ON COVERING DIMENSION AND SECTIONS OF VECTOR BUNDLES

M. C. CRABB

ABSTRACT. An elementary result in point-set topology is used, with knowledge of the mod 2 cohomology of real projective spaces, to establish classical results of Lebesgue and Knaster-Kuratowski-Mazurkiewicz, as well as the topological central point theorem of Karasev, which is applied to deduce results of Helly-Lovász, Bárány and Tverberg.

1. INTRODUCTION

Throughout this note, X will be a compact Hausdorff topological space and ξ_1, …, ξ_n ($n \geqslant 1$) will be n finite-dimensional real vector bundles over X. We write $\xi = \xi_1 \oplus \cdots \oplus \xi_n$.

Suppose that $A_1, \ldots, A_n$ are closed subspaces covering X and that, for each $k \in \{1, \ldots, n\}$, the restriction of ξ_k to A_k admits a nowhere zero section, s'_k say. By Tietze's theorem, s'_k extends to a section s_k of ξ_k on X. Then $s = (s_1, \ldots, s_n)$ is a nowhere zero section of ξ. (And conversely, if ξ admits a nowhere zero section $s = (s_k)$, we may construct such closed subspaces A_k by choosing an inner product on each ξ_k and setting $A_k = \{x \in X \mid \|s_k(x)\| = \max\{\|s_j(x)\| \mid 1 \leqslant j \leqslant n\}\}$.) The main result of this paper, modelled on a cohomological lemma [8, Lemma 3.2] of Karasev and stated as Theorem 2.1, is in the same vein as this classical observation. The methods are from elementary point-set topology. As applications we derive in Sections 3 and 4, using ideas introduced by Karasev in [7, 8], a classical result of Lebesgue and Knaster-Kuratowski-Mazurkiewisz and the more recent topological central point theorem of Karasev with, as corollaries, results of Helly-Lovász and Bárány [2].

When we discuss Euler classes, we shall use representable cohomology as, for example, in [5, Section 8].

From the foregoing summary it should be clear that most of the ideas presented in this note derive from the paper [8] of Karasev. It is hoped, nevertheless, that the elementary approach taken here may have some conceptual advantages.

2. THE PRINCIPAL RESULT

Theorem 2.1. *Let ξ_1, …, ξ_n be n finite-dimensional real vector bundles over a compact Hausdorff topological space X. Suppose that $(U_i)_{i \in I}$ is a finite open cover of X such that each point of X lies in U_i for at most n indices $i \in I$.*

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Suppose that for each $i \in I$ and $k \in \{1, \dots, n\}$ there exists a section of ξ_k with no zeros in U_i. Then $\xi = \xi_1 \oplus \cdots \oplus \xi_n$ admits a global nowhere zero section.

Proof. We begin with an argument from [10, Lemma 2.4]. Choose a partition of unity $(\varphi_i)_{i \in I}$ subordinate to the cover. For $x \in X$, define

$$J(x) = \{j \in I \mid \varphi_j(x) = \max\{\varphi_i(x) \mid i \in I\}\}.$$

By assumption $\#J(x) \leqslant n$ (and $J(x)$ is non-empty). For a non-empty subset $J \subseteq I$, we now set

$$U_J = \{x \in X \mid \text{ for all } j \in J,\ \varphi_j(x) > 0 \text{ and } \varphi_i(x) < \varphi_j(x) \text{ for all } i \in I - J\}.$$

It is clearly an open subset of $\bigcap_{j \in J} U_j$. Moreover, $x \in U_{J(x)}$, so that the sets U_J cover X. If J and J' are distinct subsets of I with $\#J = \#J'$, then $U_J \cap U_{J'} = \emptyset$. (For there exist elements $j \in J$, with $j \notin J'$, and $j' \in J'$, with $j' \notin J$.)

Choose a partition of unity (ψ_J) subordinate to the open cover $(U_J)_{J \subseteq I \,:\, 1 \leqslant \#J \leqslant n}$ of X and, for each J with $\#J = k \leqslant n$, a section s_J of ξ_k which is nowhere zero on U_J (possible because $U_J \subseteq U_i$ if $i \in J$). We can then define a section s_k of ξ_k by

$$s_k(x) = \sum_{\#J = k} \psi_J(x) s_J(x).$$

Notice that, if $\psi_J(x) \neq 0$ for some J with $\#J = k$, then $s_k(x) \neq 0$. For, if J' is a different subset with $\#J' = k$, $\psi_{J'}(x) = 0$.

The section $s = (s_1, \dots, s_n)$ of ξ is nowhere zero. □

Corollary 2.2. *Suppose that $(A_i)_{i \in I}$ is a finite closed cover of X such that each point of X lies in A_i for at most n indices $i \in I$.*

Suppose that for each $i \in I$ and $k \in \{1, \dots, n\}$ the restriction of ξ_k to A_i admits a nowhere zero section. Then ξ admits a global nowhere zero section.

Proof. It is an elementary exercise to show that there is an open cover $(U_i)_{i \in I}$ such that (i) $A_i \subseteq U_i$, (ii) for each (i, k) there exits a global section of ξ_k with no zeros in U_i, and (iii) each point of X lies in at most n of the sets U_i. Then we can apply Theorem 2.1.

(Here are the details when $\#I > n$. Consider a subset $J \subseteq I$ with $\#J > n$. The open sets $X - A_j$, $j \in J$, cover X. Choose a partition of unity $(\chi_j)_{j \in J}$ subordinate to this cover. Then we can define $U_j^J = \{x \in X \mid \#J \cdot \chi_j(x) < 1\}$. By construction $\bigcap_{j \in J} U_j^J = \emptyset$ (because $\sum_{j \in J} \chi_j(x) = 1$ for any x) and $A_j \subseteq U_j^J$ for $j \in J$.

Every nowhere zero section of ξ_k over the closed set A_i extends to a global section of ξ_k and such a section will be nowhere zero on an open neighbourhood of A_i. So it is easy to choose U_i in the intersection of the sets U_i^J with $\#J > n$ and $i \in J$ to satisfy (i) and (ii). And then, for any set J with $\#J > n$, we have $\bigcap_{i \in J} U_i = \emptyset$.) □

3. A theorem of Lebesgue

If the mod 2 cohomology Euler class $e(\xi)$ of ξ is non-zero, the *zero-set* $\mathrm{Zero}(s) = \{x \in X \mid s(x) = 0\}$ of any section s of ξ is non-empty.

Proposition 3.1. *Let $p : X \to Y$ be a continuous map from X to a compact Hausdorff space Y. Suppose that $(B_i)_{i \in I}$ is a finite cover of Y by closed sets such that any point of Y lies in at most n of the sets B_i.*

If the mod 2 cohomology Euler class $e(\xi)$ of ξ is non-zero, then there exist $i \in I$ and $k \in \{1, \dots, n\}$ such that $p(\mathrm{Zero}(s)) \cap B_i$ is non-empty for each section s of ξ_k.

Proof. We take $A_i = p^{-1}(B_i)$ in Corollary 2.2. Since $e(\xi) \neq 0$, ξ does not admit a nowhere zero section. Hence there is a pair (i,k) such that $\mathrm{Zero}(s) \cap B_i \neq \emptyset$ for every section s of ξ_k. □

For a finite set V of cardinality $n+1$, we write $\mathbb{R}[V]$ for the $(n+1)$-dimensional real vector space of maps $t : V \to \mathbb{R}$, $\Delta(V)$ for the n-simplex of maps t such that $t(v) \geqslant 0$ for all $v \in V$ and $\sum_v t(v) = 1$, $S(\mathbb{R}[V])$ for the unit n-sphere in $\mathbb{R}[V]$ of vectors t with $\sum_v t(v)^2 = 1$ and $P(\mathbb{R}[V])$ for the n-dimensional real projective space of lines $[t]$ in $\mathbb{R}[V]$ (where $t \in \mathbb{R}[V]$ is non-zero). There is a surjective map

$$\pi_V : P(\mathbb{R}[V]) \to \Delta(V)$$

defined by $(\pi_V[t])(v) = t(v)^2$ for $t \in S(\mathbb{R}[V])$. There is also an embedding

$$\sigma_V : \Delta(V) \hookrightarrow S(\mathbb{R}[V])\,,$$

defined by $(\sigma_V(t)(v) = \sqrt{t(v)}$, such that the composition

$$\Delta(V) \xrightarrow{\sigma_V} S(\mathbb{R}[V]) \to P(\mathbb{R}[V]) \xrightarrow{\pi_V} \Delta(V)$$

is the identity. The Hopf line bundle over $P(\mathbb{R}[V])$, with fibre $\mathbb{R}t \subseteq \mathbb{R}[V]$ at $[t]$, is denoted by H.

The *support*, $\mathrm{supp}(t)$, of $t \in \mathbb{R}[V]$ is the set of points $v \in V$ such that $t(v) \neq 0$. For an integer d, $0 \leqslant d \leqslant \#V$, we write $\mathcal{S}_d(V)$ for the set of finite subsets T of $\Delta(V)$ such that any two elements of T have disjoint supports and $\#V - \#T = d$. For $T \in \mathcal{S}_d(V)$, we write Δ_T for the convex hull of T; it is a simplex of codimension d in $\Delta(V)$.

Lemma 3.2. *For $T \in \mathcal{S}_d(V)$, the simplex Δ_T can be expressed as $\pi_V(\mathrm{Zero}(s))$ for some section s of the vector bundle $dH = \mathbb{R}^d \otimes H$.*

Proof. Let E_T be the codimension d vector subspace of $\mathbb{R}[V]$ spanned by $\sigma_V(T)$. The section s_T of $\mathrm{Hom}(H, \mathbb{R}[V]/E_T)$ over $P(\mathbb{R}[V])$ given by the projection $s_T([t]) = \{[t] \hookrightarrow \mathbb{R}[V] \to \mathbb{R}[V]/E_T\}$, for $t \in S(\mathbb{R}[V])$, has the property that $\pi_V(\mathrm{Zero}(s_T)) = \Delta_T$. Choose some isomorphism $\mathbb{R}[V]/E_T \cong \mathbb{R}^d$ to get the required section s. □

Theorem 3.3. *Let $(V_l)_{l=1}^m$ be a family of m finite sets with $\#V_l = d_l n_l + 1$, where $d_l \geqslant 1$ and $n_l \geqslant 1$, $l = 1, \dots, m$, are positive integers. Write $n = n_1 + \ldots + n_m$. Suppose that $(B_i)_{i \in I}$ is a finite closed cover of*

$$Y = \Delta(V_1) \times \cdots \times \Delta(V_m)$$

such that any point of Y lies in at most n of the sets B_i.

Then for some $i \in I$ and $l \in \{1, \dots, m\}$ the projection of B_i to the lth factor $\Delta(V_l)$ meets each of the codimension d_l simplices Δ_T for $T \in \mathcal{S}_{d_l}(V_l)$.

The Lebesgue theorem [8, Theorem 4.1] is the special case $n_l = 1$, $d_l = 1$, for all l; the case $m = 1$, $d_1 = 1$ is a result of Knaster, Kuratowski, Mazurkiewicz [8, Remark 2.2].

We follow the method of Karasev [8, Theorems 2.1 and 4.1].

Proof. This is an immediate consequence of Proposition 3.1. Take p to be the product

$$\pi_{V_1} \times \ldots \times \pi_{V_m} : X = P(\mathbb{R}[V_1]) \times \cdots \times P(\mathbb{R}[V_m]) \to Y = \Delta(V_1) \times \cdots \times \Delta(V_m)$$

and ξ_k to be the multiple $d_l H_l$ of the Hopf bundle H_l over $P(\mathbb{R}[V_l])$ if $n_1 + \ldots + n_{l-1} < k \leqslant n_1 + \ldots + n_l$. The Euler class $e(\xi) = e(H_1)^{d_1 n_1} \ldots e(H_m)^{d_m n_m}$ is non-zero. □

The next lemma gives a way of checking that $e(\xi)$ in the application of Proposition 3.1 is non-zero.

Lemma 3.4. *Suppose that X is a closed subspace of a compact Hausdorff space $\hat{X}$ and that ξ is the restriction of a vector bundle $\hat{\xi}$ over $\hat{X}$. Let η be a real vector bundle over $\hat{X}$ with the property that the restriction of η to each connected component of the complement $\hat{X} - X$ admits a nowhere zero section.*

If the mod 2 cohomology Euler class $e(\hat{\xi} \oplus \eta)$ is non-zero, then $e(\xi)$ is non-zero.

Proof. Fix a Euclidean metric on η. For each component C of $\hat{X} - X$ choose a continuous section s_C of the sphere bundle $S(\eta \,|\, C)$, and choose a continuous function $\rho : \hat{X} \to [0, 1]$ such that $\rho^{-1}(0) = X$. Then one can define a continuous section s of η with zero-set $\mathrm{Zero}(s) = X$ by $s(x) = \rho(x)s_C(x)$ if $x \in C$, $s(x) = 0$ if $x \in X$.

Since $e(\hat{\xi} \oplus \eta) = e(\hat{\xi}) \cdot e(\eta)$ is non-zero, the restriction $e(\xi)$ of $e(\hat{\xi})$ to $\mathrm{Zero}(s) = X$ is non-zero. See, for example, [5, Proposition 2.7]. □

This allows us to deduce Karasev's strengthened KKM theorem [8, Theorem 2.1]).

Proposition 3.5. *Let $(B_i)_{i \in I}$ be closed subsets of a simplex $\Delta(V)$ with vertex set V of cardinality $dn + r + 1$, where $d \geqslant 1$ and $r \geqslant 0$ are integers, such that any point of $Y = \bigcup_{i \in I} B_i$ lies in at most n of the sets B_i. Then, either for some $i \in I$ the subset B_i meets each codimension d simplex Δ_T for $T \in \mathcal{S}_d(V)$, or some connected component of $\Delta(V) - Y$ intersects every dn-dimensional simplex Δ_T for $T \in \mathcal{S}_r(V)$.*

Proof. Considering $\pi_V : \hat{X} = P(\mathbb{R}[V]) \to \hat{Y} = \Delta(V)$, take $X \subseteq \hat{X}$ to be $\pi_V^{-1}(Y)$ and $p : X \to Y$ to be the restriction of π_V. Let each ξ_k be the restriction of dH, so that ξ is the restriction of $\hat{\xi} = dnH$ to X. Take $\eta = rH$, so that $e(\hat{\xi} \oplus \eta) = e(H)^{dn+r}$ is non-zero.

If, for each component of $\Delta(V) - Y$, written as the image $\pi_V(C)$ of a component C of $\hat{X} - X$, there is some simplex Δ_T, where $T \in \mathcal{S}_r(V)$, such that $\pi_V(C) \cap \Delta_T = \emptyset$, then Lemma 3.2 provides a nowhere zero section of $rH = \eta$ over C. By Lemma 3.4, $e(\xi)$ is then non-zero, and we can apply Proposition 3.1 to deduce the existence of an $i \in I$ such that B_i meets each codimension d simplex. □

4. Karasev's topological central point theorem

An early result of the following type appears in [12, Lemma 3.1].

Proposition 4.1. *Let $f : X \to Z$ be a continuous map from X to a compact Hausdorff space Z with covering dimension less than n. Suppose that the mod 2 cohomology class $e(\xi)$ is non-zero.*

Then there exists a point $z \in Z$ and $k \in \{1, \ldots, n\}$ such that $z \in f(\mathrm{Zero}(s))$ for each section s of ξ_k.

Proof. Suppose that for each point $z \in Z$ there exist sections s_k^z of ξ_k, $1 \leqslant k \leqslant n$, such that $z \notin f(\mathrm{Zero}(s_k^z))$ for each k. Then the open sets $(Z - f(\mathrm{Zero}(s_1^z)) \cap \cdots \cap (Z - f(\mathrm{Zero}(s_n^z))$, $z \in Z$, cover Z. Since Z is compact with covering dimension $< n$, this open cover may be refined by a finite open cover $(W_i)_{i \in I}$ such that each point of Z lies in at most n of the sets W_i.

Set $U_i = f^{-1}(W_i)$. Then we may apply Theorem 2.1 to the open cover $(U_i)_{i \in I}$ of X to conclude that there exist i and k such that for every section s of ξ_k the zero-set $\mathrm{Zero}(s)$ meets U_i. So $W_i \not\subseteq Z - f(\mathrm{Zero}(s))$ for every section s of ξ_k. But $W_i \subseteq Z - f(\mathrm{Zero}(s_k^z))$ for some $z \in Z$. This contradiction completes the proof. $\square$

Theorem 4.2. *Let $(V_l)_{l=1}^m$ be a family of m finite sets with $\#V_l = d_l n_l + 1$, where d_l, $n_l \geqslant 1$, $l = 1, \ldots, m$ are positive integers. Write $n = n_1 + \ldots + n_m$. Suppose that*

$$g : Y = \Delta(V_1) \times \cdots \times \Delta(V_m) \to Z$$

is a continuous map to a compact Hausdorff space Z with covering dimension less than n.

Then for some $l \in \{1, \ldots, m\}$

$$\bigcap_{T \in \mathcal{S}_{d_l}(V_l)} g(\prod_{j<l} \Delta(V_j) \times \Delta_T \times \prod_{j>l} \Delta(V_j)) \neq \emptyset\,.$$

Karasev's topological central point theorem, as in [7, Theorem 1.1] and [8, Theorem 5.1], is the special case $m = 1$.

Proof. We apply Proposition 4.1 with X and ξ_k as in Theorem 3.3 and with f equal to the composition of

$$\pi_{V_1} \times \cdots \times \pi_{V_m} : X = P(\mathbb{R}[V_1]) \times \cdots \times P(\mathbb{R}[V_m]) \to Y = \Delta(V_1) \times \cdots \times \Delta(V_m)$$

with $g : Y \to Z$. We recall that $\xi_k = d_l H_l$ if $n_1 + \ldots + n_{l-1} < k \leqslant n_1 + \ldots + n_l$, so that the Euler class $e(\xi) = e(H_1)^{d_1 n_1} \cdots e(H_m)^{d_m n_m}$ is non-zero. $\square$

As an application we prove a result of Helly-Lovász [2, Theorem 3.1].

Corollary 4.3. *Suppose that $C_{l,v}$, $l = 1, \ldots, m$, $v \in V_l$, $\#V_l = d_l + 1$, are convex subsets of a real vector space E with the property that the intersection $C_{1,v_1} \cap \cdots \cap C_{m,v_m}$ is non-empty for each $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$.*

If $\dim E < m$, then, for some l, the intersection $\bigcap_{v \in V_l} C_{l,v}$ is non-empty.

Proof. For each $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$ choose $z_{v_1,\ldots,v_m} \in C_{1,v_1} \cap \cdots C_{m,v_m}$. We apply Theorem 4.2 with $n = m$, $n_l = 1$, and $Z \subseteq E$ the convex hull of the points $z_{v_1,\ldots,v_m}$. Take g to be the piecewise linear map

$$(\sum_{v_1 \in V_1} t_1(v_1), \ldots, \sum_{v_m \in V_m} t_m(v_m)) \mapsto \sum_{(v_1,\ldots,v_m) \in V_1 \times \cdots \times V_m} t_1(v_1) \cdots t_m(v_m)\, z_{v_1,\ldots,v_m} \in Z.$$

We conclude from Theorem 4.2, noting that a codimension d_l simplex in $\Delta(V_l)$ is a point, that there is some l and $z \in Z$ such that, for each $v \in V_l$ the vector z can be written as

$$z = \sum_{(v_1,\ldots,v_m) \in V_1 \times \cdots \times V_m \,:\, v_l = v} t_1(v_1) \cdots t_m(v_m)\, z_{v_1,\ldots,v_m},$$

where $\sum_{v_j \in V_j} t_j(v_j) = 1$ for each j and so $\sum_{(v_1,\ldots,v_m)} t_1(v_1) \cdots t_m(v_m) = 1$. Since each $z_{v_1,\ldots,v_m}$ with $v_l = v$ lies in the convex set $C_{l,v}$, we see that $z \in C_{l,v}$, as required. $\square$

Bárány's dual result [2, Theorem 2.1] (as formulated in [6, Theorem 3.1] and [9, Theorem 3]) can be obtained in a similar fashion.

Corollary 4.4. *Let $K \subseteq E$ be a non-empty compact convex subspace of a finite-dimensional real vector space E. Suppose that $V_1, \ldots, V_m$ are finite sets with $\#V_l = d_l + 1$, $l = 1, \ldots, m$, and that $\varphi_l : V_l \to E$ are maps with the property that for each $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$ the convex hull of $\{\varphi_1(v_1), \ldots, \varphi_m(v_m)\}$ in E is disjoint from K.*

If $\dim E < m$, then, for some $l \in \{1, \ldots, m\}$, the convex hull of $\varphi_l(V_l)$ in E is disjoint from K.

Proof. Choose a basepoint $* \in K$. Let A to be the affine space of affine linear maps $z : E \to \mathbb{R}$ such that $z(*) = -1$. Notice that the dimension of A, as affine space, is equal to $\dim E$.

For each $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$ choose an affine linear map $z_{v_1,\ldots,v_m} \in A$ taking strictly positive values on $\{\varphi_1(v_1), \ldots, \varphi_m(v_m)\}$ and strictly negative values on K. We again apply Theorem 4.2 with $n = m$, $n_l = 1$, and $Z \subseteq A$ the convex hull of the points $z_{v_1,\ldots,v_m}$. As in the proof of Helly's theorem, take g to be the piecewise linear map

$$\Big(\sum_{v_1 \in V_1} t_1(v_1), \ldots, \sum_{v_m \in V_m} t_m(v_m)\Big) \mapsto \sum_{(v_1,\ldots,v_m) \in V_1 \times \cdots \times V_m} t_1(v_1) \cdots t_m(v_m)\, z_{v_1,\ldots,v_m} \in Z.$$

Theorem 4.2 provides some l and $z \in Z$ such that, for each $v \in V_l$ the affine linear map z can be written as

$$z = \sum_{(v_1,\ldots,v_m) \in V_1 \times \cdots \times V_m \,:\, v_l = v} t_1(v_1) \cdots t_m(v_m)\, z_{v_1,\ldots,v_m},$$

where $\sum_{(v_1,\ldots,v_m)} t_1(v_1) \cdots t_m(v_m) = 1$, so that z takes a strictly positive value at each $\varphi_l(v) \in \varphi_l(V_l)$ and strictly negative values on K. □

As observed by Sarkaria [11] (and expounded in [3]), Tverberg's theorem is an easy consequence of Corollary 4.4. The following generalization, discussed in [3, Theorem 3.8] and due to Arocha, Bárány, Bracho, Fabila and Montejano [1], can be viewed as a coincidence theorem.

Corollary 4.5. *Let $r \geqslant 0$ and $m \geqslant 1$ be integers. For $l = 1, \ldots, m$, $s = 0, \ldots, r$, let $V_{l,s}$ be non-empty finite sets and $\varphi_{l,s} : V_{l,s} \to F$ be maps to a finite-dimensional real vector space F satisfying the two conditions:*

(i) *for each $l \in \{1, \ldots, m\}$, there is a non-zero vector in F that can be expressed, for each $s = 0, \ldots, r$, as a linear combination with non-negative coefficients of the elements of $\varphi_{l,s}(V_{l,s})$;*

(ii) *for each s and $v_l \in V_{l,s}$, $l = 1, \ldots, m$, the convex hull of $\{\varphi_{1,s}(v_1), \ldots, \varphi_{m,s}(v_m)\}$ is disjoint from $\{0\}$.*

Then, if $r \cdot \dim F < m$, there is a partition $\{1, \ldots, m\} = \bigsqcup_{s=0}^{r} I_s$ into $r + 1$ non-empty subsets I_s and a non-zero vector $c \in F$ such that

$$\sum_{i \in I_s} \lambda_i \varphi_{i,s}(v_i) = c \quad \text{for } s = 0, \ldots, r,$$

for some $\lambda_i \geqslant 0$ and $v_i \in V_{i,s}$ for $i \in I_s$.

If, further, there is some affine hyperplane H in F that contains all the subsets $\varphi_{l,s}(V_{l,s})$ but does not contain 0, then c may be chosen in the hyperplane H and then $\sum_{i \in I_s} \lambda_i = 1$ for each s.

Proof. Let L_r be the quotient of $\mathbb{R}^{r+1} = \bigoplus_{s=0}^{r} \mathbb{R}e_s$ by the subspace generated by $e_0 + \ldots + e_r$ and write $[e_s]$ for the coset of e_s.

We take $V_l = \bigsqcup_{s=0}^{r} V_{l,s}$, $E = L_r \otimes F$, $\varphi_l : V_l \to E$ defined by $\varphi_l(v) = [e_s] \otimes \varphi_{l,s}(v)$ for $v \in V_{l,s}$, and apply Corollary 4.4 with $K = \{0\}$. By assumption, the convex hull of each $\varphi_l(V_l)$ in E contains 0. (Notice that, if $a_1, \ldots, a_r \in F$, then $\sum_s [e_s] \otimes a_s = 0$ if and only if $a_1 = a_2 = \cdots = a_r$.)

So there exist $v_i \in V_i$, $i = 1, \ldots, m$, $\lambda_i \geqslant 0$, and a non-zero $c \in F$ such that $\sum_{i\,:\, v_i \in V_{i,s}} \lambda_i \varphi_{i,s}(v_i) = c$. Take $I_s = \{i \mid v_i \in V_{i,s}\}$.

If there is a linear form $\alpha : F \to \mathbb{R}$ taking the value 1 on all $\varphi_{l,s}(V_{l,s})$, we can scale to arrange that $\alpha(c) = 1$, and then $\sum_{i \in I_s} \lambda_i = 1$. □

The original Tverberg theorem is the case in which $V_{l,s} = \{*\}$ is a single point for all l, s and $\varphi_{l,s}(*)$ is independent of s.

Appendix A. Cohomology

It is a classical result that, if $(A_k)_{k=1}^{n}$ is a closed cover of a compact Hausdorff space X and, for each k, e_k is a mod 2 cohomology class of X that restricts to zero on A_k, then the product $e_1 \cdots e_n$ is zero. Here is the corresponding version of Corollary 2.2, which was used by Karasev in the form [8, Lemma 3.2].

Theorem A.1. *Let X be a compact Hausdorff space and let $e_1, \ldots, e_n$ by classes in the mod 2 cohomology of X.*

Suppose that $(A_i)_{i \in I}$ is a finite closed cover of X such that each point of X lies in A_i for at most n indices $i \in I$ and that for each $i \in I$ and $k \in \{1, \ldots, n\}$ the restriction of e_k to A_i is zero.

Then $e_1 \cdots e_n = 0$.

Proof. By the argument used in the proof of Corollary 2.2 one can manufacture an open cover (U_i) such that each cohomology class is represented by a map which is null (not just null-homotopic) on U_i. The construction in the proof of Theorem 2.1 gives an open cover (U_J) indexed by the non-empty subsets J of I with $\#J \leqslant n$. The map representing e_k is null on the disjoint union of the sets U_J with $\#J = k$. Since these n open sets cover X, the product $e_1 \cdots e_n$ is represented by the null map, and so the cohomology class $e_1 \cdots e_n$ is zero. □

Appendix B. Fibrewise joins

The principal result, Theorem 2.1, extends from sphere bundles to fibre bundles (understood to be locally trivial).

Lemma B.1. *Let $E \to X$ be a fibre bundle over a compact Hausdorff space X with each fibre a compact ENR. Then there is a fibrewise embedding $j : E \hookrightarrow \Omega$ into an open subspace of a trivial real vector bundle $X \times V$ admitting a fibrewise retraction $r : \Omega \to E$.*

Proof. We recall the well known argument (from, for example, [4, II, Lemma 5.8]). Choose a finite open cover $(U_i)_{i=1}^{n}$ of X, with a partition of unity (φ_i) subordinate to the cover, and trivializations of the restriction of E to each subspace U_i: $E \,|\, U_i \to U_i \times F_i : y \mapsto (x, f_i(y))$, for a point $y \in E_x$ in the fibre of E at $x \in X$, where F_i is a compact Euclidean Neighbourhood Retract (ENR) embedded as a subspace $F_i \subseteq \Omega_i \subseteq V_i$ of an open subspace Ω_i of a Euclidean space V_i with a retraction $r_i : \Omega_i \to F_i$.

Putting $V = \bigoplus_{i=1}^n V_i$, define an embedding $j : E \to X \times V$ by

$$j(y) = (x, \varphi_1(x)f_1(y), \ldots, \varphi_n(x)f_n(y))$$

for $y \in E_x$.

Let $W_i \subseteq X \times V$ be the open subset of points $(x, (v_j))$ such that either $n\varphi_i(x) < \frac{1}{2}$ or

$$\varphi_i(x) > 0 \text{ and } v_i/\varphi_i(x) \in \Omega_i.$$

Observe that $j(E) \subseteq W_i$.

Define $q_i : W_i \to X \times V$ by $q_i(x, (v_j)) = (x, (v_j))$ if $n\varphi_i(x) < \frac{1}{2}$, and

$$q_i(x, (v_j)) = (x, (t\varphi_j(x)f_j(y) + (1-t)v_j)) \text{ if } n\varphi_i(x) \geqslant \tfrac{1}{2},$$

where $r_i(v_i/\varphi_i(x)) = f_i(y)$ and $t = \min\{1, n\varphi_i(x) - 1/2\}$. So, if $n\varphi_i(x) \geqslant 1$, we have $t = 1$ and $q_i(x, (v_j)) \in j(E)$. And, because $\sum_j \varphi_j(x) = 1$, there is at least one i such that $n\varphi_i(x) \geqslant 1$.

Now take Ω and r to be the open subset

$$\Omega = \{(x, (v_j)) \in W_1 \mid q_i(q_{i-1}(\cdots q_1(x, (v_j))\cdots)) \in W_{i+1},\ i = 1, \ldots, n-1\}$$

and retraction $r(x, (v_j)) = q_n(q_{n-1}(\cdots q_1(x, (v_j))\cdots))$. □

Theorem B.2. *Let $E_1, \ldots, E_n$ be n fibre bundles, with each fibre a compact ENR, over a compact Hausdorff topological space X. Suppose that $(A_i)_{i\in I}$ is a finite closed cover of X such that each point of X lies in A_i for at most n indices $i \in I$.*

Suppose that for each $i \in I$ and $k \in \{1, \ldots, n\}$ there exists a section of $E_k \mid A_i$ over A_i. Then the fibrewise join $E = E_1 *_X \cdots *_X E_n$ admits a global section.

We shall write points of the join $F_1 * \cdots * F_n$ of spaces F_k, $k = 1, \ldots, n$, as $[(y_1, \ldots, y_n), (t_1, \ldots, t_n)]$, where $y_k \in F_k$, $t_k \in [0, 1]$, and $\sum t_k = 1$.

Proof. The proof of Corollary 2.2 using Theorem 2.1 is readily adapted.

First, using Lemma B.1 to see that a section of E_k over A_i extends to a section over an open neighbourhood of A_i in X, we construct an open cover (U_i) such that (i) $A_i \subseteq U_i$, (ii) for each (i, k) there is a section of $E_k \mid U_i$ over U_i, and (iii) each point of X lies in at most n of the sets U_i.

Then having produced the open subsets U_J as in the proof of Theorem 2.1 and chosen ψ_J, we choose sections s_J of $E_k \mid U_J$, where $\#J = k$. These combine over the finite disjoint union of the subsets U_J to give a section s_k of E_k over $\bigsqcup_{\#J=k} U_J$. A section s of the fibrewise join $E_1 *_X \cdots *_X E_n$ is given by $s(x) = [(s_1(x), \ldots, s_n(x)), (t_1(x), \ldots, t_n(x))]$, where $t_k(x) = \sum_{\#J=k} \psi_J(x)$, so that $t_1(x) + \cdots + t_n(x) = 1$. □

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Institute of Mathematics, University of Aberdeen, Aberdeen AB24 3UE, UK
Email address: m.crabb@abdn.ac.uk