Dissipative Dynamics in Open Fermionic Chains

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Abstract. By merging the Feynman-Vernon's approach with the out-of-equilibrium Keldysh-Schwinger formalism, we construct the reduced generating functional through which all the time-dependent correlation functions of an open fermionic system can be directly derived by applying the appropriate functional derivatives. As a concrete example, we investigate the transverse Ising model, we derive the covariance matrix at the steady state of the system and we investigate its critical behavior.

1. Introduction

The unavoidable influence of the environment on open quantum systems is of great importance for understanding their physical properties and for handling their practical applications. When a system is embedded in a controlled or uncontrolled environment, induced decoherence is one of the major issues for storing and processing quantum information. Among these systems, for reasons of both technical and theoretical origin, fermionic and spin chains with short-range interactions, play a central role as their dynamics reveal a quite interesting complexity even at their simple 1D versions. Isolated systems, the Hamiltonian of which is quadratic in fermionic or bosonic degrees of freedom, have been extensively studied as they are exactly solvable and the structure of their ground state offers the explanation basis for highly non-trivial phenomena as quantum phase transitions of topological character [1, 2]. However, the embedding of such a system into a bath induces major changes in the behavior of the system's correlations in a way that is not easy to calculate even if the isolated system is completely solvable. When a system is isolated, the structure of its ground state as well as the energy gap between this state and the excited ones has a central role for the system's properties [3]. In the case of an open system, the role of the ground state is played by the so-called steady state that is, the state at which the reduced density matrix relaxes at the infinite time limit [4, 5]. The study of the steady state's quantum properties as well as the rate at which this it is approached, is of great

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importance for both theoretical and practical reasons. A driven approach to the steady state has been used to deal with quantum information processing [6–9]. The issue is standardly probed in the Lindbland master equation framework in which the steady state can be defined as the (right) eigenstate of the Lindblad super-operator with zero eigenvalues [10, 11].

In a recent paper [12], we presented a scheme that combines the Feynman-Vernon's influence functional technique with the out-of-equilibrium Keldysh formalism. This construction enables the direct calculation of the environment's impact on the correlation functions of an open bosonic or fermionic quantum system. In this approach, the focus is not on the reduced density matrix but on the correlation functions per se. This permits the calculation of functions that contain more than a single time variable [13].

In the present work, we apply this technique to probe the dynamics of quadratic 1D fermionic chains that when isolated, can be mapped on a spin chain model. The focus of the calculation is on the subsystem's covariance matrix [5], the quantity that encaptures the properties of reduced density matrix. For the quadratic case, we find the steady state at which the system relaxes as well as the rate at which this state is approached.

The structure of the paper is the follows: In Section 2 we briefly present the basic ingredients of our formalism which is characterized by the the introduction of the reduced generating functional. The latter is written as a coherent state path integral over paths parametrized in terms of the Keldysh-Schwinger complex time variable. The environment is simulated by a collection of fermionic harmonic oscillators and, by assuming that it interacts linearly with the system, its degrees of freedom are integrated out. The resulting quantity, called influence functional, fully expresses the impact of the environment on the dynamics of the system. In this way, we construct the reduced generating functional through which the calculation of the covariance matrix, as well the calculation of any reduced correlation function, is immediate via functional differentiations. The results of this Section are quite general not depending on the specific system under consideration.

In Section 3, we apply the aforementioned formalism for a system that is described by a Hamiltonian which is quadratic when written in terms of Majoranas variables and we derive the general form of the generating functional through which the all-important covariance matrix can be straightforwardly calculated. We also discuss the Markovian limit, on which the analytical calculations of the next Section are based.

Section 4 refers to a fermionic system which, when isolated, is characterized by a quantum phase transition. In the framework of the present formalism, we examine its properties when it is part of a compound system. The site-translational symmetry it possesses, facilitates the diagonalization of the corresponding Hamiltonian providing, thus, a concrete example of the calculations presented in Section 3. We derive the covariance matrix and we examine its analytic properties at the steady state limit. The main result of this Section is the confirmation that the non-analyticity presented in the ground state correlations of the isolated system, remains in the steady state of the open system indicating the possible persistence of quantum correlations.

Finally, in Section 5 we present the conclusions of the current investigation. In Appendix A, we present some technical details that were left out in the main text.

2. The Reduced Generating Functional

We consider a compound fermionic system consisting of two parts: The (sub)system S and its environment E. The dynamics of the isolated $S+E$ system is controlled by a Hamiltonian of the form:

$$\hat{H} = \hat{H}_S \otimes \hat{I}_E + \hat{I}_S \otimes \hat{H}_E + \hat{H}_I \tag{1}$$

The Hamiltonian operators in the last expression are defined in terms of fermionic creation and annihilation operators:

$$\hat{H}_S = \hat{H}\left(\hat{a}_S^\dagger, \hat{a}_S\right), \quad \hat{H}_E = \hat{H}\left(\hat{a}_E^\dagger, \hat{a}_E\right), \quad \hat{H}_I = \hat{H}\left(\hat{a}_S^\dagger, \hat{a}_S; \hat{a}_E^\dagger, \hat{a}_E\right) \tag{2}$$

The physical picture we adopt is the following: Up to a moment $t_{in.}$ the environment and the system are independent of each other $\left(\hat{H}_I\left(t \leq t_{in.}\right) = 0\right)$ and the density operator of the compound system can be expressed in the product form $\hat{\rho}\left(t_{in.}\right) = \hat{\rho}_E\left(t_{in.}\right) \otimes \hat{\rho}_S\left(t_{in.}\right)$. Before the moment $t_{in.}$ the environment is in equilibrium at temperature $T = \beta^{-1}$ meaning that $\hat{\rho}_E = \left(Z_E\left(\beta\right)\right)^{-1} \exp\left(-\beta \hat{H}_E\right)$. After the initialization of the interaction, the parts of the compound system entangle, and time evolution becomes non-unitary.

As probes for the system's dynamics, one usually considers correlations of the form:

$$G_{jk}\left(\rho; t_2, t_1\right) = Tr\left[\hat{\rho}\left(t_{in.}\right) \hat{T}\left(\hat{a}_{S,j}^\dagger\left(t_2\right) \hat{a}_{S,k}\left(t_1\right)\right)\right]. \tag{3}$$

In the last expression, $\hat{a}_{S,j}\left(t\right)$ is a fermionic Heisenberg operator that refers to the system, the subscript j is a site or space index, $\hat{\rho}\left(t_{in.}\right)$ is the initial density operator of the compound system, $\hat{T}$ is the time ordering operator and the trace operation refers to both the environment's and the system's degrees of freedom.

Needless to say, the calculation of 3 and of any reduced correlation function, is not a trivial task. All the efforts, analytical or numerical, for confronting the issue are restricted in the framework of Lindblad's master equation [10, 11]. In the present study we shall adopt a more general formalism for the study of the system's dynamics, based on the introduction of a functional that generates, via the application of the appropriate derivatives, the reduced functions we are interested in.

To set the stage, we begin by noting that correlators of Heisenberg field operators are characterized by a forward-backward time structure (see Eq. 3). As long as the system is in equilibrium and isolated this structure is easily taken into account by assuming an adiabatic evolution of the non-interacting vacuum. However, when the system is out of equilibrium or open, the adiabatic hypothesis is not valid. In that case, the dynamics has been confronted, a long ago, via the Keldysh-Schwinger formalism [14]. In the present case, we examine a quantum system in contact with its surroundings or, put in other words, we focus on a (presumably small) part of an isolated compound system. As it is obvious, the situation shares a lot with the out of equilibrium dynamics. Taking this into account, we have presented in [12] a formalism based on a reduced version of the Keldysh-Schwinger correlation functions. It is based on an extension of Feynman-Vernon's technique along the Keldysh time contour, an

extension that focuses on the calculation of correlation functions bypassing the determination of the reduced density matrix per se.

The scheme begins with the interpretation of the trace operation in Eq. 3 in the (over complete) coherent state basis $|\mathbf{z}_S\rangle \equiv \otimes_{j\in S} \left|z_j\right\rangle$ for the system and $|\boldsymbol{\zeta}_E\rangle \equiv \otimes_{\mu\in E} \left|\zeta_\mu\right\rangle$ for the environment, followed by the construction of the generating functional for system's correlators.

Apart from the details about the peculiarities of defining path integration over coherent states, the key ingredient in the aforementioned construction is that the paths entering the generating functional are parametrized along the so-called Keldysh time contour [14].

The Keldysh time contour is defined on the complex plane along a closed contour P that encircles the real t axis running from $t_{in.+} \equiv t_{in.} + i0$ to $t_{in.-} \equiv t_{in.} - i0$. The contour consists of two straight lines. The first one, denoted as L_+, joins the point $t_{in.+}$ to an arbitrary time instance $T_+ = T + i0$. Along this line the time variable is denoted as t_+. The second line joins $t_- = T - i0$ and along this line, defined as L_-, time is denoted as t_-. In the case of thermal initial states, the contour is extended by a complex time line, running parallel to the imaginary axis, from $t_{in.-}$ to $t_{in.} - i\beta$, where β^{-1} is the temperature of the corresponding thermal state. This third line is denoted as L_β, while the extended contour is denoted as C.

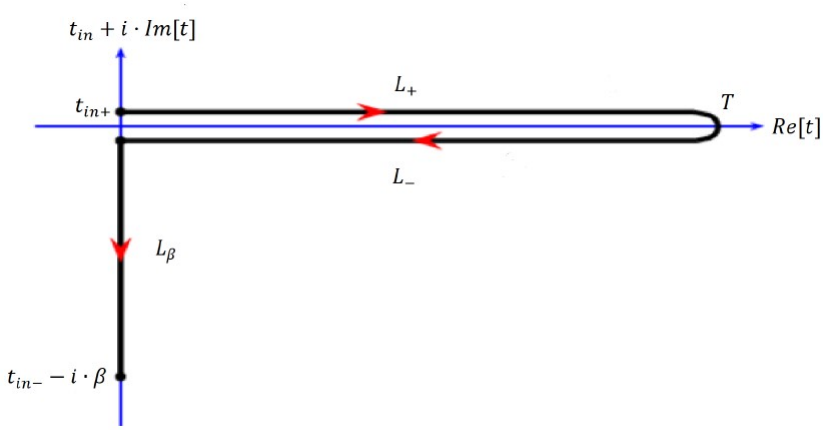

Figure 1. The Keldysh Contour

A natural ordering arises along this configuration since times along L_+ are considered to be before times along L_-, which are also considered to be before times along L_β (see Fig. 1). Taking as granted the technical details presented in Refs. [12, 13]. we quote here the result for the reduced generating functional:

$$Z_S\left[\bar{\mathbf{J}}, \mathbf{J}\right] = \int d^2\mathbf{z}_S d^2\mathbf{z}'_S \left\langle -\mathbf{z}_S\right| \hat{\rho}_S\left(t_{in.+}\right) \left|\mathbf{z}'_S\right\rangle \times \int\limits_{\mathbf{z}(t_{in.+})=\mathbf{z}_S}^{\bar{\mathbf{z}}(t_{in.-})=\bar{\mathbf{z}}'_S} D^2\mathbf{z}\left(t_P\right) e^{iS_S[\bar{\mathbf{z}},\mathbf{z}]-E(\bar{\mathbf{z}}_S,\mathbf{z}_S)+i\int\limits_P dt_P\left(\bar{\mathbf{J}}\cdot\mathbf{z}+\bar{\mathbf{z}}\cdot\mathbf{J}\right)} I_E\left[\bar{\mathbf{z}}, \mathbf{z}\right]. \tag{4}$$

At this point explanations are needed for the last expression to become transparent: The Grassmann fields $\mathbf{z}\left(t_P\right) \equiv \mathbf{z}_P\left(t\right)$ entering in the path integral are defined along the Keldysh

contour P (assuming that the system is at zero temperature):

$$\mathbf{z}\left(t_P\right) = \begin{cases} \mathbf{z}\left(t_+\right) \equiv \mathbf{z}_+ & \text{along } L_+ \\ \mathbf{z}\left(t_-\right) \equiv \mathbf{z}_- & \text{along } L_- \end{cases} . \tag{5}$$

These variables are considered as independent and they are integrated separately. The action S refers to the system and assumes the following form:

$$S_S = \int\limits_P dt_P \left[\frac{i}{2}\left(\bar{\mathbf{z}}\cdot\mathbf{z} - \bar{\mathbf{z}}\cdot\mathbf{z}\right) - H_S\left(\bar{\mathbf{z}},\mathbf{z}\right)\right]. \tag{6}$$

In Eq. 4 the action is accompanied by the surface term

$$E\left(\bar{\mathbf{z}}'_S,\mathbf{z}_S\right) = \frac{1}{2}\left[\left|\bar{\mathbf{z}}'_S\right|^2 + \left|\mathbf{z}_S\right|^2\right] - \frac{1}{2}\left[\bar{\mathbf{z}}'_S\cdot\mathbf{z}\left(t_{in.-}\right) + \bar{\mathbf{z}}\left(t_{in.+}\right)\cdot\mathbf{z}_S\right]. \tag{7}$$

The generating functional depends on the initial state of the system through the factor $\left\langle -\mathbf{z}_S\right|\hat{\rho}_S\left(t_{in.+}\right)\left|\mathbf{z}'_S\right\rangle$. When the system is initially in its ground state, $\hat{\rho}_S\left(t_{in.+}\right) = |GS\rangle\langle GS|$, this factor together with the surface term are integrated out and the generating functional reduces to the simpler form:

$$Z_S^{(0)}\left[\bar{\mathbf{J}},\mathbf{J}\right] = \int D^2\mathbf{z}\left(t_P\right) e^{iS_S[\bar{\mathbf{z}},\mathbf{z}]+i\int\limits_P dt_P\left(\bar{\mathbf{J}}\cdot\mathbf{z}+\bar{\mathbf{z}}\cdot\mathbf{J}\right)} I_E\left[\bar{\mathbf{z}},\mathbf{z}\right]. \tag{8}$$

Sources have been added in Eq. 4 to produce the system's correlation functions through functional differentiation. For example, the correlation function 3 can be derived by using the formula:

$$G_{jk}\left(\rho;t_2,t_1\right) = \left.\frac{\delta^2 \ln Z_S^{(0)}}{\delta J_j\left(t_{1+}\right)\delta\bar{J}_k\left(t_{2+}\right)}\right|_{J=0} . \tag{9}$$

The functional I_E, appearing in Eq. 4 is produced after tracing out the environmental degrees of freedom. It is the reason we consider Z_S as reduced. It incorporates the influence of the environment on the system's dynamics and has the same significance as the Feynman-Vernon's influence functional for the calculation of the reduced density matrix. It is defined as

$$I_E\left[\bar{\mathbf{z}},\mathbf{z}\right] = \frac{1}{Z_E\left(\beta\right)}\int\limits_{AP} D\boldsymbol{\zeta}\left(t_C\right) e^{i\int\limits_C dt_C\left(\bar{\boldsymbol{\zeta}}\cdot\boldsymbol{\zeta}-H_E\left(\bar{\boldsymbol{\zeta}},\boldsymbol{\zeta}\right)\right)-i\int\limits_C dt_C H_I\left(\bar{\boldsymbol{\zeta}},\boldsymbol{\zeta};\bar{\mathbf{z}},\mathbf{z}\right)} . \tag{10}$$

Here, the paths are parametrized along the full Keldysh contour $C = P \cup L_\beta$ with the agreement that interactions are absent along the thermal branch L_β. The antiperiodic boundary conditions are induced by the fermionic trace operation. As expected, the calculation of the influence functional is not a trivial task. In the present work, we adopt the simplifying

assumption that the environment can be simulated by a collection of (fermionic) harmonic oscillators which interact linearly with the system:

$$\begin{aligned} \hat{H}_E &= \sum_{\mu} E_\mu \left(\hat{a}^\dagger_{E,\mu} \hat{a}_{E,\mu} - \frac{1}{2} \right), \\ \hat{H}_I &= \sum_{j\in S, \mu\in E} \left(\lambda_{j\mu} \hat{a}^\dagger_{E,\mu} \hat{a}_{S,j} + \bar{\lambda}_{j\mu} \hat{a}^\dagger_{S,j} \hat{a}_{E,\mu} \right) \end{aligned} . \tag{11}$$

This interaction results in fermion number non-conservation and is responsible for the dissipation effects we are interested in. In this case, the influence functional can be exactly calculated [13]:

$$I_E = C_E^{-1} \exp \left[- \int_P dt_P \int_{P'} dt'_{P'} \bar{\mathbf{z}}\left(t_P\right) \mathbf{\Delta}\left(t_P - t'_{P'}\right) \mathbf{z}\left(t'_{P'}\right) \right]. \tag{12}$$

The different indexing in time integration indicates the fact that the time parameters in the last expression can run along different branches of the Keldysh contour. The constant appearing in Eq. 12, $C_E^{-1} = \prod_{\mu\in E} 2\cosh\left(\beta E_\mu/2\right)$, is of no significance for the calculation of correlation functions. The crucial quantities are the matrix elements $(\mathbf{\Delta})_{jk} = \Delta_{jk}$ which assume the form [13]:

$$\begin{aligned} \Delta_{jk}\left(t_P - t'_{P'}\right) &= \sum_{\mu\in E} \bar{\lambda}_{j\mu}\lambda_{k\mu} \left[\Theta\left(t_P - t'_{P'}\right) - \frac{1}{1+e^{\beta E_\mu}} \right] e^{-i\left(t_P - t'_{P'}\right)E_\mu} = \\ &= \int_0^\infty dE D_{jk}\left(E\right) \left[\Theta\left(t_P - t'_{P'}\right) - \frac{1}{1+e^{\beta E}} \right] e^{-i\left(t_P - t'_{P'}\right)E}. \end{aligned} \tag{13}$$

Here we defined:

$$D_{jk}\left(E\right) = \sum_{\mu\in E} \bar{\lambda}_{j\mu}\lambda_{k\mu}\delta\left(E - E_\mu\right). \tag{14}$$

In the following we shall assume that we can write:

$$\sum_{\mu\in E} \bar{\lambda}_{j\mu}\lambda_{k\mu} = g_{jk} \sum_{\mu\in E} \left|\gamma_\mu\right|^2, \quad g_{jk} = \bar{g}_{kj}. \tag{15}$$

As it is obvious the coefficients g_{jk} characterize the strength of the interaction between the sites j, k of the system, induced by the environment. The two extreme choices $g_{jk} = g\delta_{jk}$ and $g_{jk} = g \;\; \forall j, k$ correspond, the first one, to the case where each site of the system interacts independently with the environment, and the second one to the case in which all the sites of the system interact with each other with the same strength, irrespective of their distance.

Thus, the influence of the environment, as it is encoded in Eq. 12, yields a contribution to the generating functional which, in general, is nonlocal both in space and time indices.

Inserting Eq. 12 into Eq. 8 we find the expression that we indent to use for the calculation in the current investigation:

$$Z_S^{(0)}\left[\bar{\mathbf{J}},\mathbf{J}\right] = C_E^{-1}\int D^2\mathbf{z}\left(t_P\right)e^{iS_S[\bar{\mathbf{z}},\mathbf{z}]+iS_{INF.}[\bar{\mathbf{z}},\mathbf{z}]+i\int_P dt_P\left(\bar{\mathbf{J}}\cdot\mathbf{z}+\bar{\mathbf{z}}\cdot\mathbf{J}\right)} \tag{16}$$

with

$$S_{INF.} = i\int_P dt_P\int_{P'} dt'_{P'}\bar{\mathbf{z}}\left(t_P\right)\mathbf{\Delta}\left(t_P - t'_{P'}\right)\mathbf{z}\left(t'_{P'}\right). \tag{17}$$

The generating functional in Eq. 16 can, in principle at least, be exactly calculated if the system's Hamiltonian is quadratic. In such a case the covariance matrix can also be exactly calculated. We shall confront the issue in the next chapter. At this point, is worth note that through the (reduced) generating functional correlations of the form $\left\langle\hat{O}_{1,S}\left(t_1\right)\hat{O}_{2,S}\left(t_2\right)...\hat{O}_{n,S}\left(t_n\right)\right\rangle_\rho$ can be calculated. This must be contrasted to the case of calculations based on Lindblad's equation where one considers correlations that contain a single time variable [15].

3. Quadratic Hamiltonians

In this chapter we are interested in systems the dynamics of which are described by Hamiltonians that, when written in terms of Majorana variables, assume the quadratic form:

$$H_S = \frac{i}{4}\sum_{j,k=1}^{N}\sum_{u,v=0,1}\gamma_{j,u}A_{ju,kv}\gamma_{k,v} = \frac{i}{4}\sum_{\alpha,\beta=1}^{2N}\gamma_\alpha A_{\alpha\beta}\gamma_\beta \equiv \frac{i}{4}\boldsymbol{\gamma}\mathbf{A}\boldsymbol{\gamma}. \tag{18}$$

The Majorana variables entering the last expression are defined according to the rule:

$$\gamma_{2j-1} = \bar{z}_{S,j} + z_{S,j} \equiv \gamma_{j,0}, \quad \gamma_{2j} = i\left(\bar{z}_{S,j} - z_{S,j}\right) \equiv \gamma_{j,1}. \tag{19}$$

In Eq. 18 we adopted the compact notation of Ref. [5] $\alpha = (j,u)$ in which $j = 1,...,N$ and $u = 0,1$. In accordance, in the present Section, vectors and matrices are defined in a $2N$ dimensional space:

$$\boldsymbol{\gamma} \equiv \left(\gamma_1,\gamma_2,...,\gamma_{2N-1},\gamma_{2N}\right) = \left(\gamma_{1,0},\gamma_{1,1},...,\gamma_{N,0},\gamma_{N,1}\right). \tag{20}$$

The $2N\times 2N$ antisymmetric matrix $\mathbf{A} \doteq A_{\alpha\beta} = A_{jk,uv} = -A_{kj,vu} = -A_{\beta\alpha}$ describes the interaction between the sites of the system. In the quadratic case, the generating functional 16 can be exactly calculated by minimizing the action:

$$S = S_S + S_{INF.} + S_J. \tag{21}$$

Due to form 18 of the Hamiltonian, it is convenient to express S in terms of Majorana variables:

$$\begin{aligned} S =& \frac{i}{4}\int\limits_{P} dt_P \boldsymbol{\gamma}\left(t_P\right)\left(\mathbf{I}\partial_{t_P}-\mathbf{A}\right)\boldsymbol{\gamma}\left(t_P\right) \\ &+\frac{i}{4}\int\limits_{P} dt_P \int\limits_{P'} dt'_{P'}\boldsymbol{\gamma}\left(t_P\right)\mathbf{D}\left(t_P,t'_{P'}\right)\boldsymbol{\gamma}\left(t'_{P'}\right) \\ &+\frac{1}{2}\int\limits_{P} dt_P \mathbf{f}\left(t_P\right)\cdot\boldsymbol{\gamma}\left(t_P\right). \end{aligned} \tag{22}$$

In the last expression we introduced the sources:

$$f_{j,0}=\bar{J}_j-J_j,\quad f_{j,1}=i\left(\bar{J}_j+J_j\right) \tag{23}$$

and we defined

$$\mathbf{D}\doteq D_{\alpha\beta}=D_{ju,kv}=\Delta_{jk}\left(1-\sigma^y\right)_{uv}. \tag{24}$$

By minimizing Eq. 22 we get the following "classical" equation:

$$\left(\mathbf{I}\partial_{t_P}-\mathbf{A}\right)\boldsymbol{\gamma}^{cl.}\left(t_P\right)+\int\limits_{P'} dt'_{P'}\tilde{D}\left(t_P,t'_{P'}\right)\boldsymbol{\gamma}^{cl.}\left(t'_{P'}\right)=-i\mathbf{f}\left(t_P\right) \tag{25}$$

in which

$$\tilde{D}_{\alpha\beta}\left(t_P,t'_{P'}\right)=\frac{1}{2}\left(D_{\alpha\beta}\left(t_P,t'_{P'}\right)-D_{\beta\alpha}\left(t'_{P'},t_P\right)\right)=-\tilde{D}_{\beta\alpha}\left(t_P,t'_{P'}\right). \tag{26}$$

By introducing Green's function $\Gamma_{\alpha\beta}$ as the causal solution of Green's equation

$$\begin{aligned} &\int\limits_{P''} dt''_{P''}\left[\left(\mathbf{I}\partial_{t_P}-\mathbf{A}\right)\delta\left(t_P-t''_{P''}\right)+\tilde{\mathbf{D}}\left(t_P,t''_{P''}\right)\right]\boldsymbol{\Gamma}\left(t''_{P''},t'_{P'}\right)= \\ &=-\mathbf{I}i\delta\left(t_P-t'_{P'}\right). \end{aligned} \tag{27}$$

the generating functional 16 assumes the form:

$$Z_S^{(0)}\left[\mathbf{f}\right]=C_E^{-1}\exp\left[\frac{i}{4}\int\limits_{P'} dt'_{P'}\int\limits_{P} dt_P\mathbf{f}\left(t_P\right)\boldsymbol{\Gamma}\left(t_P,t'_{P'}\right)\mathbf{f}\left(t'_{P'}\right)\right]. \tag{28}$$

As it is obvious, having in hand the solution of Eq. 27 and the generating functional 28, one can calculate, via a unitary transformation of the sources, all the correlation functions of the system.

In the present study, we focus on the real and antisymmetric covariance matrix that is defined as [5]:

$$C_{jk,uv}\left(t\right)=\mathrm{Tr}\left\{\hat{\rho}\left(t\right)\frac{i}{2}\left[\hat{\gamma}_{j,u},\hat{\gamma}_{k,v}\right]\right\}=\mathrm{Tr}_{\mathrm{S}}\left\{\hat{\rho}_S^R\left(t\right)\frac{i}{2}\left[\hat{\gamma}_{j,u},\hat{\gamma}_{k,v}\right]\right\}. \tag{29}$$

The key ingredient in this expression is the reduced density matrix

$$\hat{\rho}_S^R(t) = Tr_E\left\{e^{-i(t-t_{in.})\hat{H}}\hat{\rho}(t_{in.})\, e^{i(t-t_{in.})\hat{H}}\right\} \tag{30}$$

which, in the present case, is a Gaussian state and, as such, all its properties are encoded into the covariance matrix 29.

Knowledge of the covariance matrix permits physically important issues to be addressed. One of the most challenging refers to the rate at which correlations diminish due to decoherence. Closely related to it is the approach of the reduced density matrix to the, so-called, steady state, $\hat{\rho}_S^R(t) \underset{t\to\infty}{\to} \hat{\rho}_{S,0}^R$ and the consequent approach of the covariance matrix to the corresponding steady state covariance: $C_{jk,uv}(t) \underset{t\to\infty}{\to} C_{jk,uv}^{(0)}$. At the thermodynamic limit $N \to \infty$, the knowledge of the analytic structure of these functions is of great importance as it can reveal possible quantum phase transitions.

The covariance matrix can be straightforwardly calculated through the generating functional 28 by applying functional derivatives as follows:

$$C_{jk,uv}(t) = -4i\left.\frac{\delta^2 \ln Z_S^{(0)}}{\delta f_{ju}(t_+)\,\delta f_{kv}(t_+)}\right|_{J=0}. \tag{31}$$

In the present approach, the key function for achieving the final result is the solution of Green's equation 27. However, this equation, in its general form, is nonlocal both in space and time indices, a fact that makes its analytical solution a very difficult task. Even so, it can be written in a way more convenient for calculations as the real antisymmetric matrix $A_{\alpha\beta}$ can always be diagonalized through an orthogonal transformation:

$$\mathbf{Q}^{\perp}\mathbf{A}\mathbf{Q} = \oplus_{j=1}^{N}\begin{pmatrix} 0 & \varepsilon_j \\ -\varepsilon_j & 0 \end{pmatrix}, \quad \varepsilon_j \geq 0, \quad \mathbf{Q} \in O(2N). \tag{32}$$

Thus, by using the unitary transformation

$$\mathbf{V} = \mathbf{Q} \oplus^N \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{33}$$

we can write

$$\mathbf{V}^{\dagger}\mathbf{A}\mathbf{V} = \oplus^N \begin{pmatrix} i\varepsilon_j & 0 \\ 0 & -i\varepsilon_j \end{pmatrix} \equiv i\mathbf{E}. \tag{34}$$

By writing

$$\begin{aligned} &\sum_{j,u}\gamma_{j,u}V_{jk,uv} = \psi_{k,v} \to \boldsymbol{\gamma}\cdot\mathbf{V} = \boldsymbol{\psi}, \quad \mathbf{K} = \mathbf{V}^{\dagger}\tilde{\mathbf{D}}\mathbf{V}, \\ &\mathbf{j} = \mathbf{f}\cdot\mathbf{V} \to j_{k,v} = \sum_{j,u} f_{j,u}V_{jk,uv} \end{aligned} \tag{35}$$

the action 22 can be written as follows:

$$\begin{aligned} S = & \frac{i}{4}\int\limits_{P} dt_P \boldsymbol{\psi}\left(t_P\right)\left(\mathbf{I}\partial_{t_P} - i\mathbf{E}\right)\boldsymbol{\psi}\left(t_P\right) \\ & + \frac{i}{4}\int\limits_{P} dt_P \int\limits_{P'} dt'_{P'} \boldsymbol{\psi}\left(t_P\right)\mathbf{K}\left(t_P, t'_{P'}\right)\boldsymbol{\psi}\left(t'_{P'}\right) \\ & - \frac{1}{4}\int\limits_{P} dt_P \bar{\mathbf{j}}\left(t_P\right)\cdot\boldsymbol{\psi}\left(t_P\right) - \frac{1}{4}\int\limits_{P} dt_P \bar{\boldsymbol{\psi}}\left(t_P\right)\cdot\mathbf{j}\left(t_P\right) \end{aligned} . \tag{36}$$

In this case the classical equation of motion assumes the form:

$$\left(\mathbf{I}\partial_{t_P} - i\mathbf{E}\right)\boldsymbol{\psi}^{\mathbf{cl.}}\left(t_P\right) + \int\limits_{P'} dt'_{P'}\mathbf{K}\left(t_P, t'_{P'}\right)\boldsymbol{\psi}^{\mathbf{cl.}}\left(t'_{P'}\right) = -i\mathbf{j}\left(t_P\right). \tag{37}$$

Thus, by defining the function $\mathbf{L} = \mathbf{V}^{\dagger}\boldsymbol{\Gamma}\mathbf{V}$, the Green's equation 27 is recasted as:

$$\left(\mathbf{I}\partial_{t_P} - i\mathbf{E}\right)\mathbf{L}\left(t_P, t'_{P'}\right) + \int\limits_{P''} dt''_{P''}\mathbf{K}\left(t_P, t''_{P''}\right)\mathbf{L}\left(t''_{P''}, t'_{P'}\right) = -\mathbf{I}i\delta\left(t_P - t'_{P'}\right) \tag{38}$$

The last expression is a set of equations coupled due to the presence of the kernel $\mathbf{K}$, the dissipation kernel. In the current paper we adopt an approximation that can lead to the exact solution of the Green's Eqs. 38. It is the same approximation on which the Lindbland approach is founded namely the, so-called, Markovian approximation [15]. It is based on the hypothesis that the environment acts as a memoryless bath and, consequently, that the influence functional can be considered as local in time. The validity of such an approximation is based on the existence of two characteristic time scales [16]. The first one, τ_E, refers to the decay of environmental correlations, while the second one, τ_S, characterizes the frequencies $\varepsilon_S\ \tau_S^{-1}$ after which system's dynamics are screened out. If the last scale is much larger than the first one, the function 30 behaves much like as a delta function:

$$\Delta_{jk}\left(t_P - t'_{p'}\right) \rightarrow g_{jk}\Delta_{PP'}\left(\varepsilon_S\right)\delta\left(t - t'\right) \tag{39}$$

The matrix elements $\Delta_{PP'}$ are defined as [13]:

$$\begin{aligned} \Delta_{++} &= -i\delta E + \Gamma\left(\frac{1}{2} - b\right), \\ \Delta_{--} &= i\delta E + \Gamma\left(\frac{1}{2} - b\right), \\ \Delta_{+-} &= -\Gamma b, \\ \Delta_{-+} &= \Gamma\left(1 - b\right) \end{aligned} \tag{40}$$

with:

$$\Gamma = 2\pi D\left(\varepsilon_S\right), \quad \delta E = \mathrm{Pr.}\int\limits_{0}^{\infty} dE\frac{D\left(E\right)}{E - \varepsilon_S}, \quad b = \frac{1}{1 + e^{\beta\varepsilon_S}} \tag{41}$$

The Markovian limit makes the dissipation kernel diagonal in the time variables. Even so, expression 38 is a set of $4 \times (2N)^2$ coupled equations the solution of which is a rather complicated task. However, when the Hamiltonian is translationally invariant Eqs. 38 can be exactly solved. This is the task of the next Section.

4. Kitaev' s chains

In the present Section we examine a simple -yet interesting- fermionic model in one spatial dimension, namely the $1D$ superconducting chain [1, 17, 18]:

$$\hat{H}_S = \sum_{j=1}^{N} \left[-w \left(\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j \right) + \Delta \left(\hat{a}_j \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j^\dagger \right) - \mu \left(\hat{a}_j^\dagger \hat{a}_j - \frac{1}{2} \right) \right]. \tag{42}$$

Although simple and completely solvable when isolated, this model is quite rich due to the underlying quantum phase transition. After the Jordan-Wigner transformation

$$\hat{a}_j = \prod_{k=1}^{j-1} \left(-\sigma_k^z \right) \sigma_j^-, \hat{a}_j^\dagger = \prod_{k=1}^{j-1} \left(-\sigma_k^z \right) \sigma_j^+, \sigma_j^\pm = \frac{1}{2} \left(\sigma_j^x \pm i \sigma_j^y \right) \tag{43}$$

the fermionic system is mapped onto the anisotropic XY spin system:

$$\hat{H}_S = \sum_j \left(\frac{\Delta - w}{2} \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + \frac{\Delta + w}{2} \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y - \frac{\mu}{2} \hat{\sigma}_j^z \right). \tag{44}$$

For reasons of simplicity, we choose $w = -\Delta = 1$ and we write $\mu = 2h$. In this case, the spin system 44 reduces to the transverse Ising model:

$$\hat{H}_S = -\sum_j \left(\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + h \hat{\sigma}_j^z \right) \tag{45}$$

while its fermionic ancestor reads:

$$\hat{H}_S = -\sum_{j=1}^{N} \left[\left(\hat{a}_j^\dagger - \hat{a}_j \right) \left(\hat{a}_{j+1}^\dagger + \hat{a}_{j+1} \right) + 2h \left(\hat{a}_j^\dagger \hat{a}_j - \frac{1}{2} \right) \right]. \tag{46}$$

In the presence of the environment, the dynamics of these systems become quite complicated studied mainly in the framework of the Lindbland master equation. From this point of view the model 42, and all its integrable variants, provides the ideal stage for presenting the analytic formalism introduced in Sections 2 and 3. However, a note is needed at this point: When the system we are interested in is isolated, the Hamiltonians 42 and 44 are mathematically equivalent meaning that the calculation of the relevant correlation functions can be based either on the first or the second one. When the system is part of a larger compound system, it is the mutual interaction that defines the physical degrees of freedom. In the case under study the environment is a fermionic bath meaning that the physical degrees of freedom for the description of the system are fermionic and the relevant quantum Hamiltonian is 44.

The classical Hamiltonian representing the fermionic system 46 in the integral7, can be written in the form:

$$\begin{aligned} H_S &= \sum_j \left[\left(z_j z_{j+1} - \bar{z}_j \bar{z}_{j+1} \right) + \left(z_j \bar{z}_{j+1} - \bar{z}_j z_{j+1} \right) - 2h \bar{z}_j z_j \right] = \\ &= \frac{i}{4} \sum_{jk,uv} \gamma_{j,u} A_{jk,uv} \gamma_{k,v} \end{aligned} \tag{47}$$

The matrix A entering the Majorana expression of the Hamiltonian is defined as:

$$\begin{aligned} &A_{j,j+1;0,0} = -A_{j+1,j;0,0} = 2, A_{j,j+1;0,1} = -A_{j+1,j;1,0} = -2, \\ &A_{j,j;0,1} = -A_{j,j;1,0} = 2h. \end{aligned} \tag{48}$$

The first step for obtaining the solution of the classical equation 27 is the diagonalization of system's Hamiltonian. For the isolated case, this task is accomplished via the implementation of the discrete Fourier transform [14, 18]:

$$z_j = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{i\varphi_m j} c_m. \tag{49}$$

The exact form of the phase $\varphi_m = \frac{2\pi}{N}(m + \kappa)$ depends on the number of fermions in the system. When this number is even, $\kappa = 1/2$, while $\kappa = 0$ when odd. The first case is connected with the antiperiodic boundary condition $z_{j+N} = -z_j$ and the latter with the periodic one $z_{j+N} = z_j$ [18]. When dissipation is present, the fermion number is not conserved meaning that, strictly speaking, the transformation 49 should not be used. However, at the thermodynamic limit $N \to \infty$ which we are interested in, the summation in Eq. 47 is extended over $\mathbb{Z}$ and the discrete transform 49 can be replaced by its continuum version:

$$z_j = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\varphi e^{i\varphi j} c_\varphi, \quad c_\varphi = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-i\varphi j} z_j. \tag{50}$$

These transformations are going to be helpful only if the system-environment interaction does not destroy the translational invariance presented in the isolated system. To preserve this symmetry, we shall assume that $g_{jk} = g_{j-k} = g_{k-j}$.

For the Majorana variables, the Fourier transforms read as follows:

$$\gamma_{j,u} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\varphi e^{i\varphi j} \gamma_{\varphi,u}, \quad \gamma_{\varphi,0} = \bar{c}_{-\varphi} + c_\varphi, \quad \gamma_{\varphi,1} = i\left(\bar{c}_{-\varphi} - c_\varphi\right). \tag{51}$$

Inserting expressions 51 into Eq. 47, the Hamiltonian can be rewritten in terms of the conjugate fields in the following form:

$$H_S = \int_0^{\pi} d\varphi \bar{\boldsymbol{\gamma}}_\varphi \mathbf{A}_\varphi \boldsymbol{\gamma}_\varphi. \tag{52}$$

To arrive at the last expression, we used the translational invariance of the Hamiltonian to write $A_{jk,uv} = A_{j-k,uv}$, we defined the vectors $\boldsymbol{\gamma}_\varphi = \left(\gamma_{\varphi,0}, \gamma_{\varphi,1}\right)$ and the matrices:

$$\left(\mathbf{A}_\varphi\right)_{uv} \equiv A_{\varphi,uv} = \sum_j e^{-i\varphi j} A_{j,uv} = -A_{-\varphi,vu}, A_{j,uv} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi e^{i\varphi j} A_{\varphi,uv}. \tag{53}$$

It's not difficult to find that:

$$\mathbf{A}_\varphi = -\frac{i}{2}\begin{pmatrix} 0 & h + e^{-i\varphi} \\ -\left(h + e^{i\varphi}\right) & 0 \end{pmatrix}. \tag{54}$$

By using the unitary transformation

$$\mathbf{V}_\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} e^{-i\theta_\varphi \sigma^x} \tag{55}$$

with

$$\cos 2\theta_\varphi = \frac{h + \cos\varphi}{\varepsilon_\varphi}, \quad \sin 2\theta_\varphi = \frac{\sin\varphi}{\varepsilon_\varphi}, \quad \varepsilon_\varphi = \sqrt{(h + \cos\varphi)^2 + \sin^2\varphi} \tag{56}$$

the matrix 54 can be put into diagonal form:

$$\mathbf{A}_\varphi = -\frac{1}{2}\mathbf{V}_\varphi \begin{pmatrix} \varepsilon_\varphi & 0 \\ 0 & -\varepsilon_\varphi \end{pmatrix} \mathbf{V}_\varphi^\dagger. \tag{57}$$

By defining

$$\boldsymbol{\psi}_\varphi = \mathbf{V}^\dagger{}_\varphi \boldsymbol{\gamma}_\varphi, \quad \bar{\boldsymbol{\psi}}_\varphi = \bar{\boldsymbol{\gamma}}_\varphi \mathbf{V}_\varphi \tag{58}$$

the Hamiltonian 52 simplifies as follows:

$$H_S = -\frac{1}{2}\int_0^\pi d\varphi \varepsilon_\varphi \bar{\boldsymbol{\psi}}_\varphi \sigma^z \boldsymbol{\psi}_\varphi. \tag{59}$$

Introducing the function

$$\tilde{g}_\varphi = \frac{1}{\sqrt{2\pi}}\sum_j e^{i\varphi j} g_j = \tilde{g}_{-\varphi} \tag{60}$$

the influence of the environment is written as

$$S_{INF.} = \frac{i}{2}\int_0^\pi d\varphi \int_P dt_P \int_{P'} dt'_{P'} \bar{\boldsymbol{\psi}}_\varphi\left(t_P\right) \mathbf{K}_\varphi\left(t_P, t'_{P'}\right) \boldsymbol{\psi}_\varphi\left(t'_{P'}\right) \tag{61}$$

where

$$\mathbf{K}_\varphi = \tilde{g}_\varphi \mathbf{V}_\varphi^\dagger \begin{pmatrix} \Delta^{(-)} & i\Delta^{(+)} \\ -i\Delta^{(+)} & \Delta^{(-)} \end{pmatrix} \mathbf{V}_\varphi, \quad \Delta^{(\pm)} = \Delta\left(t_P, t'_{P'}\right) \pm \Delta\left(t'_{P'}, t_P\right). \tag{62}$$

Thus, the action 21 written in terms of the fields 53 gets the form:

$$\begin{aligned} S =& \frac{1}{2}\int_0^{\pi} d\varphi \int_P dt_P \bar{\boldsymbol{\psi}}_\varphi \left(i\partial_{t_P} + \varepsilon_\varphi \sigma^z\right) \boldsymbol{\psi}_\varphi \\ &+ \frac{i}{2}\int_0^{\pi} d\varphi \int_P dt_P \int_{P'} dt'_{P'} \bar{\boldsymbol{\psi}}_\varphi \left(t_P\right) \mathbf{K}_\varphi \left(t_P, t'_{P'}\right) \boldsymbol{\psi}_\varphi \left(t'_{P'}\right) \\ &+ \frac{1}{2}\int_0^{\pi} d\varphi \int_P dt_P \left(\bar{\mathbf{j}}_\varphi \boldsymbol{\psi}_\varphi + \bar{\boldsymbol{\psi}}_\varphi \mathbf{j}_\varphi\right). \end{aligned} \tag{63}$$

For the source term we followed the notation (see Eq. 23):

$$\bar{\mathbf{j}}_\varphi = \mathbf{f}_\varphi \mathbf{V}_\varphi, \quad \mathbf{f}_\varphi = \left(f_{\varphi,0}, f_{\varphi,1}\right), \quad f_{\varphi,u} = \frac{1}{\sqrt{2\pi}} \sum_j e^{i\varphi j} f_{j,u} \tag{64}$$

By minimizing Eq. 62 we get the classical equation:

$$\left(i\partial_{t_P} + \varepsilon_\varphi \sigma^z\right) \boldsymbol{\psi}_\varphi^{cl.} \left(t_P\right) + i \int_{P'} dt'_{P'} \mathbf{K}_\varphi \left(t_P, t'_{P'}\right) \boldsymbol{\psi}_\varphi^{cl.} \left(t'_{P'}\right) = -\mathbf{j}_\varphi \left(t_P\right). \tag{65}$$

The corresponding Green's equation reads:

$$\left(i\partial_{t_P} + \varepsilon_\varphi \sigma^z\right) \mathbf{L}_\varphi \left(t_P, t'_{P'}\right) + i \int_{P''} dt''_{P''} \mathbf{K}_\varphi \left(t_P, t''_{P''}\right) \mathbf{L}_\varphi \left(t''_{P''}, t'_{P'}\right) = \mathbf{I}\delta\left(t_P - t'_{P'}\right). \tag{66}$$

Having in hand the solution of the last equation and using Eq. 58 the generating functional gets the form

$$Z_S^{(0)}\left[\mathbf{j}\right] = C_E^{-1} \exp\left[-\frac{i}{2}\int_0^{\pi} d\varphi \int_{P'} dt'_{P'} \int_P dt_P \bar{\mathbf{j}}_\varphi \left(t_P\right) \mathbf{L}_\varphi \left(t_P, t'_{P'}\right) \mathbf{j}\left(t'_{P'}\right)\right]. \tag{67}$$

Taking into account 58 the generating functional can be rewritten in the form 28 with

$$\Gamma_{jk,uv}\left(t_P, t'_{P'}\right) = \frac{1}{4\pi}\int_0^{\pi} d\varphi e^{i\varphi(j-k)} \left[\mathbf{V}_\varphi \mathbf{L}_\varphi \left(t_P, t'_{P'}\right) \mathbf{V}_\varphi^\dagger\right]_{uv}. \tag{68}$$

The Green's equation 66 with causal boundary conditions can be exactly solved. In Appendix A we present the details of the calculation. Here it is enough to quote the result for $t_+ = t'_+ = t$:

$$\begin{aligned} \mathbf{L}_\varphi =& -i\left[\frac{1}{2} - \sin^2\theta_\varphi \left(1 - e^{-\left|\tilde{g}_\varphi\right|\Gamma(t-t_{in.})}\right)\right]\sigma^z \\ &- \frac{i}{2}\frac{\left(\tilde{g}_\varphi \frac{\Gamma}{2}\right)^2}{\varepsilon_\varphi^2 + \left(\tilde{g}_\varphi \frac{\Gamma}{2}\right)^2} \sin^2\left(2\theta_\varphi\right)\left(1 - e^{-\left|\tilde{g}_\varphi\right|\Gamma(t-t_{in.})}\right)\begin{pmatrix} \cos^2\theta_\varphi & 0 \\ 0 & \sin^2\theta_\varphi \end{pmatrix} \end{aligned} \tag{69}$$

By applying Eq. 31 it is straightforward to find the covariance matrix:

$$
\begin{aligned}
C_{jk,uv}(t) = \frac{2}{\pi}\int_0^{\pi} d\varphi \begin{pmatrix} 0 & \cos\left(\varphi\,(j-k) - 2\theta_\varphi\right) \\ -\cos\left(\varphi\,(j-k) + 2\theta_\varphi\right) & 0 \end{pmatrix} \\
\times \left[\frac{1}{2} - \sin^2\theta_\varphi \left(1 - e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma(t-t_{in.})}\right)\right] \\
+ \frac{1}{2\pi}\int_0^{\pi} d\varphi \frac{\left(\tilde{g}_\varphi \frac{\Gamma}{2}\right)^2}{\varepsilon_\varphi^2 + \left(\tilde{g}_\varphi \frac{\Gamma}{2}\right)^2} \sin^2\left(2\theta_\varphi\right)\left(1 - e^{-\left|\tilde{g}_\varphi\right|\Gamma(t-t_{in.})}\right) \\
\times \begin{pmatrix} \sin\varphi\,(j-k) & \cos\left(\varphi\,(j-k) - 2\theta_\varphi\right)\cos\left(2\theta_\varphi\right) \\ -\cos\left(\varphi\,(j-k) + 2\theta_\varphi\right)\cos\left(2\theta_\varphi\right) & \sin\varphi\,(j-k) \end{pmatrix}
\end{aligned}
\tag{70}
$$

At the limit $t \to \infty$, we get the covariance at the steady state:

$$
\begin{aligned}
C^{(0)}_{jk,uv} = \frac{1}{\pi}\int_0^{\pi} d\varphi cos\left(2\theta_\varphi\right) \begin{pmatrix} 0 & \cos\left(\varphi\,(j-k) - 2\theta_\varphi\right) \\ -\cos\left(\varphi\,(j-k) + 2\theta_\varphi\right) & 0 \end{pmatrix} \\
+ \frac{1}{2\pi}\int_0^{\pi} d\varphi \frac{\left(\tilde{g}_\varphi \frac{\Gamma}{2}\right)^2}{\varepsilon_\varphi^2 + \left(\tilde{g}_\varphi \frac{\Gamma}{2}\right)^2} \sin^2\left(2\theta_\varphi\right) \\
\times \begin{pmatrix} \sin\varphi\,(j-k) & \cos\left(\varphi\,(j-k) - 2\theta_\varphi\right)\cos\left(2\theta_\varphi\right) \\ -\cos\left(\varphi\,(j-k) + 2\theta_\varphi\right)\cos\left(2\theta_\varphi\right) & \sin\varphi\,(j-k) \end{pmatrix}
\end{aligned}
\tag{71}
$$

The existence of the second term in the rhs of Eqs. 69 - 71 is due to the fact that the dissipation kernel in Eq. 62 has non-zero off-diagonal entries. An important observation is that both the functions 70 and 71 are non-analytic at the point $|h| = 1$. Firstly, we neglect the off-diagonal contribution of the dissipation kernel by assuming that $(g\Gamma/2) \ll 1$. This assumption simplifies the results without qualitatively changing them. In this case the steady-state covariance reads:

$$
C^{(0)}_{jk,uv} \simeq \frac{1}{\pi}\int_0^{\pi} d\varphi \cos\left(2\theta_\varphi\right) \begin{pmatrix} 0 & \cos\left(\varphi\,(j-k) - 2\theta_\varphi\right) \\ -\cos\left(\varphi\,(j-k) + 2\theta_\varphi\right) & 0 \end{pmatrix}
\tag{72}
$$

If the environment were absent ($\Gamma = 0$) the corresponding result would have the form (see Eq. 70):

$$
C^{(0)}_{jk,uv} \simeq \frac{1}{\pi}\int_0^{\pi} d\varphi \begin{pmatrix} 0 & \cos\left(\varphi\,(j-k) - 2\theta_\varphi\right) \\ -\cos\left(\varphi\,(j-k) + 2\theta_\varphi\right) & 0 \end{pmatrix}.
\tag{73}
$$

This function represents the covariance at the ground state of the isolated system and it is non-analytic. The integrals involved, are connected with the complete elliptic integral of the

second kind that has a branch cut at $|h| = 1$. This non-analytic behavior reflects the underlying quantum phase transition in the closed system. When the system is open the situation changes but non-analyticity persists. This is immediately confirmed by considering 72 for $j = k$:

$$C^{(0)}_{jj,uv} \simeq \frac{1}{\pi}\int_0^{\pi} d\varphi \cos^2\left(2\theta_\varphi\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{\pi}\int_0^{\pi} d\varphi \frac{\left(h+\cos\varphi\right)^2}{h^2+2h\cos\varphi+1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{74}$$

A simple calculation yields the result:

$$C^{(0)}_{jj,uv}\left(h\right) \simeq \left[\frac{1}{2}\theta\left(1-|h|\right) + \left(1-\frac{1}{2h^2}\right)\theta\left(|h|-1\right)\right]\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{75}$$

and

$$\frac{d}{d\,|h|}C^{(0)}_{jj,uv}\left(h\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{cases} 0, & |h| < 1 \\ 1/|h|^3, & |h| > 1 \end{cases} \tag{76}$$

Thus, in the presence of dissipation the steady state's covariance matrix possesses a finite discontinuity at the point $|h| = 1$. This point of non-analyticity coincides with the critical point of the, well-known, quantum phase transition that characterizes the corresponding isolated system and its equivalent, the transverse Ising model.

This result indicates that, although differently, non-analyticity remains in the steady state of the open system despite the impact of the environment.

For $j - k = L \to +\infty$ a straightforward calculation yields the result:

$$C^{(0)}_{jk,uv}\left(h\right) \underset{L\to+\infty}{\simeq} \frac{1-h^2}{2h^2}\begin{pmatrix} 0 & h^L\theta\left(1-|h|\right) \\ \frac{1}{h^L}\theta\left(|h|-1\right) & 0 \end{pmatrix} \tag{77}$$

and

$$C^{(0)}_{jk,uv}\left(h\right) \underset{L\to-\infty}{\simeq} \frac{h^2-1}{2h^2}\begin{pmatrix} 0 & \frac{1}{h^{|L|}}\theta\left(|h|-1\right) \\ h^{|L|}\theta\left(1-|h|\right) & 0 \end{pmatrix}. \tag{78}$$

Combining these expressions, we can infer that for $|L| \to \infty$ and $||h|-1| \to 0$ the steady state covariance behaves as [4, 19]:

$$C^{(0)}_{jk,uv}\left(h\right) \underset{|L|\to\infty;|h|\to 1}{\sim} ||h|-1| \exp\left(-|L|\,||h|-1|\right). \tag{79}$$

The last expression indicates the fact that the function $\frac{d}{d|h|}C^{(0)}_{jk,uv}\left(h\right)$ is characterized, at the limit $|h| \to 1$, by long range correlations the length of which has the form $\xi \sim \frac{1}{||h|-1|} \to \infty$.

5. Conclusions

We developed a framework for the construction of the time-dependent correlation functions of an open fermionic system. It is a framework, that is based on the merge of the Feynman-Vernon's technique with the Keldysh-Schwinger formalism on the ground of the coherent

state path integrals. By integrating out the environmental degrees of freedom, we defined the generating functional through which every system's correlation function can be derived by applying the appropriate functional derivatives. We presented the details of this construction in the case of a system that is quadratic when written in terms of Majorana variables. In the last Section of the work, we applied the proposed formalism to examine a model which is completely solvable when isolated namely, the transverse Ising model. We calculate the exact form of the covariance matrix at the steady state. We confirm the it is non-analytic at the same point at which the corresponding isolated model develops a quantum phase transition and we examine the specific form in which this non-analyticity is realized.

Appendix A.

In this Appendix we solve the Green's equation 66 of the main text:

$$\left(i\partial_{t_P} + \varepsilon_\varphi \sigma^z\right) \mathbf{L}_\varphi \left(t_P, t'_{P'}\right) + i \int\limits_{P''} dt''_{P''} \mathbf{K}_\varphi \left(t_P, t''_{P''}\right) \mathbf{L}_\varphi \left(t''_{P''}, t'_{P'}\right) = \mathbf{I}\delta \left(t_P - t'_{P'}\right). \tag{A.1}$$

The solution we are interested for, must obey the boundary conditions:

$$\mathbf{L}\left(t_P = t_{in.+}, t'_{P'}\right) = \mathbf{L}\left(t_P, t'_{P'} = t_{in.-}\right) = 0, \quad \mathbf{L}\left(t_P, t'_{P'}\right) \underset{|t-t'|\to+\infty}{\longrightarrow} 0. \tag{A.2}$$

The first demand expresses the causal character of propagation. The second one is connected with the decoherence encoded into the third term of Eq. A.1, the term that brings the influence of the environment into the propagation.

The dissipation kernel appearing in last equation has the form:

$$\mathbf{K}_\varphi = \tilde{g}_\varphi \begin{pmatrix} \Delta\left(t_P, t'_{P'}\right)\cos^2\theta_\varphi - \Delta\left(t'_{P'}, t_P\right)\sin^2\theta_\varphi & -\frac{i}{2}\left(\Delta\left(t_P, t'_{P'}\right) + \Delta\left(t'_{P'}, t_P\right)\right)\sin\left(2\theta_\varphi\right) \\ \frac{i}{2}\left(\Delta\left(t_P, t'_{P'}\right) + \Delta\left(t'_{P'}, t_P\right)\right)\sin\left(2\theta_\varphi\right) & \Delta\left(t_P, t'_{P'}\right)\sin^2\theta_\varphi - \Delta\left(t'_{P'}, t_P\right)\cos^2\theta_\varphi \end{pmatrix} \tag{A.3}$$

The solution of the Green's equation is a matrix of Green's functions:

$$\mathbf{L}_\varphi = \begin{pmatrix} L_{\varphi,00} & L_{\varphi,01} \\ L_{\varphi,10} & L_{\varphi,11} \end{pmatrix} \tag{A.4}$$

The dissipation kernel splits the set of equations A.1 to two groups of coupled equations:

$$\begin{aligned}
&\partial_{t_P} L_{\varphi,00}\left(t_P, t'_P\right) - i\varepsilon_\varphi L_{\varphi,00}\left(t_P, t'_P\right) + \int\limits_{P''} dt''_P K_{\varphi,00}\left(t_P, t''_P\right) L_{\varphi,00}\left(t''_P, t'_P\right) \\
&\quad + \int\limits_{P''} dt''_P K_{\varphi,01}\left(t_P, t''_P\right) L_{\varphi,10}\left(t''_P, t'_P\right) = -i\delta\left(t_p - t'_P\right) \\
&\partial_{t_P} L_{\varphi,10}\left(t_P, t'_P\right) + i\varepsilon_\varphi L_{\varphi,10}\left(t_P, t'_P\right) + \int\limits_{P''} dt''_P K_{\varphi,10}\left(t_P, t''_P\right) L_{\varphi,00}\left(t''_P, t'_P\right) \\
&\quad + \int\limits_{P''} dt''_P K_{\varphi,11}\left(t_P, t''_P\right) L_{\varphi,10}\left(t''_P, t'_P\right) = 0
\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}
&\partial_{t_P} L_{\varphi,11}\left(t_P, t'_P\right) + i\varepsilon_\varphi L_{\varphi,11}\left(t_P, t'_P\right) + \int\limits_{P''} dt''_P K_{\varphi,10}\left(t_P, t''_P\right) L_{\varphi,01}\left(t''_P, t'_P\right) \\
&\quad + \int\limits_{P''} dt''_P K_{\varphi,11}\left(t_P, t''_P\right) L_{\varphi,11}\left(t''_P, t'_P\right) = -i\delta\left(t_p - t'_P\right) \\
&\partial_{t_P} L_{\varphi,01}\left(t_P, t'_P\right) - i\varepsilon_\varphi L_{\varphi,01}\left(t_P, t'_P\right) + \int\limits_{P''} dt''_P K_{\varphi,00}\left(t_P, t''_P\right) L_{\varphi,01}\left(t''_P, t'_P\right) \\
&\quad + \int\limits_{P''} dt''_P K_{\varphi,01}\left(t_P, t''_P\right) L_{\varphi,11}\left(t''_P, t'_P\right) = 0.
\end{aligned} \tag{A.6}$$

Restoring the explicit form of the Keldysh variables each of the groups A.5 andA.6 splits into two distinct groups of four coupled equations. As the calculations for each one of the 16 groups have the same structure it is enough to present here only those groups that yield the result for $\mathbf{L}_\varphi\left(t_+, t'_+\right)$.

In the following, and for reasons of simplicity, we have set $b = 0$ and $\delta E = 0$ in expressions 41 and we have used the abbreviations:

$$A_\varphi = \tilde{g}_\varphi \frac{\Gamma}{2}\cos\left(2\theta_\varphi\right), \quad B_\varphi = \tilde{g}_\varphi \frac{\Gamma}{2}\sin\left(2\theta_\varphi\right). \tag{A.7}$$

After these, the first tetrad of equations results from Eq. A.5 and reads as follows:

$$\begin{aligned}
&\left(\partial_t - i\varepsilon_\varphi + A_\varphi\right) L_{\varphi,00}\left(t_+, t_+\right) + \Gamma\tilde{g}_\varphi \sin^2\theta_\varphi L_{\varphi,00}\left(t_-, t_+\right) \\
&\qquad - iB_\varphi\left(L_{\varphi,10}\left(t_+, t_+\right) - L_{\varphi,10}\left(t_-, t_+\right)\right) = -i\delta\left(t - t'\right), \\
&\Gamma\tilde{g}_\varphi \cos^2\theta_\varphi L_{\varphi,00}\left(t_+, t_+\right) + \left(\partial_t - i\varepsilon_\varphi - A_\varphi\right) L_{\varphi,00}\left(t_-, t_+\right) \\
&\qquad - iB_\varphi\left(L_{\varphi,10}\left(t_+, t_+\right) - L_{\varphi,10}\left(t_-, t_+\right)\right) = 0, \\
&iB_\varphi\left(L_{\varphi,00}\left(t_+, t_+\right) - L_{\varphi,00}\left(t_-, t_+\right)\right) + \left(\partial_t + i\varepsilon_\varphi - A_\varphi\right) L_{\varphi,10}\left(t_+, t_+\right) \\
&\qquad + \Gamma\tilde{g}_\varphi \cos^2\theta_\varphi L_{\varphi,10}\left(t_-, t_+\right) = 0, \\
&iB_\varphi\left(L_{\varphi,00}\left(t_+, t_+\right) - L_{\varphi,00}\left(t_-, t_+\right)\right) + \Gamma\tilde{g}_\varphi \sin^2\theta_\varphi L_{\varphi,10}\left(t_+, t_+\right) \\
&\qquad + \left(\partial_t + i\varepsilon_\varphi + A_\varphi\right) L_{\varphi,10}\left(t_-, t_+\right) = 0.
\end{aligned} \tag{A.8}$$

The set of Eqs. A.8 can be exactly solved. Without any boundary restriction it is a straightforward exercise to find:

$$L_{\varphi,00}\left(t_+,t'_+\right) = -i\cos^2\theta_\varphi e^{i\varepsilon_\varphi(t-t')-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma(t-t')}\theta\left(t-t'\right) + i\sin^2\theta_\varphi e^{i\varepsilon_\varphi(t-t')-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma(t'-t)}\theta\left(t'-t\right)$$
$$-\frac{i}{2}\frac{\left(\tilde{g}_\varphi\frac{\Gamma}{2}\right)^2}{\varepsilon_\varphi^2+\left(\tilde{g}_\varphi\frac{\Gamma}{2}\right)^2}\sin^2\left(2\theta_\varphi\right)\cos^2\theta_\varphi e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma\left|t-t'\right|}\left(\cos\left(\varepsilon_\varphi\left|t-t'\right|\right)+\tilde{g}_\varphi\frac{\Gamma}{2}\frac{\sin\left(\varepsilon_\varphi\left|t-t'\right|\right)}{\varepsilon_\varphi}\right) \tag{A.9}$$

and

$$L_{\varphi,10}\left(t_+,t'_+\right) = -\frac{1}{2}e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma(t-t')}\tilde{g}_\varphi\frac{\Gamma}{2}\frac{\sin\varepsilon_\varphi\left(t-t'\right)}{\varepsilon_\varphi}\theta\left(t-t'\right). \tag{A.10}$$

The second group we are interested for is connected with Eq. A.6 and assumes the form:

$$\begin{aligned}
&\left(\partial_t+i\varepsilon_\varphi-A_\varphi\right)L_{\varphi,11}\left(t_+,t'_+\right)+\Gamma\tilde{g}_\varphi\cos^2\theta_\varphi L_{\varphi,11}\left(t_-,t'_+\right)\\
&\qquad+iB_\varphi\left(L_{\varphi,01}\left(t_+,t'_+\right)-L_{\varphi,01}\left(t_-,t'_+\right)\right)=-i\delta\left(t-t'\right),\\
&\Gamma\tilde{g}_\varphi\sin^2\theta_\varphi L_{\varphi,11}\left(t_+,t'_+\right)+\left(\partial_t+i\varepsilon_\varphi+A_\varphi\right)L_{\varphi,11}\left(t_-,t'_+\right)\\
&\qquad+iB_\varphi\left(L_{\varphi,01}\left(t_+,t'_+\right)-L_{\varphi,01}\left(t_-,t'_+\right)\right)=0,\\
&-iB_\varphi\left(L_{\varphi,11}\left(t_+,t'_+\right)-L_{\varphi,11}\left(t_-,t'_+\right)\right)+\left(\partial_t-i\varepsilon_\varphi+A_\varphi\right)L_{\varphi,10}\left(t_+,t'_+\right)\\
&\qquad+\Gamma\tilde{g}_\varphi\sin^2\theta_\varphi L_{\varphi,01}\left(t_-,t'_+\right)=0,\\
&-iB_\varphi\left(L_{\varphi,11}\left(t_+,t'_+\right)-L_{\varphi,11}\left(t_-,t'_+\right)\right)+\Gamma\tilde{g}_\varphi\cos^2\theta_\varphi L_{\varphi,01}\left(t_+,t'_+\right)\\
&\qquad+\left(\partial_t-i\varepsilon_\varphi-A_\varphi\right)L_{\varphi,01}\left(t_-,t_+\right)=0.
\end{aligned} \tag{A.11}$$

The solution of this system can be deduced from the solution of the system A.8 by observing that the coefficients in A.11 can be produced from the coefficients in (A.10) by making the replacement $\theta_\varphi \to \theta_\varphi \pm \pi/2$. This makes easy to find:

$$L_{\varphi,11}\left(t_+,t'_+\right) = -i\sin^2\theta_\varphi e^{i\varepsilon_\varphi(t-t')-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma(t-t')}\theta\left(t-t'\right) + i\cos^2\theta_\varphi e^{i\varepsilon_\varphi(t-t')-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma(t'-t)}\theta\left(t'-t\right)$$
$$-\frac{i}{2}\frac{\left(\tilde{g}_\varphi\frac{\Gamma}{2}\right)^2}{\varepsilon_\varphi^2+\left(\tilde{g}_\varphi\frac{\Gamma}{2}\right)^2}\sin^2\left(2\theta_\varphi\right)\sin^2\theta_\varphi e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma\left|t-t'\right|}\left(\cos\left(\varepsilon_\varphi\left|t-t'\right|\right)+\tilde{g}_\varphi\frac{\Gamma}{2}\frac{\sin\left(\varepsilon_\varphi\left|t-t'\right|\right)}{\varepsilon_\varphi}\right) \tag{A.12}$$

and

$$L_{\varphi,01}\left(t_+,t'_+\right) = L_{\varphi,10}\left(t_+,t'_+\right). \tag{A.13}$$

As discussed in the main text (see Eqs. 64 and 65) the Green's matrix **L** propagates a two-component fermion field:

$$\psi^{cl.}_{\varphi,u}\left(t_P\right) = \sum_\nu \int_{P'} dt'_{P'} L_{\varphi,u\nu}\left(t_P,t'_{P'}\right) j_{\varphi,\nu}\left(t'_{P'}\right). \tag{A.14}$$

One of the components is propagating forward in time, while the second one, backwards. Based on this observation, we construct, the appropriate for the present problem, Green's functions by imposing the following boundary conditions:

$$\begin{aligned} L_{\varphi,00}\left(t_+ = t_{in.+}, t'_+\right) = 0, \quad L_{\varphi,01}\left(t_+ = t_{in.+}, t'_+\right) = 0 \\ L_{\varphi,10}\left(t_+, t'_+ = t_{in.+}\right) = 0, \quad L_{\varphi,11}\left(t_+, t'_+ = t_{in.+}\right) = 0. \end{aligned} \tag{A.15}$$

To obtain these functions we can add at the Green's functions we found, the general solution of the corresponding homogenous equation

$$L_{\varphi,u\nu} \to L_{\varphi,u\nu} + \sum_{j=1}^{4} a^{(j)}_{\varphi,u\nu} e^{ip_j t}, \quad p_j = \pm\varepsilon_\varphi \pm i\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma \tag{A.16}$$

and determine the coefficients to satisfy A.15 together with $L_{\varphi,u\nu} \underset{|t-t'|\to+\infty}{\to} 0$. In this way we find:

$$\begin{aligned} L_{\varphi,00}\left(t_+, t'_+\right) = -ie^{i\varepsilon_\varphi(t-t')-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma|t-t'|}\theta\left(t-t'\right) \\ + iF_{\varphi,00}\left(t-t'\right)\left[e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma|t-t'|} - e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma(t+t'-2t_{in.})}\right] \end{aligned} \tag{A.17}$$

and

$$\begin{aligned} L_{\varphi,11}\left(t_+, t'_+\right) = ie^{i\varepsilon_\varphi(t-t')-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma|t-t'|}\theta\left(t'-t\right) \\ - iF_{\varphi,11}\left(t-t'\right)\left[e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma|t-t'|} - e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma(t+t'-2t_{in.})}\right] \end{aligned} \tag{A.18}$$

The abbreviations in these expressions read as follows:

$$\begin{aligned} F_{\varphi,00} = \sin^2\theta_\varphi e^{i\varepsilon_\varphi(t-t')} - \frac{1}{2}\frac{\left(\tilde{g}_\varphi\frac{\Gamma}{2}\right)^2}{\varepsilon_\varphi^2 + \left(\tilde{g}_\varphi\frac{\Gamma}{2}\right)^2}\sin^2\left(2\theta_\varphi\right)\cos^2\theta_\varphi \\ \times\left(\cos\left(\varepsilon_\varphi\left|t-t'\right|\right) + \tilde{g}_\varphi\frac{\Gamma}{2}\frac{\sin\left(\varepsilon_\varphi\left|t-t'\right|\right)}{\varepsilon_\varphi}\right) \end{aligned} \tag{A.19}$$

and

$$\begin{aligned} F_{\varphi,11} = \sin^2\theta_\varphi e^{i\varepsilon_\varphi(t-t')} + \frac{1}{2}\frac{\left(\tilde{g}_\varphi\frac{\Gamma}{2}\right)^2}{\varepsilon_\varphi^2 + \left(\tilde{g}_\varphi\frac{\Gamma}{2}\right)^2}\sin^2\left(2\theta_\varphi\right)\sin^2\theta_\varphi \\ \times\left(\cos\left(\varepsilon_\varphi\left|t-t'\right|\right) + \tilde{g}_\varphi\frac{\Gamma}{2}\frac{\sin\left(\varepsilon_\varphi\left|t-t'\right|\right)}{\varepsilon_\varphi}\right). \end{aligned} \tag{A.20}$$

In the same way we find

$$L_{\varphi,01}\left(t_+, t'_+\right) = -\frac{1}{2}e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma(t-t')}\tilde{g}_\varphi\frac{\Gamma}{2}\frac{\sin\varepsilon_\varphi\left(t-t'\right)}{\varepsilon_\varphi}\theta\left(t-t'\right) \tag{A.21}$$

and

$$\begin{aligned} L_{\varphi,10}\left(t_+, t'_+\right) = \frac{1}{2}e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma|t-t'|}\tilde{g}_\varphi\frac{\Gamma}{2}\frac{\sin\varepsilon_\varphi\left|t-t'\right|}{\varepsilon_\varphi}\theta\left(t'-t\right) \\ - e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma|t-t'|}\tilde{g}_\varphi\frac{\Gamma}{2}\frac{\sin\varepsilon_\varphi\left|t-t'\right|}{\varepsilon_\varphi}\left[e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma|t-t'|} - e^{-\frac{1}{2}\left|\tilde{g}_\varphi\right|\Gamma(t+t'-2t_{in.})}\right]. \end{aligned} \tag{A.22}$$

Combining these results, we find that for $t_+ = t'_+ = t$:

$$\begin{aligned}\mathbf{L}_\varphi = -i\left[\frac{1}{2} - \sin^2\theta_\varphi\left(1 - e^{-\left|\tilde{g}_\varphi\right|\Gamma(t-t_{in.})}\right)\right]\sigma^z \\ - \frac{i}{2}\frac{\left(\tilde{g}_\varphi\frac{\Gamma}{2}\right)^2}{\varepsilon_\varphi^2 + \left(\tilde{g}_\varphi\frac{\Gamma}{2}\right)^2}\sin^2\left(2\theta_\varphi\right)\left(1 - e^{-\left|\tilde{g}_\varphi\right|\Gamma(t-t_{in.})}\right)\begin{pmatrix}\cos^2\theta_\varphi & 0 \\ 0 & \sin^2\theta_\varphi\end{pmatrix}\end{aligned} \tag{A.23}$$

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