beta_y}(vv_j)$,
- $\beta'(vu'') = \beta'_y(vu'') = 2$,
- for any edge $e \in C$:
 - if $\beta(e) = \beta'_y(vu) = 3$, then $\beta'(e) = m^{\beta_y}(u) = 2$, and
 - if $\beta(e) = m^{\beta_y}(u) = 2$, then $\beta'(e) = \beta'_y(vu) = 3$.

The coloring β' is $(G \setminus (C \cup \mathcal{X}))$ -identical to β'_y , and is $\mathcal{X}_{\geq u}$ -identical to β'_y . So the coloring β' is $(G \setminus X)$ -identical to β'_y .

Let $\gamma = \mathcal{V}^{-1}(\beta)$. For any $j \in [2, i-2]$, we have $\beta'(vv_j) = m^{\beta_y}(v_j) = m^\beta(v_j) = \gamma(vv_j)$, and $m^{\beta'}(v_j) = \beta'_y(vv_j) = \beta(vv_j) = m^\gamma(v_j)$, so the coloring β' is $(\mathcal{V} \setminus \{u', vu', u'', vu'', u, vu\})$ -identical to γ . Moreover, $\beta'(vu) = m^{\beta_y}(u') = m^\beta(u') = \gamma(vu')$ and $m^{\beta'}(u'') = \beta_y(vu'') = \beta(vu'') = m^\gamma(u'')$. So in total the coloring β' is $(\mathcal{V} \setminus \{u', vu'', u, vu\})$ -identical to the coloring γ .

The coloring β' is $\mathcal{X}_{\geq u}$ -identical to β'_y , so in particular, $\beta'(vu) = \beta'_y(vu) = m^{\beta_y}(u'')$. We also have that $\beta'(vu'') = 2 = \beta'_y(vu'')$, and $m^{\beta'}(u') = 3 = \beta'_y(vu)$.

Finally, for any edge e in C :

- if $\beta_y(e) = \beta(e) = 2$, then $\beta'(e) = 3$, and
- if $\beta_y(e) = \beta(e) = 3$, then $\beta'(e) = 2$.

So by Lemma 36, the cycle \mathcal{V} is invertible; a contradiction. □

Lemma 38. *Let $\mathcal{V} = (vv_1, \dots, vv_k)$ be a minimum cycle in a coloring β , $u = v_j$, $u' = v_{j+1}$, and $u'' = v_{j-1}$ three consecutive vertices of \mathcal{V} and \mathcal{Y} a (\mathcal{V}, u) -independent path. Then in the coloring $\beta_{\mathcal{Y}} = \mathcal{Y}^{-1}(\beta)$, the fan $\mathcal{X} = X_v(m^\beta(u)) = (vx_1, \dots, vx_s)$ is a cycle.*

Proof. By Lemma 37, the fan \mathcal{X} is not a path. To show that it is a cycle, we prove that \mathcal{X} is not a comet. Otherwise, assume that \mathcal{X} is a comet, then there exists $i < s$ such that $m(x_i) = m(x_s)$. Without loss of generality, we assume that $m^\beta(v) = 1$, $m^\beta(u) = \beta(vu') = 2$, $\beta(vu) = m^\beta(u'') = 3$ and $m^{\beta_{\mathcal{Y}}}(x_i) = m^{\beta_{\mathcal{Y}}}(x_s) = 4$. We now have to distinguish the cases.

Case 38.1 ($4 \notin \beta(\mathcal{V})$).

In the coloring β , the fan \mathcal{V} is a minimum cycle, so by Observation 21, it is tight and in particular, $u \in K_{u''}(2, 3)$. Let $C = K_{u''}(2, 3) \setminus \{vu', vu\}$. The path C is a $(2, 3)$ -bichromatic path between u'' and u' which does not contain v nor u . Since \mathcal{Y} is a (\mathcal{V}, u) -independent path, the coloring $\beta_{\mathcal{Y}}$ is C -identical to β , and thus C is still a $(2, 3)$ -bichromatic path between u'' and u' which does not contain u nor v . Let $X = C \cup E(\mathcal{V}) \cup (V(\mathcal{V}) \cup \{v\} \setminus \{u\})$. We now consider the components of $K(1, 4)$ in the coloring $\beta_{\mathcal{Y}}$. The vertices x_i and x_s are not both part of $K_v(1, 4)$. Note that we may have $x_i = u$. If x_i does not belong to $K_v(1, 4)$, then we swap the component $C_{1,4} = K_{x_i}(1, 4)$ to obtain a coloring β' X -equivalent to $\beta_{\mathcal{Y}}$ where the fan $X_v(2)$ is now a path. By Lemma 37; this is a contradiction.

So $x_i \in K_v(1, 4)$, and thus $x_s \notin K_v(1, 4)$. Similarly to the previous case, we now swap the component $K_{x_s}(1, 4)$ and obtain a coloring X -equivalent to $\beta_{\mathcal{Y}}$ where $X_v(2)$ is a path. By Lemma 37 this is again a contradiction.

Case 38.2 ($4 \in \beta(\mathcal{V})$).

In this case, we have that $x_i \in V(\mathcal{V})$. Since \mathcal{Y} is a (\mathcal{V}, u) -independent path, it does not contain any vertex missing the color 4 so β is $\{x_s\}$ -identical to $\beta_{\mathcal{Y}}$, and this vertex is still missing the color 4 in the coloring β . Since \mathcal{V} is a minimum cycle in the coloring β , by Lemma 15 it is saturated, so $x_i \in K_v(1, 4)$, and thus $x_s \notin K_v(1, 4)$. We now swap the component $C_{1,4} = K_{x_s}(1, 4)$, and denote by β' the coloring obtained after the swap. The fan \mathcal{Y} was a (\mathcal{V}, u) -independent path in the coloring β , so the coloring β' is \mathcal{Y} -equivalent to β , and \mathcal{Y} is still a (\mathcal{V}, u) -independent path in this coloring. We now invert \mathcal{Y} and obtain a coloring $\beta'_{\mathcal{Y}}$ which is $(X_v^{\beta_{\mathcal{Y}}}(2) \setminus \{x_s\})$ -equivalent to the coloring $\beta_{\mathcal{Y}}$. So now, in the coloring $\beta'_{\mathcal{Y}}$, the fan $X_v(2)$ is a path, by Lemma 37 this is a contradiction. □

Lemma 39. *Let $\mathcal{V} = (vv_1, \dots, vv_k)$ a minimum cycle in a coloring β , $u = v_j$ and $u' = v_{j+1}$ two consecutive vertices of \mathcal{V} , $\mathcal{Y} = (uy_1, \dots, uy_r)$ a (\mathcal{V}, u) -independent path, and x the extremity of $K_{y_r}(m(u), m(v))$ which is not y_r . Then the fan $X_v(\beta(uy_1))$ is a comet containing x which is missing the color $m(u)$.*

Proof. We assume that \mathcal{Y} is of minimum size such that $\mathcal{X} = X_v(\beta(uy_1))$ is not a comet containing x missing the color $m(u)$. Without loss of generality, we assume that $m(v) = 1$, $m(u) = \beta(vu') = 2$, $\beta(uv) = m^\beta(u'') = 3$, and $\beta(uy_1) = 4$.

If $|\mathcal{Y}| = 1$, then \mathcal{Y} consists of a single edge. We swap this edge, and denote by β' the coloring obtained after the swap. In the coloring β' , by Lemma 38, the fan $X_v(2)$ is a cycle. In this coloring, the vertex u is missing the color 4, so $4 \in \beta'(X_v(2))$. Let $\mathcal{X}' = (vx_1, \dots, vx_s)$ be the maximal subfan of $X_v(2)$ starting with an edge colored 4, and not containing any edge of \mathcal{V} . Note that $E(\mathcal{X}') = E(\mathcal{X})$ and $V(\mathcal{X}') = V(\mathcal{X})$. Note also that we have $m^{\beta'}(x_s) = 2$. The subfan \mathcal{X}' does not contain any edge of \mathcal{V} , thus it does not contain the vertex u , and so it does not contain any vertex missing the color 4. So the coloring β is \mathcal{X} -equivalent to the coloring β' , and thus in the coloring β , the fan $X_v(4) = (vx_1, \dots, vx_s, vu', \dots, vu)$ is a comet where

x_s and u are both missing the color 2. In the coloring β , the cycle \mathcal{V} is a minimum cycle, so it is saturated by Lemma 15, and thus $u \in K_v(1, 2)$ and $x_s \notin K_v(1, 2)$. If x_s is not in $K_{y_r}(1, 2)$, then we swap $C_{1,2} = K_{x_s}(1, 2)$, to obtain a coloring β'' which is $((\mathcal{X} \cup \mathcal{V} \cup \mathcal{Y}) \setminus \{x_s\})$ -equivalent to β . We now invert the path \mathcal{Y} , and obtain a coloring where $X_v(2)$ is a path, by Lemma 38 this is a contradiction.

So $|\mathcal{Y}| > 1$. The size of \mathcal{Y} is minimum, so for any subpath $X_u(\beta(uy_j))$ of \mathcal{Y} with $j > 1$, the fan $X_v(\beta(uy_j))$ is a comet containing x . So the fan $X_v(4)$ does not contain any vertex missing a color in $\beta(\mathcal{Y})$, otherwise it would be a comet containing x . Hence the coloring $\beta_{\mathcal{Y}} = \mathcal{Y}^{-1}(\beta)$ is \mathcal{X} -equivalent to β . In the coloring $\beta_{\mathcal{Y}}$, the fan $X_v(2)$ is a cycle by Lemma 38. Moreover, it contains the fan \mathcal{X} since u is missing the color 4 in the coloring $\beta_{\mathcal{Y}}$. Therefore, in the coloring β , the fan $\mathcal{X} = (vx_1, \dots, vx_s, vu', \dots, vu)$ is a comet containing \mathcal{V} where x_s and u are both missing the color 2. Similarly to the previous case, since \mathcal{V} is a minimum, it is saturated by Lemma 15, so $u \in K_v(1, 2)$, and thus $x_s \notin K_v(1, 2)$. If $x_s \notin K_v(1, 2)$, then we swap $C_{1,2} = K_{x_s}(1, 2)$, and obtain a coloring where $X_v(4)$ is a path. This coloring is \mathcal{Y} -equivalent to β , and thus if we invert \mathcal{Y} we obtain a coloring where $X_v(2)$ is a path, a contradiction by Lemma 38. \square

In the following section we prove the property $P(0)$.

4.2 Proof of $P(0)$

In this section we prove the following lemma.

Lemma 40. *The property $P(0)$ is true.*

To prove that $P(0)$ is true, we need the following lemma.

Lemma 41. *Let $\mathcal{V} = (vv_1, \dots, vv_k)$ be a minimum cycle in a coloring β , $u = v_j$ and $u' = v_{j'}$ two vertices of \mathcal{V} . If $uu' \in E(G) \cap$, and $\beta(uu') \neq m(v)$, then the fan $\mathcal{X} = X_v(\beta(uu'))$ is a saturated cycle.*

The following lemma is the first step of the proof of Lemma 41.

Lemma 42. *Let $\mathcal{V} = (vv_1, \dots, vv_k)$ be a minimum cycle in a coloring β , $u = v_j$ and $u' = v_{j'}$ two vertices of \mathcal{V} . If $uu' \in E(G)$ and $\beta(uu') \neq m(v)$, then the fan $\mathcal{X} = X_v(\beta(uu'))$ is not a path.*

Proof. Otherwise, assume that \mathcal{X} is a path. Without loss of generality, we assume that the vertices v, u and u' are respectively missing the colors 1, 2 and 3. Since $\beta(uu') \notin \{1, 2, 3\}$, we also assume that $\beta(uu') = 4$. Finally, we assume that \mathcal{X} is of length one, indeed if the length of \mathcal{X} is more than one, we invert it until we reach a coloring β' \mathcal{V} -equivalent to β where it has length one without changing the color of uu' .

We denote by x the only vertex of \mathcal{X} , and by β_2 the coloring obtained after swapping the edge vx . The coloring β_2 is \mathcal{V} -equivalent to β' , so \mathcal{V} is the same minimum cycle in the coloring β_2 by Observation 20. By Lemma 24, the fans $\mathcal{U} = X_u^{\beta'}(3) = (uu_1, \dots, uu_l)$ and $\mathcal{U}' = X_{u'}^{\beta_2}(3)$ are both cycles and uu' is the last edge of both of these cycles; we denote by w the vertex missing 4 in \mathcal{U} . Note that since $\beta(uu') = 4$, the vertex w is the vertex u_{l-1} , and $\mathcal{U} = (uu_1, \dots, uw, uu')$. We first remark that $4 \notin \beta(\mathcal{V})$, otherwise the fan $E(\mathcal{X}) = E(\mathcal{V})$, and the fan \mathcal{X} is a cycle and thus is not a path, as desired.

We first prove some basic properties on the fan \mathcal{U} .

Proposition 43. *The fan \mathcal{U} contains an edge colored 1, and there is no edge colored with a color in $\beta(\mathcal{V})$ between the edge colored 1 and the edge colored 4 in \mathcal{U} .*

Proof. We first prove that there is an edge colored 1 in the fan \mathcal{U} . Assume that \mathcal{U} does not contain any edge colored 1 in the coloring β' . Since the fan \mathcal{U} is a cycle, it means that it does not contain any vertex missing the color 1, and in particular it does not contain v . So the coloring β_2 is also \mathcal{U} -equivalent to the coloring β' . Therefore, $\mathcal{U} = \mathcal{U}'$ and \mathcal{U}' contains the vertex w that is still missing the color 4. The fan \mathcal{U}' is thus not entangled with \mathcal{V} , by Lemma 24 we have a contradiction.

So the fan \mathcal{U} contains an edge colored 1. Since by Lemma 24, the fan \mathcal{U} is a cycle entangled with \mathcal{V} , it contains the vertex v which is missing the color 1, and thus it contains the edge uv , and also the edge vv_{j-1} (recall that vv_{j-1} is the edge just before $vu = vv_j$ in the sequence \mathcal{V}). Note that the vertex u' and v_{j-1} may be the same vertex.

We now prove that, in the sequence \mathcal{U} , there is no edge colored with a color in $\beta(\mathcal{V})$ between the edge colored 1 and the edge colored 4. Assume on the contrary that there exists such an edge uu_t colored with a color $c \in \beta(\mathcal{V})$. Similarly to the previous proof, this means that in the coloring β_2 , the fan $X_u(c)$ is the sequence $(uu_t, uu_{t+1}, \dots, uw, uu')$ with $m(w) = m(v) = 4$. So this fan is not entangled with \mathcal{V} and by Lemma 24 we again get a contradiction. \square

Let y_1 be the neighbor of u connected to u by the edge colored 1, and y_2 the vertex just after y_1 in the sequence \mathcal{U} . Note that since $\beta'(uu') \neq 1$, the vertex y_1 is different from the vertex u' but may be equal to the vertex w . In this case, the vertices y_2 and u' are the same vertex.

Proposition 44. *The edge uy_1 belongs to the component $K_v(1, \beta(vu'))$.*

Proof. Assume that uy_1 does not belong to $K_v(1, \beta(vu'))$. If the edge vu' is just after the edge vu in the fan \mathcal{V} (i.e. if $j' = j + 1$), then it means that $\beta(vu') = 3$, and since $\beta(uy_1) = 1$, we have that the vertex u does not belong to the component $K_v(1, 3)$. So the fan \mathcal{V} is not saturated, by Lemma 15 we have a contradiction. So the edge vu' is not the edge just after the edge vu in the fan \mathcal{V} , and without loss of generality, we assume that $\beta(vu') = 5$.

Let $C_{1,5} = K_{y_1}(1, 5)$, we first prove that the vertex x belongs to this component. Since the vertex y_1 is not in $K_v(1, 5)$, we have that $K_v(1, 5) \neq C_{1,5}$. The fan \mathcal{V} is a minimum cycle, it is saturated by Lemma 15, so after swapping $C_{1,5}$, we obtain a coloring β'' \mathcal{V} -equivalent to β' . By Observation 20 the cycle \mathcal{V} is the same minimum cycle in this coloring. In the coloring β'' the edge uy_1 is now colored 5, and the fan $X_u(5)$ still contains the vertex w missing the color 4. Moreover, the vertex x is still missing the color 1, so we swap the edge vu to obtain a coloring \mathcal{V} -equivalent to β'' where $X_u(5)$ contains the vertex w which is missing the color $m(v) = 4$. So $X_u(5)$ is not entangled with \mathcal{V} , and by Lemma 24 we have a contradiction.

Therefore, the vertex x belongs to the component $C_{1,5}$. We first swap the component $C_{1,5}$ and obtain a coloring β'' \mathcal{V} -equivalent to β' . In the coloring β'' , the fan $X_u(5)$ now contains the vertex w that is still missing 4. So the vertex $X_v(5)$ contains the vertex u' and we have $X_v(5) = X_v(3)$.

Since the cycle \mathcal{V} is minimum, by Observation 21, it is tight. In the coloring β'' , the vertex x is now missing the color 5, we now apply a sequence a of Kempe swaps of the form $K_x(m(v_{t-1}), m(v_t))$ for $t \in (j' - 1, j' - 2, \dots, j + 1)$ to obtain a coloring β_3 where $m(x) = m(v_{j-1}) = 2$. Note that each of these swaps is \mathcal{V} -stable since after each swap the fan \mathcal{V} is a minimum cycle and thus is tight. Moreover, since no edge of \mathcal{U} between uy_2 and uu' is colored with a color in $\beta'(\mathcal{V})$, the coloring β_3 is $\mathcal{U}_{[y_2, w]}$ -equivalent to β'' .

Hence we have $X_u(\beta_3(uy_1))_{\leq w} = (uy_1, uy_2, \dots, uw)$. The edge uy_1 may have been recolored during the sequence of swaps, but in the coloring β_3 , uy_1 is guaranteed to be colored with a color in $\beta_3(\mathcal{V})$. In the coloring β_3 , the vertices x and u are missing the same color 2 and the vertex v is still missing the color 1. the cycle \mathcal{V} is minimum, so it is saturated by Lemma 15, and therefore $x \notin K_v(1, 2)$.

We swap the component $C_{1,2} = K_x(1, 2)$ to obtain a coloring where v and x are missing the same color 1 and where the edge vx is colored 4. We now swap the edge vx , and denote by β_4 the coloring obtained after these swaps. The coloring β_4 is \mathcal{V} -equivalent to β_3 , and is also $X_u(\beta_4(uy_1))_{[y_2, w]}$ -equivalent to the coloring β_3 . The vertices v and w are missing the same color 4, so $X_u(\beta_4(uy_1))$ and \mathcal{V} are not entangled in this coloring, and thus by Lemma 24 we have a contradiction. \square

Proposition 45. *In the coloring β_2 , the vertex x belongs to $K_x(2, 4)$.*

Proof. Otherwise, assume that it is not the case. In the coloring β_2 , the fan \mathcal{V} is a minimum cycle, so it is saturated by Lemma 15. Therefore the vertex u belongs to $K_v(2, 4)$ and the vertex w does not belong to this component. By Proposition 43 $X_u(1)$ contains the vertex w . We swap the component $C_{2,4} = K_w(2, 4)$, and obtain a coloring β'' \mathcal{V} -equivalent to β' . By Observation 20, the cycle \mathcal{V} is still the same minimum cycle in the coloring β'' , and now the vertex w is missing the color 2. The coloring β'' is also $X_u(1)_{< w}$ -equivalent to the coloring β' , so where $X_u(1)$ still contains the vertex w . The vertex x is still missing the color 4, so we swap the edge vu to obtain a coloring β_3 where $X_u(1)$ contains the vertex w missing the color 2, and thus $X_u(1)$ is a path. By Lemma 24 we have a contradiction. \square

We are now ready to prove the lemma. We need to distinguish whether or not $j = j' + 1$.

Case 45.1 ($j = j' + 1$).

In this case, we have $\beta'(vu) = m^{\beta'}(u') = 3$. In the coloring β' , the fan \mathcal{V} is saturated, so $u' \in K_v(1, 3)$ and thus $uy_1 \in K_{u'}(1, 3)$. Let $C_{1,3} = K_{u'}(1, 3) \setminus \{uy_1, vu\}$, $C_{1,3}$ is a $(1, 3)$ -bichromatic path between u' and y_1 . In the coloring β_2 , we consider the component $C_{2,4} = K_w(2, 4)$; this component contains the vertex x by Proposition 45. After swapping $C_{2,4}$ we obtain a coloring β_3 \mathcal{V} -equivalent to \mathcal{V} where the fan $X_u(1)$ is a path. By Observation 20 the fan \mathcal{V} is still the same minimum cycle in the coloring β_3 . Moreover, the coloring β_3 is $C_{1,3}$ -equivalent to the coloring β_2 , and thus $C_{1,3}$ -equivalent to the coloring β' , so $C_{1,3}$ is still a $(1, 3)$ -bichromatic path between u' and y_1 .

By Proposition 43 there is no edge in $E(X_u(1))$ colored with a color in $\beta_4(\mathcal{V})$, so we invert $X_u(1)$ to obtain a coloring β_5 that is $(C_{1,3} \cup (\mathcal{V} \setminus \{u\}))$ -equivalent to β_4 . In the coloring β_4 , the vertex y_1 is missing the color 1, so $K_{u'}(1, 3) = C_{1,3}$, and we swap this component; we denote by β_5 the coloring obtained after the swap.

In the coloring β_5 , the vertices u and u' are both missing the color 1, so we swap the edge uu' to obtain a coloring where u and u' are missing the color 4. In the coloring β_5 , the fan $X_u(2)$ is now a path that we invert to obtain a coloring β_6 . In the coloring β_6 , the edge uw is colored 2, and the vertex u is now missing the color 4, so $K_u(2, 4) = C_{2,4} \cup \{uw\}$, and we swap back this component, we denote by β_7 the coloring obtained after this swap. Note that since $|\{1, 2, 3, 4\}| = 4$, we can swap back $C_{2,4}$ before $C_{1,3}$.

In the coloring β_7 , the vertices u and v are both missing the color 2, and the edge vu is colored 3, so we swap the edge vu to obtain a coloring where u and v are both missing the color 4. In the coloring obtained after the swap, the vertices u and y_1 are both missing the color 3, so the fan $X_u(4)$ is now a path that we invert. We denote by β_8 the coloring obtained after the swap.

In the coloring β_8 , the edge uu' is colored 1, and the edge uy_1 is colored 3, so $K_u(1, 3) = C_{1,3} \cup \{uu', uy_1\}$ and this component is a $(1, 3)$ -bichromatic cycle that we swap. In the coloring obtained after the swap, the component $K_v(3, 4) = \{uu', u'\}$, and it suffices to swap this component to obtain exactly $\mathcal{V}^{-1}(\beta')$. Since \mathcal{V} is a minimum cycle, this is a contradiction.

So $j \neq j + 1$, and since the role of u and u' is symmetric, we also have that $j' \neq j + 1$. Therefore, in the cycle \mathcal{V} , there exists a vertex v_{j+1} and v_{j-1} such that $|\{u, u', v_{j-1}, v_{j'+1}\}| = 4$. Without loss of generality, we assume that $\beta'(vu') = m^{\beta'}(v_{j'+1}) = 5$, and that $\beta'(vu) = m^{\beta'}(v_{j-1}) = 6$.

Case 45.2 ($j \neq j' + 1$).

For this case, we need to distinguish the cases based on the shape of $C_{1,5} = K_{uy_1}(1, 5)$. Since \mathcal{V} is saturated in the coloring β' , by Proposition 44, $C_{1,5}$ also contains v and $v_{j'-1}$, and therefore this component is a $(1, 5)$ -path in this coloring. Moreover, the fan \mathcal{V} is tight by Observation 21, so $K_{v_{j'-1}}(2, 6)$ contains vv_{j+1} , and vu . Let $C_{2,6} = K_{v_{j'-1}}(2, 6) \setminus \{vu, vv_{j+1}\}$. The path $C_{2,6}$ is a $(2, 6)$ -bichromatic path between v_{j+1} and v_{j-1} .

There are two cases, in the coloring β' , either $C_{1,5}$ is such that u is between $v_{j'-1}$ and y_1 , or y_1 is between $v_{j'-1}$ and u . We start both cases by swapping $C_{2,4} = K_w(2, 4)$ in the coloring β_2 , by Proposition 45 the vertex w belongs to this component, and after the swap we have $m(w) = m(x) = m(u) = 2$. By Proposition 43 $X_u(1)$ is a path that we invert to obtain a coloring β_3 ($\{uu'\} \cup (\mathcal{V} \setminus \{u\})$)-equivalent to β_2 .

In the coloring β_3 , depending on the shape of $C_{1,5}$, either u is in $C = K_{v_{j'-1}}(1, 5)$, or y_1 belongs to this component. We now have to distinguish the cases. Both cases are pretty similar, their proofs rely on the same principle: apply Kempe swaps to reach a coloring where the edges of $E(\mathcal{V}) \cup \{vw'\}$ induce two fans that are cycles smaller than \mathcal{V} (and that are invertible since \mathcal{V} is minimum).

Subcase 45.2.1 (u belongs to C).

In this case, $C = K_{v_{j'-1}}(1, 5)$ is a $(1, 5)$ -bichromatic path between $v_{j'-1}$ and u and there is a $(1, 5)$ -bichromatic path C' between y_1 and u' .

From the coloring β_3 , we swap the component C to obtain a coloring β_4 where the fan $X_v(5) = (vu', vv_{j'+1}, \dots, vv_{j-1}, vu)$ is a cycle strictly smaller than \mathcal{V} , so since \mathcal{V} is minimum, this cycle is invertible. Moreover, the fan $X_v(1) = (vx, vv_{j+1}, \dots, vv_{j'-1})$ is also a cycle strictly smaller than \mathcal{V} , and so it is also invertible.

After inverting these two cycles, we obtain a coloring where the component $K_{v_{j'-1}}(1, 5) = C \cup \{vv_{j-1}, vu\}$ is $(1, 5)$ -bichromatic cycle that we swap back; we denote by β_5 the coloring obtained after the swap. Now the component $K_{y_1}(1, 5)$ is exactly C' and we swap it to obtain a coloring β_6 .

In the coloring β_6 , the fan $X_v(3) = (vu', vu, vv_{j-1}, \dots, vv_{j'+1})$ is now a cycle strictly smaller than \mathcal{V} , so we invert it. In the coloring obtained after this inversion, the $(2, 6)$ -bichromatic path $C_{2,6}$ is still a path between v_{j+1} and v_{j-1} , but now v_{j-1} is missing the color 6, and v_{j+1} is missing the color 2. So $K_{v_{j+1}}(2, 6) = C_{2,6}$, and we swap this component. Let β_7 be the coloring obtained after the swap.

In the coloring β_7 , the fan $X_v(1) = (vu', vv_{j'+1}, \dots, vv_{j-1}, vx)$ is now a cycle strictly smaller than \mathcal{V} and we invert it. In the coloring obtained after the inversion, $K_{y_1}(1, 5)$ is now exactly C' , and we swap back this component and denote by β_8 the coloring obtained after the swap.

In the coloring β_8 , the vertices y_1 and u are both missing the color 1, so the fan $X_u(2)$ is now a path that we invert to obtain a coloring where u and w are missing the color 2. In the coloring obtained after the inversion, the component $K_{v_{j+1}}(2, 6)$ is exactly $C_{2,6} \cup \{vv_{j-1}, vu\}$ and we swap back this component. In the coloring obtained after the swap, the component $K_w(2, 4)$

is exactly $C_{2,4}$, and thus after swapping back this component, we obtain exactly $\mathcal{V}^{-1}(\beta')$; a contradiction.

Subcase 45.2.2 (y_1 belongs to C).

In this case, $C = K_{v_{j'-1}}(1, 5)$ is a $(1, 5)$ -bichromatic path between $v_{j'-1}$ and y_1 and there is a $(1, 5)$ -bichromatic path C' between u and u' . From the coloring β_3 , we swap the component C to obtain a coloring where $X_v(2) = (vv_{j+1}, \dots, vv_{j'-1}, vx)$ is a cycle strictly smaller than \mathcal{V} , so it is invertible. After inverting it, we obtain a coloring where the component $K_{v_{j-1}}(2, 6)$ is exactly $C_{2,6}$. We swap this component and denote by β_4 the coloring obtained after the swap.

In the coloring β_4 , the fan $X_v(1) = (vv_{j'-1}, \dots, vv_{j+1}, vu)$ is now a cycle strictly smaller than \mathcal{V} , so it is invertible. After inverting it, the component $K_{u'}(1, 5)$ is now exactly $C' \cup \{vu', vu\}$ and so it is a $(1, 5)$ -bichromatic cycle containing vu and vu' . After swapping this component, we obtain a coloring where the fan $X_v(1) = (vu', vv_{j'+1}, \dots, vv_{j-1}, vx)$ is now a cycle strictly smaller than \mathcal{V} , and we invert it. We denote by β_5 the coloring obtained after the inversion.

In the coloring β_5 , the component $K_{v_{j'-1}}(1, 5)$ is exactly C , and we swap back this component. After the swap we obtain a coloring where the fan $X_v(5) = (vu, vv_{j+1}, \dots, vv_{j'-1})$ is now a cycle strictly smaller than \mathcal{V} , and so we invert it and denote by β_6 the coloring obtained after the swap.

In the coloring β_6 , the component $K_{u'}(1, 5)$ is now exactly C' and we swap back this component. After the swap we obtain a coloring where u and y_1 are both missing the color 1, so the fan $X_u(2)$ is now a path that we invert. We denote by β_7 the coloring obtained after the swap.

In the coloring β_7 the component $K_{v_{j-1}}(2, 6)$ is exactly $C_{2,6} \cup \{vv_{j-1}, vu\}$ and we swap it back. After the swap of this component, we obtain a coloring where $K_w(2, 4)$ is exactly $C_{2,4}$. After swapping back this component, we obtain exactly $\mathcal{V}^{-1}(\beta')$. This is a contradiction. \square

From the previous lemma we derive the following corollary.

Corollary 46. *Let $\mathcal{V} = (vv_1, \dots, vv_k)$ be a minimum cycle in a coloring β , $u = v_j$ and $u' = v_{j'}$ two vertices of \mathcal{V} . If $uu' \in E(G)$ and $\beta(uu') = m(v)$, then no fan around v is a path.*

Proof. Assume that there exists a fan \mathcal{X} around v which is a path. It suffices to swap the last edge vx of \mathcal{X} to obtain a coloring β_2 ($\mathcal{V} \cup \{uu'\}$)-equivalent to β such that $X_v(\beta_2(uu')) = \{vx\}$ is now a path (of length one). By Observation 20, the fan \mathcal{V} is a minimum cycle in the coloring β_2 , so by Lemma 42, we get a contradiction. \square

We are now ready to prove Lemma 41

Proof of Lemma 41. Let $\mathcal{V} = (vv_1, \dots, vv_k)$ be a minimum cycle in a coloring β , $u = v_j$ and $u' = v_{j'}$ two vertices of \mathcal{V} . Without loss of generality, we assume that the vertices v , u and u' are respectively missing the colors 1, 2 and 3. By Lemma 24, the fan $\mathcal{U} = X_u(m(u'))$ is a cycle entangled with \mathcal{V} , so the edge uu' is in $E(G)$. Assume the $\beta(uu') \neq 1$.

We first prove that $X_v(\beta(uu'))$ is a saturated cycle. If $\beta(uu') \in \beta(\mathcal{V})$, then $X_v(\beta(uu'))$ is exactly the fan \mathcal{V} . Since \mathcal{V} is minimum, by Lemma 15, it is saturated, so $X_v(\beta(uu'))$ is a saturated cycle as desired.

Hence assume that $\beta(uu') \notin \beta(\mathcal{V})$, and without loss of generality, say $\beta(uu') = 4$. By Lemma 42, then fan $X_v(4)$ is not a path.

We now prove that $X_v(4)$ is not a comet. Suppose that $X_v(4) = (vw_1, \dots, vw_t)$ is a comet. So there exists $i < t$ with $m(w_i) = m(w_t)$, we denote by c this color. If $c \in \beta(\mathcal{V})$, the cycle

\mathcal{V} is a subfan of the fan $X_v(4)$, and thus $w_t = M(\mathcal{V}, c) \in V(\mathcal{V})$ and $w_i \notin V(\mathcal{V})$. Since \mathcal{V} is minimum, it is saturated by Lemma 15, so $w_t \in K_v(1, c)$, and thus $w_i \notin K_v(1, c)$. We now swap the component $K_{w_i}(1, c)$ and obtain a coloring β_2 ($\mathcal{V} \cup \{uu'\}$)-equivalent to β , so the cycle \mathcal{V} is also a minimum cycle in the coloring β_2 by Observation 20. In the coloring β_2 , the fan $X_v(\beta_2(uu')) = X_v(4)$ is a now path, by Lemma 42 this is a contradiction.

So $c \notin \beta(\mathcal{V})$. The vertices w_i and w_t are not both part of $K_v(1, c)$. If w_i is not in $K_v(1, c)$, we swap $K_{w_i}(1, c)$ and obtain a coloring β_2 , ($\mathcal{V} \cup \{uu'\}$)-equivalent to β . So the coloring β_2 , by Observation 20, the fan \mathcal{V} is a minimum cycle. But the fan $X_v(4) = X_v(\beta_2(uu'))$ is now a path, a contradiction by Lemma 42.

So the vertex w_i belongs to the component $K_v(1, c)$ and thus w_t does not belong to this component. We now swap $K_{w_i}(1, c)$ and obtain a coloring β_2 which is ($\mathcal{V} \cup \{uu'\}$)-equivalent to β . So by Observation 20, the fan \mathcal{V} is still the same minimum cycle in β_2 , but the fan $X_v(4) = X_v(\beta_2(uu'))$ is now a path, again a contradiction by Lemma 42.

Therefore the fan $X_v(4)$ is a cycle. We now prove that it is saturated. Note that since $X_v(4)$ is a cycle, $\beta(X_v(4)) \cap \beta(\mathcal{V}) = \{1\}$. Assume that $X_v(4) = (vw_1, \dots, vw_t)$ is not saturated, so there exists i such that $w_i \notin K_v(1, m(w_i))$. We now have to distinguish whether $w_i = w_t$ or not.

Case 46.1 ($w_i \neq w_t$).

This case is similar to the case where $X_v(4)$ is a comet. In this case, the vertex w_i is missing a color which is not in $\{1, 2, 3, 4\}$, and we can assume without loss of generality that $m(w_i) = 5$. Since w_i does not belong to $K_v(1, 5)$, we swap the component $K_{w_i}(1, 5)$ to obtain a coloring β_2 ($\mathcal{V} \cup \{uu'\}$)-equivalent to β . In the coloring β_2 , by Observation 20, the fan \mathcal{V} is the same minimum cycle, but the fan $X_v(4) = (w_1, \dots, w_i)$ is now a path, a contradiction by Lemma 42.

Case 46.2 ($w_i = w_t$).

In this case, w_t does not belong to $K_v(1, 4)$. We first swap the component $C_{1,4} = K_{w_t}(1, 4)$. If $uu' \notin C_{1,4}$, then we obtain a coloring β_2 ($\mathcal{V} \cup \{uu'\}$) equivalent to β . So by Observation 20, the fan \mathcal{V} is a minimum cycle in the coloring β_2 , but now the fan $X_v(4) = X_v(\beta_2(uu')) = (vw_1, \dots, vw_t)$ is now a path; a contradiction by Lemma 42.

So the edge uu' is in $C_{1,4}$. After swapping $C_{1,4}$, we obtain a coloring β_2 (\mathcal{V})-equivalent to β , so \mathcal{V} is still a minimum cycle. But now $\beta_2(uu') = 1$, and $X_v(4)$ is a path, so by Corollary 46, we have a contradiction.

Hence $X_v(4)$ is a saturated cycle as desired. □

The proof of $P(0)$ is a direct consequence of the two previous lemmas.

Proof of Lemma 40. Let \mathcal{V} be a minimum cycle around a vertex v in a coloring β , u and u' two vertices of \mathcal{V} , $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_l)$, assume that $\beta(uu') \neq m(v)$ and let $\mathcal{W} = X_v(\beta(uu_l)) = (vw_1, \dots, vw_s)$. Without loss of generality, we assume that the vertices v , u and u' are respectively missing the colors 1, 2, and 3, and that the edge uu' is colored 4.

We first prove that \mathcal{W} is a saturated cycle containing u_{l-1} . By Lemma 41, the fan \mathcal{W} is a saturated cycle, and thus w_s is missing the color 4. We now prove that the fan \mathcal{W} contains the vertex u_{l-1} .

If $4 \in \beta(\mathcal{V})$, then $\mathcal{W} = \mathcal{V}$, and since \mathcal{U} is entangled with \mathcal{V} by Lemma 24, we have that $u_{l-1} = w_s \in \mathcal{V} = \mathcal{W}$. So the color 4 is not in $\beta(\mathcal{V})$.

Assume that the fan \mathcal{W} does not contain u_{l-1} , so in particular, $u_{l-1} \neq w_s$. The cycle \mathcal{W} is saturated, so $w_s \in K_v(1, 4)$, and thus $u_{l-1} \notin K_v(1, 4)$. By Lemma 49

- $u \in K_{u_{l-1}}(1, 4)$,

- there exists $j \leq l - 1$ such that $\beta(uu_j) = 1$, and
- the subfan $(uu_{j+1}, \dots, uu_{l-1})$ is a (\mathcal{V}, u) -independent subfan.

We now consider the coloring β' obtained from β after swapping the component $C_{1,4} = K_{u_{l-1}}(1, 4)$. Let $\mathcal{X} = (uu_{j+1}, \dots, uu_{l-1})$. The coloring β' is $(\mathcal{V} \cup \mathcal{W} \cup (\mathcal{X} \setminus \{u_{l-1}\}))$ -equivalent to β , so \mathcal{V} is a minimum by Observation 20, and $\mathcal{W} = X_v(4)$ is still a cycle. The vertex v is still missing the color 1, but now the vertex u_{l-1} is missing the color 1, the edge uu_j is colored 4, and the edge uu_l is colored 1. So now $X_u(4) = \mathcal{X}' = (uu_j, \dots, uu_l)$ is a (\mathcal{V}, u) -independent subfan avoiding the vertex v . By Lemma 34 the fan $X_v(4)$ is a path; a contradiction. \square

We now prove some properties of the fans around a vertex of a minimum cycle.

4.3 Fans around the vertices of a minimum cycle

We first prove that some fans around a vertex of a minimum cycle are not paths.

Proposition 47. *Let $\mathcal{V} = (v_1, \dots, v_k)$ be a minimum cycle in a coloring β , $u = v_j$ and $u' = v_{j'}$ two vertices of \mathcal{V} , and $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_l)$, and $w = u_s$ a vertex of \mathcal{U} . Then for any color $c \in \beta(\mathcal{V})$, the fan $\mathcal{W} = X_w(c) = (ww_1, \dots, ww_t)$ is not a path.*

Proof. Otherwise assume that the fan \mathcal{W} is a path. The vertex w is not a vertex of $V(\mathcal{V})$, otherwise since \mathcal{W} is a path, by Lemma 24 we have a contradiction. So the vertex w is not in $V(\mathcal{V})$.

We invert it until we reach a coloring β_2 where $m^{\beta_2}(w) \in \beta(\mathcal{V} \cup \mathcal{U}_{<s})$, we denote by c' this new missing color. Since $c \in \beta(\mathcal{W})$, the color c' is well defined. The coloring β_2 is $(\mathcal{V} \cup \mathcal{U}_{<s})$ -equivalent to β . Thus by Observation 20, the sequence \mathcal{V} is still a minimum cycle in the coloring β_2 . Let $\mathcal{U}' = X_u(m^{\beta_2}(u')) = (uu'_1, \dots, uu'_l)$. Since β_2 is $(\mathcal{U}_{<s})$ -equivalent to β , we have that $\mathcal{U}_{<s} = \mathcal{U}'_{<s}$, so the edge uw is also in $E(\mathcal{U}')$, it is exactly the edge uu'_s . If $c' \in \beta(\mathcal{V})$, then \mathcal{U}' is not entangled with \mathcal{V} in the coloring β_2 , a contradiction by Lemma 24. If $c' \in \beta\mathcal{U}_{<s}$, then \mathcal{U}' is now a comet in the coloring β_2 , again, by Lemma 24 we have a contradiction. \square

Lemma 48. *Let $i \geq 0$, \mathcal{V} be a minimum cycle in a coloring β , u and u' two vertices of \mathcal{V} , $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_l)$, and $c \in \beta(\mathcal{V}) \cup \beta(\mathcal{U}_{<u_i})$. If $u_i \notin V(\mathcal{V}) \cup \{v\}$. Then the fan $\mathcal{X} = X_{u_i}(c) = (u_ix_1, \dots, u_ix_s)$ is not a path.*

Proof. Assume $u_i \notin V(\mathcal{V})$ and that \mathcal{X} is a path. Without loss of generality, we assume that there is no edge in $\mathcal{X}_{[x_2, x_s]}$ colored with a color in $\beta(\mathcal{V}) \cup \beta(\mathcal{U}_{<u_i})$, otherwise, it suffices to consider the subfan of \mathcal{X} starting with this edge, this fan is also a path. We now invert \mathcal{X} and obtain a coloring β' where $m(u_i) = c$. The coloring β' is $(\mathcal{V} \cup \mathcal{U}_{<u_i})$ -equivalent to β . So by Observation 20, the fan \mathcal{V} is a minimum cycle in the coloring β' . If $c \in \beta(\mathcal{V})$, now the fan $X_u(m(u'))$ contains the vertex u_i which is missing the color $c \in \beta(\mathcal{V})$, so $X_u(m(u'))$ is not entangled with \mathcal{V} . If $m(u_i) \in \beta(\mathcal{U}_{<u_i})$, let $u'' = M(\mathcal{U}_{<u_i}, c)$. Then $X_u(m(u'))$ is now a comet since it contains the vertices u_i and u'' both missing the color c . In both cases, by Lemma 24 we have a contradiction. \square

We now prove a sufficient condition for a fan around a vertex of a minimum to contain an edge colored with the color missing at the central vertex of the minimum cycle.

Lemma 49. *Let \mathcal{V} be a minimum cycle in a coloring β , u and u' two vertices of \mathcal{V} , $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_l)$ and $i \leq l$. If $\beta(uu_i) \neq m(v)$, $m(u_i) \notin \beta(\mathcal{V})$ and $u_i \notin K_v(m(v), m(u_i))$, then:*

- $u \in K_{u_i}(m(v), m(u_i))$,
- there exists $j < i$ such that $\beta(uu_j) = m(v)$, and
- the subfan (uu_{j+1}, \dots, uu_i) is a (\mathcal{V}, u) -independent subfan.

Proof. Let \mathcal{V} , u , u' and \mathcal{U} be as in the lemma. Without loss of generality, we assume that the vertices v , u , and u' are respectively missing the colors 1, 2 and 3. Assume that $u_i \notin K_v(1, m(u_i))$. Since the cycle \mathcal{V} is minimum, by Lemma 15 it is saturated, so for any $u'' \in V(\mathcal{V})$, $u'' \in K_v(m(v), m(u''))$, thus $u_i \notin V(\mathcal{V})$, so without loss of generality, we may assume that u_i is missing the color 4. We first consider the component $C_{1,4} = K_{u_i}(1, 4)$. In the coloring β , by Lemma 24, \mathcal{U} is a cycle entangled with \mathcal{V} , so it does not contain any other vertex missing 4. Since $v \notin C_{1,4}$, then $V(C_{1,4}) \cap V(\mathcal{U}) = \{u_i\}$. After swapping $C_{1,4}$, we obtain a coloring β' ($V(\mathcal{U}) \setminus \{u_i\}$)-identical to β where u_i is now missing the color 1. Note that the coloring β' is also \mathcal{V} -equivalent to β , and thus \mathcal{V} is still a minimum cycle in β' . Moreover the vertex v is still missing the color 1 in β' .

We first prove that the vertex u belongs to $C_{1,4}$ and that there is an edge colored 1 in $\{uu_1, \dots, uu_{i-1}\}$. If $u \notin C_{1,4}$, or if there is no edge colored 1 in $\{uu_1, \dots, uu_{i-1}\}$, then the coloring β' is also $(E(\mathcal{U}_{\{uu_1, uu_i\}}))$ -identical to β , and so $X_u(3)$ now contains the vertex u_i which is missing the color 1, so $X_u(3)$ is not a cycle entangled with \mathcal{V} . Since the cycle \mathcal{V} is minimum, we have a contradiction by Lemma 24. So $u \in C_{1,4}$ and there is an edge uu_j with $j < i$ colored 1.

We now prove that (uu_{j+1}, \dots, uu_i) is a (\mathcal{V}, u) -independent subfan. Note that we have $j + 1 = i$ (i.e. the subfan is of length 1). Since β' is $(V(\mathcal{U}) \setminus \{u_i\})$ -identical to β , the sequence (uu_{j+1}, \dots, uu_i) is a subfan. Assume that there exists $s \in \{j + 1, \dots, i\}$ such that $\beta(uu_s) \in \beta(\mathcal{V})$. Then, in the coloring β' , $X_u(\beta(uu_s))$ contains the vertex u_i that is missing the color 1, thus it is not a cycle entangled with \mathcal{V} , by Lemma 24, this is a contradiction. \square

Lemma 50. *Let \mathcal{V} be a minimum cycle in a coloring β , u and u' two vertices of \mathcal{V} , $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_l)$ and $i \leq l$ such that $m(u_i) \notin \beta(\mathcal{V})$. Let β' be a coloring obtained from β by swapping a $(m(v), c)$ -component C that does not contain v for some color $c \notin (\beta(\mathcal{U}_{<u_i}) \cup \{m(v)\})$. If there exists a coloring β'' such that:*

- β'' is $(\mathcal{V} \cup \mathcal{U}_{<u_i})$ -equivalent to β' , and
- $m^{\beta''}(u_i) \in \beta''(\mathcal{V}) \cup \beta''(\mathcal{U}_{<u_i})$.

Then

- $u \in C$,
- there exists $j < i$ such that $\beta(uu_j) = m(v)$, and
- the subfan (uu_{j+1}, \dots, uu_i) is a (\mathcal{V}, u) -independent subfan in β .

Proof. Let \mathcal{V} , \mathcal{U} , u , u' , β' , and c be as in the lemma. Without loss of generality, we assume that the vertices v , u , and u' are respectively missing the colors 1, 2 and 3. Assume that there exists such a coloring β'' . Note that since $m(u_i) \notin \beta(\mathcal{V})$, the vertex u_i is not in \mathcal{V} . The cycle \mathcal{V} is a minimum cycle in β , so it is saturated by Lemma 15. Therefore, if $c \in \beta(\mathcal{V})$, then $M(\mathcal{V}, c) \in K_v(1, c)$, and thus $M(\mathcal{V}, c) \notin C$. So β' is $V(\mathcal{V})$ -equivalent to β . Moreover, $v \notin C$ so β' is also $(E(\mathcal{V}) \cup \{v\})$ -equivalent to β . Therefore, the coloring β' is $(\mathcal{V} \cup \{v\})$ -equivalent to β .

We first prove that the vertex u belongs to C and that there exists an edge colored 1 in $\mathcal{U}_{<u_i}$. Assume that u does not belong to C , or that there is no edge colored 1 in $\mathcal{U}_{<u_i}$ in β . We show that β'' is $(\mathcal{V} \cup \mathcal{U}_{<u_i})$ -equivalent to β . To prove it, it suffices to prove that β' is $\mathcal{U}_{<u_i}$ -equivalent to β . The swap between β and β' only changes the colors of edges colored 1 or c . Since $\{1, c\} \cap \beta(\mathcal{U}_{<u_i}) = \emptyset$ this means that the coloring β' is $\mathcal{U}_{<u_i}$ -equivalent to β . Since β' is also $(\mathcal{V} \cup \{v\})$ -equivalent to β , in total it is $(\mathcal{V} \cup \mathcal{U}_{<u_i})$ -equivalent to β . Note that the missing color of v may be different in β' and β'' . Since β'' is $(\mathcal{V} \cup \mathcal{U}_{<u_i})$ -equivalent to β' , the coloring β'' is $(\mathcal{V} \cup \mathcal{U}_{<u_i})$ -equivalent to β . Note that the missing color of v may be different in β' and β'' . Hence, in the coloring β'' , by Observation 20, the cycle \mathcal{V} is a minimum cycle and we have that $X_u(m(u'))$ now contains the vertex u_i which is missing a color in $(\beta''(\mathcal{V}) \cup \beta''(\mathcal{U}_{<u_i}))$. Let c' be this color. Since the cycle \mathcal{V} is minimum in β'' , by Lemma 24, $X_u(m(u'))$ is a cycle entangled with \mathcal{V} . If $c' \in \beta''(\mathcal{V})$, then $X_u(m(u'))$ is not entangled with \mathcal{V} , and if $c' \in \beta''(\mathcal{U}_{<u_i})$ then $X_u(m(u'))$ is a comet. In both cases, we have a contradiction. So $u \in C$, and there exists $j < i$ such that $\beta(uu_j) = 1$.

We now prove that the subfan $\mathcal{X} = (uu_{j+1}, \dots, uu_i)$ is a (\mathcal{V}, u) -independent subfan in β . Note that we have $j+1 = i$ (i.e. the subfan is of length 1). If \mathcal{X} is not a (\mathcal{V}, u) -independent subfan, then there exists $s \in \{j+1, \dots, i\}$ such that $c' \in \beta(uu_s) \in \beta(\mathcal{V})$. Recall that the coloring β' is $(\mathcal{V} \cup \{v\})$ -equivalent to β , and thus that β'' is also $(\mathcal{V} \cup \{v\})$ -equivalent to β . In the coloring β' , the edge uu_j is now colored c , and this is the only edge in $E(\mathcal{U}_{<u_i})$ that has been recolored during the swap of C . Moreover, the cycle \mathcal{V} is minimum in β , and thus by Lemma 24, then fan \mathcal{U} is a cycle, and does not contain any vertex missing the color 1 in $V(\mathcal{U}_{[u_j, u_i]})$. Since $c \notin \beta(\mathcal{U}_{<u_i})$, the coloring β' is also $V(\mathcal{U}_{[u_j, u_i]})$ -equivalent to the coloring β , and so it is \mathcal{X} -equivalent to β . The coloring β'' is $\mathcal{U}_{<u_i}$ -equivalent to β' , so it is $(\mathcal{X} \setminus \{u_i\})$ -equivalent to β and thus the fan $X_u(c')$ now starts with the edge uu_s and contains the vertex $u_i \notin V(\mathcal{V})$ which is missing a color in $\beta(\mathcal{V}) = \beta''(\mathcal{V})$. Since \mathcal{V} is also a minimum cycle in β'' , by Lemma 24, $X_v(c')$ is a cycle entangled with \mathcal{V} ; this is a contradiction. \square

In the following section we prove some properties that are guaranteed if the property P is true up to some i .

4.4 Properties guaranteed by $P(i)$

The following lemma guarantees that the last vertices of two cycles will be the same.

Lemma 51. *Let $i \geq 0$, \mathcal{V} be a minimum cycle around a vertex v in a coloring β , u and u'' two vertices of \mathcal{V} , $\mathcal{U} = X_u^\beta(m^\beta(u'')) = (uu_1, \dots, uu_i)$, β' a coloring $(\mathcal{V} \cup \mathcal{U}_{[u_{i-(i-1)}, u_i]}) \cup \bigcup_{j \in [0, i-1]} X_v(\beta(uu_{i-j}))$ -equivalent to β and $\mathcal{U}' = X_{u'}^{\beta'}(m^{\beta'}(u'')) = (uu'_1, \dots, uu'_s)$. If*

- for any $j < i$ $P(j)$ is true, and
- $\{m^\beta(v), m^{\beta'}(v)\} \cap \beta(\mathcal{U}_{[u_{i-(i-1)}, u_i]}) = \emptyset$

then for any $t \leq i$, $u_{i-t} = u'_{s-t}$.

Proof. Let i , \mathcal{V} , \mathcal{U} , \mathcal{U}' , β , β' , v , u , and u'' be as in the lemma. Assume that $P(j)$ is true for all $j < i$, that $\{m^\beta(v), m^{\beta'}(v)\} \cap \beta(\mathcal{U}_{[u_{i-(i-1)}, u_i]}) = \emptyset$ and that there exists $t \leq i$ such that $u_{i-t} \neq u'_{s-t}$, without loss of generality, we may assume that such a t is minimum. The cycle \mathcal{V} is a minimum cycle in β , and β' is \mathcal{V} -equivalent to β , so by Observation 20, the cycle \mathcal{V} is also a minimum cycle in the coloring β' . Therefore by Lemma 24, the fans \mathcal{U} and \mathcal{U}' are both cycles entangled with \mathcal{V} respectively in β and β' . Note that since $m^\beta(v) \notin \beta(\mathcal{U}_{[u_{i-(i-1)}, u_i]})$,

and that $P(j)$ is true for all $j < i$, for all $j < i$, the fan $X_v(\beta(uu_{l-j}))$ is a cycle containing u_{l-j-1} . Moreover, this also means that no vertex in $V(\mathcal{U}_{[u_{l-i}, u_l]})$ is missing the color $m(v)$, and thus none of them is v . Therefore the vertex v may be missing a different color in β and in β' . Note also that, in the coloring β , the edge uu_{l-i} may be colored $m^\beta(v)$ or $m^{\beta'}(v)$.

We first show that $t \neq 0$. Since the fans \mathcal{U} and \mathcal{U}' are both cycles we have $m^\beta(u_l) = m^{\beta'}(u_s) = m^\beta(u'')$, and moreover, \mathcal{U} and \mathcal{U}' are entangled with \mathcal{V} so $u_l = u'' = u'_s$, and thus $t \neq 0$.

Since t is minimum, $u_{l-(t-1)} = u'_{s-(t-1)}$. Moreover, β' is $\mathcal{U}_{[u_{l-(i-1)}, u_l]}$ -equivalent to β , so in particular $\beta(uu_{l-(t-1)}) = \beta'(uu_{l-(t-1)}) = \beta'(uu'_{s-(t-1)})$, without loss of generality, we assume that this color is 1. This means that both the vertices u_{l-t} and u'_{s-t} are missing the color 1. To reach a contradiction we show that both these vertices belong to a same cycle. Since $1 \neq m^\beta(v)$ and $P(t-1)$ is true, then $X^\beta(1)$ is a cycle containing u_{l-t} . Similarly $1 \neq m^{\beta'}(v)$ and $P(t-1)$ is true so $X_v^{\beta'}(1)$ is a cycle containing u'_{s-t} . However, the coloring β' is $(\bigcup_{j \in [0, i-1]} X_v(\beta(uu_{l-j})))$ -

equivalent to the coloring β , so in particular $X_v^{\beta'}(1) = X_v^\beta(1)$; we denote by \mathcal{X} this fan. The fan \mathcal{X} is a cycle and contains two vertices u_{l-t} and u_{s-t} that are both missing the color 1, this is a contradiction. \square

Now we prove that we can guarantee that there is no path around the central vertex of a minimum cycle

Lemma 52. *Let $i \geq 0$, \mathcal{V} be a minimum cycle around a vertex v in a coloring β , u and u'' two vertices of \mathcal{V} , $\mathcal{U} = X_u^\beta(m(u'')) = (uu_1, \dots, uu_i)$, and $\mathcal{X} = (vx_1, \dots, vx_s)$ a fan around v . If*

- for any $j < i$ $P(j)$ is true,
- $P_{weak}(i)$ is true, and
- $\beta(uu_{l-i}) = m(v)$,

then \mathcal{X} is not a path.

Proof. Let $i, \mathcal{V}, \mathcal{U}, \mathcal{X}, \beta, v, u$, and u'' be as in the lemma, and without loss of generality we assume that $m(v) = 1$. Assume that for any $j < i$ $P(j)$ is true, that $P_{weak}(i)$ is true, that $\beta(uu_{l-i}) = m(v)$ and that \mathcal{X} is a path. The fan \mathcal{X} is a path so the vertex x_s is also missing the color 1, without loss of generality, we assume that $\beta(vx_s) = 2$. Note that this means that $X_v(2)$ is also a path (of length 1). The cycle \mathcal{V} is minimum and by Lemma 24 the fan \mathcal{U} is a cycle entangled with \mathcal{V} . Since $\beta(uu_{l-i}) = 1$, no edge in $E(\mathcal{U}_{[u_{l-(i-1)}, u_l]})$ is colored 1. Since $P(j)$ is true for all $j < i$, $X_v(\beta(uu_{l-j}))$ is a cycle for all $j < i$; since $X_v(2)$ is a path, no edge in $E(\mathcal{U}_{[u_{l-(i-1)}, u_l]})$ is colored 2 either.

We now consider the coloring β' obtained from β by swapping the edge vx_s . Note that in the coloring β' , the vertex v is now missing the color 2, and the fan $X_v^{\beta'}(1)$ is now a path (of length 1). The coloring β' is clearly \mathcal{V} -equivalent to β so by Observation 20, the fan \mathcal{V} is a minimum cycle in the coloring β' . Let $\mathcal{U}' = X_u^{\beta'}(m(u'')) = (uu'_1, \dots, uu'_s)$. No edge in $E(\mathcal{U}_{[u_{l-(i-1)}, u_l]})$ is colored 1, so no vertex in $V(\mathcal{U}_{[u_{l-i}, u_l]})$ is missing the color 1, and thus β' is also $\mathcal{U}_{[u_{l-(i-1)}, u_l]}$ -equivalent to β . Finally since no edge in $E(\mathcal{U}_{[u_{l-(i-1)}, u_l]})$ is colored 1 and $P(j)$ is true for all $j < i$, the fans $X_v(\beta(uu_{l-j}))$ are cycles for all $j < i$. Therefore the coloring β' is also $(\bigcup_{j \in [0, i-1]} X_v(\beta(uu_{l-j})))$ -equivalent to β . By Lemma 51, for any $t \leq i$ $u_{l-t} = u'_{s-t}$,

so in particular $u_{l-i} = u'_{s-i}$. In the coloring β' the edge uu_{s-i} is still colored 1, and now the vertex v is missing the color 2. Since $P_{weak}(i)$ is true the fan $X_v^{\beta'}(1)$ is not a path, this is a contradiction. \square

The next lemma considers (\mathcal{V}, u) -independent cycles.

Lemma 53. *Let $i \geq 0$, \mathcal{V} be a minimum cycle in a coloring β , u and u' two vertices of \mathcal{V} , $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_i)$, $h \leq i$ such that $\beta(uu_{i-h}) = m(v)$, c' a color not in $\beta(\mathcal{V})$ such that $\mathcal{Y} = X_u(c') = (uy_1, \dots, uy_r)$ is a (\mathcal{V}, u) -independent cycle different from \mathcal{U} and $\mathcal{X} = X_v(c') = (vx_1, \dots, vx_s)$ is a cycle different from \mathcal{V} with $y_r = x_s = z$, and c'' a color in $\beta(\mathcal{V})$. If $P(j)$ is true for all $j \leq i$, then $\mathcal{Z} = X_z(c'')$ is not a path.*

Proof. Without loss of generality, we assume that the vertices v , u and u' are respectively missing the colors 1, 2, and 3; we also assume that $c' = 4$. Note that since \mathcal{U} and \mathcal{Y} are different cycles, we have $\beta(\mathcal{U}) \cap \beta(\mathcal{Y}) = \{m(u)\} = \{2\}$, and since \mathcal{X} and \mathcal{V} are different cycles, we have that $\beta(\mathcal{V}) \cap \beta(\mathcal{X}) = \{m(v)\} = 1$. Assume that the fan \mathcal{Z} is a path. The fan \mathcal{V} is a minimum cycle, so by Lemma 24, the fan \mathcal{U} is a cycle entangled with \mathcal{V} , and thus $u_i = u'$.

We first invert \mathcal{Z} until we reach a coloring β' where $m(z) = c \in (\beta(\mathcal{V}) \cup \beta(\mathcal{X}) \cup \beta(\mathcal{Y})) \setminus \{4\}$. The coloring β' is \mathcal{V} -equivalent to β , so by Observation 20, the cycle \mathcal{V} is the same minimum cycle in the coloring β' . The coloring β' is also \mathcal{U} -equivalent to β , so, in the coloring β' , the fan $X_u(3)$ is exactly \mathcal{U} . Since the property $P(j)$ is true for all $j \leq h$, for any $j \leq h$ such that $\beta(uu_{i-j}) \neq 1$, the fan $X_v(\beta(uu_{i-j}))$ is a saturated cycle containing u_{i-j-1} .

We first show that $c \notin \beta(\mathcal{V})$. Otherwise, assume that $c \in \beta(\mathcal{V})$, then $c \notin \beta(\mathcal{Y})$ since \mathcal{Y} is a (\mathcal{V}, u) -independent cycle, and $c \notin \beta(\mathcal{X})$ since \mathcal{X} is different from \mathcal{V} . So the coloring β' is $(\mathcal{X} \cup \mathcal{Y} \setminus \{z\})$ -equivalent to β . Hence, in the coloring β' , the fans $X_u(4)$ and $X_v(4)$ still contain the vertex z . If $c = 1$, then in the coloring β' , since the fan $X_u(4)$ still contains the vertex z , we have that $X_u(4)_{\leq z}$ is a (\mathcal{V}, u) -independent subfan avoiding v . However, the fan $X_v(4)$ is now a path containing z , by Lemma 34, we have a contradiction. So $c \neq 1$. Since the fan \mathcal{V} is a minimum cycle in the coloring β' , it is saturated by Lemma 15, thus $z \notin K_v(1, c)$. We now swap the component $C_{1,c} = K_z(1, c)$, and denote by β'' the coloring obtained after the swap. The coloring β'' is \mathcal{V} -equivalent to β' , so \mathcal{V} is still a minimum cycle in the coloring β'' by Observation 20. The coloring β'' is also $(X_u(4)_{\leq z} \cup X_v(4))$ -equivalent to β' , so the fan $X_u(4)$ still contains the vertex z which is now missing the color 1. Similarly to the previous case, the subfan $X_u(4)_{\leq z}$ is now a (\mathcal{V}, u) -independent subfan avoiding v , and $X_v(4)$ is now a path; again by Lemma 34, we have a contradiction. Without loss of generality, we assume that $c = 5$.

Case 53.1 ($5 \notin \beta(\mathcal{X})$).

In this case, the coloring β' is $(\mathcal{X} \setminus \{z\})$ -equivalent to β , and so in the coloring β' , the fan $X_v(4)$ still contains the vertex z which is now missing the color 5.

Subcase 53.1.1 ($5 \in \beta(\mathcal{U}_{<u_{i-h}})$).

Let z' be the vertex of $\mathcal{U}_{<u_{i-h}}$ missing the color 5. If the vertex z' does not belong to $K_v(1, 5)$, then we swap the component $C_{1,5} = K_{z'}(1, 5)$, and denote by β'' the coloring obtained after the swap. The coloring β'' is clearly \mathcal{V} -equivalent to β' , so by Observation 20, the cycle \mathcal{V} is still the same minimum cycle in the coloring β'' . Since in the coloring β' , there is no edge colored 1 or 5 in $\mathcal{U}_{<u_{i-h}}$, the coloring β'' is also $\mathcal{U}_{<u_{i-h}}$ -equivalent to β' . So in the coloring β'' , the fan $X_u(3)$ still contains the vertex z' which is now missing the color 1, and thus $X_u(3)$ is not entangled with \mathcal{V} . By Lemma 24, we have a contradiction. So the vertex z' belongs to $K_v(1, 5)$, and thus the vertex z does not belong to $K_v(1, 5)$. We now swap the component $C_{1,5} = K_z(1, 5)$, and denote by β'' the coloring obtained after the swap. The coloring β'' is also \mathcal{V} -equivalent to β' , so the fan \mathcal{V} is a minimum cycle in the coloring β'' . Moreover, since $5 \in \beta(\mathcal{U}_{<u_{i-h}})$, $5 \notin \beta(\mathcal{Y})$. We also have that $5 \notin \beta(\mathcal{X})$, so in total, the coloring β'' is $(\mathcal{Y} \cup \mathcal{X} \setminus \{z\})$ -equivalent to the coloring β' . This means that in the coloring β'' , the fan $X_u(4)$ still contains the vertex z which is now missing the color 1, so $X_u(4)_{\leq z}$ is a (\mathcal{V}, u) -independent

subfan avoiding v . We also have that the fan $X_v(4)$ is now a path containing the vertex z , so by Lemma 34 we have a contradiction.

Subcase 53.1.2 ($5 \in \beta(\mathcal{U}_{\geq u_{l-h}})$).

Let s be such that $m(u_{l-s}) = 5$. The fan $X_v(5)$ is a saturated cycle containing u_{l-s} , so the vertex u_{l-s} belongs to the component $K_v(1, 5)$, and the vertex z does not belong to this component. We now swap the component $K_z(1, 5)$ and denote by β'' the coloring obtained after the swap. Since the color 5 is not in $\beta(X_v(4))$, the coloring β'' is $(X_v(4) \setminus \{z\})$ -equivalent to β' . So in the coloring β'' , the fan $X_v(4)$ still contains the vertex z which is missing the color 1, so it is now a path. Since the color 5 is in $\beta(\mathcal{U}_{>u_{l-h}})$, it is not in $\beta(\mathcal{V})$, so the coloring β'' is $(X_u(4) \setminus \{z\})$ -equivalent to β' , and thus $X_u(4)$ still contains the vertex z . So the subfan $X_u(4)_{\leq z}$ is now a (\mathcal{V}, u) -independent subfan avoiding v . By Lemma 34, the fan $X_v(4)$ is a path not containing z ; a contradiction.

Subcase 53.1.3 ($5 \in \beta(\mathcal{Y})$).

Let z' be the vertex of \mathcal{Y} missing the color 5 in the coloring β . The vertices z and z' are both missing the color 5 in the coloring β' , so at least one of them is not in $K_v(1, 5)$. If the vertex z is not in $K_v(1, 5)$, then we swap the component $C_{1,5} = K_z(1, 5)$, and denote by β'' the coloring obtained after the swap. The coloring β'' is \mathcal{V} -equivalent to β' , so the cycle \mathcal{V} is the same minimum cycle in the coloring β'' by Observation 20.

If the vertex u does not belong to $C_{1,5}$, then the fan $X_u(5)$ now contains the vertex z which is missing the color 1. Thus $X_u(5)_{\leq z}$ is now a (\mathcal{V}, u) -independent subfan avoiding v , and by Lemma 34, the fan $X_v(5)$ is a path. However, $\beta''(uu_{l-h}) = \beta(uu_{l-h}) = 1$, and the property $P(j)$ is true for all $j \leq h$, so by Lemma 52, there is no path around v . This is a contradiction.

So the vertex u belongs to $C_{1,5}$, and now $X_u(1)$ contains the vertex z which is missing the color 1. So $X_u(1)$ is not entangled with \mathcal{V} , and by Lemma 24, we also have a contradiction.

Case 53.2 ($5 \in \beta(\mathcal{X})$). Let z' be the vertex of \mathcal{X} missing the color 5 in the coloring β .

Subcase 53.2.1 ($5 \in \beta(\mathcal{U}_{<u_{l-h}})$).

Let z'' be the vertex of $\mathcal{U}_{<u_{l-h}}$ missing the color 5 in the coloring β . Note that we may have $z'' = z'$. The vertices z and z'' are both missing the color 5 in the coloring β' , so they are not both part of $K_v(1, 5)$. If z'' is not in $K_v(1, 5)$, then we swap $K_{z''}(1, 5)$, and denote by β'' the coloring obtained after the swap. The coloring β'' is \mathcal{V} -equivalent to β' , so by Observation 20, the cycle \mathcal{V} is the same minimum cycle in the coloring β'' . Since there is no edge colored 1 or 5 in $E(\mathcal{U}_{\leq z''})$, the coloring β'' is also $\mathcal{U}_{\leq z''}$ -equivalent to the coloring β' , so $X_u(3)$ still contains the vertex z'' which is now missing the color 1, so it is not entangled with \mathcal{V} , by Lemma 24, this is a contradiction.

So the vertex z'' belongs to $K_v(1, 5)$, and thus the vertex z does not belong to this component. We swap the component $C_{1,5} = K_z(1, 5)$, and denote by β'' the coloring obtained after the swap. In the coloring β'' , the fan $X_v(5)$ still contains the vertex z which is missing the color 1, so this fan is now a path. If the vertex u does not belong to $C_{1,5}$, then $\beta''(uu_{l-h}) = 1$. Since the property $P(j)$ is true for all $j \leq h$, there is no path around v , a contradiction. So the vertex u belongs to $C_{1,5}$. Now in the coloring β'' , the fan $\mathcal{U}' = X_u(3) = (uu'_1, \dots, uu'_l)$ is smaller, but for any $j \leq h$, we still have that $u_{l-j} = u'_{l-j} \in V(\mathcal{U}')$. Note that we have $z'' = u'_{l-h-1}$. So we have $\mathcal{U}' = (uu'_1, \dots, uu'_{l-h-1} = uz'', uu_{l-h}, \dots, uu_l)$. Since the property $P(j)$ is true for all $j \leq h$, the fan $X_v(\beta(uu_{l-h})) = X_v(5)$ is a cycle, a contradiction.

Subcase 53.2.2 ($5 \in \beta(\mathcal{U}_{>u_{l-h}})$).

Let s be such that $m^\beta(u_{l-s}) = 5$. In the coloring β , since the property $P(s)$ is true, the fan $X_v(5)$ is a saturated cycle containing u_s . But the color 5 is in \mathcal{X} , so the fan $X_v(5)$ also contains the vertex z , and thus $X_v(5) = \mathcal{X}$. In the coloring β' , the fan $X_v(5)$ still contains the vertex z

which is now missing the color 5. Since the property $P(s)$ is true, the cycle $X_v(5)$ is a cycle containing the vertex u_{l-s} , so we have a contradiction.

Subcase 53.2.3 ($5 \in \beta(\mathcal{Y})$).

Let z'' be the vertex of \mathcal{Y} missing the color 5 in the coloring β . Note that we may have $z'' = z'$. Since the vertices z and z'' are both missing the color 5 in the coloring β' , they are not both part of $K_v(1, 5)$. If z is not in $K_v(1, 5)$, then we swap the component $K_z(1, 5) = C_{1,5}$ and denote by β'' the coloring obtained after the swap. The coloring β'' is \mathcal{V} -equivalent to β' , so by Observation 20, the cycle \mathcal{V} is the same minimum cycle in the coloring β'' . Moreover, the coloring β'' is also $(X_v(5) \setminus \{z\})$ -equivalent to β' , so this fan is now a path containing z .

If the vertex u does not belong to $C_{1,5}$, then the coloring β'' is $(X_u(5) \setminus \{z\})$ -equivalent to β' , and thus $X_u(5)$ still contains the vertex z which is now missing the color 1. So the subfan $X_u(<)_{\leq z}$ is now a (\mathcal{V}, u) -independent subfan avoiding v , and it also contains z , by Lemma 34, the fan $X_v(5)$ is a path that does not contain z , a contradiction.

So the vertex u belongs to $C_{1,5}$, and now in the coloring β'' , the fan $X_u(1)$ contains the vertex z which is missing the color 1. So this fan is not entangled with \mathcal{V} , by Lemma 24, we also have a contradiction.

Subcase 53.2.4 ($5 \notin (\beta(\mathcal{Y}) \cup \beta(\mathcal{U}))$).

The vertices z and z' are both missing the color 5, so at least one of them is not in $K_v(1, 5)$.

If the vertex z is not in $K_v(1, 5)$, since $P(j)$ is true for all $j \leq h$, then for all $j \leq h$, the vertex z is not in $X_v(\beta(uu_{l-j}))$. We now swap the component $C_{1,5} = K_z(1, 5)$ to obtain a coloring β'' where the subfan $X_u(4)$ is now a (\mathcal{V}, u) -independent subfan avoiding v . The coloring β'' is \mathcal{V} -equivalent to β' , so by Observation 20 the cycle \mathcal{V} is the same minimum cycle in the coloring β' . By Lemma 34, the fan $X_v(4)$ is now a path that does not contain z . Since for all $j \leq h$, $P(j)$ is true and z does not belong to $X_v(\beta(uu_{l-j}))$, the coloring β'' is also $(\bigcup_{j \in [0, h]} X_v(\beta(uu_{l-j})))$ -equivalent to the coloring β' .

If the vertex u does not belong to $C_{1,5}$, then the coloring β'' is also \mathcal{U} equivalent to the coloring β' . The edge uu_{l-h} is still colored 1, and the property $P(j)$ is true for all $j \leq h$. By Lemma 52 there is no path around v , a contradiction. So the vertex u belongs to the component $C_{1,5}$, and the edge uu_{l-h} is now colored 5. Let $\mathcal{U}' = X_u^{\beta''}(3) = (uu'_1, \dots, uu'_p)$. Since $z \notin X_v(\beta(uu_{l-j}))$ for all $j \leq h$, the coloring β'' is $(\bigcup_{j \in [0, h]} X_v(\beta(uu_{l-j})))$ -equivalent to the coloring β' , and it is also $\mathcal{U}_{>u_{l-h}}$ -equivalent to the coloring β' , so by Lemma 51, for all $j \leq h$, we have $u'_{l-j} = u_{l-j}$. In particular, $u'_{l-h} = u_{l-h}$. The coloring β'' is $X_u(5)_{<z}$ -equivalent to the coloring β' , so in the coloring β'' , the fan $X_u(5)$ still contains the vertex z which is now missing the color 1, therefore the fan $X_u(5)$ is a path. Since $P(h)$ is true, we have a contradiction.

So the vertex z belongs to $K_v(1, 5)$, and the vertex z' does not belong to this component. We now swap the component $C_{1,5} = K_{z'}(1, 5)$, and obtain a coloring β'' that is \mathcal{V} -equivalent to β' . By Observation 20, the fan \mathcal{V} is the same minimum cycle in the coloring β'' . Since the property $P(j)$ is true for all $j \leq h$, and z' is not in $K_v(1, 5)$, then for all $j \leq h$, the vertex z' is not in $X_v(\beta(uu_{l-j}))$, and the coloring β'' is $(\bigcup_{j \in [0, h]} X_v(\beta(uu_{l-j})))$ -equivalent to the coloring

β' . In the coloring β'' , the fan $X_v(4)$ is now a path, so similarly to the previous case, the vertex u belongs to the component $C_{1,5}$. Therefore, in the coloring β'' , the edge uu_{l-h} is colored 5.

Let $\mathcal{U}' = (uu'_1, \dots, uu'_p)$. Since the coloring β'' is $(\bigcup_{j \in [0, h]} X_v(\beta(uu_{l-j})))$ -equivalent to the coloring β' , by Lemma 51, for any $j \leq h$ we have $u'_{l-j} = u_{l-j}$. The coloring β'' is $\mathcal{U}_{\leq v}$ equivalent to β' , so v is in \mathcal{U}' , and thus we have $X_u(1) = X_u(5) = \mathcal{U}'$. So there exists a vertex z'' missing the color 5 in the fan $X_u(1)$. Note that since $m(z'') = 1$ and $m(z') = 5$ we have

$z \neq z''$, however, we may have $z'' = z$, and in this case there exists a vertex in $X_u(1)$ missing a color in $\beta(X_u(4)_{\leq z})$. We now have to distinguish the cases.

Subsubcase 53.2.4.1 ($z'' \neq z$).

We consider the coloring β' . In this coloring, the vertex z'' is in $X_u(5)$ since u is in $C_{1,5}$. If the vertex z'' also belongs to $C_{1,5}$, then now $X_u(5)_{\leq z''}$ is a subfan avoiding v . If there is an edge uu'' in $E(X_u(5)_{\leq z''})$ colored with a color in $\beta'(\mathcal{V})$ then the fan $X_u(\beta'(uu''))$ is not entangled with \mathcal{V} , and by Lemma 24 we have a contradiction. So the subfan $X_u(5)_{\leq z''}$ is a (\mathcal{V}, u) -independent subfan avoiding v . By Lemma 34 the fan $X_v(5)$ is a path, however, in the coloring β' the fan $X_v(5)$ still contains the vertex z that is missing the color 5, so $X_v(5)$ is a cycle, a contradiction.

So the vertex z'' does not belong to $C_{1,5}$, and thus is still missing the color 5 in the coloring β' . We now swap the component $K_{z''}(1, 5)$ to obtain a coloring β_f where $X_u(5)_{\leq z''}$ is a (\mathcal{V}, u) -independent subfan avoiding v , and where $X_u(5)$ is a cycle. Again by Lemma 34 we have a contradiction.

Subsubcase 53.2.4.2 ($z'' = z'$).

So there exists a vertex w in $X_u(5)$ such that $m(w) \in \beta'(X_u(4)_{\leq z})$ and $w \notin V(X_u(4)_{\leq z})$. We now need to distinguish whether or not $m(w) = 4$.

Subsubsubcase 53.2.4.2.1 ($m(w) \neq 4$).

In this case, without loss of generality, assume that $m(w) = 6$, and let w' be the vertex of $X_v(4)_{\leq z}$ missing the color 6. The vertices w and w' are both missing the color 6, so they are not both part of $K_v(1, 6)$.

If w' is not in $K_v(1, 6)$, then we swap $C_{1,6} = K_{w'}(1, 6)$ to obtain a coloring β'' where $X_u(4)_{\leq w'}$ is a (\mathcal{V}, u) -independent subfan avoiding v . The coloring β'' is \mathcal{V} -equivalent to \mathcal{V} , so the fan \mathcal{V} is still the same minimum cycle in the coloring β'' by Observation 20. So by Lemma 34 the fan $X_v(4)$ is a path that does not contain w' . If the vertex u does not belong to $C_{1,6}$, then the coloring β'' is \mathcal{U} -equivalent to β' , the property $P(j)$ is true for all $j \leq h$ and $\beta''(uu_{l-h}) = 1$ so by Lemma 52 there is no path around v , a contradiction.

So the vertex u belongs to $C_{1,6}$, and we have $\beta''(uu_{l-h}) = 6$. Let $\mathcal{U}' = X_u^{\beta''}(3) = (uu'_1, \dots, uu'_l)$. The coloring β'' is $(\bigcup_{j \in [0, h]} X_v(\beta''(uu_{l-j})))$ -equivalent to the coloring β' and is $\mathcal{U}_{>u_{l-h}}$ -equivalent to the coloring β' so by Lemma 51, for all $j \leq h$ we have $u'_{l-j} = u_{l-j}$. In the coloring β' , the fan $X_v(4)$ contains the vertex z' missing the color 5, and the fan $X_v(5)$ is a cycle that contains the vertex z , and in the coloring β'' , the fan $X_u(4)$ is a path. So there exists a vertex w'' in $X_v(4)$ that is missing the color 6 in the coloring β' and that belongs to $C_{1,6}$. This vertex is now missing the color 1 in the coloring β'' . If w'' is in $X_v^{\beta''}(4)_{\leq z'}$, then the fan $X_v^{\beta''}(6)$ is now a comet containing two vertices (z and z') missing the color 5. Since the property $P(h)$ is true, we have a contradiction. So the vertex w'' is in $X_v^{\beta''}(5)$. But now, in the coloring β'' , the fan $X_v(6)$ contains the vertex z which is still missing the color 5, and the fan $X_v(5)$ is a path, so the fan $X_v(6)$ is a path. Again since $P(h)$ is true, we have a contradiction. So the vertex w' belongs to $K_v(1, 6)$.

Therefore, the vertex w does not belong to $K_v(1, 6)$, and we swap the component $C_{1,6} = K_w(1, 6)$ to obtain a coloring β'' where $X_u(5)_{\leq w}$ is a subfan avoiding v . The coloring β'' is \mathcal{V} -equivalent to the coloring β' , so by Observation 20, the fan \mathcal{V} is the same minimum cycle in the coloring β'' . Similarly to the previous case, the subfan $X_u(5)_{\leq w}$ is a (\mathcal{V}, u) -independent subfan avoiding v , so $X_v(5)$ is a path, and the vertex u belongs to $C_{1,6}$. Since the vertex z is still missing the color 5, it means that in the coloring β'' the fan $X_u(1)$ now contains the vertex w which is missing the color 1, and so it is not entangled with \mathcal{V} . By Lemma 24 we have a contradiction.

Subsubsubcase 53.2.4.2.2 ($m(w) = 4$).

Since the fan $\mathcal{V} = X_z(5)$ is a path in the coloring β , the fan $X_z(4)$ is a path in the coloring β' . In the coloring β' we invert the path $X_z(5)$ until we reach a coloring where $m(z) \in \beta'(X_u(5)_{\leq w})$ and denote by β'' the coloring obtained after the inversion. Note that since $4 \in \beta'(X_z(5)) \cap \beta(X_u(5)_{\leq w})$ the inversion is well defined. The coloring β'' is \mathcal{V} -equivalent to β' so the fan \mathcal{V} is the same minimum cycle in the coloring β'' . The coloring β'' is also \mathcal{U} -equivalent to β' , so $X_u^{\beta''}(3) = \mathcal{U}$ and since $P(j)$ is true for all $j \leq h$, the fan $X_v(\beta''(uu_{l-j}))$ is a saturated cycle if $\beta''(uu_{l-j}) \neq 1$. The coloring β'' is $(X_v(4) \setminus \{z\})$ -equivalent to β' , so the fan $X_v(4)$ still contains the vertex z' which is missing 5, and the vertex z . Finally, the coloring β'' is $X_u(5)_{\leq w}$ -equivalent to the coloring β' . Let c_z be the missing color of z in β'' , and let w' be the vertex of $X_u(5)_{\leq w}$ missing the color c_z . Note that if $c_z = 4$, then we have $w' = w$.

The proof is similar to the previous case, and we now consider the components of $K(1, c_z)$. The vertices z and w' are both missing the color c_z so at least of them is not in $K_v(1, c_z)$. If the vertex z is not in $K_v(1, c_z)$, then we swap the component $C_{1, c_z} = K_z(1, c_z)$ to obtain a coloring β_f that is \mathcal{V} -equivalent to β'' . By Observation 18 the fan \mathcal{V} is the same minimum cycle in the coloring β_f , and now $X_u(4)_{\leq z}$ is a (\mathcal{V}, u) -independent subfan avoiding v , so by Lemma 34 the fan $X_v(4)$ is now a path not containing z . Note that the coloring β_f is also $(\bigcup_{j \in [0, h]} X_v(\beta(uu_{l-j})))$ -equivalent to the coloring β'' . If the vertex u does not belong to C_{1, c_z} , then the coloring β_f is \mathcal{U} -equivalent to β'' , and in particular $\beta_f(uu_{l-h}) = 1$. Since $P(j)$ is true for all $j \leq h$, by Lemma 52 there is no path around v , a contradiction.

So the vertex u belongs to C_{1, c_z} , and now $\beta_f(uu_{l-h}) = 6$. Since the fan $X_v(4)$ is now a path that does not contain z , it means that in the coloring β'' there is a vertex w'' in $X_v(4)$ which is missing the color c_z and which also belongs to C_{1, c_z} . It means that in the coloring β_f , the fan $X_v(c_z)$ is now a path containing z . Let $\mathcal{U}' = X_u^{\beta_f}(3) = (uu'_1, \dots, uu'_l)$. The coloring β_f is $\mathcal{U}_{>u_{l-h}}$ -equivalent to β'' and is also $(\bigcup_{j \in [0, h]} X_v(\beta(uu_{l-j})))$ -equivalent to β'' , so

for all $j \leq h$, we have $u'_{l-j} = u_{l-j}$ by Lemma 51. In particular $u'_{l-h} = u_{l-h}$. Since $P(h)$ is true, and $\beta_f(uu'_{l-h}) = c_z$, the fan $X_v(c_z)$ is a cycle, a contradiction.

So the vertex z belongs to $K_v(1, c_z)$ and the vertex w' does not belong to this component. We now swap the component $C_{1, c_z} = K_w(1, c_z)$ and denote by β_f the coloring obtained after the swap. The coloring β_f is \mathcal{V} -equivalent to β'' so by Observation 18 the fan \mathcal{V} is the same minimum cycle in the coloring β_f . The coloring β_f is also $(\bigcup_{j \in [0, h]} X_v(\beta(uu_{l-j})))$ -equivalent

to the coloring β'' . In the coloring β_f the subfan $X_u(5)_{\leq w'}$ is now a subfan avoiding v . If there is an edge uw'' in $E(X_u(5)_{\leq w'})$ colored with a color in $\beta_f(\mathcal{V})$, then $X_u(\beta_f(uw''))$ is not entangled with \mathcal{V} and by Lemma 24 we have a contradiction. So the subfan $X_u(5)_{\leq w'}$ is a (\mathcal{V}, u) -independent subfan avoiding v and thus by Lemma 34 the fan $X_v(5)$ is now a path that does not contain w' . Similarly to the previous case, this means that the vertex u belongs to the component C_{1, c_z} , and thus that $\beta_f(uu_{l-h}) = c_z$. The fan $X_v(5)$ still contains the vertex z which is missing the color c_z , and the fan $X_v(5)$ is a path, so the fan $X_v(c_z)$ is a path. Let $\mathcal{U}' = X_u^{\beta_f}(3) = (uu'_1, \dots, uu'_l)$. Since $P(j)$ is true for all $j \leq h$ and since the coloring β_f is $(\bigcup_{j \in [0, h]} X_v(\beta(uu_{l-j})))$ -equivalent to the coloring β'' , by Lemma 51 for $j \leq h$, we have

$u'_{l-j} = u_{l-j}$. In particular, $u'_{l-h} = u_{l-h}$. The edge uu'_{l-h} is now colored c_z and the property $P(h)$ is true, so the fan $X_v(c_z)$ is a cycle. This is a contradiction. □

Before proving the induction step of the proof we need to introduce a new property implied by $P(i)$.

4.5 The property $Q(i)$

Definition 54. Let $i \geq 0$, we define the property $Q(i)$ as follows:

For any minimum cycle \mathcal{V} in a coloring β , for any pair of vertices u and u' of \mathcal{V} , let $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_i)$. If $\beta(uu_{i-1}) \neq m(v)$, then for any color $c \in \beta(\mathcal{V})$, the fan $X_{u_{i-1}}(c)$ is a cycle entangled with \mathcal{V} and $\mathcal{U}_{\geq u_{i-2}}$.

We now prove that the property $Q(i)$ is implied by the property $P(i)$. And we first prove the following lemma concerning saturated cycles around the central vertex of a minimum cycle.

Lemma 55. Let $\mathcal{V} = (vv_1, \dots, vv_k)$ be a minimum cycle in a coloring β , $u = v_j$ and $u' = v_{j'}$ two vertices of \mathcal{V} and $\mathcal{W} = (vw_1, \dots, vw_t)$ a saturated cycle around v . Then the fans \mathcal{W} and $\mathcal{U} = X_u(m(u'))$ are entangled.

Proof. By Lemma 24, the fan \mathcal{U} is a cycle entangled with \mathcal{V} , so if $\mathcal{W} = \mathcal{V}$, the fans \mathcal{W} and \mathcal{U} are entangled as desired. So assume that $\mathcal{W} \neq \mathcal{V}$ and that \mathcal{W} is not entangled with \mathcal{U} . Without loss of generality, we assume that the vertices v , u and u' are respectively missing the colors 1, 2, and 3. Since $\mathcal{W} \neq \mathcal{V}$ and \mathcal{W} is centered at v , we have that $\beta(\mathcal{W}) \cap \beta(\mathcal{V}) = \{1\}$. Since \mathcal{W} and \mathcal{U} are not entangled, there exists $c \in \beta(\mathcal{U}) \cap \beta(\mathcal{W})$ such that $M(\mathcal{U}, c) \neq M(\mathcal{W}, c)$. Without loss of generality, since $c \notin \{1, 2, 3\}$, we assume that $c = 4$ and that $u_i = M(\mathcal{U}, 4)$ is the first such vertex in \mathcal{U} ; up to shifting the indices in \mathcal{W} , we also assume that $m(w_t) = 4$, and thus that $\mathcal{W} = X_v(4)$.

Since the cycle \mathcal{W} is saturated, the vertex w_t belongs to $K_v(1, 4)$, so the vertex z does not belong to $K_v(1, 4)$. We swap the component $C_{1,4} = K_z(1, 4)$ and denote by β_2 the coloring obtained after the swap.

If $u \notin C_{1,4}$, or there is no edge colored 1 in $\mathcal{U}_{<i}$, then the coloring β_2 is $(\mathcal{V} \cup \mathcal{W} \cup \mathcal{U}_{<i})$ -equivalent to β . Hence, in the coloring β_2 , the fan \mathcal{V} is a minimum cycle by Observation 20, but now the fan $X_u(m(u')) = (uu_1, \cdot, uu_i)$ is now a path, by Lemma 24, this is a contradiction.

So $u \in C_{1,4}$, and there is an edge colored 1 in $\mathcal{U}_{<i}$. Since by Lemma 24, the cycle \mathcal{U} is entangled with \mathcal{V} , the edge uv_{j-1} and the edge uv are in $E(\mathcal{U})$. We denote by x the vertex connected to u by the edge colored 1 and by c_{j-1} the missing color of v_{j-1} in β . Note that we may have $v_{j-1} = u'$, and thus $c_{j-1} = 2$. The fan \mathcal{U} is of the form $(uu_1, \dots, uv_{j-1}, uv, ux, \dots, uu_i, \dots, uu')$. The coloring β_2 is (\mathcal{V}) -equivalent to β , so by Observation 20, the cycle \mathcal{V} is a minimum cycle in β_2 . But now the fan $X_u(4)$ is a comet where v and u_i are missing the same color 1, more precisely, $X_u(4) = (ux, \dots, uu_i, \dots, uu', uu_1, \dots, uv_{j-1}, uv)$. Note that $X_u(3)$ is a cycle which is a subsequence of $X_u(4)$. If there is an edge colored with a color $c \in \beta(\mathcal{V})$ in $X_u(4)$ between the edges ux and uu_i , then the fan $X_v(c)$ is a comet, which is a contradiction by Lemma 24.

So there is no edge colored with a color $c \in \beta(\mathcal{V})$ in $X_u(4)$ between the edges ux and uu_i . Since the fan \mathcal{V} is a minimum cycle, it is saturated by Lemma 15, so $u \in K_v(1, 2)$, and thus $u_i \notin K_v(1, 2)$. We now swap the component $K_{u_i}(1, 2)$ to obtain a coloring β_3 . The coloring β_3 is $(\mathcal{V} \cup \mathcal{W})$ -equivalent to β_2 , so the fan \mathcal{V} is a minimum cycle in β_3 by Observation 20.

We now show that \mathcal{V} is invertible in the coloring β_3 . The cycle \mathcal{V} is tight by Observation 21, so the vertex u belongs to the component $C_{2,j-1} = K_{v_{j-1}}(2, c_{j-1})$, thus the edges vu and vv_{j+1} also belong to $C_{2,j-1}$. In the coloring β_3 , the fan $X_u(4)$ is now a path that we invert until we reach a coloring β_4 where $m(u) \in \beta(\mathcal{W})$. Note that since $4 \in \beta(\mathcal{W})$, the inversion is well-defined and moreover, since β_3 is also (\mathcal{W}) -equivalent to β , we have $\beta_3(\mathcal{W}) = \beta(\mathcal{W})$. Since $u \notin \mathcal{W}$, by Observation 17, the coloring β_4 is (\mathcal{W}) -equivalent to β_3 , so \mathcal{W} is still the same cycle in β_4 . Moreover, since $\beta_3(X_u(4)) \cap \beta_3(\mathcal{V}) = \{2\}$, the coloring β_4 is $(\mathcal{V} \setminus \{u\} \cup C_{2,j-1})$ -equivalent to β_3 .

We denote by w_s the vertex of \mathcal{W} such that $m^{\beta_4}(u) = m^{\beta_4}(w_s)$, and we denote by c_s this missing color. Note that we may have that $w_t = w_s$, and thus $c_s = 4$. The vertices u and w_s are missing the same color c_s , so they are not both part of the component $K_v(1, c_s)$ and we now have to distinguish the cases.

Case 55.1 ($u \notin K_v(1, c_s)$). In this case, we swap the component $C_{1, c_s} = K_u(1, c_s)$ and obtain a coloring that we denote by β_5 . Since $\{1, c_s, 2, c_{j-1}\} = 4$, the coloring β_5 is $(C_{2, j-1})$ -equivalent to β_4 , so it is $(C_{2, j-1})$ -equivalent to β_3 . In the coloring β_5 , the vertex u is now missing the color 1, so the fan $X_v(m(u)) = (vv_{j+1}, \dots, vv_{j-1}, vu)$ is now a path that we invert, we denote by β_6 the coloring obtained after the inversion. In the coloring β_6 , the vertices v_{j+1} and v are missing the color 2, and the vertex u is missing the color c_{j-1} . So now the component $C'_{2, j-1} = K_{v_{j-1}}$ is exactly $C_{2, j-1} \cup \{vv_{j-1}\} \setminus \{vu, vv_{j+1}\}$ and we swap it. After this swap, the vertices v and u are missing the same color c_{j-1} , and the edge uv is colored 1; we swap this edge and we denote by β_7 the coloring obtained after the swap. In the coloring β_7 , the vertex u is missing the color 1, so the component $K_u(1, c_s)$ is now exactly C_{1, c_s} , so we swap back this component. Note that since $\{1, 2, c_s, c_{j-1}\} = 4$, we can swap back C_{1, c_s} before $C'_{2, j-1}$. In the coloring obtained after the swap, the fan $X_u(2)$ is now a path that we invert, and we denote by β_8 the coloring obtained after the inversion. In the coloring β_8 , the vertex u is now missing the color 2, so the component $K_{v_{j-1}}(2, c_{j-1})$ is now exactly $C'_{2, j-1} \cup \{uv\}$. After swapping back this component we obtain exactly $\mathcal{V}^{-1}(\beta_3)$, a contradiction.

Case 55.2 ($u \in K_v(1, c_s)$). The principle is the same as in the previous case, but instead of changing the missing color of u , we will change the missing color of v using the fan $X_v(c_s)$ to transform \mathcal{V} into a path. As u belongs to $K_v(1, c_s)$, the vertex w_s does not belong to this component. So we swap the component $C_{1, c_s} = K_{w_s}(1, c_s)$ to obtain a coloring where $X_v(c_s)$ is now a path that we invert; we denote by β_5 the coloring obtained after the inversion. Note that since $X_v(c_s)$ was a cycle in β_4 , we have $\beta_4(X_v(c_s)) \cap \beta_4(\mathcal{V}) = \{1\}$, and so $\{2, c_{j-1}\} \cap \beta_4(X_v(c_s)) = \emptyset$. Hence the coloring β_5 is $(C_{2, j-1})$ -equivalent to the coloring β_4 . In the coloring β_5 , the fan $X_v(2) = (vv_{j+1}, \dots, vu)$ is now a path that we invert, and we denote by β_6 the coloring obtained after the swap. Similarly to the previous case, in the coloring β_6 , the vertices v and v_{j+1} are missing the color 2, and the vertex u is missing the color c_{j-1} . So in the coloring β_6 , the component $C'_{2, j-1} = K_{v_{j-1}}(2, c_{j-1})$ is exactly $C_{2, j-1} \cup \{vv_{j-1}\} \setminus \{vv_{j+1}, vu\}$, and we swap it to obtain a coloring where the vertices u and v are missing the color c_{j-1} and where the edge uv is colored c_s . After swapping the edge uv , we obtain a coloring where, the fan $X_v(1)$ is now a path that we invert, we denote by β_7 the coloring obtained after the inversion. In the coloring β_7 , the component $K_{w_s}(1, c_s)$ is exactly C_{1, c_s} and we swap back this component. Note that since $|\{1, 2, c_s, c_{j-1}\}| = 4$, we can swap back this component before $C'_{2, j-1}$. In the coloring obtained after the swap, the fan $X_u(2)$ is now a path that we invert, we denote by β_8 the coloring obtained after the swap. In the coloring β_8 , the vertex u is now missing the color 2, so the component $K_{v_{j-1}}(2, c_{j-1})$ is now exactly $C'_{2, j-1} \cup \{vu\}$ and we swap back this component to obtain $\mathcal{V}^{-1}(\beta_3)$ as desired. □

Lemma 56. *Let $i \geq 0$, if $P(i)$ is true for all $j \leq i$, then $Q(j)$ is true for all $j \leq i$.*

Proof. Let $i \geq 0$, \mathcal{V} be a minimum cycle in a coloring β , u and u' two vertices of \mathcal{V} , $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_i)$, and assume that $P(j)$ is true for all $j \leq i$. Without loss of generality, we assume that the vertices v , u and u' are respectively missing the colors 1, 2, and 3. Let $t \leq i$, and $z = u_{i-t-1}$. We prove that $Q(t)$ is true.

Claim 57. *The vertex z is not missing a color in $\beta(\mathcal{V})$.*

Proof. Otherwise, assume that $m(z) \in \beta(\mathcal{V})$. The fan \mathcal{V} is a minimum cycle in β so by Lemma 24, then fan \mathcal{U} is a cycle entangled with \mathcal{V} .

If $m(z) \neq 1$, then since \mathcal{U} is a cycle entangled with \mathcal{V} by Lemma 24, we have $z \in V(\mathcal{V})$ so by Lemma 24 for any color $c \in \beta(\mathcal{V})$, $X_z(c)$ is a cycle entangled with \mathcal{V} . Moreover, since the property $P(t)$ is true, so $X_v(\beta(uz)) = (vw_1, \dots, w_x)$ is a saturated cycle, and by Lemma 55 is entangled with $\mathcal{U} = X_u(m(u'))$ and $X_z(m(u)) = (zz_1, \dots, zz_r)$, and thus $u_{l-t-2} = w_x = z_{r-1}$, so $Q(t)$ is true.

If $m(z) = 1$, then since \mathcal{U} is entangled with \mathcal{V} , we have $z = v$. So for any $c \in \beta(\mathcal{V})$, $X_z(c) = \mathcal{V}$ and thus is a cycle entangled with \mathcal{V} . Moreover, this means that $\beta(uz) = \beta(uv)$ and thus $m(u_{l-t-2}) = \beta(uv)$, so u_{l-t-2} is the vertex just before u in the cycle \mathcal{V} . By definition of \mathcal{V} , the fan $X_z(m(u))$ contains this vertex, and thus $Q(t)$ is true. In both cases, we have a contradiction. \square

Claim 58. *There is no edge in $E(\mathcal{U}_{>z})$ colored with a color beta(\mathcal{V}).*

Proof. We first prove that there is no vertex in $V(\mathcal{U}_{>z}) \setminus \{u'\}$ missing a color $c \in \beta(\mathcal{V})$. Otherwise, assume that there exists such a vertex z' . The cycle \mathcal{V} is minimum in $\beta(\mathcal{V})$, so by Lemma 24, the fan \mathcal{U} is entangled with \mathcal{V} . If $c \neq 1$, then $z' \in V(\mathcal{V})$. By Lemma 24, the fan $\mathcal{U}' = X_u(m(z')) = (uu'_1, \dots, uu'_r)$ is a cycle entangled with \mathcal{V} , so $u'_r = z'$ and $V(\mathcal{U}) = V(\mathcal{U}')$. Thus there exists $t' < t$ such that $z = ul' - t' - 1$. Since t is minimum, $Q(t')$ is true, and thus $Q(t)$ is true.

If $c = 1$, then $z' = v$ since \mathcal{U} is entangled with \mathcal{V} , and $\beta(uz') = \beta(uv)$. Let z'' be the vertex just before z' in \mathcal{U} . Since $\beta(uz') = \beta(uv)$, then $m(z'') = \beta(uv) \in \beta(\mathcal{V})$. Since $m(z) \notin \beta(\mathcal{V})$, we have that $z'' \neq z$. This means that z'' is a vertex in $V(\mathcal{U}_{>z}) \setminus \{u'\}$ missing a color in $\beta(\mathcal{V})$, this is a contradiction. \square

Let $c \in \beta(\mathcal{V})$, we prove that $\mathcal{Z} = X_z(c) = (zz_1, \dots, zz_r)$ is a cycle entangled with \mathcal{V} and $\mathcal{U}_{\geq z}$.

By Claim 57 $m(z) \notin \beta(\mathcal{V})$, so without loss of generality, we assume that z is missing the color 4. By Lemma 48 the fan \mathcal{Z} is not a path. Before proving that \mathcal{Z} is not a comet, we first prove that it is entangled with \mathcal{V} and $\mathcal{U}_{\geq u_{l-t-2}}$.

Proposition 59. *The fan \mathcal{Z} is entangled with \mathcal{V} and $\mathcal{U}_{\geq u_{l-t-2}}$.*

Proof. Otherwise, assume that there exists s such that $m(z_s) \in \beta(\mathcal{V}) \cup \beta(\mathcal{U}_{\geq z})$ and $z_s \notin V(\mathcal{V}) \cup V(\mathcal{U}_{\geq z})$. Without loss of generality, we assume that such an s is minimum. We also assume that there is no edge colored with a color in $\beta(\mathcal{V})$ in $E(\mathcal{Z}_{[z_2, z_{s-1}]})$. Otherwise, if such an edge zz_x exists, it suffices to consider the fan $X_z(\beta(zz_x)) = (zz_x, \dots, zz_s)$. We now have to distinguish the cases.

Case 59.1 ($m(z_s) = 1$).

In this case, since $P(t)$ is true, $X_v(4)$ is a saturated cycle containing z , so $v \in K_z(1, 4)$, and thus $z_s \notin K_z(1, 4)$. We now swap the component $C_{1,4} = K_{z_s}(1, 4)$, and denote by β' the coloring obtained after the swap. In the coloring β' , the fan $X_z(c)$ is now a path. The coloring β' is \mathcal{V} -equivalent to β , so \mathcal{V} is still a minimum cycle in β' . If the coloring β' is also $\mathcal{U}_{<z}$ -equivalent to β (i.e., $C_{1,4}$ does not contain u or there is no edge colored 1 in $\mathcal{U}_{<z}$), then z is still a vertex of $\mathcal{U} = X_u(3)$, and the fan $X_z(c)$ is now a path, by Lemma 48 this is a contradiction. So the vertex u belongs to $C_{1,4}$, and there is an edge uu_h colored 1 in $\mathcal{U}_{<z}$. So in the coloring β' , the edge uu_h is now colored 4, and the edge u_{l-t} is now colored 1. The fan $X_v(4)$ is still a saturated cycle containing z , but now the fan $X_u(4)$ is also a cycle containing z . In this coloring the fan $X_z(c)$ is a path, so by Lemma 53, we have a contradiction.

Case 59.2 ($m(z_s) = c' \in \beta(\mathcal{V}) \setminus \{1\}$).

In this case, since \mathcal{V} is minimum, it is saturated by Lemma 15, thus $z_s \notin K_v(1, c')$. We now swap the component $C_{1,c'} = K_{z_s}(1, c')$, and denote by β' the coloring obtained after the swap. This coloring is \mathcal{V} -equivalent to β , so \mathcal{V} is still a minimum cycle in the coloring β' . By Claim 58, there is no edge with a color in $\beta(\mathcal{V})$ in $\mathcal{U}_{>z}$, so β' is $\mathcal{U}_{>z}$ -equivalent to β . Moreover, let $\mathcal{U}' = X_u^{\beta'}(3) = (uu'_1, \dots, uu'_t)$; the coloring β' is also $(\bigcup_{j \in [0,t]} X_v(\beta(uu_j)))$ -equivalent to β

since each of these fans are saturated cycles, and the vertex z_s does not belong to any of them. So by Lemma 51, in the coloring β' , for any $j \leq (t+1)$, $u'_{t-j} = u_{t-j}$. In particular, $u'_{t-1} = u_{t-1} = z$. But now $X_z(c)$ is not entangled with $\{v\}$ since it contains the vertex z_s which is missing the color 1. This case is similar to the previous one.

Case 59.3 ($m(z_s) = c' \in \beta(\mathcal{U}_{>z})$).

Let u_{l-h} be the vertex of $\mathcal{U}_{>z}$ missing the color c' . In this case, since $P(j)$ is true for all $j \leq t$, the fan $X_v(\beta(uu_{l-j}))$ is a saturated cycle. In particular, the vertex u_{l-h} belongs to the component $K_v(1, c')$, and so z_s does not belong to this component. We now swap the component $C_{1,c'} = K_{z_s}(1, c')$, and denote by β' the coloring obtained after the swap. Let $\mathcal{U}' = X_u^{\beta'}(3) = (uu'_1, \dots, uu'_t)$. If the coloring β' is $\mathcal{U}_{>z}$ -equivalent to β , then for the same reason as in the previous case, z is exactly the vertex u'_{t-1} , and $X_z(c')$ is now not entangled with $\{v\}$ since it contains the vertex z_s that is missing the color 1. This case is similar to the first one. So β' is not $\mathcal{U}_{>z}$ -equivalent to β , and thus since it is $\{u_{l-h}\}$ -equivalent to β , the component $C_{1,c'}$ contains the vertex u . We now have to distinguish whether or not, in the coloring β there is an edge uu_p colored 1 in $\mathcal{U}_{<z}$.

Subcase 59.3.1 (There an edge uu_p colored 1 in $\mathcal{U}_{<z}$).

In this case, in the coloring β' , the edge uu_p is now colored c' , and the edge uu_{l-h+1} is now colored 1. In the coloring β' , the fan $X_u(4)$ is now a cycle since it contains the vertex u_{l-h} which is still missing the color c' , and $X_v(c')$ now contains the vertex z which is still missing the color 4. The fan $X_v(4)$ is still a cycle containing also the vertex z , and the fan \mathcal{U}' now contains an edge uu_{l-p} colored 1 such that $p \leq t$.

We now consider the components of $K(1, 4)$. If the vertex z does not belong to the component $K_v(1, 4)$, then we swap it to obtain a coloring β'' where $X_v(4)$ is now a path. Let $\mathcal{U}'' = X_u^{\beta''}(3) = (uu''_1, \dots, uu''_t)$. If $u \notin K_v(1, 4)$, then $\beta''(uu_{l-p}) = \beta''(uu''_{l-t}) = 1$, but $p \leq t$, and $P(j)$ is true for all $j \leq t$, so by Lemma 52 we have a contradiction. Similarly, if $u \in K_v(1, 4)$, then now $\beta''(uu_{l-t}) = 1$. Since β'' is $(\bigcup_{j \in [0,t-1]} X_v(\beta(uu_j)))$ -equivalent to β' , by

Lemma 51, for any $j \leq t$, $u''_{t-j} = u'_{t-j}$. So the edge uu_{l-t} is exactly the edge uu''_{l-t} . This edge is colored 1, and $P(j)$ is true for all $j \leq t$, so by Lemma 52, we have a contradiction.

So the vertex z belongs to $K_v(1, 4)$, and therefore the vertex z_s does not belong to this component. We now swap the component $C_{1,4} = K_{z_s}(1, 4)$, and denote by β'' the coloring obtained after the swap. Let $\mathcal{U}'' = X_u^{\beta''}(3) = (uu''_1, \dots, uu''_t)$. Whether or not the vertex u belongs to the component $C_{1,4}$, the fan $X_u(3)$ contains an edge uu''_{l-j} colored 1 where $j \leq t$ (if u belongs to the component, $\beta''(uu''_{l-t}) = 1$, and $\beta''(uu''_{l-p}) = 1$). Moreover, we have that the fan $X_u(4)$ is a cycle containing z , the fan $X_v(4)$ is a cycle containing z , the fan $X_z(c)$ is a path, and, and the property $P(j)$ is true for all $j \leq t$, so by Lemma 53, we have a contradiction.

Subcase 59.3.2 (There is no edge colored 1 in $\mathcal{U}_{<z}$).

In this case, the coloring β' is $\mathcal{U}_{<z}$ -equivalent to β . We now consider the components of $K(1, 4)$. If z does not belong to $K_v(1, 4)$, then we swap the component $K_z(1, 4)$ and obtain a coloring where $X_u(3)$ still contains the vertex z which is now missing the color 1. In the coloring, the cycle \mathcal{V} is still a minimum cycle since β' is \mathcal{V} -equivalent to β , so by Lemma 24,

we have a contradiction.

So the vertex z belongs to $K_v(1, 4)$, and thus z_s does not belong to this component. We now swap the component $C_{1,4} = K_{z_s}(1, 4)$, and denote by β'' the coloring obtained after the swap. The coloring β'' is $\mathcal{U}_{\leq z}$ -equivalent to β' , so $z \in X_u(3)$. However, now the fan $X_z(c)$ is a path, by Lemma 48, we have a contradiction.

Case 59.4 ($m(z_s) = c' = m(u_{l-t-2})$).

In this case, since $c' \notin \beta(\mathcal{V})$, without loss of generality, we assume that $c' = 5$. We now consider the components of $K(1, 5)$. If u_{l-t-2} does not belong to $K_z(4, 5)$, then we swap the component $C_{4,5} = K_{u_{l-t-2}}(4, 5)$, and denote by β' the coloring obtained after the swap. Let $\mathcal{U}' = X^{\beta'}(3) = (uu'_1, \dots, uu'_l)$. The coloring β' is $(\mathcal{V} \cup \mathcal{U}_{> z})$ -equivalent to β , and for any $j \leq t$, $u'_{l-j} = u_{l-j}$, and $u'_{l-j} = u_{l-j-1}$ otherwise. Note that this means that $l' = l - 1$, i.e. $|\mathcal{U}'| = |\mathcal{U}| - 1$. If the color 5 is not in $X_v^\beta(4)$, then $X_v(4)$ is still a cycle containing z , and thus it does not contain $u_{l-t-2} = M(X_u(3), 4)$, since the property $P(t)$ is true, we have a contradiction.

So the color 5 is in $X_v^\beta(4)$. If v belongs to $C_{4,5}$, then we are in a case similar to the previous one where $X_v^{\beta'}(4)$ is a cycle containing z , and thus which does not contain $u_{l-t-2} = M(X_u(3), 4)$. Since $P(t)$ is true, we have a contradiction. So v does not belong to $C_{4,5}$, and now, in the coloring β' , the fan $X_v(5)$ is a comet containing the vertices z and u_{l-t-2} that are both missing the color 4. We now consider the components of $K(1, 4)$. Since the property $P(t)$ is true, $X_v(4)$ is a saturated cycle, so u_{l-t-2} belongs to $K_v(1, 4)$, and thus z does not belong to this component. We now swap the component $K_z(1, 4)$, and obtain a coloring where $\{uz\}$ is a (\mathcal{V}, u) -independent subfan avoiding v , and where $X_v(5)$ is a path containing z , by Lemma 34 we have a contradiction.

So the vertex u_{l-t-2} belongs to the component $K_z(4, 5)$, and therefore, the vertex x_s does not belong to this component. We now swap the component $K_{z_s}(4, 5)$, to obtain a coloring $(\mathcal{V} \cup \mathcal{U})$ -equivalent to β , where $X_z(c)$ is now a path, by Lemma 48, we again get a contradiction. \square

So the fan \mathcal{Z} is entangled with \mathcal{V} and $\mathcal{U}_{\leq u_{l-t-2}}$. We now prove that it is not a comet. Assume that \mathcal{Z} is a comet, then there exists $h < r$ such that $m(z_h) = m(z_r) = c$. By Proposition 59, $c \notin \beta(\mathcal{V}) \cup \beta(\mathcal{U}_{\geq u_{l-2-t}})$. Without loss of generality, we assume that $c = 5$, and we now consider the components of $K(1, 5)$. The vertices z_h and z_r are not both part of $K_v(1, 5)$.

If z_h does not belong to $K_v(1, 5)$, then we swap $C_{1,5} = K_v(1, 5)$, and denote by β' the coloring obtained after the swap. Let $\mathcal{U}' = X_u^{\beta'}(3) = (uu'_1, \dots, uu'_l)$. The property $P(j)$ is true for all $j \leq t$, so the coloring β' is $(\bigcup_{j \in [0, p]} X_v(\beta(uu_{l-j})))$ -equivalent to β since each of these fans are saturated cycle. Hence by Lemma 51, for any $j \leq (t + 1)$, $u'_{l-j} = u_{l-j}$. In particular, $z = u'_{l-t-1}$. If the vertex z does not belong to $C_{1,5}$ or $c \neq 1$, then the coloring β' is $\mathcal{Z}_{< z_h}$ equivalent to β . The fan $X_z(c)$ now contains the vertex z_h which is missing the color 1, by Proposition 59 we have a contradiction. So the vertex z belongs to $C_{1,5}$, and $c = 1$. Thus, in the coloring β' , the edge zz_1 is now colored 5, and the edge zz_{s+1} is now colored 1. If the vertex z_r belongs to the component $C_{1,5}$, it is now missing the color 1 in the coloring β' , and $X_z(1)$ is now a fan that contains this vertex. So the fan $X_v(1)$ is not entangled with \mathcal{V} , a contradiction by Proposition 59. If the vertex z_r does not belong to the component, then the fan $X_z(1)$ now contains the vertex z_s which is missing the color 1, again, a contradiction by Proposition 59.

So the vertex z_h belongs to $K_v(1, 5)$, and thus the vertex z_r does not belong to the component. We now swap the component $C_{1,5} = K_{z_s}(1, 5)$ and denote by β' the coloring obtained after the swap. Similarly to the previous case, If the vertex z does not belong to $C_{1,5}$, or if

$c \neq 1$, then the coloring β' is $\mathcal{Z}_{<z_r}$ -equivalent to the coloring β , and now $X_z(c)$ contains the vertex z_r missing the color 1, by Proposition 59 this is a contradiction. So the vertex z belongs to $C_{1,5}$, and $c = 1$. In this case, the fan $X_z(1)$ stills contains the vertex z_r which is missing the color 1. Again by Proposition 59, this is a contradiction.

Therefore, the fan \mathcal{Z} is a cycle entangled with \mathcal{V} and $\mathcal{U}_{\geq u_{l-t-2}}$ and thus $Q(t)$ is true as desired. \square

We are now ready to prove that $P(i)$ is true for all i .

4.6 Proof of $P(i)$

Proof of Lemma 33. Let $i \geq 0$, \mathcal{V} be a minimum cycle in a coloring β , u and u' two vertices of \mathcal{V} , $\mathcal{U} = X_u(m(u')) = (uu_1, \dots, uu_l)$, and assume that $P(i)$ not verified. Without loss of generality, we assume that i is minimum, and that the vertices v , u and u' are respectively missing the colors 1, 2 and 3. By Lemma 40, the property $P(0)$ is true, so $i > 0$. Assume that $\beta(uu_{l-i}) \neq 1$ and let $\mathcal{X} = X_v(\beta(uu_{l-i}))$.

Claim 60. *There is no edge in $E(\mathcal{U}_{>u_{l-i}})$ colored with a color in $\beta(\mathcal{V})$*

Proof. The proof is similar to the proof of Claim 58 of Lemma 56. \square

We first prove that $P_{weak}(i)$ is true (i.e. that \mathcal{X} is not a path).

Claim 61. *The property $P_{weak}(i)$ is true.*

Proof. Assume that $\beta'(uu_{l-i}) \neq 1$ and that $\mathcal{X} = X_v(\beta(uu_{l-i}))$ is a path. Then we have that $\beta(uu_{l-i}) \notin \beta(\mathcal{V})$. Without loss of generality, we assume that $\beta(uu_{l-i}) = 4$. Moreover, we have that $m(u_{l-i}) \neq 1$. Since $P(j)$ is true for all $j < i$, for all $j < i$, if $\beta(uu_{l-j}) \neq 1$, then $X_u(\beta(uu_{l-j}))$ is a saturated cycle. We now invert \mathcal{X} until we reach a coloring where $X_v(4)$ is a path of length 1; we denote by z the only vertex of this coloring. Up to a relabeling of the colors, we assume that v is also missing the color 1 in β' . The coloring β' is \mathcal{V} -equivalent to the coloring β , so \mathcal{V} is the same minimum cycle in the coloring β . So by Lemma 24 the fan $\mathcal{U}' = X_u(m(u')) = (uu'_1, \dots, uu'_l)$ is a cycle entangled with \mathcal{V} . Moreover, the coloring β' is $(\bigcup_{j < i} X_v(\beta(uu_{l-j})))$ -equivalent to β , so by Lemma 51, for any $j \leq i$, $u'_{l-j} = u_{l-j}$, the fan $X_v(\beta'(uu'_{l-j}))$ is a saturated cycle containing u'_{l-j-1} . So in particular, $uu'_{l-i} \in E(\mathcal{U}')$, and there is a vertex missing the color 4 in \mathcal{U}' . Let z' be this vertex. Note that since $X_v(4)$ is a path, for all $j < i$, $X_v(\beta'(uu'_{l-j}))$ does not contain the vertex z' .

We now swap the edge vz , and denote by β'' the coloring obtained after the swap. If the coloring β'' is \mathcal{U}' -equivalent to β' , then it means that $v \notin V(\mathcal{U}')$. So in the coloring β'' the fan $X_u(3) = \mathcal{U}'$ contains the vertex z' which is still missing the color 4. This color is also the missing color of the vertex v . Thus, \mathcal{U}' is not entangled with \mathcal{V} , and by Lemma 24, we have a contradiction.

So the vertex v belongs to $V(\mathcal{U}')$, and in the coloring β' , the fan $X_u(1)$ contains the vertex z' which is missing the color 4. If there is an edge uu'' of $E(V_u(1)_{\leq z'})$ colored with a color of β'' , then $X_u(\beta''(uu''))$ is not entangled with \mathcal{V} , so by Lemma 24, we have a contradiction. Therefore, the subfan $X_u(1)_{\leq z'}$ is a (\mathcal{V}, u) -independent subfan avoiding v . The coloring β'' is $\mathcal{U}'_{\leq v}$ -equivalent to the coloring β' , so in the coloring β'' , the fan $\mathcal{U}'' = X_u(3)$ is equal to $(uu'_1, \dots, uv, uu'_{l-i}, \dots, uu'_l = uu')$.

Since $P(j)$ is true for all $j < i$, for all $j < i$ the fan $X_v(\beta''(uu'_{l-j}))$ is a saturated cycle containing u'_{l-j-1} . In particular, the fan $X_v(\beta''(uu'_{l-(i-1)}))$ is a saturated cycle containing

u'_{l-i} . Without loss of generality, we assume that $m(u'_{l-i}) = 5$. The vertex u'_{l-i} belongs to the component $K_v(4, 5)$, so the vertex z' does not belong to this component. We now swap the component $C_{4,5} = K_{z'}(4, 5)$, and denote by β_3 the coloring obtained after the swap. Note that β_3 is \mathcal{V} -equivalent to β'' , so by Observation 20 the cycle \mathcal{V} is the same minimum cycle in the coloring β_3 . The coloring β_3 is also \mathcal{U}'' -equivalent to β'' , so we still have that $X_u(3) = (uu'_1, \dots, uv, uu'_{l-i}, \dots, uu'_l = uu')$.

If the vertex z does not belong to $C_{4,5}$, then we can swap back the edge vz . The fan $X_u(3) = X_u(1)$ still contains the vertex z' which is missing the color 5, and $X_v(\beta''(uu'_{l-i-1}))$ is still a saturated cycle containing the vertex u'_{l-i} . Since $P(i-1)$ is true, we have a contradiction. So the vertex z belongs to $C_{4,5}$, and in the coloring β_3 the vertex z is missing the color 5.

Since the property $P(j)$ is true for all $j < i$, by Lemma 56, the property $Q(i-1)$ is true, and so the fan $X_{u'_{l-i}}(2)$ is a cycle containing z' , and therefore there is an edge $u'_{l-i}z$. We denote by c' the color of this edge. We now swap this edge, and denote by β_4 the coloring obtained after the swap. The coloring β_4 is \mathcal{V} -equivalent to β_3 , so the fan \mathcal{V} is the same minimum cycle in the coloring β_4 by Observation 20. The coloring β_4 is also $X_u(3)_{<u'_{l-i}}$, so the vertex u'_{l-i} is still in $X_u(3)$. Note that now the vertex u'_{l-i} and z' are both missing the color c' . We now have to distinguish the case.

Case 61.1 ($c' = 1$).

In this case, the fan $X_u(1)$ contains the vertex z' missing the color 1, and the fan $X_u(3)$ contains the vertex u'_{l-i} missing the color 1. So the fan $X_u(3)$ is a comet containing two vertices missing the color 1, so by Lemma 24, we have a contradiction.

Case 61.2 ($c' \in \beta_3(\mathcal{V})$).

In this case, since $u'_{l-i} \in V(X_u(3))$, the fan $X_u(3)$ is not entangled with \mathcal{V} , so by Lemma 24, we have a contradiction.

Without loss of generality, we now assume that $c' = 6$.

Case 61.3 ($6 \in \beta_3(X_u(3)_{<u'_{l-i}})$).

In this case, the fan $X_u(3)$ is now a comet where two vertices are missing the color 6, thus by Lemma 24, we also have a contradiction.

Case 61.4 ($6 \in \beta_3(X_u(3)_{>u'_{l-i}})$).

Let $t < i$ such that $m^{\beta_3}(u'_{l-t}) = 6$. Since $P(t-1)$ is true, in the coloring β_3 , the fan $X_v(6)$ is a cycle containing u'_{l-t} . Since $P(i-1)$ is true, the fan $X_v(5)$ is a cycle containing u'_{l-i} . We first prove that in the coloring β_3 , we have $X_v(5) = X_v(6)$. In the coloring β_4 , the vertex u'_{l-i} is missing the color 6, so the fan $X_u(3)$ is equal to $(uu'_1, \dots, uv, uu'_{l-i}, uu'_{l-(t-1)}, \dots, uu'_l = uu')$, and u'_{l-i} is now the vertex missing the color 6 in this cycle. Since $P(t-1)$ is true, the fan $X_v(6)$ is now a cycle containing u'_{l-i} . The only vertices whose missing color is different in β_3 and β_4 are the vertices u'_{l-i} and z' . In the coloring β_3 , since $z' \notin X_v(6)$, if $u'_{l-i} \notin V(X_v(6))$ the coloring β_4 is $X_v(6)$ -equivalent to the coloring β_3 . This means that in the coloring β_4 , the fan $X_v(6)$ is a cycle containing the vertex u'_{l-t} , and not containing u'_{l-i} , a contradiction. So in the coloring β_3 , the vertex u'_{l-i} belongs to $X_v(6)$, and thus $X_v(5) = X_v(6)$ as desired.

So, in the coloring β_3 , the cycle $X_v(5)$ contains the vertex u'_{l-t} which is missing the color 6. We now consider the coloring β_4 . The fan $X_v(5)$ still contains the vertex u'_{l-t} which is still missing the color 6. The fan $X_v(6)$ is a saturated cycle containing the vertex u'_{l-i} , so the fan $X_v(5)$ is a comet containing $X_v(6)$ as a subfan. The cycle $X_v(6)$ is saturated, so u'_{l-i} belongs to $K_v(4, 6)$, and thus z' does not belong to this component.

We now swap the component $C_{4,6} = K_{z'}(4, 6)$, and denote by β_5 the coloring obtained after the swap. The coloring β_5 is \mathcal{V} -equivalent to β_4 , so the fan \mathcal{V} is the same minimum cycle in the coloring β_5 . Since the vertex $u'_{l-i} \notin C_{4,6}$, and $\beta_4(uu'_{l-}) = 4$, the vertex u does not

belong either to $C_{4,6}$, and therefore the coloring β_5 is $X_u(3)$ -equivalent to the coloring β_4 . The fan $X_u(1)$ still contains the vertex z' which is now missing the color 4, so the subfan $X_u(1)_{\leq z'}$ is a (\mathcal{V}, u) -independent subfan avoiding v . By Lemma 34, the fan $X_v(1)$ is a path that does not contain z' . In the coloring β_5 , the vertex z is still missing the color 5, and we still have $\beta_5(vz) = 1$. If the vertex u'_{i-t} does to belong to $C_{4,6}$, then the coloring β_5 is $X_v(5)$ -equivalent to β_4 , and therefore the fan $X_v(1)$ is a comet containing $X_v(6)$ as a subfan. So the vertex u'_{i-t} belongs to the component $C_{4,6}$, and it is now missing the color 4.

In the coloring β_5 , the fan $X_u(5)$ still contains the vertex u'_{i-t} which is now missing the color 4. So there is no edge uu'' in $E(X_u(5)_{\leq u'_{i-t}})$ colored with a color in $\beta_5(\mathcal{V})$, otherwise, $X_u(\beta_5(uu''))$ is not entangled with \mathcal{V} , and by Lemma 24 we have a contradiction. So the subfan $X_u(5)_{\leq u'_{i-t}}$ is a (\mathcal{V}, u) -independent subfan avoiding v . By Lemma 34, the fan $X_v(5)$ is a path that does not contain u'_{i-t} , a contradiction.

Case 61.5 ($6 \notin \beta_3(X_u(3)) \cap \beta_3(\mathcal{V}) \cup \{1\}$).

In the coloring β_4 , the vertex u'_{i-i} is missing the color 6, and $uu'_{i-i} \in E(X_u(3))$. So we have $X_u(6) = X_u(3)$. Since the fan $X_u(5)$ also contains the vertex u' , either $X_u(5) = X_u(6) = X_u(3)$, or $X_u(5)$ is a comet which contains $X_u(3)$ as a subfan.

Subcase 61.5.1 ($X_u(5) = X_u(3)$).

Let z'' be the vertex of $X_u(3)$ missing the color 5. Note that we may have $z'' = z$. Since $P(i-1)$ is true, the fan $X_v(5)$ is now a cycle containing z'' . But u'_{i-i} is the only vertex whose missing color is different in β_3 and β_4 , so in the coloring β_4 , the fan $X_v(5)$ still contains the vertex u'_{i-i} which is now missing the color 6. Therefore, the fan $X_v(6)$ is equal to the fan $X_v(5)$ and is a saturated cycle containing z'' and u'_{i-i} . The vertex z is still missing the color 5, so the fan $X_v(1)$ is now a comet containing $X_v(5)$ as a subfan.

Since the fan $X_v(6)$ is saturated, the vertex u'_{i-i} belongs to the component $K_{v4,6}$, and thus the vertex z' does not belong to this component. We now swap the component $C_{4,6} = K_{z'}(4,6)$, and denote by β_5 the coloring obtained after the swap. The coloring β_5 is \mathcal{V} -equivalent to β_4 , so the fan \mathcal{V} is a minimum cycle in this coloring. Now the fan $X_u(1)$ still contains the vertex z' which is now missing the color 4, so it is a (\mathcal{V}, u) -independent subfan avoiding v so by Lemma 34, the fan $X_v(1)$ is a path. But the coloring β_5 is also $X_v(1)$ -equivalent to the coloring β_4 , so the fan $X_v(1)$ is a comet. This is a contradiction.

Subcase 61.5.2 ($X_u(5)$ is a comet containing $X_u(3)$).

Let u'_{i-t} be the first vertex of $X_u(5)$ which is not in $X_u(3)$, and let z'' be the vertex of $X_u(3)$ missing the color $c_t = m(u'_{i-t})$. In the coloring β_3 , since $P(t-1)$ is true, the fan $X_v(c_t)$ is a saturated cycle containing u'_{i-t} . If the coloring β_4 is $X_v(c_t)$ -equivalent to the coloring β_3 , then in the coloring β_4 the fan $X_v(c_t)$ still contains the vertex u'_{i-t} , and thus does not contain the vertex z'' . Since $P(t-1)$ is true, we have a contradiction.

So the coloring β_4 is not $X_v(c_t)$ -equivalent to the coloring β_3 . Since u'_{i-i} and z' are the only vertices whose missing color are different in β_3 and β_4 , and $z' \notin V(X_v(c_t))$, we have that $u'_{i-i} \in V(X_v(c_t))$. In the coloring β_3 the vertex u'_{i-i} is also in $X_v(5)$, so in this coloring we have $X_v(5) = X_v(c_t)$. Therefore, the vertex u'_{i-t} also belongs to $X_v(5)$ in the coloring β_4 .

In the coloring β_4 , the vertex u'_{i-i} is now missing the color 6, and since $P(t-1)$ is true, the fan $X_v(c_t)$ is a saturated cycle containing the vertex z'' . So in this coloring, we have $X_v(c_t) = X_v(6)$. However, in this coloring, the vertex u'_{i-t} still belongs to $X_v(5)$, it also belongs to $X_u(5)$ and is still missing the color c_t . The cycle $X_v(c_t)$ is saturated, so the vertex z'' belongs to the component $K_v(4, c_t)$, and thus the vertex u'_{i-t} does not belong to this component. We now swap the component $K_{u'_{i-t}}(4, c_t)$ and denote by β_5 the coloring obtained after the swap.

The coloring β_5 is \mathcal{V} -equivalent to β_4 , so by Observation 20, the fan \mathcal{V} is a minimum cycle

in the coloring β_5 . The coloring β_5 is also $((X_u(5) \cup X_v(5)) \setminus \{u'_{l-t}\})$ -equivalent to the coloring β_4 , so the vertex u'_{l-t} still belongs to both $X_u(5)$ and $X_v(5)$. So the subfan $X_u(5)_{\leq u'_{l-t}}$ is a (\mathcal{V}, u) -independent subfan avoiding v , by Lemma 34, the fan $X_v(5)$ is a path that does not contain u'_{l-t} . Again we have a contradiction. \square

By the previous claim, we have that \mathcal{X} is not a path, we now prove that it is not a comet. Assume that $\mathcal{X} = (vx_1, \dots, vx_t)$ is a comet where x_s and x_t are missing the same color c_s . Since x_s and x_t are both missing the color c_s at least one of them is not in $K_v(1, c_s)$. Since $P(j)$ is true for all $j < i$, for all $j < i$, if $\beta(uu_{l-j}) \neq 1$, then $X_v(\beta(uu_{l-j}))$ is a cycle, so $\beta(\mathcal{X}) \cap (\bigcup_{j \in [0, i-1]} \beta(X_v(\beta(uu_{l-j})))) = \emptyset$.

Case 61.6 ($c_s \notin \beta(\mathcal{U}_{u > l-1})$).

If x_s is not in $K_v(1, c_s)$, then we swap the component $C_{1, c_s} = K_{x_s}(1, c_s)$ and obtain a coloring β' which is \mathcal{V} -equivalent to β , so the fan \mathcal{V} is the same minimum cycle in the coloring β' . In the coloring β' , the fan $X_v(4)$ is now a path. Moreover, $c_s \notin \beta(\mathcal{U}_{\geq u_{l-i}})$, and by Claim 60, so $1 \notin \beta(\mathcal{U}_{\geq u_{l-i}})$. Therefore, the coloring β' is $\mathcal{U}_{\geq u_{l-i}}$ -equivalent to the coloring β . Let $\mathcal{U}' = X_u^{\beta'}(3) = (uu'_1, \dots, uu'_l)$. Since $\beta(\mathcal{X}) \cap (\bigcup_{j \in [0, i-1]} \beta(X_v(\beta(uu_{l-j})))) = \emptyset$, the coloring β' is

$(\bigcup_{j \in [0, i-1]} \beta(X_v(\beta(uu_{l-j}))))$ -equivalent to β . So by Lemma 51, for all $j \leq i$, we have $u'_{l-i} = u_{l-i}$.

In particular $u'_{l-i} = u_{l-i}$. Since $\beta'(uu'_{l-i}) = 4$, and $P_{weak}(i)$ is true, the fan $X_v(4)$ is not a path.

Similarly, if $x_t \notin K_v(1, c_s)$, we swap the component $C_{1, c_s} = K_{x_t}(1, c_s)$. Note that \mathcal{V} is a minimum cycle, so it is saturated by Lemma 15, and thus $x_t \notin V(\mathcal{V})$. The coloring β' is therefore \mathcal{V} -equivalent to \mathcal{V} , so the fan \mathcal{V} is the same minimum cycle in this coloring. The fan $X_v(4)$ is now a path the coloring β' . Let $\mathcal{U}' = X_u^{\beta'}(3) = (uu'_1, \dots, uu'_l)$. Similarly to the previous case, the coloring β' is $\mathcal{U}_{> u_{l-i}}$ -equivalent to β and $(\bigcup_{j \in [0, i-1]} \beta(X_v(\beta(uu_{l-j}))))$ -

equivalent to β . So by Lemma 51, for all $j \leq i$, we have $u'_{l-j} = u_{l-j}$. In particular, $u'_{l-i} = u_{l-i}$, and $\beta'(uu'_{l-i}) = 4$. Since $P_{weak}(i)$ is true, the fan $X_v(4)$ is not a path, a contradiction.

Case 61.7 ($c_s \in \beta(\mathcal{U}_{> u_{l-1}})$).

Let t' be such that $m(u_{l-t'}) = c_s$. Since $P(j)$ is true for all $j < i$, the fan $X_v(c_s)$ is saturated cycle containing $u_{l-t'} = x_t$. So the vertex x_s does not belong to $K_v(1, c_s)$. We now swap the component $C_{1, c_s} = K_{x_s}(1, c_s)$ to obtain a coloring β' where $X_v(4)$ is now a path. The coloring β' is \mathcal{V} -equivalent to β , so by Observation 20, the cycle \mathcal{V} is the same minimum cycle in the coloring β' . If the vertex u does not belong to C_{1, c_s} , then the coloring β' is also \mathcal{U} -equivalent to β , and thus $X_u(3) = \mathcal{U}$. Since $\beta'(uu_{l-i}) = 4$, and $P_{weak}(i)$ is true, the fan $X_v(4)$ is not a path. this is a contradiction.

So the vertex u belongs to C_{1, c_s} , and in the coloring β' , the edge $uu_{l-(t'-1)}$ is now colored 1. Let $\mathcal{U}' = X_u^{\beta'}(3)$. The coloring β' is $\mathcal{U}_{> u_{l-t'}}$ -equivalent to β . The coloring β' is also $(\bigcup_{j \in [0, t'-1]} \beta(X_v(\beta(uu_{l-j}))))$ -equivalent to β , so by Lemma 51, for any $j \leq t'$ we have $u'_{l-j} = u_{l-j}$. In particular, $u'_{l-(t'-1)} = u_{l-(t'-1)}$. Now the edge $uu'_{l-(t'-1)}$ is colored 1, and the fan $X_v(4)$ is a path. Since $P(j)$ is true for all $j \leq t'$, by Lemma 52 there is not path around v , a contradiction.

So the fan $\mathcal{X} = (vx_1, \dots, vx_t)$ is a cycle, we now prove that it is saturated. Otherwise, there exists x_s such that $x_s \notin K_v(1, m(x_s))$. Note that since $P(j)$ is true for all $j < i$, for

all $j < i$, the fan $X_v(\beta(uu_{l-j}))$ is a saturated cycle, so $\beta(X_v(\beta(uu_{l-j}))) \cap \beta(\mathcal{X}) = \emptyset$, and in particular $x_s \notin \beta(\mathcal{U}_{>u_{l-i}})$.

Case 61.8 ($m(x_s) \neq 4$).

Without loss of generality, assume that $m(x_s) = 5$. Since x_s does not belong to $K_v(1, 5)$, we swap the component $C_{1,5} = K_{x_s}(1, 5)$ and obtain a coloring β' where $X_v(4)$ is a path. The coloring β' is \mathcal{V} -equivalent to β , so by Observation 20, the cycle \mathcal{V} is the same minimum cycle in the coloring β' . Let $\mathcal{U}' = X_u^{\beta'}(3) = (uu'_1, \dots, uu'_l)$. Moreover, $5 \notin \beta(\mathcal{U}_{\geq u_{l-i}})$, and by Claim 60, the color 1 does not appear either in $\mathcal{U}_{>u_{l-i}}$. The coloring β' is also $(\bigcup_{j \in [0, i-1]} \beta(X_v(\beta(uu_{l-j}))))$ -

equivalent to β , so by Lemma 51, for any $j \leq i$, we have $u'_{l-j} = u_{l-j}$. In particular, $u'_{l-i} = u_{l-i}$. The edge uu'_{l-i} is still colored 4 in the coloring β' and the property $P_{weak}(i)$ is true, so $X_v(4)$ is not a path, a contradiction.

Case 61.9 ($m(x_s) = 4$).

In this case, we swap the component $C_{1,4} = K_{x_s}(1, 4)$ and denote by β' the coloring obtained after the swap. If the vertex u does not belong to this component, then we are in a coloring similar to the previous case. So the vertex u belongs to $C_{1,4}$, and we have $\beta'(uu_{l-i}) = 1$. In the coloring β' is \mathcal{V} -equivalent to β , so by Observation 20, the cycle \mathcal{V} is the same minimum cycle in this coloring. The fan $X_v(4)$ is now a path in the coloring β' . Let $\mathcal{U}' = X_u^{\beta'}(3) = (uu'_1, \dots, uu'_l)$. The coloring β' is $\mathcal{U}_{>u_{l-i}}$ -equivalent to β , and is also $(\bigcup_{j \in [0, i-1]} \beta(X_v(\beta(uu_{l-j}))))$ -

equivalent to β . So by Lemma 51 for all $j \leq i$, we have $u'_{l-j} = u_{l-j}$. In particular $u'_{l-i} = u_{l-i}$. The property $P(j)$ is true for all $j < i$, and $P_{weak}(i)$ is also true, so by Lemma 52 there is no path around v . This is a contradiction.

So the fan $\mathcal{X} = (vx_1, \dots, vx_t)$ is a saturated cycle, and thus $x_t \in K_v(1, 4)$. Since $P(i)$ is false, we have $x_t \neq u_{l-i-1}$. So the vertex u_{l-i-1} which is also missing the color 4 does not belong to $K_v(1, 4)$. We now swap the component $C_{1,4} = K_{u_{l-i-1}}(1, 4)$ and denote by β' the coloring obtained after the swap. By Lemma 50, the vertex u belongs to $C_{1,4}$, there is an edge uu'' colored 1 in $\mathcal{U}_{<u_{l-i}}$, and the subfan $X_u(1)_{\leq u_{l-i}}$ is a (\mathcal{V}, u) -independent subfan. So in the coloring β' , the vertex u_{l-i} is now missing the color 1, the edge uu'' is now colored 4, and the subfan $X_u(4)_{\leq u_{l-i}}$ is a (\mathcal{V}, u) -independent subfan avoiding v . By Lemma 34, the fan $X_v(4)$ is a path. However, the coloring β' is $X_v(4)$ -equivalent to the coloring β , so the fan $X_v(4)$ is a cycle, a contradiction. \square

5 Cycles interactions

In this section we prove Lemma 23.

Proof. We first prove that all the three cycles are tight and saturated.

Claim 62. *The cycles \mathcal{V} , \mathcal{X} , and \mathcal{Y} are saturated and tight.*

Proof. As the fan \mathcal{V} is not invertible, it is saturated by Lemma 15. If \mathcal{X} or \mathcal{Y} are not saturated (without loss of generality, we can assume that \mathcal{X} is not saturated), then we swap a component $K_u(c_v, c_u)$ with u in \mathcal{X} and $u \notin K_v(c_v, c_u)$ to transform β into a coloring where \mathcal{V} is still a cycle of the same size, and where a fan around v is a path, by Lemma 22, \mathcal{V} is invertible in this coloring, and so it is in the original coloring. Similarly, assume that \mathcal{X} or \mathcal{Y} is not tight, without loss of generality, we can assume that \mathcal{X} is not tight. Then we can find two consecutive vertices of \mathcal{X} , u_i , and u_{i-1} such that the component $K_{u_{i-1}}(m(u_i), m(u_{i-1}))$ does not contain u_i . If we swap this component, we obtain a coloring where \mathcal{V} is still a cycle of the same size,

and where a fan around v is a comet, again by Lemma 22, \mathcal{V} is invertible in this coloring, and so it is in the coloring β . \square

By Lemma 24, we already have that if $(z, z') \in \mathcal{V}^2$, then $X_z(c_{z'})$ is a cycle containing z' , so we now assume that (z, z') is not in \mathcal{V}^2 .

Claim 63. *The fan \mathcal{Z} is not a path.*

Proof. As \mathcal{Z} is a path, we invert it until we reach a coloring where $m(z) \in (\beta(\mathcal{V}) \cup \beta(\mathcal{X}) \cup \beta(\mathcal{Y})) \setminus \{m_\beta(z)\}$. In this coloring, the fan \mathcal{V} is still a cycle of the same size, and, there is a fan around v which is a path or a comet, by Lemma 22, this is a contradiction. \square

Claim 64. *The fan \mathcal{Z} is entangled with \mathcal{V} , \mathcal{X} , and \mathcal{Y} .*

Proof. Let us assume that there exists $z'' \in \mathcal{Z} \setminus (\mathcal{V} \cup \mathcal{Y} \cup \mathcal{X})$ with $m(z'') \in (\beta(\mathcal{V}) \cup \beta(\mathcal{Y}) \cup \beta(\mathcal{X})) \setminus \{c_z\}$. If $m(z'') = c_v$, since the cycles are saturated by Claim 62, $K_{z''}(c_z, c_v)$ does not contain any vertex of $(\mathcal{V} \cup \mathcal{Y} \cup \mathcal{X})$, and after swapping it, we obtain a coloring where \mathcal{V} is still a cycle of the same size and where \mathcal{Z} is a path, by Claim 63, this is a contradiction. If $m(z'') \neq c_v$, then, since the cycles are saturated, the component $K_{z''}(c_v, m(z''))$ does not contain any vertex of $(\mathcal{V} \cup \mathcal{Y} \cup \mathcal{X})$, so if we swap it, we obtain a coloring which corresponds to the previous case. \square

Claim 65. *The fan \mathcal{Z} is not a comet.*

Proof. Assume that \mathcal{Z} is a comet, there exist z_1 and z_2 with $m(z_1) = m(z_2) = c$. By the previous claim, we have that $c \notin (\beta(\mathcal{V}) \cup \beta(\mathcal{X}) \cup \beta(\mathcal{Y}))$, otherwise, \mathcal{Z} is not entangled with one of these cycles. Hence, the component $K_z(c, m(z))$ either contains z_1 or z_2 , and without loss of generality we can assume that $z_1 \notin K_z(c, m(z))$. If we swap $K_{z_1}(c, m(z))$ we obtain a coloring where no edge of $(\mathcal{V} \cup \mathcal{X} \cup \mathcal{Y})$ has changed and where \mathcal{Z} is a path, by Claim 63 this is a contradiction. \square

By the previous claims, \mathcal{Z} is a cycle, and as it is entangled with the three other cycles, it contains z' ; this concludes the proof. \square

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