On Degeneracy in the P-Matroid Oriented Matroid Complementarity Problem

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Abstract

We investigate degeneracy in the P-Matroid Oriented Matroid Complementarity Problem (P-OMCP) and its impact on the reduction of this problem to sink-finding in Unique Sink Orientations (USOs). On one hand, this understanding of degeneracies allows us to prove a linear lower bound for sink-finding in *P-matroid USOs*. On the other hand, it allows us to prove a promise preserving reduction from P-OMCP to USO sink-finding, where we can drop the assumption that the given P-OMCP is non-degenerate. This places the promise version of P-OMCP in the complexity class PromiseUEOPL.

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1 Introduction

Degenerate input can be an issue in structural analysis and algorithm design for many algebraic and geometric problems. It is often swept under the rug by assuming the input to be non-degenerate. For example, one often assumes all input points of a geometric problem to be in general position. In some problems (e.g., the minimum convex partition [10]), such an assumption is inappropriate as it makes the problem considerably easier. In other cases, degenerate inputs can be solved easily by resolving degeneracy using *perturbation* techniques.

In this paper, we investigate degeneracy in the context of the P-Matroid Oriented Matroid Complementarity Problem (P-OMCP). Assuming non-degeneracy, this problem can be solved by converting it into a Unique Sink Orientation of the hypercube graph, and finding a sink within that orientation. Oriented matroids are abstractions for many types of configurations of geometric objects, such as (pseudo-)hyperplane arrangements or point configurations. Just like these geometric configurations, oriented matroids can exhibit degeneracies. In this paper, we analyze the effects of these degeneracies on the reduction from P-OMCP to Unique Sink Orientation sink-finding.

Both the P-OMCP as well as Unique Sink Orientations are combinatorial abstractions of the P-Matrix linear complementarity problem (P-LCP). The complexity status of the P-LCP remains an interesting and relevant open question, since the problem can be used to solve many optimization problems, such as Linear Programming [9], and binary Simple Stochastic Games [8, 13]. Sink-finding in Unique Sink Orientations can also be used to solve geometric problems such as the problem of finding the smallest enclosing ball of a set of balls [6].

2 Background

2.1 Oriented Matroids

For a more extensive introduction to oriented matroids, we refer the reader to the comprehensive textbook by Björner et al. [1]. We consider oriented matroids $\mathcal{M} = (E, \mathcal{C})$ in *circuit representation*, where E is called the *ground set*, and $\mathcal{C}$ is the collection of circuits of $\mathcal{M}$.

A *circuit* $X \in \{-, 0, +\}^E$ is a *signed set* represented by a tuple of sets $X = (X^+, X^-)$ where for all $e \in X^+\colon X_e = +$ and $e \in X^-\colon X_e = -$. We write $-X$ for the inversed signed set $-X = (X^-, X^+)$. The *support* is defined as the set of non-zero elements $\underline{X} := X^+ \cup X^-$.

▶ **Definition 1** (Circuit axioms). *For an oriented matroid $\mathcal{M} = (E, \mathcal{C})$ on the ground set E the following set of axioms are satisfied for $\mathcal{C}$:*

(C0) $(\emptyset, \emptyset) \notin \mathcal{C}$.
(C1) $X \in \mathcal{C} \Leftrightarrow -X \in \mathcal{C}$.
(C2) *For all* $X, Y \in \mathcal{C}$, *if* $\underline{X} \subseteq \underline{Y}$, *then* $X = Y$ *or* $X = -Y$.
(C3) *For all* $X, Y \in \mathcal{C}$, $X \neq -Y$, *and* $e \in X^+ \cap Y^-$ *there is a* $Z \in \mathcal{C}$ *such that*
- $Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\}$ *and*
- $Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}$.

A *basis* $B \subseteq E$ of an oriented matroid $\mathcal{M} = (E, \mathcal{C})$ is an inclusion-maximal subset of E such that B contains no circuit. The *rank* of an oriented matroid is the size of its bases. An oriented matroid is called *uniform*, if all subsets of E with cardinality equal to the rank of the oriented matroid are bases.

The *cocircuits* $\mathcal{C}^*$ of an oriented matroid $\mathcal{M}$ are the circuits of the *dual* oriented matroid $\mathcal{M}^*$. To understand duality, we need the following notion of orthogonality:

▶ **Definition 2.** *Two signed sets X, Y are said to be* orthogonal, *if $\underline{X} \cap \underline{Y} = \emptyset$, or there exist $e, f \in \underline{X} \cap \underline{Y}$, such that $X_e Y_e = -X_f Y_f$.*

In other words, two signed sets are orthogonal if their supports either do not intersect at all, or if they agree (same non-zero sign) and disagree (opposite non-zero sign) on at least one element.

▶ **Lemma 3** ([1]). *Let $X \in \mathcal{C}$ be a circuit and $Y \in \mathcal{C}^*$ be a cocircuit of some oriented matroid $\mathcal{M}$. Then, X and Y are orthogonal.*

Given the set of circuits, the set of cocircuits can be computed, since the cocircuits are exactly the inclusion-minimal non-empty signed sets that are orthogonal to all circuits. Since duality of oriented matroids is self-inverse, the opposite holds too.

▶ **Definition 4.** *In an oriented matroid $\mathcal{M} = (E, \mathcal{C})$, given a basis B and an element $e \notin B$, the* fundamental circuit *$C(B, e)$ is the unique circuit X with $X_e = +$ and $\underline{X} \subseteq B \cup \{e\}$.*

▶ **Definition 5.** *In an oriented matroid $\mathcal{M} = (E, \mathcal{C})$, given a basis B and an element $e \in B$, the* fundamental cocircuit *$C^*(B, e)$ is the unique cocircuit D with $D_e = +$ and $\underline{D} \cap (B \setminus \{e\}) = \emptyset$.*

An oriented matroid $\widehat{\mathcal{M}} = (E \cup \{q\}, \widehat{\mathcal{C}})$ is called an *extension of* $\mathcal{M}$, if its *minor* $\widehat{\mathcal{M}} \setminus q := (E, \{X \mid X \in \widehat{\mathcal{C}} \text{ and } X_q = 0\})$ is equal to $\mathcal{M}$. A *localization* is a way to describe an extension $\widehat{\mathcal{M}} = (E \cup \{q\}, \widehat{\mathcal{C}})$ of $\mathcal{M} = (E, \mathcal{C})$.

▶ **Definition 6.** *Given an oriented matroid $\mathcal{M}$ on ground set E and with cocircuits $\mathcal{C}^*$, a function $\sigma : \mathcal{C}^* \to \{-, 0, +\}$ defines the following family of signed sets*

$$\begin{aligned}\widehat{\mathcal{C}}^* := &\{(Y,\ \sigma(Y)) : Y \in \mathcal{C}^*\} \cup \\ &\{(Y_1 \circ Y_2,\ 0) : Y_1, Y_2 \in \mathcal{C}^*,\ \text{for adjacent } Y_1, Y_2 \text{ with} \\ &\ \sigma(Y_1) = -\sigma(Y_2) \neq 0\},\end{aligned}$$

where the notation (X, s) denotes the signed set where all elements of E have the same sign as in X, and the new element q gets sign s. For the definition of adjacency, we refer the reader to [1]. For the further discussion, only the first of the two sets forming $\widehat{\mathcal{C}}^*$ is relevant.

The function σ is called a localization, *if $\widehat{\mathcal{C}}^*$ is a valid set of cocircuits. Then, the oriented matroid $\widehat{\mathcal{M}}$ on the ground set $E \cup \{q\}$ with cocircuits $\widehat{\mathcal{C}}^*$ is called the* extension of $\mathcal{M}$ specified by σ.

2.2 P-OMCP

We consider oriented matroids $\mathcal{M} = (E_{2n}, \mathcal{C})$ on the ground set $E_{2n} = S \cup T$, which is made up of two parts $S = \{s_1, \dots, s_n\}$ and $T = \{t_1, \dots, t_n\}$, $S \cap T = \emptyset$. We call a set $J \subseteq E_{2n}$ *complementary*, if it contains no *complementary pair* s_i, t_i.

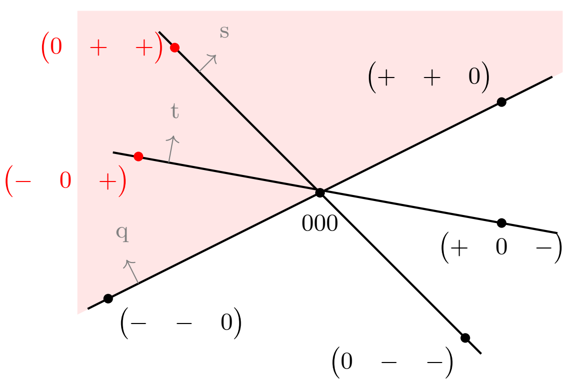
▶ **Definition 7** (P-matroid). *An oriented matroid $\mathcal{M} = (E_{2n}, \mathcal{C})$ is a* P-matroid *if S is a basis and there is no* sign-reversing circuit. *A sign-reversing circuit is a circuit X such that for each complementary pair s_i, t_i contained in $\underline{X}$, $X_{s_i} = -X_{t_i}$.*

▶ **Example 8.** The matroid $\mathcal{M} = (\{s, t\}, \mathcal{C})$ is a P-Matroid. The matroid $\mathcal{M}' = (\{s, t\}, \mathcal{C}')$ is *not* a P-Matroid, since both of its circuits are sign-reversing.

$$\mathcal{C} = \{(+\quad +), (-\quad -)\}, \qquad \mathcal{C}' = \{(+\quad -), (-\quad +)\}.$$

Let q be such that $q \notin E_{2n}$. Then $\widehat{E_{2n}} := S \cup T \cup \{q\}$, and we write $\widehat{\mathcal{M}} = (\widehat{E_{2n}}, \widehat{\mathcal{C}})$ for an *extension of* $\mathcal{M}$.

▶ **Example 9.** $\widehat{\mathcal{M}} = (\{s, t, q\}, \widehat{\mathcal{C}})$ and $\widehat{\mathcal{M}'} = (\{s, t, q\}, \widehat{\mathcal{C}'})$ are both valid extensions of the P-Matroid $\mathcal{M}$ from Example 8 for $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{C}'}$ as given below. Figures 1 and 2 show the realizations of the corresponding oriented matroids as arrangements of oriented hyperplanes through the origin; each one-dimensional cell of these arrangements corresponds to a circuit.



$$\begin{aligned}\widehat{\mathcal{C}} = \{&(+\quad +\quad 0), \\ &(-\quad -\quad 0), \\ &(+\quad 0\quad -), \\ &(-\quad 0\quad +), \\ &(0\quad +\quad +), \\ &(0\quad -\quad -)\}\end{aligned}$$

Figure 1 Realization of $\widehat{\mathcal{M}}$.

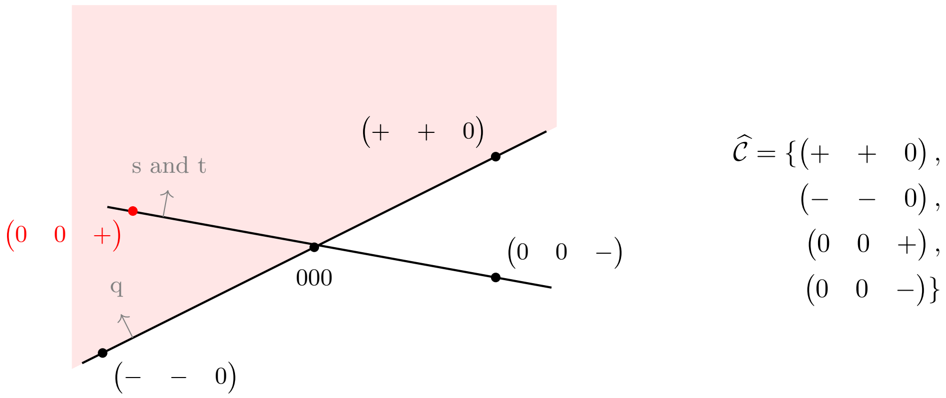


Figure 2 Realization of $\widehat{\mathcal{M}}'$.

Given an extension $\widehat{\mathcal{M}} = (\widehat{E_{2n}}, \widehat{\mathcal{C}})$ of a P-matroid, the goal of the *P-Matroid Oriented Matroid Complementarity Problem (P-OMCP)* is to find a circuit $X \in \widehat{\mathcal{C}}$ with $X \geq 0$, $X_q = +$, and $X_{s_i} X_{t_i} = 0$ for every $i \in [n]$. The matroid extension is given to an algorithm by a circuit oracle, which given a set $B \subset \widehat{E_{2n}}$ and another element $e \in \widehat{E_{2n}} \setminus B$ either returns that B is not a basis of $\widehat{\mathcal{M}}$, or returns the fundamental circuit $C(B, e)$ (recall that this is the unique circuit $X \in \widehat{\mathcal{C}}$ with $X_e = +$ and $\underline{X} \subseteq B \cup \{e\}$). It is known that in P-matroids and P-matroid extensions, every complementary set $B \subset S \cup T$ of size n is a basis [11]. Every P-OMCP instance has a unique solution [17]. The unique solution of an P-OMCP instance with $\widehat{\mathcal{M}}$ of Example 9 as input is $\begin{pmatrix}0 & + & +\end{pmatrix}$, the unique solution in $\widehat{\mathcal{M}}'$ is $\begin{pmatrix}0 & 0 & +\end{pmatrix}$.

A P-matroid extension (a P-OMCP instance) is *non-degenerate*, if for every complementary basis B, the circuit $C(B, q)$ is non-zero on all elements in $B \cup \{q\}$.

▶ **Example 10.** The red shaded area in Figures 1 and 2 denotes the areas where q is positive. The circuits marked in red are the fundamental circuits $C(\{s\}, q)$ and $C(\{t\}, q)$. As can be seen, the P-Matroid extension $\widehat{\mathcal{M}}$ in Example 9 is non-degenerate, whereas $\widehat{\mathcal{M}}'$ is degenerate.

2.3 Unique Sink Orientation (USO)

The *n-dimensional hypercube graph Q_n (n-cube)* is the undirected graph on the vertex set $V(Q_n) = \{0, 1\}^n$, where two vertices are connected by an edge if they differ in exactly one coordinate. An *orientation* $O : V(Q_n) \to \{-, +\}^n$ assigns each vertex an orientation of its incident edges, where $O(v)_i = +$ denotes an outgoing edge from vertex v in dimension i and $O(v)_i = -$ denotes an incoming edge. A *Unique Sink Orientation (USO)* is an orientation, such that every non-empty subcube contains exactly one sink, i.e., a unique vertex v with $O(v)_i = -$ for all dimensions i in the subcube [16].

▶ **Lemma 11** (Szabó-Welzl Condition [16]). *An orientation O of Q_n is USO if and only if for all pairs of distinct vertices $v, w \in V(Q_n)$, we have: $\exists i \in [n] : (v_i \neq w_i) \wedge (O(v)_i \neq O(w)_i)$.*

The classical algorithmic problem associated to USOs is that of finding the unique global sink v with $\forall i : O(v)_i = -$, with as few as possible queries to an oracle computing O.

2.4 Classical Reduction

Todd [17] showed that a non-degenerate P-OMCP given by a matroid $\widehat{\mathcal{M}} = (\widehat{E_{2n}}, \widehat{\mathcal{C}})$ can be translated to an USO of the n-cube. Every vertex v of the cube is associated with a complementary basis $B(v) \subset S \cup T$. For each $i \in [n]$, $s_i \in B(v)$ if $v_i = 0$, otherwise $t_i \in B(v)$. The orientation $O(v)$ is then computed using the fundamental circuit $C := C(B(v), q)$:

$$O(v)_i := \begin{cases} + & \text{if } C_{s_i} = - \text{ or } C_{t_i} = -, \\ - & \text{if } C_{s_i} = + \text{ or } C_{t_i} = +. \end{cases}$$

As the P-OMCP instance is non-degenerate, no other case can occur. Todd showed that the computed orientation O is USO, and that its sink v corresponds to a fundamental circuit $C(B(v), q)$ which is positive on all elements and thus a solution to the P-OMCP instance.

▶ **Example 12.** Recall the P-Matroid extension $\widehat{\mathcal{M}}$ from Example 9. Figure 3 shows the USO created by this reduction, where $B(0) = s$ and $B(1) = t$.

$$C(B(0), q) = \begin{pmatrix} - & 0 & + \end{pmatrix} \qquad C(B(1), q) = \begin{pmatrix} 0 & + & + \end{pmatrix}$$

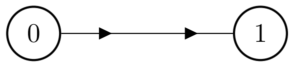


Figure 3 USO created by the reduction from $\widehat{\mathcal{M}}$.

3 The Effect of Degeneracy on the Resulting USOs

In the above reduction, if the P-OMCP instance is degenerate, we can sometimes not decide which way to orient an edge since $C_{s_i} = C_{t_i} = 0$. For now, we leave these edges unoriented. This leads to a *partial orientation* of the hypercube, which is a function $O : V(Q_k) \to \{-, 0, +\}^k$ where $O(v)_i = 0$ denotes an unoriented edge. We call such a partial orientation arising from a degenerate P-OMCP a *partial P-matroid USO (PPU)*. In this section we aim to understand the structure of unoriented edges in PPUs.

Not every partial orientation can be turned into an USO by directing the unoriented edges. We thus state the following condition inspired by the Szabó-Welzl condition:

▶ **Definition 13.** *A partial orientation O is said to be* partially Szabó-Welzl *if for any two distinct vertices $v, w \in V(Q_k)$, either*

$$O(v)_i = O(w)_i = 0 \textit{ for all } i \textit{ with } v_i \neq w_i, \textit{ or} \tag{1}$$

$$\exists i : v_i \neq w_i \wedge \big((O(v)_i = + \wedge O(w)_i = -) \vee (O(v)_i = - \wedge O(w)_i = +)\big). \tag{2}$$

▶ **Lemma 14.** *A partial orientation O which is partially Szabó-Welzl can be extended to an USO by orienting all unoriented edges towards the endpoint with fewer* 1*s, i.e., downwards.*

Proof. By orienting all unoriented edges of O from the vertex with more 1s to the vertex with fewer 1s (i.e., "downwards"), any two vertices that previously fulfilled condition (1) of Definition 13 now fulfill the classic Szabó-Welzl condition as in Lemma 11. Note that condition (2) of Definition 13 is equivalent to this classic condition on full (non-partial) orientations. We conclude that all pairs of vertices must now fulfill the Szabó-Welzl condition as in Lemma 11. ◀

▶ **Lemma 15.** *A partial P-matroid USO is partially Szabó-Welzl.*

Proof. Assume two vertices v, w in a PPU O failed both conditions of Definition 13. Let $V = C(B(v), q)$ and $W = C(B(w), q)$ be the fundamental circuits used to derive $O(v)$ and

$O(w)$. Since v and w violate the first condition of Definition 13, $V \neq W$. Applying circuit axiom (C3) to V and $-W$ to eliminate the element q shows that there exists a circuit Z with certain properties. Since $q \notin \underline{Z}$, Z must contain both s_i and t_i for at least one $i \in [k]$ (since all complementary sets are independent in a P-matroid extension). As we assumed that v and w violate the second condition of Definition 13, we know that s_i and t_i must have opposite signs in Z. Since this holds for any $i \in [k]$, Z is a sign-reversing circuit of the underlying P-matroid, which contradicts Definition 7. We conclude that no two vertices can fail Definition 13. ◀

▶ **Lemma 16.** *In a partial P-matroid USO, the unoriented edges form a set of vertex-disjoint faces. In each such face, the orientation is the same at every vertex.*

Proof. Let v be a vertex of a PPU, and let w be another vertex within the face spanned by the unoriented edges incident to v. Then, the fundamental circuit $C(B(v), q)$ fulfills all the conditions that a circuit has to fulfill to be the fundamental circuit $C(B(w), q)$. Since fundamental circuits are unique in all oriented matroids [11], we must have $C(B(v), q) = C(B(w), q)$ and thus v and w must be oriented the same way, which implies the lemma. ◀

Lemmas 14–16, and [14, Corollary 6] imply that the unoriented subcubes of a PPU can in fact be oriented according to *any* USO:

▶ **Corollary 17.** *Let O be a PPU and let O' be the orientation obtained by independently orienting each unoriented face f of O according to some USO of the same dimension as f. Then, O' is USO.*

Proof. By Lemmas 14 and 15, O can be extended to some USO. By Lemma 16, each unoriented face is a hypervertex; a face where all edges of the same dimension leaving the face are oriented the same way. By [14, Corollary 6], each such hypervertex can be reoriented according to an arbitrary USO while preserving that the whole orientation is USO. ◀

▶ **Example 18.** Recall the P-Matroid extension $\widehat{\mathcal{M}}$ from Example 9. Figure 4 shows the USO created by this reduction, where $B(0) = s$ and $B(1) = t$.

$$C(B(0), q) = \begin{pmatrix} 0 & 0 & + \end{pmatrix} \qquad C(B(1), q) = \begin{pmatrix} 0 & 0 & + \end{pmatrix}$$

Figure 4 USO created by the reduction from the degenerate P-Matroid extension $\widehat{\mathcal{M}}'$.

4 Constructions Based on Degeneracy and Perturbations

In this section we show how existing constructions of oriented matroid extensions can be interpreted as constructions of (partial) P-matroid USOs. An extension $\widehat{\mathcal{M}}$ of an oriented matroid $\mathcal{M}$ can be uniquely described by a *localization*, a function σ from the set $\mathcal{C}^*$ of cocircuits of $\mathcal{M}$ to the set $\{-, 0, +\}$. We give some more background about localizations in Section 2.1. Note that not every function $f : \mathcal{C}^* \to \{-, 0, +\}$ describes a valid extension and thus not every such function is a localization. The following lemma connects a localization to the circuits relevant to the resulting (partial) P-matroid USO.

▶ **Lemma 19.** *Let $\mathcal{M}$ be a P-matroid and let σ be a localization for $\mathcal{M}$ describing the extension $\widehat{\mathcal{M}}$. Then, for any complementary basis B of $\mathcal{M}$ (and thus also of $\widehat{\mathcal{M}}$), and every element $e \in B$, the sign of e in the fundamental circuit $C(B,q)$ of $\widehat{\mathcal{M}}$ is the opposite of the sign assigned by σ to the fundamental cocircuit $C^*(B,e)$ of $\mathcal{M}$.*

Proof. $D := C^*(B,e)$ is a cocircuit of $\mathcal{M}$. By Definition 6, $\widehat{D} := (D, \sigma(D))$ must be a cocircuit of $\widehat{\mathcal{M}}$.

By Definition 5, $\underline{\widehat{D}}$ must be a subset of $\widehat{E_{2n}} \setminus (B \setminus \{e\})$ and $\widehat{D}_e = +$. On the other hand, the support of $C := C(B,q)$ must be a subset of $B \cup \{q\}$, and $C_q = +$.

Lemma 3 says that C and $\widehat{D}$ must be orthogonal, i.e., their supports either do not intersect, or they must agree and disagree on at least one element. Since $\underline{C} \cap \underline{\widehat{D}} \subseteq \{e,q\}$, the first case only occurs if $C_e = \widehat{D}_q = 0$. The second case can only occur if $C_e = -\widehat{D}_q$, since $C_q = \widehat{D}_e = +$. ◀

Las Vergnas [12] showed that the set of localizations is closed under composition, i.e., given two localizations σ_1, σ_2, the following function is a localization too:

$$\forall c \in \mathcal{C}^* : (\sigma_1 \circ \sigma_2)(c) := \begin{cases} \sigma_1(c), & \text{if } \sigma_1(c) \neq 0, \\ \sigma_2(c), & \text{otherwise.} \end{cases}$$

Lemma 19 allows us to understand the effect of such composition on the resulting (partial) P-matroid USO: For localizations σ_1, σ_2 and their corresponding PPUs O_1, O_2, the PPU O' given by the localization $\sigma_1 \circ \sigma_2$ is

$$\forall v \in V(Q_k), i \in [k] : O'(v)_i = \begin{cases} O_1(v)_i, & \text{if } O_1(v)_i \neq 0, \\ O_2(v)_i, & \text{otherwise.} \end{cases}$$

This can be seen as filling in all unoriented subcubes of O_1 with the orientation O_2.

Furthermore, Las Vergnas [12] describes *lexicographic extensions* of oriented matroids.

▶ **Definition 20** (Lexicographic extension [12]). *Let $\mathcal{M} = (E, \mathcal{C})$ be an oriented matroid. Given an element $e \in E$ and a sign $s \in \{-,0,+\}$, the function $\sigma : \mathcal{C}^* \to \{-,0,+\}$ given by*

$$\sigma(D) := \begin{cases} s \cdot D_e, & \textit{if } D_e \neq 0, \\ 0, & \textit{otherwise,} \end{cases}$$

is a localization. The extension of $\mathcal{M}$ specified by this localization is called the lexicographic extension *of $\mathcal{M}$ by $[s \cdot e]$.*

Lexicographic extensions of *uniform* P-matroids give rise to PPUs in which all edges of some dimension are oriented the same way, and one half is left unoriented while the other half is completely oriented (see Figure 5).

▶ **Lemma 21.** *Let $\mathcal{M} = (E_{2n}, \mathcal{C})$ be a P-matroid. Let $\widehat{\mathcal{M}}$ be the lexicographic extension of $\mathcal{M}$ by $[+ \cdot t_i]$. Then, in the partial P-matroid USO O defined by $\widehat{\mathcal{M}}$, the upper i-facet (the facet of vertices v with $v_i = 1$) is an unoriented subcube, and all i-edges point towards this facet. Furthermore, if $\mathcal{M}$ is uniform, the lower i-facet is completely oriented.*

Proof. We first prove that all edges in dimension i are oriented from the vertices with $v_i = 0$ to the vertices with $v_i = 1$. As t_i is positive in all cocircuits $C^*(B, t_i)$ for B such that $t_i \in B$, σ assigns $+$ to all such cocircuits. By Lemma 19, for each vertex v with $v_i = 1$, we have $O(v)_i = -$.

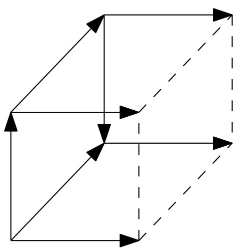


Figure 5 The form of the PPU given by a lexicographic extension of a uniform P-matroid.

Furthermore, since $\mathcal{M}$ is a P-matroid, every complementary set of n elements is a basis; thus, also every such set is a cobasis. We therefore know that every fundamental cocircuit $C^*(B, e)$ for a complementary set B must be non-zero on both elements of the complementary pair which includes e.

For any basis B with $s_i \in B$, t_i is thus non-zero in $C^*(B, s_i)$, and σ assigns a non-zero sign to this cocircuit. By [11, Theorem 5.4], a P-matroid contains no *sign-preserving* cocircuit, so s_i and t_i must have opposite signs in this cocircuit. Thus, t_i must be negative in $C^*(B, s_i)$, and σ assigns $-$ to it. We conclude that for each vertex v with $v_i = 0$, we have $O(v)_i = +$.

Next, we prove that the facet of vertices v with $v_i = 1$ is unoriented. Let B be a basis with $t_i \in B$, and let $e \in B \setminus \{t_i\}$ be some element. Now, note that $t_i \notin \underline{C^*(B, e)}$. Thus, σ assigns 0 to these circuits, and therefore $C(B, q)_e = 0$, showing that this facet is unoriented.

Lastly, we prove that if $\mathcal{M}$ is uniform, the facet of vertices v with $v_i = 0$ is completely oriented. When $\mathcal{M}$ is uniform, all subsets $B \subset E$ of size n are bases and cobases. Thus, $|\underline{C^*(B, e)}| = n + 1$, and for every complementary B such that $s_i \in B$ and any $e \in B$, we have that $t_i \in \underline{C^*(B, e)}$ and therefore σ assigns a non-zero sign to that circuit. This shows that $|\underline{C(B, q)}| = n + 1$ too, proving that all edges around a vertex v with $v_i = 0$ are oriented. ◀

Of course this lemma symmetrically also applies to lexicographic extensions where $s = -$ or $e = s_i$. Switching t_i out for s_i swaps the role of the two facets, and switching the sign makes all i-edges point to the oriented facet instead of the unoriented one.

We can use these two construction techniques to prove a lower bound on the number of queries needed by deterministic sink-finding algorithms on P-matroid USOs. In essence, we successively build a localization by composition with lexicographic extensions. The construction keeps the invariant that there exists an unoriented subcube guaranteed to contain the global sink. The dimension of this subcube is reduced by at most one with every query, thus at least n queries are required.

▶ **Theorem 22.** *Let $\mathcal{M} = (E_{2n}, \mathcal{C})$ be a uniform P-matroid. Then, for every deterministic sink-finding algorithm $\mathcal{A}$, there exists a non-degenerate extension $\widehat{\mathcal{M}}$ of $\mathcal{M}$ such that $\mathcal{A}$ requires at least n queries to find the sink of the P-matroid USO given by $\widehat{\mathcal{M}}$.*

Proof. We specify an adversarial procedure which iteratively builds up a localization σ for $\mathcal{M}$. At any point of this procedure, the current localization describes an extension $\widehat{\mathcal{M}}$ for which the PPU O contains exactly one unoriented subcube U, and all edges incident to U are oriented into U. Thus, the global sink of O must lie in U, but its exact location has not been determined yet.

At the beginning of the procedure, σ is set to be all-zero, i.e., O is completely unoriented and U is the whole cube. Now, whenever the sink-finding algorithm queries a vertex v, the adversarial procedure must return the complete orientation around this vertex. If v lies outside of U, it is already completely oriented, and its orientation can simply be returned.

Otherwise, if v lies in U, the localization σ has to be changed. To do this, we pick one dimension i which spans U. If $v_i = 0$, we change σ to $\sigma' := \sigma \circ [+ \cdot t_i]$, i.e., σ is combined with the lexicographic extension $[+ \cdot t_i]$. On O this has the effect that some edges in U are oriented. By Lemma 21, all edges in the lower i-facet of U are oriented, and all i-edges in U are pointed away from this facet. Thus, v is now completely oriented, and U has shrunk by one dimension. The orientation of v can thus be returned.

Note that if $v_i = 1$, the lexicographic extension would be picked to be $[+ \cdot s_i]$, the rest of the procedure staying the same.

Since U shrinks by only one dimension with every query, and U has n dimensions at the beginning, the first n vertices queried by the algorithm are never the sink. Thus, it takes at least n queries to determine the sink. ◀

Previously, the best known lower bound for sink-finding on P-matroid USOs was $\Omega(\log n)$ queries [18]. In contrast, the stronger, almost-quadratic lower bound of Schurr and Szabó [14] does not apply to P-matroid USOs (for a proof of this see Lemma 30 in Appendix B).

The P-Matrix Linear Complementarity Problem (P-LCP) is an algebraic analogue of P-OMCP. We discuss our results (Sections 3 and 4) in the context of P-LCPs in Appendix A.

5 The Search Problem Complexity of P-OMCP

An instance of Unique End of Potential Line consists of an implicitly given exponentially large graph G, in which each vertex has a positive cost and in- and out-degree at most one. Thus, the graph is a collection of directed paths called *lines*. The computational task is as follows: if the nodes of G form a single line (that starts in some given start vertex) with strictly increasing cost, then find the unique end node of this line — a *sink*. Otherwise, either find *some* sink in G or a *violation certificate* that shows that G does not consist of a single line. Unique End of Potential Line is a total search problem, i.e., there always exists a sink or a violation. Note that there might exist a sink and a violation simultaneously.

▶ **Definition 23.** *The search complexity class Unique End of Potential Line (UniqueEOPL) contains all problems that can be reduced in polynomial time to Unique End of Potential Line. Thus, the complexity class UniqueEOPL captures all total search problems where the space of candidate solutions has the structure of a unique line with increasing cost.*

UniqueEOPL was introduced in 2018 by Fearnley et al. [4]. UniqueEOPL is a subclass of PPAD ∩ PLS [3]. Problems in UniqueEOPL are not known to be solvable in polynomial time.

The promise version of a total search problem with violations is to find a solution under the promise that no violations exist for the given instance. PromiseUEOPL is the promise version of the search problem class UniqueEOPL.

A search problem reduction from a problem R to a problem T is *promise preserving*, if every violation of T is mapped back to a violation of R and every valid solution of T is mapped back to a valid solution or a violation of R. Promise preserving reductions are transitive. When containment of a search problem R in UniqueEOPL is shown via a polynomial time, promise preserving reduction, the promise version of R is contained in PromiseUEOPL.

We now state the problem of USO sink-finding as a total search problem with a violation.

▶ **Definition 24.** *Given an orientation function $O : \{0,1\}^n \to \{+,-\}^n$, the task of the total search problem Unique Sink Orientation Sink-Finding (USO-SF) is to find:*

(U1) *A vertex $v \in \{0,1\}^n$ such that $\forall i \in [n] : O(v)_i = -$. The vertex v is a sink.*

(UV1) *Two distinct vertices $v, w \in \{0,1\}^n$ with $\forall i \in [n] : (v_i = w_i) \vee (O(v)_i = O(w)_i)$. The orientation O does not fulfill the Szabó-Welzl condition and thus is not USO.*

▶ **Lemma 25** ([4])**.** *USO-SF is in UniqueEOPL and its promise version is in PromiseUEOPL.*

Next, we define the P-OMCP problem as a total search problem with violations.

▶ **Definition 26.** *Let $\widehat{\mathcal{M}} = (\widehat{E_{2n}}, \widehat{\mathcal{C}})$ be an oriented matroid with the set S being a basis. The task of the total search version of P-OMCP is to find one of the following:*

(M1) *A circuit $X \in \widehat{\mathcal{C}}$ such that $X \geq 0$, $X_q = +$ and $\forall i \in [n] : X_{s_i} X_{t_i} = 0$.*
(MV1) *A circuit $Z \in \mathcal{C}$ which is sign-reversing.*
(MV2) *A complementary set $B \subset E_{2n}$ of size n which is not a basis of $\widehat{\mathcal{M}}$.*
(MV3) *Two distinct, complementary circuits $X, Y \in \mathcal{C}$ with $X_q = Y_q = +$ and $\forall i \in [n] : X_{s_i} Y_{t_i} = X_{t_i} Y_{s_i} = 0$, or $X_{s_i} = Y_{t_i}$ and $X_{t_i} = Y_{s_i}$.*

The definition of the violation *(MV3)* may look unintuitive, but the following lemma shows that it correctly implies that $\widehat{\mathcal{M}}$ is not a P-matroid extension.

▶ **Lemma 27.** *A violation of type* (MV3) *implies that $\widehat{\mathcal{M}}$ is not a P-matroid extension.*

Proof. Suppose we are given such a violation, i.e., two distinct complementary circuits $X, Y \in \mathcal{C}$ with $X_q = Y_q = +$ and $\forall i \in [n] : X_{s_i} Y_{t_i} = X_{t_i} Y_{s_i} = 0$ or $X_{s_i} = Y_{t_i}$ and $X_{t_i} = Y_{s_i}$.

As X, Y are distinct, $X \neq Y$. Since $X_q = Y_q = +$, it holds that $X \neq -Y$. We can thus apply circuit axiom (C3) on circuits X and $-Y$ and element $q \in X^+ \cap (-Y)^-$. It follows that there must exist some circuit Z with:

- $Z^+ \subseteq X^+ \cup (-Y)^+ \setminus \{q\}$ and
- $Z^- \subseteq X^- \cup (-Y)^- \setminus \{q\}$.

If $\underline{Z}$ contained no complementary pair, it would be a complementary set. Any complementary set $B \supseteq \underline{Z}$ of size n can not be a basis, since $\underline{Z}$ is a circuit. This is a violation of type *(MV2)* and implies that $\widehat{\mathcal{M}}$ is not a P-matroid extension.

Otherwise, $\underline{Z}$ must contain at least one complementary pair s_i, t_i. As X and Y are complementary, s_i and t_i are each only contained in one of the two circuits, w.l.o.g. $s_i \in \underline{X}$ and $t_i \in \underline{Y}$. Therefore, s_i and t_i are each only contained in one of the two sets $X^+ \cup (-Y)^+ \setminus \{q\}$ and $X^- \cup (-Y)^- \setminus \{q\}$. Since $X_{s_i} = Y_{t_i}$, they are both in different sets, and thus $Z_{s_i} = -Z_{t_i}$. Since this holds for every complementary pair in $\underline{Z}$, we conclude that Z is sign-reversing. Thus Z is a violation of type *(MV1)*, and $\widehat{\mathcal{M}}$ can not be a P-matroid extension.

Note that even if we cannot find Z explicitly in polynomial time, we can check the conditions on X and Y in polynomial time. ◀

Technically, the violation *(MV1)* would be enough to make this search problem total, but our reduction to USO-SF detects only violations of type *(MV2)* and *(MV3)*. Note that as Fearnley et al. [5] already observed, there may be a difference in the complexity of a total search problem depending on the violations chosen. There is no trivial way known to the authors to transform a violation of type *(MV3)* or *(MV2)* to a violation of type *(MV1)*.

With the help of Lemmas 14 and 15 we now adapt Todd's reduction of non-degenerate P-OMCP instances to USO (recall Section 2.4) to also work with degenerate instances and their respective total search versions.

Given a P-OMCP instance $\widehat{\mathcal{M}} = (\widehat{E_{2n}}, \widehat{\mathcal{C}})$ (note that $\widehat{\mathcal{M}}$ is possibly not a P-matroid extension, or degenerate), we define the orientation $O : V(Q_n) \to \{+,-\}^n$:

$$O(v)_i := \begin{cases} - & \text{if } B(v) \text{ is not a basis,} \\ - & \text{if } v_i = 0 \text{ and } C_{s_i} = 0, \\ + & \text{if } v_i = 1 \text{ and } C_{t_i} = 0, \\ + & \text{if } v_i = 0 \text{ and } C_{s_i} = - \\ & \text{or } v_i = 1 \text{ and } C_{t_i} = -, \\ - & \text{otherwise,} \end{cases}$$

with $B(v)$ and $C := C(B(v), q)$ defined as in Section 2.4. Furthermore, using Lemmas 14 and 15 we know that O is USO if $\widehat{\mathcal{M}}$ is a P-matroid extension.

▶ **Theorem 28.** *The construction above is a polynomial time, promise preserving reduction from P-OMCP to USO-SF.*

Proof. Given a P-OMCP instance $\widehat{\mathcal{M}} = (\widehat{E_{2n}}, \widehat{\mathcal{C}})$, let (Q_n, O) be an USO-SF instance with O as defined above.

Polynomial time For the reduction we build an orientation oracle O for USO-SF from the given circuit oracle for P-OMCP. Note that this does not mean that we have to compute the output of O for every vertex, we simply have to build the algorithm (usually represented by a logical circuit) computing O from the algorithm computing the circuit oracle.

Since O merely computes $B(v)$ from a given vertex, invokes the circuit oracle, and then performs a case distinction, it can clearly be built and queried in polynomial time in n.

Correctness To prove correctness of this reduction being promise preserving, we must show that every violation of USO-SF can be mapped back to a violation of P-OMCP and every valid solution of USO-SF can be mapped back to a valid solution or a violation of P-OMCP.

A solution of type *(U1)* Let $v \in V(Q_n)$ be a solution to the USO-SF instance, i.e., a sink.

It might be that v is a sink because $B(v)$ is not a basis, and thus $O(v)_i = -$ for all i. To map this back to a violation or solution of P-OMCP, we first check if the P-OMCP oracle returns that $B(v)$ is not a basis for the input $C(B(v), q)$. If so, we found a violation of type *(MV2)*. Otherwise, the fundamental circuit $C(B(v), q)$ is a solution to the P-OMCP instance: Since v is a sink, there is no index at which the fundamental circuit is negative. All entries of $C(B(v), q)$ are positive and the complementarity condition is fulfilled by construction of $B(v)$.

A violation of type *(UV1)* If a violation is found, we have two distinct vertices v and w with $\forall i \in [k] : (v_i = w_i) \vee (O(v)_i = O(w)_i)$. We first again check whether $B(v)$ or $B(w)$ are bases, if not we map this violation to a violation of type *(MV2)*.

Otherwise, we show that there are two distinct complementary circuits $X, Y \in \mathcal{C}$ with $X_q = Y_q = +$ and $\forall i \in [n] : X_{s_i} Y_{t_i} = X_{t_i} Y_{s_i} = 0$ or $X_{s_i} = Y_{t_i}$ and $X_{t_i} = Y_{s_i}$, i.e., a violation of type *(MV3)*. We claim that the circuits $X := C(B(v), q)$ and $Y := C(B(w), q)$ fulfill these conditions.

First, we need to show that $X \neq Y$. If the two circuits were equal, they would have to be degenerate on all dimensions spanned by v and w. Then by construction of O, v and w could not fail Szabó-Welzl (see Lemma 14).

Next, we see that by definition $C(B(v), q)_q = +$ and $C(B(w), q)_q = +$ and both circuits are complementary.

Finally, we show that for each dimension i, either (i) $X_{s_i} Y_{t_i} = X_{t_i} Y_{s_i} = 0$ or (ii) $X_{s_i} = Y_{t_i}$ and $X_{t_i} = Y_{s_i}$. For every dimension i for which $v_i = w_i$ (w.l.o.g. both are 0), both $X_{t_i} = 0$ and $Y_{t_i} = 0$. Therefore, condition (i) holds. For a dimension i for which $v_i \neq w_i$, if at least one of the i-edges incident to v_i and w_i is degenerate, we have $X_{s_i} = X_{t_i} = 0$ (or $Y_{s_i} = Y_{t_i} = 0$). Thus, condition (i) also holds in this case. For a dimension i in which both are non-degenerate, since v and w are a violation of type *(UV1)*, $O(v)_i = O(w)_i$. By construction of O it must hold that $X_{s_i} = Y_{t_i}$ and $X_{t_i} = Y_{s_i}$, i.e., condition (ii) holds.

Therefore, the circuits $C(B(v), q)$ and $C(B(w), q)$ form a violation of type *(MV3)*. ◀

It follows that P-OMCP as defined in Definition 26 is in UniqueEOPL and its promise version is in PromiseUEOPL.

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A P-Matrix Linear Complementarity

We only give a short overview over the P-Matrix Linear Complementarity problem here, and refer the interested reader to the comprehensive textbook by Cottle, Pang and Stone [2], as well as the seminal paper of Stickney and Watson [15] on solving P-LCPs using sink-finding in USOs.

A *P-matrix* is a matrix $M \in \mathbb{R}^{n\times n}$ where every principal minor is positive. Given a P-matrix M and a vector $q \in \mathbb{R}^n$, the *P-Matrix Linear Complementarity Problem (P-LCP)* given by (M, q) is to find two vectors $w, z \in \mathbb{R}^n$ with (i) $w - Mz = q$, (ii) $w^T z = 0$, and (iii) $w, z \geq \mathbf{0}$. Every P-LCP has a unique solution.

Any pair of vectors w, z fulfilling condition (i) and $w_i = 0 \vee z_i = 0$ for all i is called a *candidate solution*. Given a choice from $\{w_i, z_i\}$ for each index i, there is a unique candidate solution for which the chosen entries of w and z are 0. This unique candidate solution can be computed with basic linear algebra. We say that (M, q) is non-degenerate, if all candidate solutions contain n non-zero elements.

It is well-known that a P-LCP can be translated to a P-OMCP. This is achieved by considering the matrix $[I; -M; -q]$, associating the first n columns (those of I) with S, the next n columns (those of $-M$) with T, and the last column with element q. The circuits of the P-matroid extension $\widehat{\mathcal{M}}$ are then given by the minimal linear dependencies of these column vectors.

Non-degenerate P-LCPs can thus be reduced to sink-finding in USOs (by this detour through P-OMCP, but also more directly by an equivalent reduction [15]). USOs that are obtained through this reduction are called *P-Cubes* and are heavily studied. Degenerate P-LCPs yield degenerate P-OMCPs, and we call their corresponding partial orientations *partial P-Cubes*. All structural results from Section 3 also hold for partial P-Cubes, since they are a (strict) subset of the partial P-matroid USOs.

The constructions from Section 4 can also be translated to work with P-LCPs. The composition of localizations can be replaced by the composition of vectors q. This composition is computed as $q' := q_1 + \epsilon \cdot q_2$ for some $\epsilon > 0$, where ϵ is chosen to be small enough that no sign of a non-zero element of any candidate solution is flipped. The lexicographic extensions correspond to setting q to be equal to vectors $e_i, -e_i, M_i$, or $-M_i$. Armed with constructions equivalent to both compositions and lexicographic extensions, we can restate Theorem 22 in the context of P-LCP:

▶ **Theorem 29.** *Let M be an $n \times n$ P-matrix such that the matrix $[I; -M]$ has no linear dependencies of fewer than $n+1$ columns. Then, for every deterministic sink-finding algorithm $\mathcal{A}$, there exists a vector q such that $\mathcal{A}$ requires at least n queries to find the sink of the P-Cube given by the P-LCP instance (M, q).*

B Schurr and Szabó's Lower Bound

On an intuitive level, Schurr and Szabó's adversarial construction yielding the $\Omega(n^2/\log n)$ lower bound for deterministic sink-finding algorithms [14] works as follows: In the first phase, the construction answers $n - \lceil \log_2 n \rceil$ queries of the algorithm. After these queries, it is guaranteed that there exists some face of dimension $n - \lceil \log_2 n \rceil$ in which no vertex has been queried yet. The queries in the first phase are answered such that this face is a hypersink (all edges are incoming) and can thus be filled in with any USO. The lower bound then follows from a recursive argument.

In more detail, in the first phase, the construction keeps a set of dimensions L and an USO $\tilde{s}$ of the cube spanned by the dimensions in L. Queried vertices are oriented according to their projected location in $\tilde{s}$ for the dimensions in L, and always outgoing for the dimensions in $[n] \setminus L$. As an invariant, no two queried vertices can be the same vertex when projected onto the cube spanned by L, thus a dimension is added to L (and $\tilde{s}$ is adapted by a defined procedure) whenever this condition would be violated.

▶ **Lemma 30.** *There exists a deterministic sink-finding algorithm $\mathcal{A}$, against which Schurr and Szabó's adversarial construction from the proof of [14, Theorem 9] produces an USO that is not a P-matroid USO.*

Proof. We describe an algorithm $\mathcal{A}$ that forces the construction to make $\tilde{s}$ a fixed 3-dimensional subcube which is not a P-matroid USO. For this strategy to work, we require five queries. We set the dimension of the final cube to be at least 8, such that the construction stays in the first phase for at least five queries.

All vertices queried by our algorithm have a zero in all coordinates except the first three; we therefore omit writing these additional zeroes in their coordinates. The algorithm begins by querying the following two vertices:

$$v_1 = 000, \ v_2 = 111$$

After the second query, the construction has to add one of the first three dimensions to the set L, since otherwise v_1 and v_2 have the same coordinates within the (empty) set L. Note that the algorithm can detect this choice ℓ, as the only incoming edge of v_2 is in this dimension ℓ. W.l.o.g., we assume the algorithm picks $\ell = 1$, i.e., $L = \{1\}$.

The algorithm continues by querying $v_3 = 011$. Once again, the construction has to pick either the second or third dimension to be added to L, as otherwise v_1 and v_3 have the same coordinates within the dimensions in L. Again this choice can be detected by the algorithm, and w.l.o.g. we assume that now $L = \{1, 2\}$.

The algorithm now queries $v_4 = 100$, and L does not change, since all vertices $v_1, \dots, v_4$ have different coordinates in the dimensions $\{1, 2\}$.

The final query is $v_5 = 001$, and now L must be changed to $\{1, 2, 3\}$. The USO $\tilde{s}$ on the cube spanned by L evolves with these queries as shown in Figure 6. Note that at no point the construction has any choice in how to orient the edges in the newly added dimension, since all edges of dimensions not in L incident to queried vertices must be oriented away from the queried vertex by definition of the construction. Thus, the construction is forced to build this orientation when confronted with our algorithm.

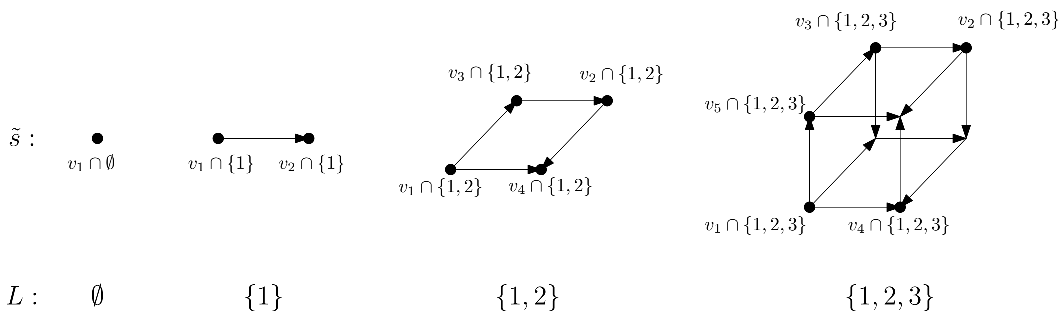

Figure 6 The orientation $\tilde{s}$ built by Schurr and Szabó's adversarial construction. The notation $v \cap L$ describes projection of v onto the cube spanned by the dimensions in L.

By [7], every n-dimensional P-Cube must contain n vertex-disjoint paths from the source to the sink (a property called *Holt-Klee*). The orientation $\tilde{s}$ clearly does not fulfill this, as it does not have three edges from the lower 3-facet (containing the source) to the upper 3-facet (containing the sink). Thus, $\tilde{s}$ is not a P-Cube. In 3 dimensions, the set of P-Cubes is the same as the set of P-matroid USOs [11] (this follows from every oriented matroid of 7 elements being realizable). We thus also know that $\tilde{s}$ is not a P-matroid USO. As both P-Cubes and P-matroid USOs are closed under the operation of taking subcubes [11], and because the final orientation constructed by the construction of Schurr and Szabó contains $\tilde{s}$ as the subcube spanned by v_1 and v_2, it cannot be a P-Cube or a P-matroid USO, either. ◀