UNIVERSAL CONSTANTS, LAW OF INERTIA AND EMERGENT GEOMETRY

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ABSTRACT. In this paper, we only treat the law of inertia as the first principle, then a nontrivial geometry emerges by introducing more universal constants, in which the main ideas appearing in deformed special relativity (DSR), (Anti-)de Sitter special relativity [(A)dSSR] and bimetric gravity (BMG) have been contexture.

1. INTRODUCTION

The de Sitter and anti-de Sitter spacetimes are of the most symmetric solutions of Einstein's field equations including the cosmological constant. For this reason, they are important for general relativity. After 1998, these spacetimes have drawn attention of high energy physicists due to the conjectured (anti-)de Sitter space/conformal field theory [(A)dS/CFT] correspondence. In this letter, we propose a new mechanism to produce de Sitter and anti-de Sitter spacetimes from the law of inertia of massive free particles.

Our initial motivation is to consider a fundamental theory of relativity that admits more universal constants. In Galilei relativity there is no observer-independent scale, and Einstein's special relativity (SR) introduced the first observer-independent relativistic scale: the velocity scale c identified with the speed of light. Naturally, the second observer-independent scale could be considered as length. It is clear that there is an inevitable price to pay for admitting at the same time relativity principle and the observer-independent scale of length. That is, analogous to the Galilei→Einstein transition, one should deform Poincaré group which has already deformed Galilean group through the contraction limit of infinity c.

There are two possible scenarios: in order to describe ultra-short-distance or ultra-large-distance physics, we might have to set aside SR and replace it with a new relativity theory with two characteristic invariant scales. The extra universal constant with dimension of length is denoted by ℓ.

– ℓ is very small identified with the Planck length $\sim 10^{-35}m$, therefore the modified theory, called deformed special relativity or doubly special relativity (DSR), may be rooted in quantum gravity [1, 2, 3, 4]. The best developed approach to DSR is realized based on the so-called ℓ-Poincaré algebra and ℓ-Minkowski spacetime. Here, the usual Poincaré algebra has been deformed into a quantum Hopf algebra which can be understood as the symmetry algebra of a noncommutative deformation of usual Minkowski spacetime [5, 6]. In the low-energy limit, i.e. ℓ tending to zero, everything returns to the standard SR.

– ℓ is very large identified with the radius of (observable) universe $\sim 10^{26}m$, then things could turn into relatively easy. At hand, we have the (Anti)-de Sitter group, which is interpreted as a particular deformation of the Poincaré group through the contraction limit of infinity ℓ. Hence, conceptually, one should establish special relativity with invariance under (Anti)-de Sitter group [(A)dSSR]. To our knowledge, this theory was first suggested by Dyson in his famous paper [7] and independently by the authors of [8], and was was further developed in [9, 10, 11, 12, 13, 14, 15].

– The above two scenarios can be combined together to construct an extension of SR characterized by three invariant scales: in addition to c, two universal constants ℓ_1, ℓ_2 with dimensions of length are included, which are identifies with Planck length and radius of universe, respectively. Such theory will reduce to DSR when ℓ_2 goes to infinity and reduce to (A)dSSR when ℓ_1 tends to zero. A proposal called triply special relativity has been described by a new nonlinear deformation of Poincaré algebra in [16].

Our method is very different. More precisely, we start with the usual spacetime $\mathbb{R}^4$ equipped with the Minkowski metric $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$, then we allow extra universal constant ℓ to appear in the dynamical part of theory. If the background is fixed to be Minkowski spacetime without any modification, it seems that it is the unique approach to introduce the new universal constants. We will show a nontrivial geometry emerges from the dynamical structure. For simplicity, we only work within the single particle sector of the theory. Analogous to the standard action for a free particle with mass m in SR

$$S = -m \int dt \sqrt{|\eta_{\mu\nu} v^\mu v^\nu|}. \tag{1.1}$$

where $c = 1$ is set, and $v^\mu = \frac{dx^\mu}{dt}$, we write a new action

$$S = -m \int dt \sqrt{|B_{\mu\nu} v^\mu v^\nu|}, \tag{1.2}$$

where the symmetric second-order tensor $B_{\mu\nu}$ is not chosen a priori to be equal to $\eta_{\mu\nu}$. In other words, the geometry of the background and the dynamics of matter are separately considered, and described by two different (but maybe relevant) symmetric second-order tensors $\eta_{\mu\nu}$ and $B_{\mu\nu}$, respectively. Actually, this idea is similar with the theory of bimetric gravity (BMG). That theory also consists of two metric-like symmetric second-order tensors which play different roles [17]. The first one is surely the dynamical metric that describe the geometry of spacetime and thus the gravitational field, and the second one can be non-dynamical or dynamical. For example, in Rosen-type theory, the second metric refers to the Minkowski metric and describes the inertial forces [18], and in Hassan-Rosen-type theory which is free from the Boulware-Deser ghost and propagates seven degrees of freedom, the introduction of the second metric nonlinearly coupled to the spacetime metric allows for a description of a massive spin-2 field [19, 20, 21].

Coming back to our theory, now $B_{\mu\nu}$ does not arise from background geometry any more, but is still tied down by dynamics. Specifically speaking, the dynamical content of the massive free particle in Minkowski spacetime is just the law of inertia, hence, if the action (1.2) correctly produces the dynamics of massive free particle according to the least action principle, the law of inertia will continue to hold true. This provides a constraint on the form of $B_{\mu\nu}$. We will see that under some suitable domain in Minkowski spacetime, $B_{\mu\nu}$ can be exactly viewed as a “metric" on the maximally symmetric spacetime with nonzero curvature (cosmological constant), i.e. (A)dS spacetime such that at the level of practice, this dynamics is equivalent to (A)dSSR. In this sense, we say that a nontrivial geometry [i.e. (A)dS geometry (or (A)dSSR) in the present case] emerges from the dynamics of massive free particles – the law of inertia. We can call such emergent metric the inertial metric, analogous to inertial force.

It is noteworthy that the above process is only valid for the large scale ℓ, namely the theory makes no sense if ℓ goes to zero. On the other hand, in principle, the choice of our action for a massive free particle has a lot of freedom, as long as it produces the right law of inertia. This reveals that considering more general action would allow us to introduce more universal constants, in particular, those involved the small scale. Of course, some dynamical symmetries will disappear.

An effortless manner is using the pair $(\eta_{\mu\nu}, B_{\mu\nu})$ to construct the following bipartite-Finsler-like action

$$S = -m\int dt(\sqrt{|\eta_{\mu\nu}v^\mu v^\nu|} + \xi\sqrt{|B_{\mu\nu}v^\mu v^\nu|}). \tag{1.3}$$

Now ξ is a dimensionless constant such that to kill the dimension we can simultaneity introduce more universal constants with the same dimension. For example, we can puts $\xi = \frac{\ell_1}{\ell_2}$, where ℓ_1, ℓ_2 are the universal constants identified with Planck length and radius of universe, respectively. Now the theory returns to the usual one when ℓ_1 tend to zero or ℓ_2 tends to infinity. If one picks $B_{\mu\nu}$ as the inertial metric mentioned previously, it is obvious that this action can also describe the massive free particle in physical domains in the sense of preserving the law of inertia. The emergent geometry from the action (1.3) can be described by the Finsler metric, which is exactly the second-order derivative of the corresponding Lagrangian with respect to the 4-velocity. Then the dynamical effects can be studied under the framework of Finsler geometry [22]. It's also worth mentioning that the action of type (1.3) has been used to investigate the Lorentz violation in [23, 24, 25], and Finsler geometry also provides a geometric tool in some modified special relativity theories (DSR [26, 27, 28], very special relativity (VSR) [29, 30]).

In conclusion, we only treat the law of inertia as the first principle, then a nontrivial geometry emerges by introducing more universal constants, in which the main ideas appearing in DSR, (A)dSSR and BMG have been contexture.

2. Solutions of Law of Inertia with Two Universal Constants

We need to determine the general form of $B_{\mu\nu}$ based on some reasonable assumptions. Firstly, according to our target, $B_{\mu\nu}$ should involve the universal constants c and ℓ. Secondly, when ℓ tends to infinity or zero, $B_{\mu\nu}$ should revert to $\eta_{\mu\nu}$. Then, taking into account the dimension, we pick the following very general ansatz

$$B_{\mu\nu} = A_0\eta_{\mu\nu} + \sum_{I=1}^{d} A_I \frac{(x\cdot v)^{a_I-2}(v\cdot v)^{b_I}\eta_{\mu\alpha}\eta_{\nu\beta}x^\alpha x^\beta}{\ell^{a_I}}, \tag{2.1}$$

where for the two 4-vectors Θ^μ, Ξ^ν, one defines

$$\Theta\cdot\Xi = \eta_{\mu\alpha}\Theta^\alpha\Xi^\mu,$$

and the integers a_I, b_I satisfy

- $a_I + 2b_I = 2$,
- $a_I \neq 0$,
- all a^I have the same sign.

The above action is recognized to be Finsler-like, and the corresponding Finsler function [22] is exactly the Lagrangian given by

$$\begin{aligned} &L(t,x^i,v^i)\\ =&-m\sqrt{|A_0 v\cdot v+\sum_{I=1}^{d}A_I(x\cdot v)^{a_I}(v\cdot v)^{b_I}|}, \end{aligned} \tag{2.2}$$

where as before, we put $\ell=1$ for convenience.

Note that $B_{\mu\nu}$ generally dose not only depend on the coordinates on $\mathbb{R}^4$, even is not necessary to be well-defined over the entire $\mathbb{R}^4$. The domain lying $\mathbb{R}^4$ such that $B_{\mu\nu}$ makes sense (for example, well-defined, non-degenerate and has the suitable signature) is called the physical domain.

If a massive free particle is assumed to be subject to our new action (1.2), then the corresponding Euler-Lagrange equation should imply the the law of inertia, that is the acceleration of particle has to vanish. The Euler-Lagrange equation reads

$$\frac{\partial L}{\partial x^i}=\frac{\partial^2 L}{\partial t\partial v^i}+v^j\frac{\partial^2 L}{\partial x^j\partial v^i}+\frac{\mathrm{d}v^j}{\mathrm{d}t}\frac{\partial^2 L}{\partial v^j v^i}, \tag{2.3}$$

therefore we must have

$$\frac{\partial L}{\partial x^i}-\frac{\partial^2 L}{\partial t\partial v^i}-v^j\frac{\partial^2 L}{\partial x^j\partial v^i}=0, \tag{2.4}$$

$$\det(\frac{\partial^2 L}{\partial v^j\partial v^i})\neq 0. \tag{2.5}$$

Substituting the expression (2.2) into the equation (2.4) gives rises to

$$\begin{aligned} &2[\partial_i A_0(v\cdot v)+\sum_{I=1}^{d}\partial_i A_I(x\cdot v)^{a_I}(v\cdot v)^{b_I}][A_0(v\cdot v)+\sum_{I=1}^{d}A_I(x\cdot v)^{a_I}(v\cdot v)^{b_I}]\\ =&2[2(v\cdot\partial A_0)v^i+\sum_{I=1}^{d}(v\cdot\partial A_I)a_I(x\cdot v)^{a_I-1}(v\cdot v)^{b_I}x_i\\ &+\sum_{I=1}^{d}A_I a_I(a_I-1)(x\cdot v)^{a_I-2}(v\cdot v)^{b_I+1}x_i+2\sum_{I=1}^{d}(v\cdot\partial A_I)b_I(x\cdot v)^{a_I}(v\cdot v)^{b_I-1}v_i\\ &+2\sum_{I=1}^{d}A_I a_I b_I(x\cdot v)^{a_I-1}(v\cdot v)^{b_I}v_i][A_0(v\cdot v)+\sum_{I=1}^{d}A_I(x\cdot v)^{a_I}(v\cdot v)^{b_I}]\\ &-[2A_0 v^i+\sum_{I=1}^{d}A_I a_I(x\cdot v)^{a_I-1}(v\cdot v)^{b_I}x_i+2\sum_{I=1}^{d}A_I b_I(x\cdot v)^{a_I}(v\cdot v)^{b_I-1}v_i]\\ &\times[(v\cdot\partial A_0)(v\cdot v)+\sum_{I=1}^{d}(v\cdot\partial A_I)(x\cdot v)^{a_I}(v\cdot v)^{b_I}+\sum_{I=1}^{d}A_I a_I(x\cdot v)^{a_I-1}(v\cdot v)^{b_I+1}], \end{aligned} \tag{2.6}$$

where the following notations are employed

$$\begin{aligned} \partial_i A&=\frac{\partial A}{\partial x^i},\\ \partial A&=(-\frac{\partial A}{\partial t},\frac{\partial A}{\partial x^1},\frac{\partial A}{\partial x^2},\frac{\partial A}{\partial x^3}), \end{aligned}$$

and $\Xi_\mu=\eta_{\mu\alpha}\Xi^\alpha$ for a 4-vector Ξ^μ. Comparing the monomials of the both sides of the equation (2.6) with the same type, we find that only one index I can survive such that (2.6) is simplified to the following equations on A_0, A_1 with $a_1=2, b_1=0$:

$$\begin{aligned} \partial_i A_0&=2A_1 x_i,\\ (2A_1^2 x_i+\partial_i A_1 A_0)(x\cdot v)&=[2(v\cdot\partial A_1)A_0-(v\cdot\partial A_0)A_1]x_i,\\ \partial_i A_1(x\cdot v)&=(v\cdot\partial A_1)x_i,\\ 2(v\cdot\partial A_0)A_1&=4A_1^2(x\cdot v)=(v\cdot\partial A_1)A_0. \end{aligned}$$

These equations leads to

$$\partial A_0=2A_1 x, \tag{2.7}$$

$$\partial A_1 A_0=4A_1^2 x. \tag{2.8}$$

whose general solutions are given by

$$A_0 = \frac{A}{B + C\eta_{\mu\nu}x^\mu x^\nu}, \tag{2.9}$$

$$A_1 = -\frac{AC}{(B + C\eta_{\mu\nu}x^\mu x^\nu)^2} \tag{2.10}$$

for constants A, B, C. Consequently, we obtain

$$B_{\mu\nu} = \frac{A}{B + C\frac{\eta_{\gamma\delta}x^\gamma x^\delta}{\ell^2}}\eta_{\mu\nu} - \frac{AC}{(B + C\frac{\eta_{\gamma\delta}x^\gamma x^\delta}{\ell^2})^2}\frac{\eta_{\mu\alpha}\eta_{\nu\beta}x^\alpha x^\beta}{\ell^2}, \tag{2.11}$$

where the universal constant ℓ is restored.

A useful observation is that $B_{\mu\nu}$ can be written in terms of the combination of two projection operators as

$$B_{\mu\nu} = \frac{A}{B + C\frac{\eta_{\mu\nu}x^\mu x^\nu}{\ell^2}}\Phi_{\mu\nu} + \frac{AB}{(B + C\frac{\eta_{\mu\nu}x^\mu x^\nu}{\ell^2})^2}\Psi_{\mu\nu}, \tag{2.12}$$

where

$$\Phi_{\mu\nu} = \eta_{\mu\nu} - \frac{x_\mu x_\nu}{\eta_{\alpha\beta}x^\alpha x^\beta}, \tag{2.13}$$

$$\Psi_{\mu\nu} = \frac{x_\mu x_\nu}{\eta_{\alpha\beta}x^\alpha x^\beta} \tag{2.14}$$

satisfy the projection relations

$$\eta^{\nu\alpha}\Phi_{\mu\nu}\Phi_{\alpha\beta} = \Phi_{\mu\beta}, \tag{2.15}$$

$$\eta^{\nu\alpha}\Psi_{\mu\nu}\Psi_{\alpha\beta} = \Psi_{\mu\beta}, \tag{2.16}$$

$$\eta^{\nu\alpha}\Phi_{\mu\nu}\Psi_{\alpha\beta} = 0. \tag{2.17}$$

Since it is required that $B_{\mu\nu}$ is non-degenerate and it tends to $\eta_{\mu\nu}$ up to an insignificant constant conformal scalars when ℓ goes to infinity, we have

- $A \neq 0, B \neq 0,$
- $AB > 0.$

To check the condition (2.4), we only need to show that under the limit $l \to \infty$, which is straightforward calculated as

$$\lim_{\ell\to\infty} \det(\frac{\partial^2 L}{\partial v^j \partial v^i}) = m^3 \frac{(\frac{A}{B})^{\frac{3}{2}}}{(|v \cdot v|)^{\frac{5}{2}}} \neq 0.$$

Obviously, if $C = 0$ everything essentially goes back to the classical theory with $B_{\mu\nu} = \eta_{\mu\nu}$. Therefore, we consider $C \neq 0$, and it can be assumed to be 1.

To determined the signature of $B_{\mu\nu}$, we need to consider the signs of

$$\begin{aligned}
B_{00} &= -\frac{A(B + \boldsymbol{x}\cdot\boldsymbol{x})}{(B - t^2 + \boldsymbol{x}\cdot\boldsymbol{x})^2}, \\
\widetilde{B}_{11} &= \frac{A(B + (x^2)^2 + (x^3)^2)}{(B - t^2 + \boldsymbol{x}\cdot\boldsymbol{x})(B + \boldsymbol{x}\cdot\boldsymbol{x})}, \\
\det&\begin{pmatrix} \widetilde{B}_{11} & \widetilde{B}_{12} \\ \widetilde{B}_{12} & \widetilde{B}_{22} \end{pmatrix} \\
&= \frac{A^2(B + (x^3)^2)}{(B - t^2 + \boldsymbol{x}\cdot\boldsymbol{x})^2(B + \boldsymbol{x}\cdot\boldsymbol{x})}, \\
\det&\begin{pmatrix} \widetilde{B}_{11} & \widetilde{B}_{12} & \widetilde{B}_{13} \\ \widetilde{B}_{12} & \widetilde{B}_{22} & \widetilde{B}_{23} \\ \widetilde{B}_{13} & \widetilde{B}_{23} & \widetilde{B}_{33} \end{pmatrix} \\
&= \frac{A^3 B}{(B - t^2 + \boldsymbol{x}\cdot\boldsymbol{x})^3(B + \boldsymbol{x}\cdot\boldsymbol{x})},
\end{aligned}$$

where $\widetilde{B}_{ij} = B_{ij} - \frac{B_{0i}B_{0j}}{B_{00}}$, $\boldsymbol{x}\cdot\boldsymbol{x} = (x^1)^2 + (x^2)^2 + (x^3)^2$. One easily finds

TABLE 1. Signature of $B_{\mu\nu}$

Condition	Signature
$A>0, B>0, B-t^2+\boldsymbol{x}\cdot\boldsymbol{x}>0$	$(1,3)$
$A>0, B>0, B-t^2+\boldsymbol{x}\cdot\boldsymbol{x}<0$	$(4,0)$
$A<0, B<0, B+\boldsymbol{x}\cdot\boldsymbol{x}<0$	$(1,3)$
$A<0, B<0, B-t^2+\boldsymbol{x}\cdot\boldsymbol{x}>0$	$(2,2)$
$A<0, B<0, B-t^2+\boldsymbol{x}\cdot\boldsymbol{x}<0, B+\boldsymbol{x}\cdot\boldsymbol{x}>0$	$(1,3)$

,

where for a nondegenrate symmetric matrix, we denote its signature by (n_-, n_+) if it has n_- negative eigenvalues and n_+ positive eigenvalues, respectively. In particular, if $B_{\mu\nu}$ is required to have the the signature $(1,3)$ as a spacetime-metric, we pick the physical domains

- (I): $B-t^2+\boldsymbol{x}\cdot\boldsymbol{x}>0$ for $A>0, B>0$,
- (II): $B-t^2+\boldsymbol{x}\cdot\boldsymbol{x}<0$ for $A<0, B<0$.

3. EMERGENT $(A)dS_4$ GEOMETRY

Now we explain why $(A)dS_4$ geometry emerges from the above framework. It is known that $(A)dS_4$ is defined by a hypersurface in 5-dimensional space $\mathbb{R}^5$ with the Minkowski metric $\eta^{(5)} = \mathrm{diag}(-1,1,1,1,1)$ (or the metric $\widetilde{\eta}^{(5)} = \mathrm{diag}(-1,-1,1,1,1)$) via the following equation [31]

$$-T^2+X^2+Y^2+Z^2+bW^2 = 1 (b>0),$$

or

$$-T^2-bW^2+X^2+Y^2+Z^2 = -1 (b>0).$$

Define the following coordinates which cover the half domain $\{W>0\}$ or $\{W<0\}$ in $(A)dS_4$

$$x^0 = \frac{T}{W}, x^1 = \frac{X}{W}, x^2 = \frac{Y}{W}, x^3 = \frac{Z}{W}, \tag{3.1}$$

Then the induced metric from $-dT^2+dX^2+dY^2+dZ^2 \pm b dW^2$on this hypersurface is given by in terms of the coordinate system $\{x^0,x^1,x^2,x^3\}$

$$g_{\mu\nu} = \frac{\eta_{\mu\nu}}{b+\eta_{\alpha\beta}x^\alpha x^\beta} - \frac{\eta_{\mu\alpha}\eta_{\nu\beta}x^\alpha x^\beta}{(b+\eta_{\alpha\beta}x^\alpha x^\beta)^2}, \tag{3.2}$$

or

$$g'_{\mu\nu} = \frac{\eta_{\mu\nu}}{b-\eta_{\alpha\beta}x^\alpha x^\beta} + \frac{\eta_{\mu\alpha}\eta_{\nu\beta}x^\alpha x^\beta}{(b-\eta_{\alpha\beta}x^\alpha x^\beta)^2}. \tag{3.3}$$

They exactly coincide with our $B_{\mu\nu}$ over physical domains (I) and (II), respectively, up to constant conformal scalars, in other words, the physical domains equipped with $B_{\mu\nu}$ can be viewed as the model of $(A)dS_4$-geometry. In some literature [10, 13], $B_{\mu\nu}$ is called the Beltrami metric for $(A)dS_4$-geometry.

From this viewpoint, we immediately conclude that the coordinate transformations preserve $B_{\mu\nu}$ form the group $O(1,4)$ or $O(2,3)$. By contrast, this group is a dynamical symmetry group other than geometric symmetry group as in (A)dSSR. For our theory, Poincáre group $ISO(1,3)$ is still the geometric symmetry group, and the overlap of these two classes of symmetry groups is exactly Lorentz group $O(1,3)$. Then we can call Lorentz group the inertial group since it consists of transformations preserving the inertial motions in physical domain. By decomposing a matrix belongs to the group $O(1,4)$ or $O(2,3)$ as

$$\lambda \begin{pmatrix} N & P \\ \mp\frac{P^T\eta N}{\sqrt{1\mp\eta_{\mu\nu}P^\mu P^\nu}} & \sqrt{1\mp\eta_{\mu\nu}P^\mu P^\nu} \end{pmatrix},$$

with matrices $N=(N^\mu{}_\nu)$ and $P=(P^0,P^1,P^2,P^3)^T$ satisfing the relation

$$N^T\eta N = \eta + \frac{N^T\eta P P^T \eta N}{\mp 1+\eta_{\mu\nu}P^\mu P^\nu}, \tag{3.4}$$

where $\mp$ correspond $O(1,4)$ and $O(2,3)$ respectively, and λ is fixed to 1 or -1, then we can explicitly write these coordinate transformations as fractional linear transformations

$$x^\mu \mapsto \frac{N^\mu{}_\nu x^\nu + \sqrt{b}P^\mu}{\mp\frac{\eta_{\alpha\beta}N^\beta{}_\gamma P^\alpha x^\gamma}{\sqrt{1\mp\eta_{\mu\nu}P^\mu P^\nu}} + \sqrt{b}\sqrt{1\mp\eta_{\mu\nu}P^\mu P^\nu}}, \tag{3.5}$$

which come back to Poincáre transformations when ℓ tends to infinity. By Norther method, we can easily obtain the corresponding ten conserved charges for a massive free particle [11, 13].

By symmetry breaking, we can also construct the some other actions with less symmetries, which are closed related to the violation of the law of inertia. The Lie bracket among the basis $\{M_{AB} = -M_{BA}, A, B = 0, \cdots, 4\}$ of Lie algebra $\mathfrak{o}(1,4)$ of de Sitter group $O(1,4)$ is given by

$$[M_{AB}, M_{CD}] = \eta^{(5)}_{AD}M_{BC} + \eta^{(5)}_{BC}M_{AD} - \eta^{(5)}_{AC}M_{BD} - \eta^{(5)}_{BD}M_{AC}.$$

Let $J_\mu = \frac{M_{\mu 4}}{\ell}$, $\mu = 0, \cdots, 3$, then

$$\begin{aligned}
[J_\mu, J_\nu] &= -\frac{M_{\mu\nu}}{\ell^2},\\
[J_\mu, M_{\alpha\beta}] &= \eta_{\mu\alpha}J_\beta - \eta_{\mu\beta}J_\alpha,\\
[M_{\mu\nu}, M_{\alpha\beta}] &= \eta_{\mu\beta}M_{\nu\alpha} + \eta_{\nu\alpha}M_{\mu\beta} - \eta_{\mu\alpha}M_{\nu\beta} - \eta_{\nu\beta}M_{\mu\alpha}.
\end{aligned}$$

These relations can be realized via the following differential operators

$$\begin{aligned}
J_\mu &= \partial_\mu + \frac{\eta_{\mu\alpha}x^\alpha x^\nu\partial_\nu}{\ell^2},\\
M_{\mu\nu} &= \eta_{\mu\alpha}x^\alpha\partial_\nu - \eta_{\nu\alpha}x^\alpha\partial_\mu = \eta_{\mu\alpha}x^\alpha J_\nu - \eta_{\nu\alpha}x^\alpha J_\mu.
\end{aligned}$$

Let us introduce the following symbols

$$\begin{aligned}
K^\pm_{\mathrm{i}} &= \frac{1}{\sqrt{2}}(M_{0i} \pm M_{1i}), \mathrm{i} = 2, 3\\
F^\pm_i &= \frac{1}{\sqrt{2}}(\frac{M_{0i}}{\ell} \pm J_i), i = 1, 2, 3,\\
L_i &= \frac{1}{2}\epsilon_{ijk}M_{jk}, i, j, k = 1, 2, 3,\\
P^\pm &= \frac{1}{\sqrt{2}}(J_0 \pm J_1),\\
R &= M_{01}, T = M_{23}.
\end{aligned}$$

The maximal Lie subalgebras of $\mathfrak{o}(1,4)$ are of 7 dimensions, which are exhibited in the following list

TABLE 2. Maximal Subgroups of $O(1,4)$

	Generators	Algebraic Relations
Type I	$\{K^\pm_2, K^\pm_3, J_2, J_3, P^\pm, R, T\}$	$[K^\pm_{\mathrm{i}}, K^\pm_{\mathrm{j}}] = 0, [J_{\mathrm{i}}, J_{\mathrm{j}}] = \epsilon_{\mathrm{ij}}\frac{T}{\ell^2}, [K^\pm_{\mathrm{i}}, J_{\mathrm{j}}] = \delta_{\mathrm{ij}}P^\pm,$ $[K^\pm_{\mathrm{i}}, P^\pm] = 0, [K^\pm_{\mathrm{i}}, R] = -K^\pm_{\mathrm{i}}, [K^\pm_{\mathrm{i}}, T] = \epsilon_{\mathrm{ij}}K^\pm_{\mathrm{j}},$ $[J_{\mathrm{i}}, P^\pm] = \frac{K^\pm_{\mathrm{i}}}{\ell^2}, [J_{\mathrm{i}}, R] = 0, [J_{\mathrm{i}}, T] = \epsilon_{\mathrm{ij}}J_j.$ $[P^\pm, R] = \mp P^\pm, [P^\pm, T] = 0, , [R, T] = 0,$
Type II	$\{F^\pm_1, F^\pm_2, F^\pm_3, L_1, L_2, L_3, J_0\}$	$[F^\pm_i, F^\pm_j] = 0, [L_i, L_j] = -\epsilon_{ijk}L_k, [F^\pm_i, L_j] = -\epsilon_{ijk}F^\pm_k,$ $[F^\pm_i, J_0] = \pm\frac{F^\pm_i}{\ell^2}, [L_i, J_0] = 0.$

The little groups in $O(1,4)$ corresponding to these two types of Lie subalgebras are denoted by $\mathcal{G}$ and $\mathcal{H}$ respectively. When the parameter ℓ tends to infinity, $\mathcal{G}$ are subgroups of $ISIM(2)$, which are 8-dimensional maximal subgroups of the Poincaré group generated by $\{K^\pm_2, K^\pm_3, J_1, J_2, P^+, P^-, R, T\}$, and $\mathcal{H}$ are isomorphic to the semiproduct of $O(3)$ and 4-dimensional translation group $\mathbb{T}(4)$. Note that there are no new invariant tensors for the groups $\mathcal{G}$ or $\mathcal{H}$, therefore we should consider the subgroups of $\mathcal{G}$ and $\mathcal{H}$.

Example 3.1. Consider the subgroup $\mathcal{S}$ whose Lie algebra is generated by K_2^+, K_3^+, P^+, T. By means of the following matrix representations of generators

$$K_2^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, K_3^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P^+ = \frac{1}{\ell} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

we find a second-order non-degenerate symmetric invariant tensor with respect to $\mathcal{H}$

$$C = \begin{pmatrix} \mathsf{a} & \mathsf{b} & 0 & 0 & 0 \\ \mathsf{b} & 2\mathsf{b}-\mathsf{a} & 0 & 0 & 0 \\ 0 & 0 & \mathsf{b}-\mathsf{a} & 0 & 0 \\ 0 & 0 & 0 & \mathsf{b}-\mathsf{a} & 0 \\ 0 & 0 & 0 & 0 & \mathsf{b}-\mathsf{a} \end{pmatrix} \tag{3.6}$$

with two constants $\mathsf{a}<\mathsf{b}$, thus a metric

$$C = \mathsf{a}dT^2 + 2\mathsf{b}dTdX + (2\mathsf{b}-\mathsf{a})dX^2 + (\mathsf{b}-\mathsf{a})dY^2 + (\mathsf{b}-\mathsf{a})dZ^2 + b(\mathsf{b}-\mathsf{a})dW^2 \tag{3.7}$$

Then the coordinate transformations (3.1) give rises to an induced metric

$$C_{\mu\nu}dx^\mu dx^\nu = (\mathsf{b}-\mathsf{a})g_{\mu\nu}dx^\mu dx^\nu + \mathsf{b}\frac{[(b+\eta_{\alpha\beta}x^\alpha x^\beta)(dx^0+dx^1) - (x^0+x^1)\eta_{\alpha\beta}x^\alpha dx^\beta]^2}{(b+\eta_{\alpha\beta}x^\alpha x^\beta)^3}. \tag{3.8}$$

Hence the $\mathcal{S}$-invariant action can be chosen as

$$S = -\int \sqrt{C_{\mu\nu}dx^\mu dx^\nu}. \tag{3.9}$$

where the dimensionless constant b is set to be very small characterizing the violation of the law of inertia.

Example 3.2. Consider the subgroup $\mathcal{V}$ whose Lie algebra is generated by F_1^+, F_2^+, F_3^+. By means of the following matrix representations of generators

$$F_1^+ = \frac{1}{\sqrt{2}\ell} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, F_2^+ = \frac{1}{\sqrt{2}\ell} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, F_3^+ = \frac{1}{\sqrt{2}\ell} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

an invariant vector

$$V = (\mathsf{a}, 0, 0, 0, -\mathsf{a})^T \tag{3.10}$$

with a constant a. Therefore we can consider a Finsler-type $\mathcal{V}$-invariant action [29, 30]

$$S = \int (g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1-\delta}{2}}(V_\mu dx^\mu)^\delta, \tag{3.11}$$

where

$$V_0 = \mathsf{a}(b+\eta_{\mu\nu}x^\mu x^\nu)^{-\frac{3}{2}}(b + \boldsymbol{x}\cdot\boldsymbol{x} - x^0), \tag{3.12}$$

$$V_i = \mathsf{a}(b+\eta_{\mu\nu}x^\mu x^\nu)^{-\frac{3}{2}}x^i(1-x^0), i = 1, 2, 3, \tag{3.13}$$

and the dimensionless constant δ is set to be very small characterizing the violation of the law of inertia.

4. Emergent Finsler Geometry

We have seen that our method in previous sections cannot admit a universal constant tending to zero. As mentioned in Introduction, we consider the bipartite-Finsler-like action (1.3). Here, we set $\xi = \frac{\kappa}{\ell}$ for a new universal constant κ identified with Planck length. Once again, one imposes the law of inertia on the corresponding Euler-Lagrange equation. Obviously, one can choose $B_{\mu\nu}$ to be the solution providing by (2.11). Then the dynamical symmetry group breaks down to the Lorentz group $O(1,3)$.

For simplicity, we only work on the physical domain $B - t^2 + \boldsymbol{x}\cdot\boldsymbol{x} > 0$ with $A>0, B>0$. A Finsler metric emerges from the action (1.3) as [24, 25]

$$\begin{aligned} g^{\mathrm{F}}_{\mu\nu} :=& -\frac{1}{2}\frac{\partial^2 \tilde{L}^2}{\partial v^\mu \partial v^\nu} \\ =& -[\frac{\tilde{L}}{\lambda}\eta_{\mu\nu} + \xi\frac{\tilde{L}}{\sigma}B_{\mu\nu} + \xi\lambda\sigma k_\mu k_\nu], \end{aligned} \tag{4.1}$$

where $k_\mu = \frac{\eta_{\mu\nu}v^\nu}{\lambda^2} - \frac{B_{\mu\nu}v^\nu}{\sigma^2}, \tilde{L} = \frac{L}{m} = -\lambda - \xi\sigma$ for $\lambda = \sqrt{-\eta_{\mu\nu}v^\mu v^\nu}$ and $\sigma = \sqrt{-B_{\mu\nu}v^\mu v^\nu}$. Some dynamical effects can been studied via this Finsler metric.

For example, we derive the new dispersion relation for the massive free particle. We need to calculate the inverse $(g^{\mathrm{F}})^{\mu\nu}$ of $g^{\mathrm{F}}_{\mu\nu}$. In general, it is quite difficult. However, fortunately, for our case, taking advantage of (2.12)-(2.17), we can explicitly obtain

$$\begin{aligned} (g^{\mathrm{F}})^{\mu\nu} =& -\{(\frac{\tilde{L}}{\lambda} + \xi A\chi\frac{\tilde{L}}{\sigma})^{-1}\Phi^{\mu\nu} + (\frac{\tilde{L}}{\lambda} + \xi AB\chi^2\frac{\tilde{L}}{\sigma})^{-1}\Psi^{\mu\nu} \\ & - \frac{\xi\lambda\sigma}{1+\xi\lambda\sigma k^2}[(\frac{\tilde{L}}{\lambda} + \xi A\chi\frac{\tilde{L}}{\sigma})^{-1}\Phi^{\mu\alpha} + (\frac{\tilde{L}}{\lambda} + \xi AB\chi^2\frac{\tilde{L}}{\sigma})^{-1}\Psi^{\mu\alpha}] \\ & \cdot [(\frac{\tilde{L}}{\lambda} + \xi A\chi\frac{\tilde{L}}{\sigma})^{-1}\Phi^{\nu\beta} + (\frac{\tilde{L}}{\lambda} + \xi AB\chi^2\frac{\tilde{L}}{\sigma})^{-1}\Psi^{\nu\beta}]k_\alpha k_\beta\}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \chi &= \frac{1}{B + \eta_{\mu\nu}x^\mu x^\nu}, \\ \Phi^{\mu\nu} &= \eta^{\mu\alpha}\eta^{\nu\beta}\Phi_{\alpha\beta}, \Psi^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}\Psi_{\alpha\beta}, \\ k^2 &= [(\frac{\tilde{L}}{\lambda} + \xi A\chi\frac{L}{\sigma})^{-1}\Phi^{\mu\nu} + (\frac{\tilde{L}}{\lambda} + \xi AB\chi^2\frac{L}{\sigma})^{-1}\Psi^{\mu\nu}]k_\mu k_\nu. \end{aligned}$$

Then the dispersion relation is given by

$$(g^{\mathrm{F}})^{\mu\nu}\mathbb{P}_\mu\mathbb{P}_\nu = -m^2, \tag{4.3}$$

where

$$\mathbb{P}_\mu = -\frac{\partial L}{\partial v^u} = \mathbb{P}^\eta_\mu + \xi\mathbb{P}^B_\mu \tag{4.4}$$

is the canonical 4-momentum in the sense of dynamics with

$$\mathbb{P}^\eta_\mu = m\frac{\eta_{\mu\alpha}v^\alpha}{\lambda},\ \mathbb{P}^B_\mu = m\frac{B_{\mu\alpha}v^\alpha}{\sigma}.$$

Expanding the identity (4.3) until the first order in ξ, we get

$$\lambda^2\eta^{\mu\alpha}\eta^{\nu\beta}(k_\alpha k_\beta - \frac{1}{\sigma^2}B_{\alpha\beta})\mathbb{P}^\eta_\mu\mathbb{P}^\eta_\nu + 2\frac{\lambda}{\sigma}\eta^{\mu\nu}\mathbb{P}^\eta_\mu\mathbb{P}^B_\nu = -m^2. \tag{4.5}$$

More discussions on kinematics and dynamics in the general bipartite-Finsler geometry can be found in [24, 25].

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