Multi player Parrondo games with rigid coupling

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May 16, 2023

Abstract

In the original Parrondo game, a single player combines two losing strategies to a winning strategy. In this paper we investigate the question what happens, if two or more players play Parrondo games in a coordinated way. We introduce a strong coupling between the players such that the gain or loss of all players in one round is the same. We investigate two possible realizations of such a coupling. For both we show that the coupling increases the gain per player. The dependency of the gain on the various parameters of the games is determined. The coupling can not only lead to a larger gain, but it can also dominate the driving mechanism of the uncoupled games. Which driving mechanism dominates, depends on the type of coupling. Both couplings are set side by side and the main similarities and differences are emphasised.

Keywords: Noise induced transport; Parrondo's paradox; Markov chains; multiplayer Parrondo games; collective coupling effect

1 Introduction

A Parrondo game [1, 2] consists of two simple games, typically realized by flipping biased coins, which are played in some regular or randomly alternating sequence. The interesting effect occurring here is that even if the two games lead to a systematic loss if played sufficiently long, the combination yields a systematic win.

Originally, Parrondo invented these games to illustrate the occurrence of noise induced transport in so called Brownian motors, for a review see [3]. Brownian motors have been proposed first by Magnasco [4] as a model for intra-cellular transport created by motor proteins like kinesin, which move along micro-tubuli. In the simplest form, a Brownian motor is a Brownian particle moving in a time-dependent periodic potential without inversion symmetry. The Brownian particle is driven by a white noise process as usual, representing a finite temperature of the system. The time dependence of the potential can be either periodic, see *e*.g. [3] or stochastic. In the case of a stochastic additive or multiplicative noise added to the potential, this additional noise process must have a finite correlation time, but can be otherwise an arbitrary noise process, see *e*.g. [5] for the additive noise and [6] for the multiplicative noise process. The combination of the broken inversion symmetry and the additional additional periodic or stochastic time dependence yields a non-vanishing stationary current. The Parrondo games can be viewed as a discretized version of Brownian motors [7]. But the study of Parrondo games is an interesting topic independently of that initial motivation. Not only are Parrondo games probably the most simple systems where the coupling of systems with detailed balance yields a new system where detailed balance is broken and therefore a stationary current occurs, but they also can have direct link to living systems. Lai and Cheong [8] for example connect the Parrondo games with "societal ideas of redistribution, cooperation, voting, performance, and resource growth to bring about 'winning' outcomes in a social group." [8] Furthermore, Cheong *et al.* [9] investigate the winning strategy in bacteriophages because of Parrondo's Paradox.

Motivated by the fact that motor proteins like kinesin have two heads which couple to the micro-tubuli, Klumpp *et al.* [10] investigated the noise-induced transport of two strongly coupled particles. They showed that the transport of a system of two strongly coupled particles is more efficient than the

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	capital	game A	game B	game C (random)
p_0	$x \bmod 3 = 0$	$\frac{1}{2} - \epsilon$	$\frac{1}{10} - \epsilon$	$q_0 \cdot \frac{1}{2} + q_1 \cdot \frac{1}{10} - \epsilon$
p_1	$x \bmod 3 \neq 0$		$\frac{3}{4} - \epsilon$	$q_0 \cdot \frac{1}{2} + q_1 \cdot \frac{3}{4} - \epsilon$

Table 1: Winning probabilities $p_{0,1}$ for the original Parrondo games [2]. The variable $q_{0,1}$ defines the probability to choose game A,B, respectively, ϵ is the so called bias parameter. Parrondo originally chose $\epsilon = 0.005$ and $M = 3$.

transport by a single particle. This motivates us to study coupled multi-player Parrondo games. The idea is to introduce two or more coupled players, each playing the same Parrondo game. To our knowledge, such multi-player Parrondo games with correlated, interacting players have not been studied so far. As in [10], we concentrate on strong coupling where in each step all players win or lose. The aim is to investigate, how such a coupling can be realized, whether in such a coupled system the gain or loss is higher than in the uncoupled system, and if due to the coupling new mechanisms for the creation of a stationary current occur, which are not present for a single player.

Our paper is organized as follows. We first introduce and review ordinary Parrondo games, mainly to fix the notation. In Sect. 3 we introduce two different couplings for two or more players. These coupled systems can be easily investigated by simulations or by methods using discrete-time Markov chains (DTMC) to calculate the stationary current. We discuss the results and investigate the mechanisms that yield to the higher stationary current in the case of coupled players. Finally, in Sect. 5 we summarize our results, give an outlook and propose some future research in this area.

2 Parrondo games

Let us first introduce some basics about usual Parrondo games, mainly to fix the notation and to introduce the methods of DTMC used to investigate the capital current. First, we state the original definition of the Parrondo games given by Abbott and Harmer [1, 2]. Let $x_0 \in \mathbb{N}_0$ be the initial capital of a player. The player can win or lose one capital unit $\Delta x = \pm 1$ in every round (negative capital should be possible) and hence obtains the capital $x(n) \in \mathbb{Z}$ after n rounds. The winning probability p depends on the choice of game and is periodic as a function of capital with period M. Game A is a homogeneous process (the same coin is used in every round), game B is capital dependent (one of two coins is chosen depending on the capital). Parrondo originally chose the probabilities given in table 1. In general, game C is the result of periodically or randomly switching between games A and B.

We consider only random switching. We implement that via the dichotomous random process $z \in \{0, 1\}$ indicating the choice of game A and B, respectively, at the beginning of every round with the probabilities

$$P(\text{play A}) = P(z = 0) =: q_0 \,, \tag{1}$$

$$P(\text{play B}) = P(z = 1) =: q_1 = 1 - q_0 \,. \tag{2}$$

The variable $p_{0,1}^{\text{A,B,C}}$ denotes the winning probability for the games A, B and C for (0) a capital multiple of M and (1) otherwise.

Since the transition probabilities are periodic functions of the capital, we can restrict the discussion to the reduced state space $\mathbb{Z}/M\mathbb{Z}$ and choose periodic boundary conditions. Indeed, since the winning probabilities only depend on the current capital and hence on the current state, Parrondo games are DTMC. The transition matrix in the reduced state space $Q = (Q_{ij})_{i,j \in \mathbb{Z}/M\mathbb{Z}}$ with

$$Q_{ij}(n) \coloneqq P(x_{n+1} = i | x_n = j) \tag{3}$$

is finite and time-homogeneous and has the explicit form [11],

$$Q(n) = Q = \begin{pmatrix} 0 & 1 - p_1 & & & p_1 \\ p_0 & 0 & \ddots & & \\ & p_1 & \ddots & 1 - p_1 & \\ & & \ddots & 0 & 1 - p_1 \\ 1 - p_0 & & & p_1 & 0 \end{pmatrix} . \tag{4}$$

Let $\bar{x} = x \bmod M$ be the reduced capital and P be the probability distribution on the reduced state space, i.e. P_i be the probability for $\bar{x} = i$. We then obtain for the time evolution

$$P_i(n+1) = \sum_{j=0}^{M-1} Q_{ij} P_j(n) \Longleftrightarrow P(n+1) = QP(n)\,. \tag{5}$$

Especially, the stationary distribution π of a time-homogeneous DTMC is given by $\pi = Q\pi$.

Under certain conditions, DTMC converge towards such a stationary distribution [12], p. 150. In fact, one can show that the DTMC of the original Parrondo games converges against a unique stationary distribution in the sense of $\lim_{n\to\infty} \left|P_i(n) - \pi_i\right| = 0$ for odd M. This is true if the games are *irreducible*, *aperiodic*, and *positive recurrent*, see [12], p. 118ff, 150.

Since every state communicates with every other one by simply winning or losing D times if the capital difference is D, the DTMC of the games is *irreducible* and consists of only one communicating class (the mutual communication induces an equivalence relation) [12], p. 80.

Furthermore, the period of state i [12], p. 84.

$$d_i = \gcd\left\{n \geq 1 : P(x_n = i | x_0 = i) > 0\right\}\,, \tag{6}$$

which is the same for communicating states and hence can be defined for the whole class, becomes 1 for an odd M. This is obvious since the capital can return to its initial value after two rounds and after winning M times in a row. Therefore, the games with an odd M are *aperiodic*.

The last property is *positive recurrence*, which is defined by [12], p. 111

$$P(\tau_i < \infty) = 1 \wedge E(\tau_i) < \infty\,, \tag{7}$$

where we used the first recurrence time $\tau_i \coloneqq \min\{n \geq 1 : x_n = i | x_0 = i\}$. Brémaud [12], p. 122 proves that an irreducible, time-homogeneous DTMC with a finite state space is positive recurrent, hence positive recurrence is shown for the reduced Parrondo games with an odd M. For time-homogeneous, ergodic (irreducible, aperiodic and positive recurrent) DTMC with a finite state space, there is not only an unique stationary distribution [12], p. 118ff, but also a convergence theorem stating the given convergence [12], p. 150. Since all properties apply to the reduced Parrondo games with odd M, the convergence is proven.

We now know that there is a unique probability distribution against which every initial distribution converges in the long-run limit. From this we can compute a stationary capital current giving us the long-run capital difference per round. Indeed, since the stationary current is independent of the initial distribution, we can use it to evaluate and compare different games. This strategy is important: As can be seen for game B, a game does not have to be a martingale but nevertheless might be fair in the long-run limit corresponding to a vanishing capital current [2, 13]. The reason for this is that a martingale must be balanced at every single step. This is obviously not the case for game B. However, the game can be balanced on average anyway. This unusual effect is described in more detail in [14].

Let

$$p_j^{\text{win}} = \begin{cases} Q_{j+1,j} & | \ j \neq M-1 \\ Q_{0,M-1} & | \ j = M-1 \end{cases} \tag{8}$$

be the winning probabilities. The stationary current can be obtained from the stationary distribution π. Toral *et al.* [15] derive, using the corresponding master equation, a discrete form for the probability current, which reduces for the Parrondo games to [15]

$$J_i(n) = -(1 - p_i^{\text{win}})P_i(n) + p_{i-1}^{\text{win}} P_{i-1}(n)\,. \tag{9}$$

Summing over all states and considering periodic boundary conditions ($p_{-1}^{\text{win}} = p_{M-1}^{\text{win}},\ P_{-1} = P_{M-1}$), this gives

$$\begin{aligned} J = \sum_{i=0}^{M-1} J_i &= \sum_{i=0}^{M-1} \left[-(1 - p_i^{\text{win}})\pi_i + p_i^{\text{win}}\pi_i\right] \\ &= \sum_{i=0}^{M-1} 2p_i^{\text{win}}\pi_i - 1 = 2E(p^{\text{win}}) - 1 \end{aligned} \tag{10}$$

in the stationary case.

In particular we observe that $J = 0$ is equivalent to $E(p^{\text{win}}) = \frac{1}{2}$ which is the definition of fairness given by Abbott and Harmer [11]. Indeed, for the original Parrondo games with $M = 3$ it is possible

	capital x	$M=3$	$M=5$	$M=7$
p_0	$x \bmod M = 0$	0.04200574	0.03467694	0.02181704
p_1	$x \bmod M \neq 0$	0.82685756	0.69669249	0.65335836
	$M=11$	$M=19$	$M=29$	$M=49$
p_0	0.00849192	0.00135921	$1.45686849 \cdot 10^{-4}$	$1.88929755 \cdot 10^{-6}$
p_1	0.61680554	0.59064647	0.57822635	0.56821420

Table 2: Adapted winning probabilities $p_{0,1}$ for game B for different M with $\epsilon = 0$. For $\epsilon \neq 0$, ϵ has to be subtracted. Game A does not change, $p_0 = p_1 = 0.5 - \epsilon$.

	capital x	$d=1$	$d=2$	$d=3$	$d=4$
p_0	$x \bmod M < d$	0.02213766	0.04204007	0.03145708	0.00312982
p_1	$x \bmod M \geq d$	0.55241902	0.59092394	0.65533964	0.85097252

Table 3: Adapted winning probabilities $p_{0,1}$ for game B for different d with $\epsilon = 0$ and $M = 19$. For $\epsilon \neq 0$, ϵ has to be subtracted. Game A does not change, $p_0 = p_1 = 0.5 - \epsilon$.

to compute the stationary distribution analytically and therefore obtain an analytic expression for the stationary capital current which leads to

$$J = \frac{3\left(2p_0p_1^2 - 2p_0p_1 + p_0 - p_1^2 + 2p_1 - 1\right)}{2p_0p_1 - p_0 + p_1^2 - 2p_1 + 3}\,. \tag{11}$$

Inserting the probabilities from table 1 with $q_0 = 0.5$, one can easily see that there is a range of ϵ for which $J_A, J_B < 0$ but $J_C > 0$ leading to Parrondo's Paradox. Hence, the formalism for investigating the Parrondo games with one player is simple: Given the transition matrix for a DTMC, we obtain the stationary distribution by computing the eigenvector for the eigenvalue 1 and with equations 8 and 10 calculate the capital current for the games. It has to be noted that this does not only apply to the original Parrondo games but to every transition matrix corresponding to a one-dimensional DTMC with states sorted by their capital.

For the coupled Parrondo games we want to compare the capital current for different parameters, especially the period M and the width of the barrier d which is given by the number of capitals within one period corresponding to a small winning probability in game B. For the original Parrondo games, one can see in table 1 that $d = 1$. In order to only investigate the coupling effect, the uncoupled games should lead to the same current when varying the parameters. Therefore, we modify the winning probabilities accordingly. Since game A is independent of the capital, it does not change. However, the probabilities for game B have to be adapted so that it is always fair and game C with the random combination for $q_0 = 0.5$ induces a current $J = J_{\text{const}}$ for all parameters. This is achieved numerically by introducing the probabilities as variables and computing the roots of the function $[J_B, J_C - J_{\text{const}}]$. We implemented the numerics in Python using the package *scipy.optimize.fsolve*. We choose those roots satisfying $p_0 \in (0, 0.5)$ and $p_1 \in (0.5, 1)$. For the variation of M and d we choose $J_{\text{const}} = 0.05$ and $J_{\text{const}} = 0.02$, respectively. The results are listed in tables 2 and 3.

With these probabilities and $\epsilon = 0.5 \cdot 10^{-6}$, game A and B are losing games for all periods and widths of the barrier, respectively. However, game C with $q_0 = 0.5$ is a winning game, we obtain Parrondo's Paradox. For $\epsilon \neq 0$ there is a slight deviation of the current J_{const} for the uncoupled games. However, considering the change caused by the coupling, this is irrelevant for small bias parameters.

3 The coupled Parrondo games

We are now in the position to introduce coupled Parrondo games. The idea is to have more than one player and to couple the players so that they do not play independently. In this paper we only investigate the rigid coupling where in each round all players win or lose the same amount. We first consider two players I and II. We denote their capital as $x_{\text{I,II}} \in \mathbb{Z}$. They win or lose one capital unit in every round. The individual winning probabilities are defined by the Parrondo games for a single player. However, the players are not independent. The probability $P(x_{\text{I}}(n+1) = i, x_{\text{II}}(n+1) = k \mid x_{\text{I}}(n) = j, x_{\text{II}}(n) = l)$ is obtained by combining the individual probabilities according to the couplings, which will be explained in the following.

As above, we consider the state space $\mathbb{Z}/M\mathbb{Z}$. Since we will show that the Parrondo games with rigid coupling can be reduced to the ordinary Parrondo games with modified transition probabilities, they will also have a stationary distribution in this state space. In order to find a transition matrix, we define a projection between both capitals and a single variable

$$\begin{array}{ccc} (i,k) \in \{\mathbb{Z}/M\mathbb{Z}\}^2 & \longleftrightarrow & a \in \{\mathbb{Z}/M^2\mathbb{Z}\} \\ \bar{x}_{\mathrm{I}} = i\ ,\ \bar{x}_{\mathrm{II}} = k & \longleftrightarrow & \bar{x}_{\mathrm{I}} = \lfloor a/M \rfloor\ ,\ \bar{x}_{\mathrm{II}} = a \bmod M \end{array} \tag{12}$$

with $\bar{x}_{\mathrm{I,II}} := x_{\mathrm{I,II}} \bmod M$ and the floor function $\lfloor x \rfloor$. Especially, the variable a is obtained by $a = M \cdot i + k$. We will always use this transformation for $a \leftrightarrow i, k$ and $b \leftrightarrow j, l$. The transition matrix then becomes:

$$Q_{ab} = P(\bar{x}_{\mathrm{I}}(n+1) = i,\ \bar{x}_{\mathrm{II}}(n+1) = k \ \mid\ \bar{x}_{\mathrm{I}}(n) = j,\ \bar{x}_{\mathrm{II}}(n) = l)\,. \tag{13}$$

For calculating this matrix, we need to introduce the coupling between both players. In the following we define two couplings that combine both players' winning or losing probabilities for a collective gain or loss.

The first approach is the double-play coupling. Both players play the individual Parrondo games and have to win or lose at the same time for a collective gain or loss. Additionally, all the other cases are excluded by setting their transition probabilities to zero and renormalizing the others. This coupling is motivated by Brownian motion of two coupled particles: In an infinitesimal time interval, both particles simultaneously have to move to the right or to the left.

The second approach is the single-play coupling. One of the players is chosen at the beginning of every round with a certain probability and plays the individual Parrondo games. The result will then also be applied to the other player. This coupling is motivated by certain motor proteins: The heads of a motor protein are coupled, but can attach and detach independently during the ATP hydrolysis. This corresponds to the random choice of one player.

We adapt the notation of the original Parrondo games by adding the super- or subscripts I, II, e.g. p_{I}^X is the winning probability of player I when playing game $X \in \{A, B\}$, which is of course capital dependent itself. The individual winning and losing probabilities are obtained from the single Parrondo games in table 2. However, when varying the width of the barrier, we use the probabilities in table 3. Since the external force is the same for both particles in the continuous case, we always set $\epsilon_{\mathrm{I}} = \epsilon_{\mathrm{II}}$.

With this we obtain the combined winning probabilities for the double-play games as

$$p^{XY} = \frac{p_{\mathrm{I}}^X p_{\mathrm{II}}^Y}{p_{\mathrm{I}}^X p_{\mathrm{II}}^Y + (1 - p_{\mathrm{I}}^X)(1 - p_{\mathrm{II}}^Y)} \tag{14}$$

when player I, II plays game $X, Y \in \{A, B\}$, respectively. Let Q^{XY} be the corresponding transition matrix. The entries for possible transitions result from the collective winning and losing probabilities similar to equation 8 (note that they depend on the current capitals of the players!), the remaining entries are set to zero. By combining the different possibilities for the games we obtain

$$Q = q_0^{\mathrm{I}} q_0^{\mathrm{II}} \cdot Q^{AA} + q_0^{\mathrm{I}} q_1^{\mathrm{II}} \cdot Q^{AB} + q_1^{\mathrm{I}} q_0^{\mathrm{II}} \cdot Q^{BA} + q_1^{\mathrm{I}} q_1^{\mathrm{II}} \cdot Q^{BB}\,. \tag{15}$$

For the single-play coupling we define the probability p_{I} of choosing player I and $p_{\mathrm{II}} = 1 - p_{\mathrm{I}}$. Since the second player always wins or loses simultaneously, the collective winning probability is

$$p^{\mathrm{win,win}} = p_{\mathrm{I}} p_{\mathrm{I}}^{\mathrm{win}} + p_{\mathrm{II}} p_{\mathrm{II}}^{\mathrm{win}} = p_{\mathrm{I}} q_0^{\mathrm{I}} p_{\mathrm{I}}^A + p_{\mathrm{I}} q_1^{\mathrm{I}} p_{\mathrm{I}}^B + p_{\mathrm{II}} q_0^{\mathrm{II}} p_{\mathrm{II}}^A + p_{\mathrm{II}} q_1^{\mathrm{II}} p_{\mathrm{II}}^B\,. \tag{16}$$

Here $p_{\mathrm{I}}^{\mathrm{win}}$ is the single winning probability for the random alternation between game A and B for player I. The same applies to the transition matrices.

We now demonstrate that the Parrondo games with rigid coupling can be reduced to the original Parrondo games with modified transition probabilities. For the case of two players, one state is defined either by both capitals or by one capital and the capital difference. Since rigid coupling induces constant capital differences, the state is only determined by the capital of one player when a certain initial condition is given. Hence, we can separate the Markov chain of the coupled Parrondo games for different capital differences. This is due to the reducibility of the Markov chain. The different states with the same capital difference form an irreducible equivalence class which is equivalent to the single Parrondo games but with modified transition probabilities because of the coupling. In fact, considering the capital differences $D = (\bar{x}_{\mathrm{II}} - \bar{x}_{\mathrm{I}}) \mod M \in \{0, ..., M-1\}$ (convention: $\bar{x}_{\mathrm{II}} > \bar{x}_{\mathrm{I}}$), the transition matrix of the coupled games has the form ($Q_D \in \mathbb{R}^{M \times M},\ D \in \{0,\ ...,\ M-1\}$, all other entries vanish)

$$Q' = \begin{pmatrix} Q_0 & & \\ & \ddots & \\ & & Q_{M-1} \end{pmatrix}. \tag{17}$$

It is obvious that Q_D is the transition matrix of the class D and with the corresponding probability distribution $P_D(n)$ at step n we can write

$$P_D(n+1) = Q_D P_D(n)\,. \tag{18}$$

For the different equivalence classes and hence the different capital differences we can then use the existence of a stationary probability distribution in order to determine the stationary capital current. For an initial probability distribution that contains different capital differences with nonzero probability, the different solutions for the stationary capital current have to be added with the corresponding stochastic weights.

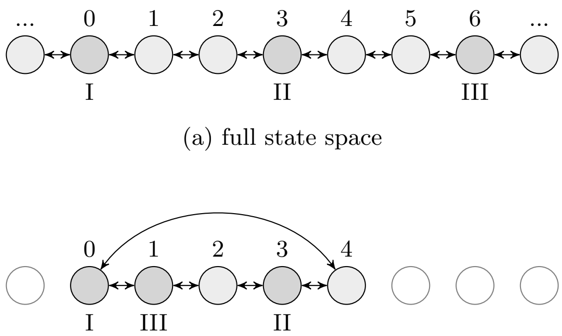
Hence we have already found a formalism for investigating the coupled games: The transition matrices are calculated according to the different couplings and dependent on certain parameters such as the period, noise parameters or width of the barrier. They can then be reduced to the transition matrices of the equivalence classes for which we can compute the stationary distribution and probability current.

It may be mentioned that we can always treat the probability current and capital current as equivalent since the probability current reflects the change of states and the discrete capital difference is set to 1. Especially, since the winning of both players only corresponds to a single state change, the capital current in the coupled case can be directly compared to the capital current in the single case.

We also want to investigate the Parrondo games with multiple ($N > 2$) players coupled. Therefore we have to slightly modify the formalism. The state is now given by all capitals, $x = (x_{\mathrm{I}}, x_{\mathrm{II}}, ...) \in \{\mathbb{Z}/M\mathbb{Z}\}^N$. It may be interesting to mention that one state is again determined by the capital of one player and the capital differences between two consecutive players in a certain order. Since the equivalence classes are determined by the capital differences due to rigid coupling, we have again reduced the multiple-coupled Parrondo games to the single Parrondo games with modified transition probabilities. For the double-play coupling, all winning (losing) probabilities are multiplied and renormalized as well as stochastically weighted with the probabilities for choosing the different games A and B for each player. For the single-play coupling, the individual transition probabilities are stochastically weighted with the probabilities $p_{\mathrm{I,II,...}}$ of choosing the players $\mathrm{I}, \mathrm{II}, ...$ at the beginning of every round.

One can once more define a projection onto one variable. The resulting transition matrix can then be reduced to the various submatrices of the equivalence classes corresponding to different capital differences for which we can compute the stationary capital current. However, since the projection does not effect the stationary capital current, we will not go into detail here.

The multiple-player state space is illustrated in figure 1 for $M = 5$, $N = 3$, and $D = 3$ between two consecutive players. In the reduced state space, the capitals of the first and third player are only one capital unit apart. This effect will be important when interpreting the results for multiple players below.



(a) full state space

(b) reduced state space

Figure 1: Illustration of the multiplayer state space. Here the state for $M = 5$ and $N = 3$ as well as $D = 3$ between two consecutive players and $x_{\mathrm{I}} = 0$ is shown. The circles are states, the numbers above the circles reflect the capital and the roman numbers underneath the circles reflect the players in a certain order.

4 The stationary current of multi-player games

4.1 Double-play games

We first analyse the double-play games. As a first step, we simulate the games. We consider the capital flow of the different classes and therefore always choose initial conditions that only contain states of the same class. The simulation is done for $M = 5$, the results are shown in figure 2.

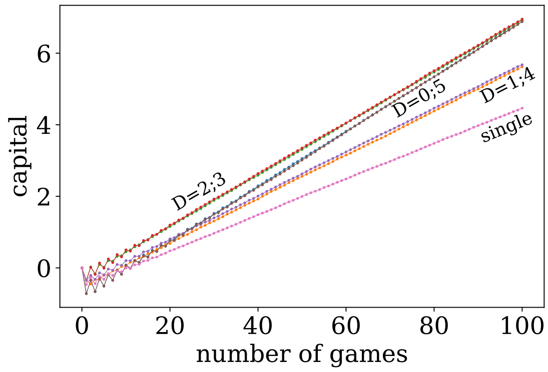


Figure 2: Simulation of the double-play games averaged over 50000 repetitions for $\epsilon_{\mathrm{I}} = \epsilon_{\mathrm{II}} = 0.5 \cdot 10^{-6}$, $M = 5$ and $q_0^{\mathrm{I}} = q_0^{\mathrm{II}} = 0.5$ compared to the uncoupled (single) case with $q_0 = 0.5$. The capital of player I is shown.

It can be seen that during the first rounds, some capital is gained and lost alternately. This is due to the initial capital $\bar{x}_{\mathrm{I}} = 0$. After several rounds, the distribution converges against the stationary one and the capital gradient approaches a positive constant.

Additionally, the slopes of D and $M - D$ are similar, respectively, even though one can observe slight offsets which are the result of the initial condition.

The next step is to analyse the games with methods of DTMC and to compare the results with the uncoupled Parrondo games. In particular, the dependence on the capital difference D and the noise parameters $q_0^{\mathrm{I,II}}$ is examined, the width of the barrier and the number of players are varied.

First, we change the capital difference for different periods. We choose the same parameters as for the simulation 2. The results are portrayed in figure 3. It is obvious that the capital current in the uncoupled case is the same for all periods, this is due to the modification of the probabilities accordingly.

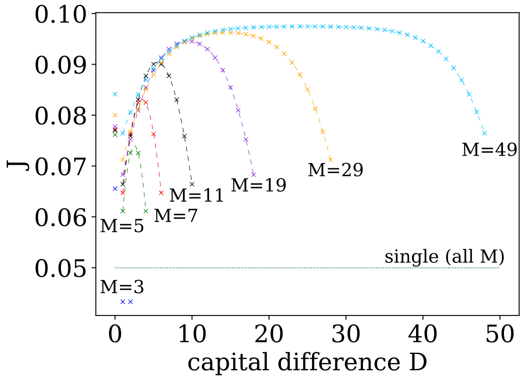


Figure 3: Capital current of the double-play games for different periods. We choose $\epsilon_{\mathrm{I}} = \epsilon_{\mathrm{II}} = 0.5 \cdot 10^{-6}$, $q_0^{\mathrm{I}} = q_0^{\mathrm{II}} = 0.5$ and $q_0 = 0.5$ for the uncoupled (single) case. We fit a parabola for $M \in \{5, 7, 11\}$ and a polynomial of degree $M - 2$ for $M \in \{19, 29, 49\}$, $D = 0$ is not taken into account.

The long-run current in the simulation can be estimated by a linear fit after several rounds. The results obtained that way coincide with the stationary current calculated using DTMC.

It can be observed that the capital current of the uncoupled games takes the value $J = 0.05$ up to a small deviation due to $\epsilon \neq 0$. Moreover, the assumption is verified that the capital currents of D and $M - D$ are equal. Therefore, we could show the accordance of the simulation and computation of the stationary capital current. In figure 3 we can observe the following:

1a) The current for $M > 3$ is always larger than in the uncoupled case. In the uncoupled case the capital accumulates in front of $\bar{x} = 0$ when playing game B because the winning probability at $\bar{x} = 0$ is smaller (a small or big winning probability is in the following always referred to game B) than

0.5, but the winning probability before is larger than 0.5. This is a kind of barrier. The switch between games A and B allows the capital to cross the barrier more likely. When two players are coupled with a positive capital difference, at least one player has a big winning probability and hence helps the other player to cross the barrier since both winning probabilities are multiplied. This is the driving mechanism of the double-play games.

However, the period $M = 3$ does not match with this observation. Here the capital current for a positive capital difference is smaller than in the uncoupled case. A possible explanation could be that for $M = 3$ and $D > 0$ there are more states with at least one player having a small winning probability than states with no player having a small winning probability at all. The same argument leads to the result that the capital current of $D = 0$ is always larger than for $D = 1$. Interestingly, the coupling with $D = 0$ has a vanishing extent but nonetheless leads to an increase in the capital current compared to the uncoupled case. This is the result of nonlinear effects within the multiplication of the winning probabilities. We will later see that this occurs because the width of the barrier is chosen to be 1. When this width is enlarged, small capital differences can not have any positive effect on the capital current.

1b) The current has a maximum around $M/2$ for positive capital differences. For $D < \lfloor M/2 \rfloor$ and $D > \lceil M/2 \rceil$ the current decreases symmetrically. For large periods some kind of saturation current is reached and the maximum of the former dependence flattens. For small periods parabolas are fitted and amazingly agree with the data, for larger periods polynomials of degree $M - 2$ are fitted in order to clarify the curve. The symmetry of the curve is a result of the choice of parameters: Since the parameters are the same for both players, the players are indistinguishable and the long-run behaviour of the games is the same for D and $M - D$ (the convention of the direction of the coupling was only important for the definition of the classes but does not effect the winning probabilities).

Particularly interesting is the maximum at $D \in \{\lfloor M/2 \rfloor, \lceil M/2 \rceil\}$. In these cases, the capitals are distributed as widely as possible over the state space considering the periodic boundary conditions. When the first player is located at a barrier ($\bar{x} = 0$), the second player can help him cross the barrier as efficiently as possible. This observation has also been found in the continuous case analysed by Klumpp *et al.* [10]. They chose the asymmetry of the potential the other way around and therefore the probability current is negative in their case.

1c) The maximum increases with M. This is a result of the coupling. In the last observation we deduced that the capital current is maximal for a wide distribution of capitals over the state space. A larger period increases this effect and hence the driving mechanism. However, there is some kind of saturation effect for large periods.

We now analyse the dependence on the noise parameters. First, we compute the capital current as a function of the noise parameters q_0^{I} and q_0^{II}. As an example, we choose $M = 7$, $\epsilon_{\mathrm{I}} = \epsilon_{\mathrm{II}} = 0.02$ and $D = 3$. The results are shown in figure 4.

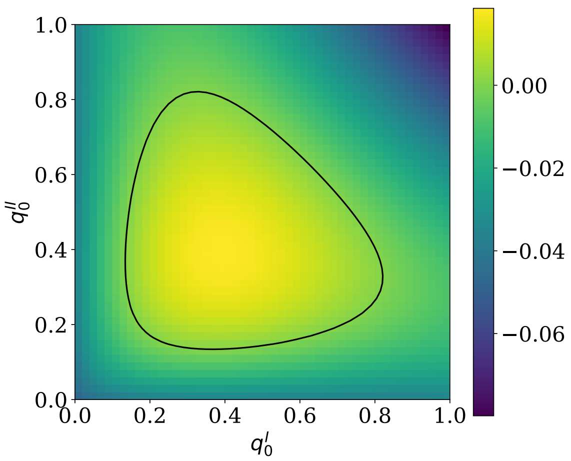


Figure 4: Capital current of the double-play games as a function of $q_0^{\mathrm{I,II}}$ for $M = 7$, $\epsilon_{\mathrm{I,II}} = 0.02$ and $D = 3$. The black line shows the contour line for $J = 0$.

We observe the following characteristics, which hold for other values of M and D as well:

1d) For sufficiently large values of ϵ, the area in which the current is positive is finite. This is one of the essential results and tells us that the switch between games A and B is important for the double-play games: This switch is the only noise process in the system and is therefore essential for the games and for crossing the barrier.

1e) For $\epsilon \to 0$ the area of positive current enlarges and reaches the boundaries $q_0^{\mathrm{I,II}} \in \{0,1\}$. This was observed when varying the bias parameter for the current as a function of the noise parameters but is not displayed here. The expansion of the area is a result of the increasing winning probabilities for $\epsilon \to 0$. It is quite interesting that a positive current can be achieved even though one player always plays the same game.

1f) The four double-deterministic points in the corners are the last ones to reach a positive current for a shrinking bias parameter. This confirms the result in observation **1d)**: When there are no noise processes and both players always play the same game, respectively, the capital current is minimal. However, it can be shown that all points but $q_0^{\mathrm{I}} = q_0^{\mathrm{II}} = 1$ can attain a positive current. This is due to the homogeneous probabilities of game A which, for a positive bias parameter, can never lead to a winning game.

Another interesting observation can be made when varying the width of the barrier. Therefore we choose the probabilities in table 3 as well as $q_0^{\mathrm{I}} = q_0^{\mathrm{II}} = 0.5$ and $\epsilon_{\mathrm{I}} = \epsilon_{\mathrm{II}} = 0.5 \cdot 10^{-6}$. The results are portrayed in figure 5.

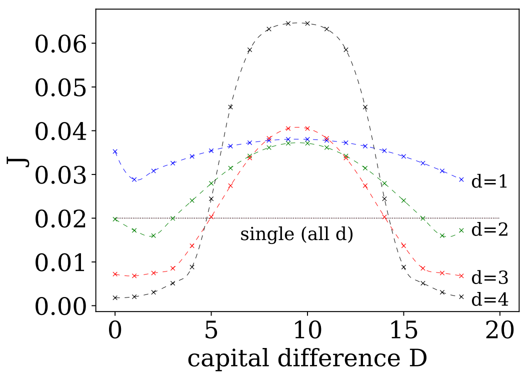


Figure 5: Capital current of the double-play games for different widths of the barrier d for $M = 19$, $q_0^{\mathrm{I}} = q_0^{\mathrm{II}} = 0.5$, $\epsilon_{\mathrm{I}} = \epsilon_{\mathrm{II}} = 0.5 \cdot 10^{-6}$ and $q_0 = 0.5$ for the uncoupled (single) case. The curve is clarified with a spline interpolation of second degree.

It is obvious that observation **1b)** applies. Nevertheless, there is a change in the characteristics at the edges of the plot:

1g) For $d > 1$ the capital current at the edge of the curve is smaller than in the uncoupled case and the slope of the curve flattens. A minimum distance between the capitals is required so that the coupling can have a constructive effect. This is the same observation as in [10]. Klumpp *et al.* [10] argue that the coupling is a positive driving mechanism only if the equilibrium distance between the particles is larger than the potential barrier. Then one particle can help the other crossing the barrier as can be seen in [10], Fig. 4. This quantitative conclusion can not be deduced in our discrete case. For example, the capital current for $d = 3$ does not exceed the current of the uncoupled case until $D = 5$.

Up to now we analysed the behaviour of the capital current as a function of different parameters for two players. We now want to consider more than two players. We restrict the investigation to equidistant capitals, hence the capital difference $D_{i,i+1}$ between two consecutive players i and $i+1$ in a certain order is a constant, $D_{i,i+1} = D \in \{0, ..., M-1\}\ \forall i \in \{1, ..., N-1\}$. Since we will always choose the same parameters for all players, the exact order is irrelevant. The results for $M = 49$ are illustrated in figure 6.

Of course many of the observations for two players can also be found here. Particularly important is the behaviour dependent on the number of players:

1h) For $M > 7$ the number of extreme points increases with N or stays constant. Especially, one can observe that the current is maximal when the capitals are distributed as widely as possible over the state space and minimal for multiple capitals being close together and hence contributes to the conclusion of observation **1b)**. We illustrate this taking $N = 4$ as example: The capital current increases until $D = 12$. If the first player has a capital multiple of M, the fourth player has a capital difference of $D_{1,4} = 3D = 36$ to the first one and reaches a multiple of M after 13 other capital units. For an increasing D and taking into account the periodic boundary conditions, the fourth capital approaches

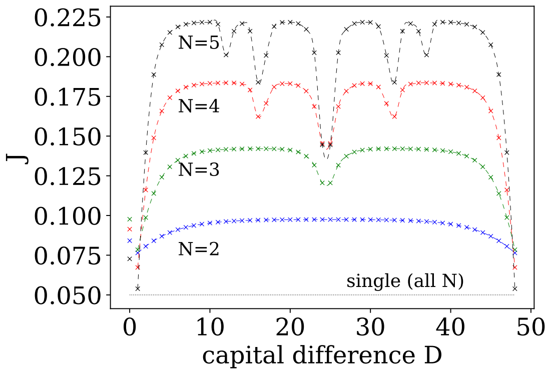


Figure 6: Capital current of the double-play games for a varying number of players N for $M = 49$, $\epsilon = 0.5 \cdot 10^{-6}$ and $q_0 = 0.5$ for all players and in the uncoupled (single) case. The curve is clarified with a spline interpolation of second degree.

the first one and is as near as possible for $D = 16$, the current is minimal. This repeats several times dependent on M and D. The effect is also illustrated in figure 1.

1i) For $D = M/2$ there is a minimum for $N > 2$ and a maximum for $N = 2$. This is a direct result of observation **1h)**: The capitals are as widely distributed as possible for $N = 2$ and concentrated at two points for $N > 2$, the current is maximal and minimal, respectively.

It may be mentioned that these effects are only that clear because M is chosen quite large. For smaller periods, the effects of the discretization are visible. However, we do not want to go into detail here.

4.2 Single-play games

We now investigate the single-play games in a similar manner. As before we simulate the games and subsequently analyse them with methods of DTMC so that we can compare the results with the uncoupled case.

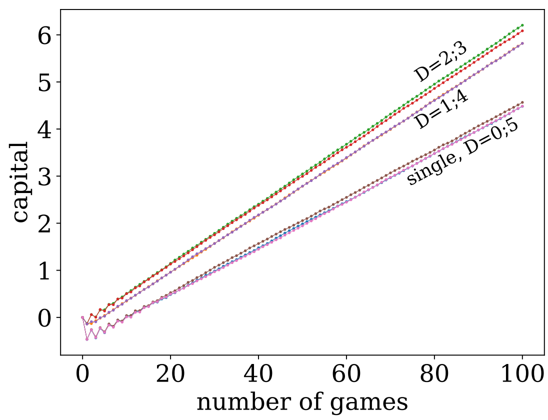


Figure 7: Simulation of the single-play games averaged over 50000 repetitions for $\epsilon_{\mathrm{I}} = \epsilon_{\mathrm{II}} = 0.5 \cdot 10^{-6}$, $M = 5$, $q_0^{\mathrm{I}} = q_0^{\mathrm{II}} = 0.5$ and $p_{\mathrm{I}} = 0.5$ compared to the uncoupled game with $q_0 = 0.5$. The capital of player I is shown.

For the simulation we choose initial distributions only containing elements of one class, the result is portrayed in figure 7.

The simulation shows the same behaviour as in the double-play games. The fluctuation during the first rounds is a result of the initial condition and we observe the convergence against a stationary capital current. Moreover, the capital differences M and $M - D$ produce the same capital current in the long-time limit, small offsets can be explained by the initial conditions.

As before, we vary different parameters. First, we can again compute the capital current for different periods and capital differences. We choose the same parameters as in the simulation. The results are shown in figure 8.

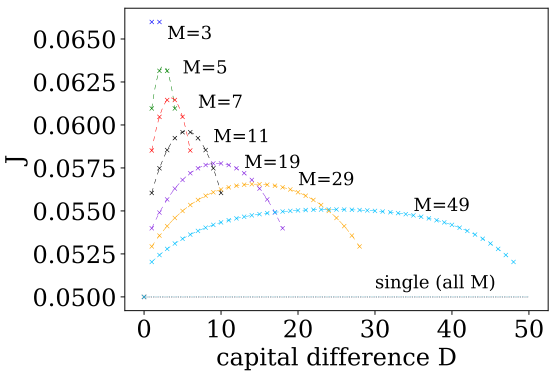


Figure 8: Capital current of the double-play games for different periods. We choose $\epsilon_{\mathrm{I}} = \epsilon_{\mathrm{II}} = 0.5 \cdot 10^{-6}$, $q_0^{\mathrm{I}} = q_0^{\mathrm{II}} = 0.5$, $p_{\mathrm{I}} = 0.5$ and $q_0 = 0.5$ for the uncoupled case. We fit a parabola for $M \in \{5, 7, 11\}$ and a polynomial of degree $M - 2$ for $M \in \{19, 29, 49\}$, $D = 0$ is not taken into account (here all data points overlap).

The simulation agrees with the computation via DTMC. We can again verify that the differences D and $M - D$ show the same current. Figure 8 gives us the following insights:

2a) The current is larger than in the uncoupled case. Hence the coupling has a positive effect on the capital current. However, the explanation differs from the double-play coupling: In the double-play games both winning probabilities are multiplied and therefore the extra driving mechanism is that both players always determine the winning probability together. The single-play coupling introduces a new noise process, the choice of the active player in each round. For this reason, even if one of the players is at a barrier, there is a chance that the other player is chosen and helps him crossing the barrier. This is the new driving mechanism of the single-play games.

Again, $D = 0$ is a special case: Since all parameters are chosen to be the same for both players, this is then equivalent to the individual Parrondo games as can be observed in figure 8 (all points overlap at that point).

2b) The current has a maximum around $M/2$ for positive capital differences. This observation is the same as observation **1b)** and can be explained equally since the effect does not depend on the coupling but on the capital distribution over the state space.

2c) The maximum decreases with M. Apparently, the driving mechanism of the single-play games has a different impact for varying periods and is, in contrast to the double-play games, more efficient the closer the barriers are.

We analyse the dependence on the noise parameters. Figure 9 shows an example of the capital current as a function of q_0^{I} and q_0^{II} for $M = 7$, $\epsilon_{\mathrm{I}} = \epsilon_{\mathrm{II}} = 0.02$, $D = 3$ and $p_{\mathrm{I}} = 0.5$. Furthermore, we compute the capital current as a function of q_0^{I} and q_0^{II} for different values of p_{I} with $M = 5$, $D = 2$ and $\epsilon_{\mathrm{I,II}} = 0.02$, the results are portrayed in figure 10.

We observe the following:

2d) For $p_{\mathrm{I}} = 0.5$ the maximum current is reached for $q_0^{\mathrm{I}} = q_0^{\mathrm{II}} = 0$. The area containing a positive current is not finite, the current decreases with q_0^{I} and q_0^{II}. This is one of the most important observations: The maximum current is reached when both players always play game B and there is no diffusion-like behaviour anymore. Even in game B, there is always a chance that the other player is chosen at the beginning of a round and the barrier is crossed more likely. However, the symmetry within the choice of the players is important. For $p_{\mathrm{I}} \neq p_{\mathrm{II}}$ the diffusion-like game A becomes important again since one player is chosen more often than the other one. This is explained in observation **2f**. For a symmetric single-play game, the driving mechanism of the single-play coupling is more efficient than the diffusion-like driving mechanism of the individual games. Hence we found a coupling that dominates the original driving mechanism!

2e) For $\epsilon \to 0$ the area of positive current enlarges. This is again due to the definition of the bias parameter. It may be interesting to mention that the capital current is always positive for $\epsilon = 0$ and $q_0^{\mathrm{I,II}} < 1$ and only vanishes for $q_0^{\mathrm{I}} = q_0^{\mathrm{II}} = 1$ since the symmetric game A can not produce any directed transport.

2f) The capital current for $p_{\mathrm{I}} = 0(1)$ is independent of $q_0^{\mathrm{I}}(q_0^{\mathrm{II}})$. Since then only one player is chosen at once, this is equivalent to the individual Parrondo games. However, the transition $p_{\mathrm{I}} \in (0, 1)$

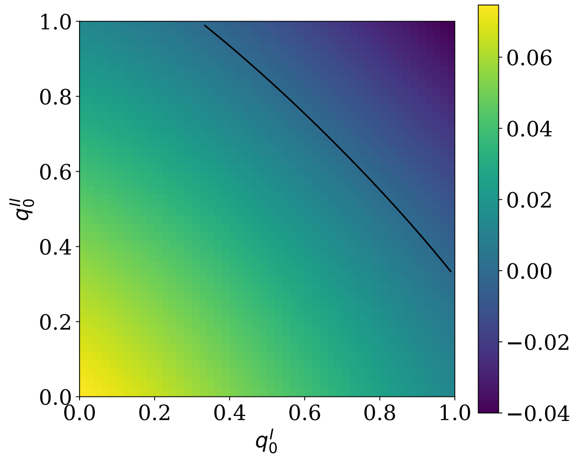


Figure 9: Capital current of the single-play games as a function of $q_0^{\mathrm{I,II}}$ for $M = 7$, $\epsilon_{\mathrm{I,II}} = 0.02$, $D = 3$ and $p_{\mathrm{I}} = 0.5$. The black line shows the contour line for $J = 0$.

is of importance which is displayed in figure 10. For $p_{\mathrm{I}} \neq 0.5$ there is an asymmetry in the games and the single-play driving mechanism is weakened since one player is chosen more often and therefore the diffusion-like game A becomes more important again for this player. Hence there is some kind of balance between the driving mechanism of the uncoupled and single-play games which is dependent on the noise parameters.

We vary the width of the barrier and compute the capital current for $M = 19$, $q_0^{\mathrm{I}} = q_0^{\mathrm{II}} = 0.5$, $\epsilon_{\mathrm{I}} = \epsilon_{\mathrm{II}} = 0.5 \cdot 10^{-6}$ and $p_{\mathrm{I}} = 0.5$. The winning probabilities are chosen according to table 3. The results are displayed in figure 11.

Again, the observation **2b** can be made, but there is also a change at the edges of the plot:

2g) The capital current for $D > 0$ is larger than in the uncoupled case. Nevertheless, the slope flattens at the edges of the curve. The single-play games differ from the double-play games in this point: For $D = 0$ the single-play games are equivalent to the uncoupled games since the parameters are chosen to be the same for all players and the current becomes larger for $D > 0$, hence the driving mechanism is constructive then. The reason for this might be that the the small winning probabilities in the barrier have a larger effect for the current when being multiplied instead of being convex combined in the single-play coupling.

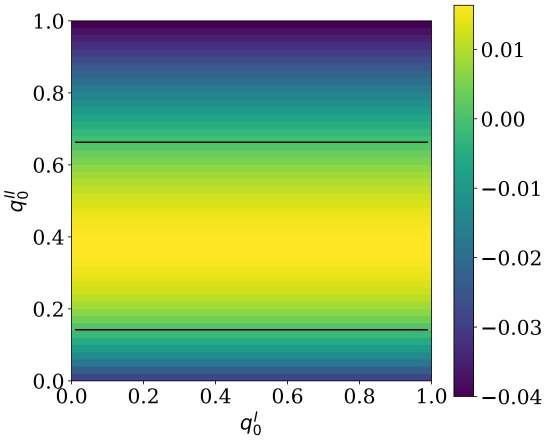
We now look at multiple players for the single-play games and restrict the discussion to equidistant capitals between consecutive players in a certain order again. The results for $M = 49$ are shown in figure 12. Many of the observations for two players can be made. Indeed, the observations **1h** and **1i** are also the same here. The reason is that this effect is not dependent on the coupling but only on the distribution of capitals over the state space.

5 Discussion and Outlook

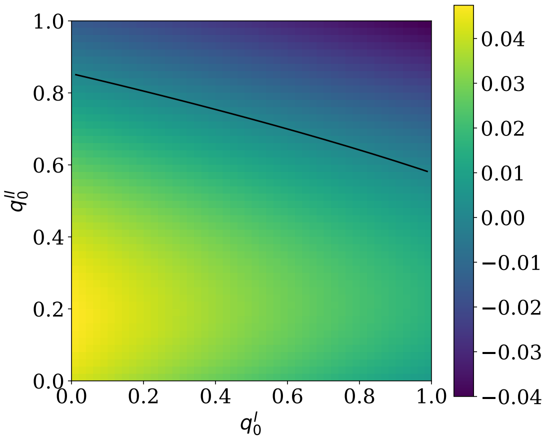
In this paper we investigate coupled Parrondo games. We restrict the discussion to rigid coupling and define two different couplings: The double-play coupling assumes that both players play the individual Parrondo games separately and have to win or lose at the same time, respectively. This is motivated from the rigid coupling in the continuous case. The single-play coupling, on the other hand, is motivated by biology: At the beginning of every round one player, whose result effects the capital of both players at the same time, is selected randomly. In the first case the individual probabilities are multiplied, in the second case convex combined.

The key to analyze the multi-player games is to show that for fixed capital differences, i.e. rigid coupling, they can be reduced to usual Parrondo games with modified parameters and can therefore be treated in the same way. The games converge to a stationary distribution and this can be used to calculate the stationary current. This allows to study the effect of the coupling and to vary the different parameters which determine the games.

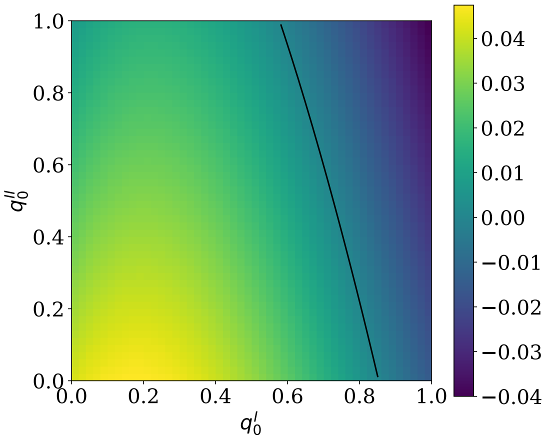
For both couplings we show that for $M > 3$ a cooperative effect occurs. The stationary capital current for $M > 3$ is larger than in the uncoupled case, hence the driving mechanism of the uncoupled games is



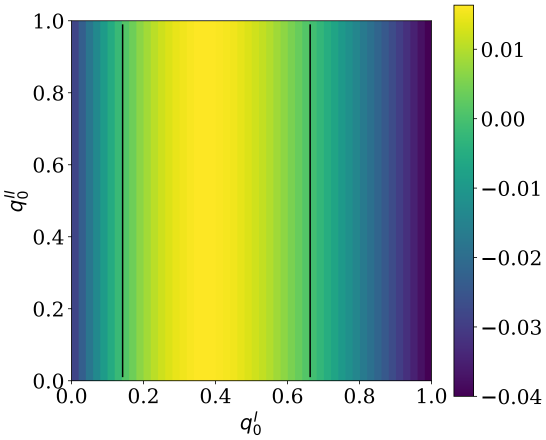
(a) $p_{\mathrm{I}} = 0$



(b) $p_{\mathrm{I}} = 0.2$



(c) $p_{\mathrm{I}} = 0.8$



(d) $p_{\mathrm{I}} = 1$

Figure 10: Capital current of the single-play games as a function of $q_0^{\mathrm{I,II}}$ for $M = 5$, $D = 2$ and $\epsilon_{\mathrm{I,II}} = 0.02$. The black lines reflect the contour lines for $J = 0$.

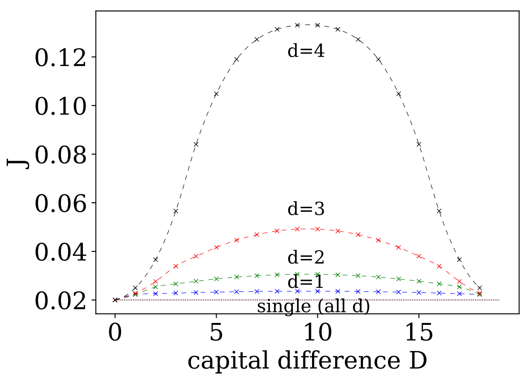


Figure 11: Capital current of the single-play games for different widths of the barrier d for $M = 19$, $q_0^{\mathrm{I}} = q_0^{\mathrm{II}} = 0.5$, $\epsilon_{\mathrm{I}} = \epsilon_{\mathrm{II}} = 0.5 \cdot 10^{-6}$, $p_{\mathrm{I}} = 0.5$ and $q_0 = 0.5$ for the uncoupled case. The curve is clarified with a spline interpolation of second degree.

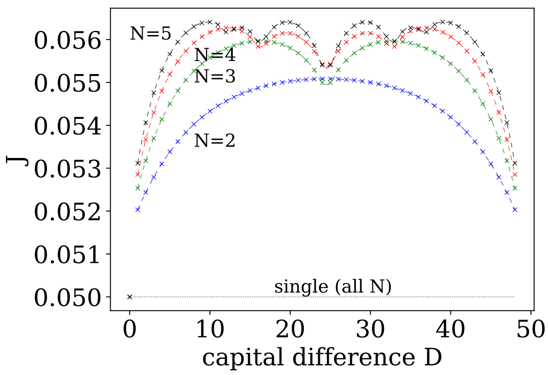


Figure 12: Capital current of the single-play games for a varying number of players N for $M = 49$, $\epsilon = 0.5 \cdot 10^{-6}$, $p_{\mathrm{I}} = p_{\mathrm{II}} = ... = 1/N$ and $q_0 = 0.5$ for all players and in the uncoupled case. The curve is clarified with a spline interpolation of second degree.

supported. For the double-play coupling the reason is the multiplication of the winning probabilities, for the single-play coupling the choice of players at the beginning of every round.

On the other hand, there are many differences between the two couplings. The double-play games show a deviation of the behaviour for $M = 3$ which is not the case in the single-play games. The change of maxima of the capital current for different periods is positive for the double-play games but negative for the single-play games. These and other effects can easily be explained by the different coupling mechanism.

Apparently, the effectiveness of both couplings is dependent on the period in different ways. The double-play coupling is more efficient for larger, the single-play coupling for smaller periods. This always has to be investigated in comparison to the uncoupled case which produced almost the same current for all periods due to the modified winning probabilities. The maximum of the capital current for $M > 3$ occurs at $D \in \{\lfloor M/2 \rfloor, \lceil M/2 \rceil\}$ for both couplings. Therefore the couplings are most efficient for the capitals being as widely distributed over the state space as possible (considering the symmetry).

This can be compared to the continuous case of two coupled Brownian particles. Klumpp *et al.* [10] determine the probability current for a dichotomous, multiplicative noise process as a function of the particle distance in the limit of rigid coupling, see Fig. 3 in their paper. At first it can be seen that the absolute value of the current shows a maximum when the particle difference is half the period. This is, neglecting the saturation effects, the same behaviour as in our case. One can also observe the symmetry between the distances l and $L - l$. However, in [10] an other effect becomes important: For particle distances smaller than the width of the potential barrier, the current hardly changes. The reason is that the coupling then can not act over the potential barrier.

Particularly interesting is the dependence on the noise parameters. For the double-play games the coupled driving mechanism does not dominate but only supports the uncoupled one: The maximum of the capital current is always found for $q_0^{\mathrm{I,II}} \notin \{0, 1\}$. The single-play games show a different result: The maximum capital current is found when both players only play game B. The coupling dominates the noise effect of switching the games in this case. The reason is the new noise process, the random change between both players. The new noise process in the single-play coupling can dominate the original one, depending on the choice of the respective parameters. The double-play games do not have any additional noise process. The original noise process is the only one in this case and there is no current without noise for a vanishing bias parameter ($\epsilon = 0$).

In the end we investigate the rigid couplings for more than two players with a constant capital difference between two consecutive players in a certain order. The results are easy to understand: Depending on the period and the number of players, the capital current shows different extreme points as a function of the capital difference. We deduce that the capital current is maximal for the capitals being as widely distributed over the state space as possible, taking the symmetry between the players into account.

There are many interesting questions that have not been investigated in this work. The most interesting question is eventually what happens if we soften the rigid coupling. A discrete analogy of the harmonic coupling in [10] is difficult to be analysed with our methods since neither the periodic state space nor the reduction can be used. It is possible to introduce a periodic harmonic coupling, but such a coupling has no direct physical interpretation. The first question one therefore needs to answer is how a physically meaningful non-rigid coupling could look like which pertains the periodicity.

Parrondo *et al.* [16] study games which are not capital but history dependent. Those games can also be played by multiple coupled players. The first question is how to introduce the coupling in that case. The second then is whether there occur similar cooperative effects.

Another interesting aspect is the optimal choice of games and players in every round. Dinis [17] examines the original Parrondo games with Markov Decision processes, Dinis an Parrondo [18] prove that a short-range optimization of the games can lead to a long-term loss for a positive bias parameter. The investigation of the coupled games with these methods could lead to interesting results, too.

Harmer *et al.* [19, 20, 13] considered the recurrence and transience of the original Parrondo games and derived conditions for the parameters leading to Parrondos Paradox. Perhaps it may be possible to find similar conditions for the coupled Parrondo games.

Other directions of further research are the relationship between Parrondo games and lattice gas automata [21] or quantum versions of Parrondo games [22] where the effect of two or more players and the new mechanisms we found can be present as well. Multiple player Parrondo games thus offer a broad variety of open questions which may be investigated in the future.

This work is based on the bachelor's thesis by Sandro Breuer. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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