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Correlated Noise and Critical Dimensions

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In equilibrium, the Mermin-Wagner theorem prohibits the continuous symmetric breaking for all dimensions $d \leq 2$. In this work, we discuss that this limitation can be circumvented in non-equilibrium systems driven by spatially or temporally long-range anticorrelated noise. We first compute the lower and upper critical dimensions of the ϕ^4 model driven by spatio-temporally correlated noise by means of the dimensional analysis. Next, we consider the spherical model, which corresponds to the large n limit of the $O(n)$ model and allows us to compute the critical dimensions analytically. Both results suggest that the critical dimensions increase when the noise is positively correlated in space and time, and decrease when anticorrelated. We also report that the spherical model with the correlated noise shows the hyperuniformity and giant number fluctuation even well above the critical point.

I. INTRODUCTION

The minimum dimension required for a phase transition to occur is known as the lower critical dimension d_c [1]. For systems with quenched randomness, Imry and Ma predicted that the lower critical dimension is $d_c = 2$ for discrete symmetry breaking and $d_c = 4$ for continuous symmetry breaking [2]. Recent studies have reported that d_c can be reduced by introducing anticorrelation to the quenched randomness. For example, in Ref. [3], the authors studied the random field Ising model with anticorrelated random fiend and showed that the ordered phase arises on the ground state even in $d = 2$. Ref. [4] reported a first-order transition of the Potts model on a random Voronoi lattice in $d = 2$. This is the consequence of the strong anticorrelation in the coordination number of the random Voronoi lattices, which reduces the lower critical dimension [5].

Interestingly, a recent numerical study revealed that the anticorrelation of the time dependent noise can also cause the similar reduction of d_c. In equilibrium, the Mermin-Wagner theorem prohibits the continuous symmetric braking in all dimensions $d \leq 2$ [6]. However recently, Ref. [7] reported the appearance of a crystalline phase even in $d = 2$ in a particle system driven by center of mass conserving (COMC) dynamics. Phenomenological arguments and field theory suggest that in COMC dynamics, the fluctuation of the effective noise is suppressed by the strong long-range anticorrelation, which reduces the lower critical dimension [8].

Based on these results, it is tempting to conjecture that the anticorrelation of the noise generally reduces the lower critical dimension. To test this conjecture here we investigate the effects of the correlated noise on the second order phase transition. Interestingly, our analysis reveals that the temporal anticorrelation, as well as spatial anticorrelation, can also reduce the lower critical dimension.

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For concreteness, we consider model-A and B dynamics [1, 9] with the correlated noise $\xi(\boldsymbol{x}, t)$ of zero mean and variance

$$\langle \xi(\boldsymbol{x}, t)\xi(\boldsymbol{x}', t')\rangle = 2TD(\boldsymbol{x} - \boldsymbol{x}', t - t'), \tag{1}$$

where $D(\boldsymbol{x}, t)$ represents the spatio-temporal correlation of the noise. The Fourier transform of $D(\boldsymbol{x}, t)$ w.r.t. $\boldsymbol{x}$ and t is given by

$$D(\boldsymbol{q}, \omega) = |\boldsymbol{q}|^{-2\rho} |\omega|^{-2\theta}, \tag{2}$$

where $\boldsymbol{q}$ denotes the wave vector, and ω denotes the frequency. The same correlation function has been considered in previous works to investigate the effects of the long-range spatio-temporal correlation on the Kardar-Parisi-Zhang (KPZ) equation [10–13]. When $\rho = \theta = 0$, the noise can be identified with the white noise in equilibrium. The positive values of ρ and θ represent the positive power-law correlation in the real space: $D(\boldsymbol{x}, t) \sim |\boldsymbol{x}|^{2\rho-d} |t|^{2\theta-1}$, where d denotes the spatial dimension. In the limit $\theta \to 1/2$, the noise does not decay and can be identified with the correlated quenched randomness. The negative values of ρ and θ imply the existence of the anticorrelation because $D(\boldsymbol{q} = 0, \omega = 0) = \int d\boldsymbol{x} \int dt D(\boldsymbol{x}, t) = 0$. Therefore, the model can smoothly connect the white noise ($\rho = \theta = 0$), quenched randomness ($\theta \to 1/2$), positively correlated noise ($\theta > 0$ and $\rho > 0$), and anticorrelated noise ($\theta > 0$ and $\rho > 0$). In this work, we discuss that the positive correlation increases the lower and upper critical dimensions d_l and d_u, and the anticorrelation reduces d_l and d_u.

The structure of the paper is as follows. In Sec. II, we investigate the ϕ^4 model driven by the model-A dynamics with the correlated noise by means of the dimensional analysis. Sec. III, we investigate the spherical model, which is the $n \to \infty$ limit of the $O(n)$ model. Interestingly, our analysis shows that the model exhibits the giant number fluctuation and hyperuniformity even well above the critical point. In Sec. IV, we discuss the behavior of the conserved order parameter driven by the model-B dynamics with the correlated noise. In Sec. V, we summarize the work.

II. DIMENSIONAL ANALYSIS

Here we derive the upper and lower critical dimensions of the ϕ^4 model driven by the model-B dynamics with the correlated noise.

A. Model

Let $\phi(\boldsymbol{x}, t)$ be a non-conserved order parameter such as the magnetization. The time evolution of $\phi(\boldsymbol{x}, t)$ may follow the model-A dynamics [9]:

$$\frac{\partial \phi(\boldsymbol{x}, t)}{\partial t} = -\Gamma \frac{\delta F[\phi]}{\delta \phi(\boldsymbol{x}, t)} + \xi(\boldsymbol{x}, t), \tag{3}$$

where Γ denotes the damping coefficient, ξ denotes the noise, and d denotes the spatial dimension. $F[\phi]$ denotes the standard ϕ^4 free-energy [1]:

$$F[\phi] = \int d\boldsymbol{x} \left[\frac{(\nabla \phi)^2}{2} + \frac{\varepsilon \phi^2}{2} + \frac{g \phi^4}{4} \right], \tag{4}$$

where k denotes the stiffness, ε denotes the linear distance to the transition point, and g denotes the strength of the non-linear term. The mean and variance of the noise $\xi(\boldsymbol{x}, t)$ are

$$\begin{aligned} &\langle \xi(\boldsymbol{x}, t) \rangle = 0, \\ &\langle \xi(\boldsymbol{x}, t) \xi(\boldsymbol{x}', t') \rangle = 2T\Gamma D(\boldsymbol{x} - \boldsymbol{x}', t - t'), \end{aligned} \tag{5}$$

where $D(\boldsymbol{x}, t)$ represents the correlation of the noise. We assume that the correlation in the Fourier space is written as [10–13]

$$D(\boldsymbol{q}, \omega) = |\boldsymbol{q}|^{-2\rho} |\omega|^{-2\theta}. \tag{6}$$

To ensure the existence of the Fourier transform of $D(\boldsymbol{q}, \omega)$, the values of ρ and θ are constrained to $\rho < d/2$ and $-1/2 < \theta < 1/2$. The noise can be generated, for instance, by integrating uncorrelated white noise $\eta(\boldsymbol{x}, t)$ with a proper kernel $K(\boldsymbol{x}, t)$:

$$\xi(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} dt \int d\boldsymbol{x} K(\boldsymbol{x} - \boldsymbol{x}', t - t') \eta(\boldsymbol{x}', t'), \tag{7}$$

where $K(\boldsymbol{x}, t)$ satisfies $K(\boldsymbol{x}, t) = 0$ for $t < 0$ and $|K(\boldsymbol{q}, \omega)| \sim |\boldsymbol{q}|^{-\rho} |\omega|^{-\theta}$ in the Fourier space [10]. The model satisfies the fluctuation dissipation theorem only when $\rho = \theta = 0$ [14]. In this case, the model exhibits the Ising universality at the critical point $\varepsilon = 0$ [1]. For $\rho \neq 0$ or $\theta \neq 0$, the model does not satisfy the detailed balance, and the steady state distribution would be different from the Maxwell-Boltzmann distribution.

B. Critical dimensions

From Eqs. (3) and (4), we get

$$\dot{\phi} = -\Gamma(-\nabla^2 \phi + \varepsilon \phi + g \phi^3) + \xi \tag{8}$$

Now we consider the following scaling transformations: $x \to bx$, $t \to b^{z_t} t$, $\phi \to b^{z_\phi} \phi$, $g \to b^{z_g} g$ [1]. To calculate the scaling dimension of the noise, we observe the fluctuation induced by the noise in $d+1$ dimensional Euclidean space $[0, l]^d \times [0, t]$ [15]:

$$\sigma(l, t)^2 \equiv \left\langle \left(\int_{\boldsymbol{x}' \in [0, l]^d} d\boldsymbol{x}' \int_0^t dt' \xi(\boldsymbol{x}', t') \right)^2 \right\rangle. \tag{9}$$

The asymptotic behavior for $l \gg 1$ and $t \gg 1$ is

$$\sigma(l, t)^2 \sim t^{1+2\theta} \left(a l^{d+2\rho} + b l^{d-1} \right), \tag{10}$$

where a and b denote constants, and the second term in RHS accounts for the surface contribution [15, 16]. Eq. (10) implies $\xi(\boldsymbol{x}, t) \to b^{z_t(2\theta-1)/2} b^{(-1-d)/2} \xi(\boldsymbol{x}, t)$ for $\rho < -1/2$, and $\xi(\boldsymbol{x}, t) \to b^{z_t(2\theta-1)/2} b^{(2\rho-d)/2} \xi(\boldsymbol{x}, t)$ for $\rho > -1/2$. Assuming the scaling invariance of the dynamics Eq. (8), we get [1, 17]

$$\begin{aligned} z_t &= 2, \\ z_g &= -2z_\phi - z_t, \\ z_\phi &= \begin{cases} 1 + \frac{-1-d}{2} + 2\theta & \rho < -1/2 \\ 1 + \frac{2\rho - d}{2} + 2\theta & \rho > -1/2. \end{cases} \end{aligned} \tag{11}$$

The simplest way to calculate the lower critical dimension d_l is to observe the fluctuation of the order parameter:

$$\langle \delta \phi^2 \rangle \sim b^{2z_\phi}. \tag{12}$$

To ensure the stability of the ordered phase, z_ϕ must be negative; otherwise, the fluctuation of the order parameter diverges in the thermodynamic limit $b \to \infty$, which destroys the long-range order. Therefore, the lower-critical dimension can be determined by setting $z_\phi = 0$, leading to

$$d_l = \begin{cases} 1 + 4\theta & \rho < -1/2 \\ 2 + 2\rho + 4\theta & \rho > -1/2. \end{cases} \tag{13}$$

When $z_g < 0$, the non-linear term is negligible for very large system $b \gg 1$, and vice versa. Therefore, the upper critical dimension is obtained by setting $z_g = 0$, leading to

$$d_u = \begin{cases} 3 + 4\theta & \rho < -1/2 \\ 4 + 2\rho + 4\theta & \rho > -1/2. \end{cases} \tag{14}$$

When $\rho = \theta = 0$, we get $d_l = 2$ and $d_u = 4$, which is consistent with the standard ϕ^4 model in equilibrium [1].

C. Correlated random Field

In the limit $\theta \to 1/2$, the noise does not decay with time $D(\boldsymbol{x}, t) \sim t^{2\theta-1} \to t^0$ and can be identified with the correlated random field. In this case, we get

$$d_l^{\mathrm{RF}} = \begin{cases} 3 & \rho < -1/2 \\ 4 + 2\rho & \rho > -1/2. \end{cases} \tag{15}$$

It would be instructive to compare the above results with the standard Imry-Ma argument for the lower critical dimension [2, 3, 16]. In a domain of linear size l, the typical fluctuation induced by the correlated random field is $\sigma^2 \equiv \left\langle \left(\int_{\boldsymbol{x}\in[0,l]^d} d\boldsymbol{x} h \right)^2 \right\rangle \sim a l^{d+2\rho} + b l^{d-1}$ [3, 16]. The domain wall energy is $\gamma \sim l^{d-1}$ for discrete variables, and $\gamma \sim l^{d-2}$ for continuous variables [2]. When $\sigma \gg \gamma$, the fluctuation of the random field destroys the ordered phase, and vise versa. Therefore, on the lower critical dimension d_l, $\sigma \sim \gamma$, leading to [3]

$$d_l^{\mathrm{I.M.}} = \begin{cases} \begin{cases} 1 & \rho < -1/2 \\ 2+2\rho & \rho > -1/2 \end{cases} & \text{(discrete)} \\ \\ \begin{cases} 3 & \rho < -1/2 \\ 4+2\rho & \rho > -1/2 \end{cases} & \text{(continuous)} \end{cases} . \tag{16}$$

The result for the continuous variable is consistent with that of the dimensional analysis Eq. (15).

III. SPHERICAL MODEL

Here we analyze the spherical model driven by the model-A dynamics with the correlated noise.

A. Model

Now we consider the spherical model [18]. The effective free-energy of the model is

$$F[\phi] = \int d\boldsymbol{x} \left[\frac{(\nabla\phi)^2}{2} + \frac{\mu\phi^2}{2} \right], \tag{17}$$

where μ denotes the Lagrange multiplier to impose the spherical constraint:

$$\int d\boldsymbol{x} \left\langle \phi(\boldsymbol{x})^2 \right\rangle = N. \tag{18}$$

In equilibrium the model has a critical point at finite temperature T_c, on which the correlation length and relaxation time diverge. The universality class at T_c can be identified with the large n limit of the $O(n)$ model [1].

B. Steady-state solution

By substituting Eq. (17) into Eq. (3) we get a linear differential equation:

$$\dot{\phi} = -\Gamma(-\nabla^2\phi + \mu\phi) + \xi. \tag{19}$$

This can be easily solved in the Fourier space:

$$\phi(\boldsymbol{q},\omega) = \frac{\xi(\boldsymbol{q},\omega)}{i\omega + \Gamma(q^2+\mu)}, \tag{20}$$

where

$$\mathcal{O}(\boldsymbol{q},\omega) = \int dt \int d\boldsymbol{x} e^{-i\boldsymbol{q}\cdot\boldsymbol{x} - i\omega t} \mathcal{O}(\boldsymbol{x},t). \tag{21}$$

The density-density correlation is calculated as

$$\langle \rho(\boldsymbol{q},\omega)\rho(\boldsymbol{q}',\omega') \rangle = (2\pi)^{d+1} \delta(\boldsymbol{q}+\boldsymbol{q}')\delta(\omega+\omega') S(q,\omega), \tag{22}$$

where

$$\begin{aligned} S(q,\omega) &\equiv \int dt \int d\boldsymbol{x} e^{i\boldsymbol{q}\cdot\boldsymbol{x} + i\omega t} \langle \rho(\boldsymbol{x},t)\rho(0,0) \rangle \\ &= \frac{2T\Gamma D(\boldsymbol{q},\omega)}{\omega^2 + \Gamma^2(kq^2+\mu)^2}. \end{aligned} \tag{23}$$

C. Correlation length and relaxation time

Since we are interested in the critical behaviors in a large spatio-temporal scale, here we analyze the scaling behavior of the correlation function for $|\boldsymbol{q}| \ll 1$ and $\omega \ll 1$. After some manipulations, we get

$$S(q,\omega) = T\mu^{-2-\rho-2\theta} \mathcal{S}(\mu^{-1/2} q, \mu^{-1}\omega), \tag{24}$$

where

$$\mathcal{S}(x,y) = \frac{2\Gamma x^{-2\rho} y^{-2\theta}}{y^2 + \Gamma^2(x^2+1)^2}. \tag{25}$$

The scaling Eq. (24) implies that the correlation length ξ and relaxation time τ behave as

$$\xi \sim \mu^{-1/2}, \ \tau \sim \xi^z, \tag{26}$$

with the dynamic critical exponent

$$z = 2. \tag{27}$$

The correlation length and relaxation time diverge in the limit $\mu \to 0$, meaning that $\mu = 0$ defines the critical point.

D. Static structure factor

The static structure factor $S(\boldsymbol{q})$ is calculated as

$$S(\boldsymbol{q}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega S(\boldsymbol{q},\omega) = \frac{ATq^{-2\rho}}{(q^2+\mu)^{1+2\theta}} \tag{28}$$

where

$$A = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|x|^{-2\theta} dx}{x^2+1} = \sec(\pi\theta). \tag{29}$$

$S(q)$ shows the power low behavior for $q \ll \mu^{1/2} \approx \xi^{-1}$:

$$S(q) \sim q^{-2\rho}. \tag{30}$$

For $\rho > 0$, $S(q) \to \infty$ for small q, leading to the power-low correlation

$$G(\boldsymbol{x}) = \langle \rho(\boldsymbol{x})\rho(0)\rangle \sim |\boldsymbol{x}|^{2\rho-d}. \tag{31}$$

As a consequence, the fluctuation of the order parameter in the d-dimensional square box $[0,l]^d$ behaves as [15]

$$\sigma(l)^2 \equiv \left\langle \left(\int_{\boldsymbol{x}\in[0,l]^d} d\boldsymbol{x}\phi(\boldsymbol{x}) \right)^2 \right\rangle \sim l^{d+2\rho}, \tag{32}$$

which is much greater than the naive expectation from the central limit theorem $\sigma^2 \sim l^d$. This anomalous enhancement of the fluctuation is referred to as the giant number fluctuation [19]. For $\rho < 0$, $S(q) \to 0$ in the limit $q \to 0$. In this case, the fluctuation of the order parameter Eq. (32) is highly suppressed. This anomalous suppression of the fluctuation is referred to as the hyperuniformity [15].

E. Lagrange multiplier

The remaining task is to determine the Lagrange multiplier μ by the spherical constraint:

$$N = \int d\boldsymbol{x} \left\langle \rho(\boldsymbol{x},t)^2 \right\rangle = \frac{V}{(2\pi)^d} \int d\boldsymbol{q} S(q), \tag{33}$$

where $V = \int d\boldsymbol{x}$ denotes the volume of the system. Substituting Eq. (50) into Eq. (33), we get

$$1 = TA' \int_0^{q_D} dq \frac{q^{d-1-2\rho}}{(q^2+\mu)^{1+2\theta}}, \tag{34}$$

where q_D denotes the cut-off and

$$A' = \frac{\Omega_d A}{(2\pi)^d \phi}. \tag{35}$$

Substituting $\mu = 0$ into Eq. (34), one can calculate the critical temperature T_c as follows:

$$T_c = \begin{cases} 0 & d \le d_l \\ (d-d_l)/A' q_D^{d-d_l} & d > d_l \end{cases}, \tag{36}$$

where we have defined the lower critical dimension as

$$d_l = 2 + 2\rho + 4\theta. \tag{37}$$

This is consistent with the result of the dimensional analysis Eq. (13) for $\rho > -1/2$. For $\rho < -1/2$, the results are inconsistent. Unfortunately, we currently lack an intuitive argument to explain this discrepancy. The detailed analysis of Eq. (34) near T_c leads to (see Appendix. A)

$$\mu \sim (T - T_c)^\gamma \tag{38}$$

with

$$\gamma = \begin{cases} \frac{2}{d-d_l} & d_l < d < d_u \\ 1 & d > d_u, \end{cases}, \tag{39}$$

where the upper critical dimension d_u is

$$d_u = 4 + 2\rho + 4\theta. \tag{40}$$

Again the result is consistent with that of the dimensional analysis for $\rho > -1/2$. Substituting this result into Eq. (26), we can determine the critical exponent:

$$\xi \sim (T - T_c)^{-\nu} \tag{41}$$

with

$$\nu = \begin{cases} 1/(d-2-2\rho-4\theta) & d_l < d < d_u \\ 1/2 & d > d_u \end{cases}. \tag{42}$$

Therefore, we get a different value of the critical exponent from that in equilibrium if $2\rho + 4\theta \neq 0$, in other words, the long-range spatio-temporal correlation of the noise changes the universality class.

Although there is a slight difference in the results when $\sigma < -1/2$, the results from the dimensional analysis of the ϕ^4 model and the analytical solution for the spherical model both predict that positively correlated noise ($\rho > 0$, $\theta > 0$) will increase the critical dimensions d_l and d_u and anticorrelated noise ($\rho < 0$, $\theta < 0$) will decrease d_l and d_u. We hope that future studies will verify this prediction by using, for instance, renormalization group calculations and numerical simulations.

IV. CONSERVED ORDER PARAMETER

Let $\phi(\boldsymbol{x},t)$ be a conserved order parameter such as density. The time evolution of $\phi(\boldsymbol{x},t)$ may follow the model-B dynamics [9]:

$$\frac{\partial\phi(\boldsymbol{x},t)}{\partial t} = \Gamma\nabla^2 \frac{\delta F[\rho]}{\delta\phi(\boldsymbol{x},t)} + \nabla\cdot\boldsymbol{\xi}(\boldsymbol{x},t), \tag{43}$$

where Γ denotes the damping coefficient, $\boldsymbol{\xi} = \{\xi_a\}_{a=1,\dots,d}$ denotes the noise, and d denotes the spatial dimension. The mean and variance of the noise $\xi_a(\boldsymbol{x},t)$ are given by

$$\begin{aligned} &\langle \xi_a(\boldsymbol{x},t)\rangle = 0, \\ &\langle \xi_a(\boldsymbol{x},t)\xi_b(\boldsymbol{x}',t')\rangle = 2T\delta_{ab}\Gamma D(\boldsymbol{x}-\boldsymbol{x}',t-t'), \end{aligned} \tag{44}$$

where the Fourier transform of $D(\boldsymbol{x},t)$ is given by Eq. (6).

A. Dimensional analysis for ϕ^4 model

Substituting the ϕ^4 free-energy Eq. (4) into Eq. (43), we get

$$\dot{\phi} = \Gamma\nabla^2(-\nabla^2\phi + \varepsilon\phi + g\phi^3) + \nabla\cdot\boldsymbol{\xi}. \tag{45}$$

As before, we consider the scaling transformations: $x \to bx$, $t \to b^{z_t}t$, $\phi \to b^{z_\phi}\phi$, $g \to b^{z_g}g$ [1]. Assuming the scaling invariance of the dynamic equation Eq. (45), we get

$$\begin{aligned} z_t &= 4, \\ z_g &= 2 - 2z_\phi - z_t, \\ z_\phi &= \begin{cases} 1 + \frac{-1-d}{2} + 4\theta & \rho < -1/2 \\ 1 + \frac{2\rho - d}{2} + 4\theta & \rho > -1/2. \end{cases} \end{aligned} \tag{46}$$

As before, the lower critical dimension is calculated by setting $z_\phi = 0$, leading to

$$d_l = \begin{cases} 1 + 8\theta & \rho < -1/2 \\ 2 + 2\rho + 8\theta & \rho > -1/2 \end{cases}. \tag{47}$$

The upper critical dimension is obtained by setting $z_g = 0$, leading to

$$d_l = \begin{cases} 3 + 8\theta & \rho < -1/2 \\ 4 + 2\rho + 8\theta & \rho > -1/2 \end{cases}. \tag{48}$$

When $\theta = 0$, the results are consistent with those of the model-A, see Sec. II, while when $\theta \neq 0$, we get different results. Although such a minor difference, the qualitative conclusion remains the same: the positive correlation of the noise ($\rho > 0$ and $\theta < 0$) increases the critical dimensions d_l and d_u, while the anticorrelation ($\rho < 0$ and $\theta < 0$) reduces d_l and d_u.

B. Spherical model

Substituting the free-energy of the spherical model Eq. (17) into Eq. (43), we get

$$\dot{\phi} = \Gamma \nabla^2 (-\nabla^2 \phi + \mu \phi) + \nabla \cdot \boldsymbol{\xi}. \tag{49}$$

one can solve it easily since this is a linear equation. For instance, the static structure factor $S(\boldsymbol{q})$ in the steady state is calculated as

$$S(\boldsymbol{q}) = \frac{BTq^{-2\rho - 4\theta}}{(kq^2 + \mu)^{1+2\theta}} \tag{50}$$

where B denotes a constant.

$$B = \frac{1}{\Gamma^{2\theta}\pi} \int_{-\infty}^{\infty} \frac{|x|^{-2\theta}\, dx}{x^2 + 1} = \frac{\sec(\pi\theta)}{\Gamma^{2\theta}}. \tag{51}$$

$S(q)$ shows the power low behavior for $q \ll \mu^{1/2} \approx \xi^{-1}$:

$$S(q) \sim q^{-2\rho - 4\theta}. \tag{52}$$

For $2\rho + 4\theta > 0$, $S(q) \to \infty$ for small q, leading to the giant number fluctuation [19]. On the contrary, for $2\rho + 4\theta < 0$, $S(q) \to 0$ for small q, leading to the hyperuniformity [15]. Interestingly, the giant number fluctuation and hyperuniformity appear even without the spatial correlation of the noise $\rho = 0$.

The Lagrange multiplier μ is to be determined by the spherical constraint $N = \int d\boldsymbol{x} \left\langle \rho(\boldsymbol{x})^2 \right\rangle$. As before, the detailed analysis of this equation allows us to calculate the lower and upper critical dimensions (see also Appendix. A):

$$\begin{aligned} d_l &= 2 + 2\rho + 8\theta, \\ d_u &= 4 + 2\rho + 8\theta. \end{aligned} \tag{53}$$

For $\rho > -1/2$, the results are consistent with the dimensional analysis in the previous sub-section. On the contrary, for $\rho < -1/2$, we get inconsistent results. Further studies would be beneficial to elucidate the origin of this discrepancy. Aside from such a minor difference, the both ϕ^4 and spherical models predict that the positive correlation of the noise ($\rho > 0$ and $\theta > 0$) increases the critical dimensions, d_l and d_u, while the anticorrelation reduces d_l and d_u.

C. Center of mass conserving dynamics

An interesting application is for the systems driven by the center of mass conserving (COMC) dynamics [8, 20]. In Ref. [8], Hexner and Levine introduced a stochastic dynamics that conserves the center of mass of particles in addition to the density, as a phenomenological model of periodically shared particles [21]. They argued that the noise term of the COMC dynamics should be $\nabla^2 \xi$, instead of $\nabla \xi$ in the standard model-B dynamics [8, 20]. Assuming ξ is a white noise, this modification leads to

$$\left\langle \nabla^2 \xi(\boldsymbol{x}, t) \nabla^2 \xi(\boldsymbol{x}', t') \right\rangle \sim \nabla^4 \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t'), \tag{54}$$

which is tantamount to set $D(\boldsymbol{q}, \omega) \sim q^2$ in our model, *i.e.*, $\rho = -1$ and $\theta = 0$. Although, the precise value of d_l depends on the model, $d_l = 1$ for the ϕ^4-model and $d_l = 0$ for the spherical-model, the both models predict that the continuous symmetric breaking can occur in $d = 2$, in contrast with the equilibrium systems for which the Mermin-Wagner theorem prohibits the phase transition. To the best of our knowledge, the effects of the COMC dynamics on the critical phenomena have not been fully investigated numerically. But somehow related numerical work appeared quite recently [7], where the authors studied the crystallization of a particle system driven by the COMC dynamics and reported the emergence of the crystal phase even in $d = 2$. This result implies that the COMC dynamics indeed reduces the lower critical dimension.

V. SUMMARY AND DISCUSSIONS

In this work, we calculated the lower and upper critical dimensions, d_l and d_u, of the ϕ^4 and spherical models driven by the model A and B dynamics with the correlated noise $\xi(\boldsymbol{x}, t)$, for summary see Table. I. The correlation of the noise is written in the Fourier space as

TABLE I. Critical dimensions

Model-A	d_l	d_u
ϕ^4 model, $\rho \leq -1/2$	$1+4\theta$	$3+4\theta$
ϕ^4 model, $\rho > -1/2$	$2+2\rho+4\theta$	$4+2\rho+4\theta$
Spherical model	$2+2\rho+4\theta$	$4+2\rho+4\theta$
Model-B	d_l	d_u
ϕ^4 model, $\rho \leq -1/2$	$1+8\theta$	$3+8\theta$
ϕ^4 model, $\rho > -1/2$	$2+2\rho+8\theta$	$4+2\rho+8\theta$
Spherical model	$2+2\rho+8\theta$	$4+2\rho+8\theta$

$D(\boldsymbol{q},\omega) = |\boldsymbol{q}|^{-2\rho} |\omega|^{-2\theta}$. Our results imply that the positive correlation of the noise ($\rho > 0$ and $\theta > 0$) increases the critical dimensions, d_l and d_u, while the anticorrelation, ($\rho < 0$ and $\theta < 0$) reduces d_l and d_u. We also found that the static structure factor $S(q)$ in the paramagnetic phase exhibits the power-low behavior for small wave number $S(q) \sim q^\alpha$ with $\alpha = -2\rho$ for the non-conserved order parameter (model-A) and $\alpha = -2\rho - 4\theta$ for the conserved order parameter (model-B), leading to the giant number fluctuation for $\alpha < 0$, and hyperuniformity for $\alpha > 0$.

The temporally correlated noise for $-1/2 < \theta < 1/2$ has been studied extensively in the context of the anomalous diffusion in crowded environments, because a free particle driven by the noise $\dot{x} = \xi$ exhibits the subdiffusion $\left\langle x(t)^2 \right\rangle \sim t^{1+2\theta}$ for $-1/2 < \theta < 0$ and superdiffusion for $0 < \theta < 1/2$, see Refs. [22, 23] for reviews. However, relatively few studies have been done on the effects of the temporal correlation on the critical phenomena. For example, in Refs. [17, 24, 25], the authors studied the effect of exponentially correlated noise on the φ^4 model and found the same universality as the equilibrium Ising model. In Ref. [26], the authors studied the $O(n)$ model with the power-low correlated noise, but the noise was introduced in a way that preserves the detailed balance. Thus, the critical dimensions and the static critical exponents are unchanged from those in equilibrium. On the contrary, non-equilibrium noises, such as the $1/f$ noises, often show the power-low frequency dependence of the power spectrum, naturally leading to the long-range temporal correlation [27–31]. Therefore, it is important to investigate the effects of the long-range power-low correlation of the noise on the critical phenomena to understand the non-equilibrium phase transitions. Our research has demonstrated the emergence of novel phenomena, such as giant number fluctuation, hyperuniformity, and changes in critical dimensions in systems driven by such long-range temporally correlated noise. We hope that our findings will motivate further investigation into the fascinating properties of these systems.

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Appendix A: Scaling of μ

To determine μ, one should solve the following self-consistent equation:

$$1 = TF(\mu) \equiv TA \int_0^{q_D} dq \frac{q^{d-1+m}}{(q^2+\mu)^n}, \tag{A1}$$

where A, n, and m are constants. We want to derive the scaling behavior of μ near the critical point:

$$T_c = \left[A \int_0^{q_D} dq q^{d-1+m-2n} \right]^{-1}. \tag{A2}$$

For $d+m-2n > 0$, the denominator of Eq. (A2) diverges, and thus the model does not have the critical point at finite T. This implies that the lower critical dimension is

$$d_l = 2n - m. \tag{A3}$$

When $d > 2n - m + 2$, $F(\mu)$ can be expanded as

$$\begin{aligned} \frac{1}{T} &= F(0) + \mu F'(0) + \cdots \\ &= \frac{1}{T_c} + \mu F'(0) + \cdots, \end{aligned} \tag{A4}$$

leading to

$$\mu \sim (T - T_c)^1. \tag{A5}$$

On the contrary, if $d \in (2n-m, 2n-m+2)$, $F'(\mu)$ for small μ behaves as

$$F'(\mu) \sim \mu^{\frac{d+m-(2n+2)}{2}}, \tag{A6}$$

implying

$$F(\mu) - F(0) = \int_0^\mu d\mu' F'(\mu') \sim \mu^{\frac{d+m-2n}{2}}, \tag{A7}$$

leading to

$$\frac{1}{T} = F(\mu) = \frac{1}{T_c} - B\mu^{\frac{d+m-2n}{2}}, \tag{A8}$$

where B is a constant. Therefore, the scaling of μ for $\mu \ll 1$ is

$$\mu \sim (T - T_c)^{\frac{2}{d+m-2n}} \sim (T - T_c)^{\frac{2}{d-d_l}}. \tag{A9}$$

The above results imply that the upper critical dimension is

$$d_u = 2n - m + 2 = d_l + 2. \tag{A10}$$

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