Quantifying separability in limit groups via representations

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Abstract

We show that for any finitely generated subgroup H of a limit group L there exists a finite-index subgroup K containing H, such that K is a subgroup of a group obtained from H by a series of extensions of centralizers and free products with $\mathbb{Z}$. If H is non-abelian, the K is fully residually H. We also show that for any finitely generated subgroup of a limit group, there is a finite-dimensional representation of the limit group which separates the subgroup in the induced Zariski topology. As a corollary, we establish a polynomial upper bound on the size of the quotients used to separate a finitely generated subgroup in a limit group. This generalizes the results in [10]. Another corollary is that a hyperbolic limit group satisfies the Geometric Hanna Neumann conjecture.

1 Introduction

A group is said to *retract* onto a subgroup if the inclusion map of the subgroup into the group admits a left-inverse. In which case, the left-inverse is called a *retraction* and the subgroup a *retract*. In [22], Wilton proves that if H is a finitely generated subgroup of a limit group L, $g \in L - H$, then H is a retract of some finite-index subgroup $K \leq L$ which contains H but not g [22]. We will refer to the smallest set of groups containing all finitely generated free groups that is closed under extensions of centralizers as ICE. By [6], limit groups are precisely the finitely generated subgroups of groups from ICE. We will modify the construction, from [22], of a finite-index subgroup $K \leq G$, where G is an ICE group, in such a way that not only is there a retraction $K \to H$, but, for a non-abelian H, a discriminating family of retractions (for each finite set S of non-trivial elements in K, there is a retraction from K onto H that is injective on S). In other words, K is fully residually H. This finite-index subgroup K will be a group obtained from H by a finite chain of groups $H = K_0 < \ldots < K_n = K$, where K_{i+1} is either $K_i * F$, where F is some free group or K_{i+1} is an extension of a centralizer in K_i. We will call a group obtained by such a chain an $H-$GICE group. (If H is free, then the classes of H-GICE groups and ICE groups containing H, coincide.) It is well known that an extension of a centralizer of a limit group G is fully residually G. This was first proved in [11] (one can find a detailed proof, for example, in [16, Lemma 3.7].) It is also known that a free product of a non-abelian limit group G and a free group is fully residually G. Therefore, each group in the chain used to construct K is fully residually H.

Theorem 1. *Let G be an ICE-group, H be a finitely generated subgroup, and $g \in L - H$. Then, there exists a finite-index subgroup K of G such that $H \leq K$, K is an H-GICE group, and $g \notin K$.*

Corollary 2. *Let L be a limit group, H be a finitely generated subgroup, and $g \in L - H$. Then, there exists a finite-index subgroup K of L such that $H \leq K$, K is a subgroup of an H-GICE group, and $g \notin K$.*

This theorem implies the following result.

Theorem 3. *Let L be a limit group, H be a finitely generated non-abelian subgroup, and $g \in L - H$. Then, there exists a finite-index subgroup K of L such that $H \leq K$, K is fully residually H, and $g \notin K$.*

Theorem 3 is also true when L is abelian and, therefore, H is abelian (see Remarks at the end of the proof). In the case when H is abelian and L is non-abelian a finite-index subgroup of L cannot be fully residually H.

Theorem 4. *Let L be a limit group. If H is a finitely generated non-abelian subgroup of L, then there is a faithful representation $\rho_H : L \to GL(V)$ such that $\overline{\rho_H(H)} \cap \rho_H(L) = \rho_H(H)$, where $\overline{\rho_H(H)}$ is the Zariski closure of $\rho_H(H)$.*

Likewise, this theorem is true when L is abelian.

Corollary 5. *Let L be a limit group and S be a finite generating set for L. If $H \leq L$ is a finitely generated subgroup, then there exists a constant $N > 0$ such that for each $g \in L - H$, there exist a finite group Q and a homomorphism $\varphi : L \longrightarrow Q$ such that $\varphi(g) \notin \varphi(H)$ and $|Q| \leq ||g||_S^N$. If $K = H \ker \varphi$, then K is a finite-index subgroup of L whose index is at most $|Q| \leq ||g||_S^N$ with $H \leq K$ and $g \notin K$. Moreover, the index of the normal core of the subgroup K is bounded above by $|Q|$.*

To use Theorem 4 for the proof of this corollary in the case when L is non-abelian and H is abelian we can take instead of H a non-abelian subgroup $H_1 = H * \langle x \rangle$ for a suitable element x.

Our Theorem 4 and Corollary 5 generalize results for free and surface groups from [10]. We use [10] to deduce Corollary 5 from Theorem 4. Corollary 5 establishes polynomial bounds on the size of the normal core of the finite index subgroup used in separating g from H. The constant N explicitly depends on the subgroup H and the dimension of V in Theorem 4. For a general finite index subgroup, the upper bound for the index of the normal core is factorial in the index of the subgroup. It is for this reason that we include the statement about the normal core of K at the end of the corollary.

Recently, several effective separability results have been established; see [2]-[10], [12]-[14], [17]-[21]. Most relevant here are papers [10], [9]. The methods used in [9] give linear bounds in terms of the word length of $|g|$ on the index of the subgroup used in the separation but do not produce polynomial bounds for the normal core of that finite index subgroup. We can also obtain bounds on the index of the separating subgroup on the order of magnitude $C|g|$, where C is a constant depending on L and H.

In Section 6 we will formulate the Geometric Hanna Neumann conjecture by Antolin and Jaikin-Zapirain for limit groups and give a proof (due to Jaikin-Zapirain) that Theorem 1 implies the conjecture for hyperbolic limit groups (Theorem 29).

2 Preliminaries

Definition 6. A family $\mathcal{F}$ of H-homomorphisms (identical on H) from a group G onto a subgroup H is called a *discriminating family* if for any finite set S of non-trivial elements in G there exists a homomorphism $\psi \in \mathcal{F}$ such that for any $g \in S$, $\psi(g) \neq 1$. We say G is *fully residually* H if there exists a discriminating family of H-homomorphisms from G to H.

Definition 7. Let G be a group and $C_G(u)$ denote the centralizer of an element $u \in G$. An *extension of a centralizer* of G is the group

$$(G, u) = \langle G, t_1, \ldots, t_k \mid [c, t_i], c \in C_G(u), [t_i, t_j], i, j = 1, \ldots, k\rangle.$$

Similarly, if we extend centralizers of several non-conjugated elements $u_1, \ldots, u_m$ in G we denote the obtained group by $(G, u_1, \ldots, u_m)$.

An *iterated extension of centralizers* is obtained by finitely many applications of this construction to a finitely generated free group and is called an ICE-group. In this case we can assume that each centralizer is extended only once. In other words, on each step $C_G(u)$ is cyclic.

Let

$$F = G_0 < G_1 = (G, u_1) < \ldots < G_n = (G_{n-1}, u_n) = G \tag{1}$$

be a chain of centralizer extensions to obtain an ICE-group G. Then we always assume that in this chain centralizers in G_i are extended before centralizers in G_{i+1}. We can modify this chain the following way

$$F = G_0 < G_{i_1} < \ldots < G_{i_k} = G, \tag{2}$$

where $G_{i_1} = (G_0, u_1, \ldots, u_{i_1})$, where $u_1, \ldots, u_{i_1}$ are in G_0 is obtained from G_0 by extending all the centralizers of elements from G_0 that appear in the first chain. Similarly $G_{i_{j+1}}$ is obtained from G_{i_j} by extending all the centralizers of elements in G_{i_j} that were extended in the first chain.

Definition 8. Let G be an ICE group. Then associated with G is a finite $K(G, 1)$ space, called an *ICE space*, which is constructed as follows:

1. If G is free, then take $X = K(G,1)$ to be a compact graph of suitable rank.

2. If G is obtained from a group G' by an extension of a centralizer and $Y = K(G',1)$, then given an essential closed curve $\partial_+ : S^1 \longrightarrow Y$ representing a generator of $C_{G'}(g')$ and a coordinate circle $\partial_- : S^1 \longrightarrow T$, where T is a torus, take
$$X = Y \sqcup \left([-1,1] \times S^1\right) \sqcup T,$$
identifying $(\pm 1, \theta) \in [-1,1] \times S^1$ with $\partial_\pm(\theta)$.

Remark 9. Associated to each ICE space X is a graph of spaces decomposition whose vertices are Y and T and edges are circles.

Definition 10. A group G is an *H-GICE group* if it is obtained from H by a series of free products with free groups and extensions of centralizers. Here, GICE stands for generalized iterated centralizer extension.

If H is a non-abelian limit group, then any H-GICE group and its subgroups containing H are fully residually H, see, for example [16]. Therefore, Theorem 1 implies Theorem 3.

Definition 11. Let each of the following spaces have a chosen basepoint, and suppose that the maps are basepoint preserving. Let $\rho : (B', b') \to (B, b)$ be a covering map. Let $\delta : (A, a) \to (B, b)$ be a map, where A is a connected complex (in our case A will be a loop). Let $\kappa : (A', a') \to (A, a)$ be the smallest cover of (A, a) such that the map $\delta \circ \kappa$ has a lift δ'. We call $\delta' : (A', a') \to (B', b')$ the *elevation* of δ.

Two elevations ${\delta_1}' : {A_1}' \to B'$ and ${\delta_2}' : {A_2}' \to B'$ are *isomorphic* if there exists a homeomorphism $\iota : {A_1}' \to {A_2}'$ covering the identity map on A, such that ${\delta_1}' = {\delta_2}' \circ \iota$.

For more information on elevations we refer to [22, Section 2].

Definition 12. Let X and X' be graphs of spaces (X' is not assumed to be connected). A *pre-covering* is a locally injective map $X' \longrightarrow X$ that maps vertex spaces and edge spaces of X' to vertex spaces and edge spaces of X respectively and restricts to a covering on each vertex space and each edge space. Furthermore, for each edge space e' of X' mapping to an edge space e of X, the diagram of edge maps

$$\begin{array}{ccc} e' & \xrightarrow{\bar{\partial}_\pm} & \bar{V}_\pm \\ \downarrow & & \downarrow \\ e & \xrightarrow{\partial_\pm} & V_\pm \end{array}$$

is required to commute. The domain X' is called a *pre-cover*.

Definition 13. ([22, Definition 3.1]) Let X be a complex, $X' \longrightarrow X$ be a covering, and

$$\mathcal{L} = \{\delta_i : C_i \longrightarrow X\}$$

be a finite collection of independent, essential loops. The cover X' is said to be *tame over* $\mathcal{L}$ if the following holds: let $\Delta \subset X'$ be a finite subcomplex and

$$\mathcal{L}' = \{\delta'_j : C'_j \longrightarrow X'\}$$

be a finite collection of pairwise non-isomorphic infinite degree elevations, each of which is an elevation of some loop in $\mathcal{L}$. Then for all sufficiently large positive integers d there exists an intermediate finite-sheeted covering

$$X' \longrightarrow \hat{X} \longrightarrow X$$

such that

1. each δ'_j descends to some degree d elevation $\hat{\delta}_j$
2. the $\hat{\delta}_j$ are pairwise non-isomorphic,
3. Δ embeds into $\hat{X}$, and
4. there exists a retraction $\rho : \pi_1(\hat{X}) \longrightarrow \pi_1(X')$ such that

$$\rho(\hat{\delta}_{j*}(\pi_1(\hat{C}_j))) \subset \delta'_{j*}(\pi_1(C'_j))$$

for each j.

Remark. We will also say a covering $X' \longrightarrow X$ is tame over a given set of finite independent, essential loops whenever its domain X' is.

Notice, that covers of tori are tame over coordinate circles, see [22, Lemma 3.3].

Definition 14. The cover X' is *strongly tame over* $\mathcal{L}$ if it is tame over $\mathcal{L}$ and $\pi_1(\hat{X})$ is a $(\pi_1(X') * F)$-GICE group, where F is a free group with basis $\{\hat{\delta}_{j*}(\pi_1(\hat{C}_j))\}$.

Definition 15. A group G is said to admit a *local GICE structure* if for each finitely generated subgroup $H \leq G$ and a finite set of elements $g_i \notin H$ one can construct a finite-index subgroup K containing H and not containing these elements such that K is a H-GICE group.

3 Proof of Theorems 1, 3

We will follow the construction in [22] changing it a couple of times to prove a theorem similar to [22, Theorem 3.8]. One difference is that we will use induction on the number of steps in chain (2) while [22, Theorem 3.8] is proved by induction on the number of steps in chain (1).

Let X be an ICE space constructed by gluing several tori $T_1, \ldots, T_k$ to a simpler ICE space Y with edge spaces being loops. Let $H \subset \pi_1(X)$ be a finitely generated subgroup and $X^H \to X$ be the corresponding covering. Then X^H inherits a graph of spaces decomposition, with vertex spaces the connected components of the pre-images of the vertex spaces of X and edge spaces and maps given by all the (isomorphism classes of) elevations of the edge maps to the vertex spaces of X^H. Let $X' \subseteq X^H$ be a core of X^H. A **core** is a connected subgraph of spaces with finite underlying graph such that the inclusion map is a π_1-isomorphism. Since H is finitely generated, a core exists. Let $\Delta \subset X^H$ be a finite subcomplex. Enlarging X' if necessary we can assume $\Delta \subset X'$.

Replacing the tameness hypothesis in [22, Proposition 3.4] by strong tameness, we have the following.

Proposition 16. *(Passing to finite-sheeted pre-covers) Let X be an ICE space constructed by gluing several tori $T_1, \ldots, T_k$ to a simpler ICE space Y with edge spaces being loops, Let $X' \to X$ be a pre-covering with finite underlying graph. Every vertex space V' of X' covers some vertex space V of X. Assume that each Y' is strongly tame over the set of edge maps incident at Y. Let $\Delta \subset X'$ be a finite subcomplex. Then there is a finite-sheeted intermediate pre-covering*

$$X' \to \bar{X} \to X$$

such that

1. *Δ embeds into $\bar{X}$; and*

2. *$\pi_1(\bar{X})$ is a $\pi_1(X')$-GICE group.*

Proof. Let Δ_0 be a finite complex that contains Δ and all the compact edge spaces of X'. Let V' be a vertex space of X' covering the vertex V of X. Set $\Delta_{V'} = V' \cap \Delta_0$ and consider the edge maps $\partial'_i : e_i \to V'$ of edges e_i incident at V' that are infinite-degree elevations of $\partial_\pm : e \to V$. Since each V' is tame over the set of edge maps incident at V and each Y' is strongly tame over the set of edge maps incident at Y, for all sufficiently large d there exists a an intermediate finite-sheeted covering

$$V' \to \bar{V} \to V$$

such that

1. $\Delta_{V'}$ embeds into $\bar{V}$,

2. each ∂'_i descends to some degree d elevation $\bar{\partial}_i$ of $\partial_\pm$.

If d is large enough, we can take it to be the same d over all vertex spaces of X'. Let $\bar{X}$ be the graph of spaces with the same underlying graph as X', but with the corresponding $\bar{V}$ in place of V'. If e' is an edge space of X' then the edge map

$$\partial_\pm : e' \to V'$$

descends to a finite-degree map $\partial_\pm : \bar{e}_\pm \to \bar{V}$. Because $\bar{e}_+ \to e$ and $\bar{e}_- \to e$ are coverings of e with the same degree, we have a finite-sheeted pre-cover $\bar{X}$. By construction, Δ embeds into $\bar{X}$. Since the compact edge spaces are added to Δ, non-isomorphic finite degree elevations are mapped into non-isomorphic elevations. This implies that $\pi_1(\bar{X})$ decomposes as a graph of groups, with the same underlying graph as the decomposition of $\pi_1(X')$.

Consider a non-abelian vertex group $\pi_1(\bar{V})$ of $\pi_1(\bar{X})$ (this means $V = Y$). To obtain $\pi_1(\bar{V})$ we first take $\pi_1(V'') = \pi_1(V') * F$, the free product with cyclic groups corresponding to elevations of degree d obtained from infinite degree elevations of edge maps, and then by a series of extensions of centralizers and free products with free groups. A cyclic fundamental group of an elevation of degree d obtained from an infinite degree elevation of an edge map extends the abelian fundamental group of an infinite cover of some torus T_i. On the group level this corresponds to the extension of the centralizer of an abelian free factor of $\pi_1(X')$ (and, therefore, extension of a centralizer of $\pi_1(X')$ itself, because the extending element is in the free factor F of $\pi_1(V') * F$. So, to obtain $\pi_1(\bar{X})$ we first extend centralizers of $\pi_1(X')$ corresponding to abelian free factors. We also extend centralizers of all $\pi_1(T')$, where T' covers some T_i so that all $\bar{T}$'s become finite covers. Denote by X'' the pre-cover that is obtained from X' by replacing the covers of tori by finite covers as above and replacing V' by V'' for each V that is not a torus. Second, we notice that the free constructions that were applied to each $\pi_1(V') * F$ to obtain $\pi_1(\bar{V})$, for V' covering the vertex $V = Y$, can be thought as applied to the whole group $\pi_1(X'')$. Replacing each V'' by $\bar{V}$ for covers V' of the non-abelian vertex group $V = Y$ we obtain $\bar{X}$. $\square$

Lemma 17. *([22, Lemma 3.5] Let T be a torus and $\delta : S^1 \to T$ be an essential loop. Then for every positive integer d there exists a finite-sheeted covering $\hat{T}_d \to T$ so that δ has a single elevation $\hat{\delta}$ to $\hat{T}_d$ and $\hat{\delta}$ is of degree d.*

Lemma 18. *(cf [22, Lemma 3.6]) Let Y be a space such that $\pi_1(Y)$ has local GICE structure and $\delta : S^1 \to Y$ be a based essential loop. Then for every positive integer d there exists a finite-sheeted covering $\hat{Y}_d \to Y$ so that δ has an elevation $\hat{\delta}$ of degree d to $\hat{Y}_d$ and $\pi_1(\hat{Y}_d)$ is an $\langle\hat{\delta}\rangle$-GICE group.*

Proof. Because $\pi_1(Y)$ has local GICE structure, for every positive integer d there exists a finite-sheeted covering $\hat{Y}_d \to Y$ so that $\hat{Y}_d \to Y$ is a $\langle\delta^d\rangle$-GICE group. Note that $\delta^k \notin \pi_1(\hat{Y}_d)$ for $0 < k < d$, therefore $\hat{\delta}$ is an elevation of degree d. $\square$

Proposition 19. *(cf [22, Proposition 3.7])(Completing a finite-sheeted pre-cover to a cover) Let X be an ICE space constructed by gluing together tori $T_1, \dots, T_k$ and a simpler ICE space Y, as above. Assume that $\pi_1(Y)$ admits a local GICE structure. Let $\bar{X} \to X$ be a finite-sheeted connected pre-covering.*

Then there exists an inclusion $\bar{X} \hookrightarrow \hat{X}$ extending $\bar{X} \to X$ to a covering $\hat{X} \to X$ such that $\pi_1(\hat{X})$ is a $\pi_1(\bar{X}) - GICE$ group.

Proof. Follows the proof of [22, Proposition 3.7]. The addition of copies of $T_{i,d}$ correspond to extensions of centralizers. The addition of Y_d's correspond by Lemma 18 to taking a free product with infinite cyclic group and then a GICE over the obtained group. Indeed, $\pi_1(Y)$ has a local GICE-structure, therefore Y_d is $C - GICE$ group, where C is a cyclic group generated by the boundary element. □

A collection of elements $g_1, \dots, g_n$ of a group G is called *independent* if whenever there exists $h \in G$ such that g_i^h and g_j commute, then, in fact, $i = j$.

Proposition 20. *(cf [22, Proposition 3.8]) Let X be an ICE space constructed by gluing together tori $T_1, \dots, T_k$ and a simpler ICE space Y. Let $H \leq \pi_1(X)$ be a finitely generated subgroup and let $X^H \to X$ be the corresponding covering. Suppose $\mathcal{L}$ is a (possible empty) set of hyperbolic loops that generate maximal cyclic subgroups of $\pi_1(X)$. Then X^H is strongly tame over $\mathcal{L}$.*

Proof. The proof is an induction on the length of the chain (2). Notice, that the induction basis holds by [22, Corollary 1.8]. Indeed, if H is a finitely generated subgroup of $\pi_1(X)$, where X is a graph, then the cover X^H is strongly tame over the set of independent elements $\{\gamma_i\}$ such that each generate a maximal cyclic subgroup, because it is tame and for a finite-sheeted intermediate covering

$$X^H \to \hat{X} \to X$$

$\pi_1(\hat{X}) = H * F$, where F is a free group.

Fix a finitely generated non-abelian subgroup $H \subset \pi_1(X)$, and let $X^H \to X$ be the corresponding covering. There exists a core $X' \subseteq X^H$. Let $\Delta \subset X^H$ be a finite subcomplex. Enlarging X' if necessary we can assume $\Delta \subset X'$, infinite degree elevations of hyperbolic loops $\{\delta_i\}$ are first restricted to elevations $\{\delta_i{}'\}$ and then made disparate. This is possible by [22, Lemma 2.24] without changing the fundamental group.

As in the proof of [22, Theorem 3.8], X' is extended to a pre-cover $\bar{X}$ where elevations $\{\delta_i{}'\}$ are extended to full elevations $\bar{\delta}_j : \bar{D}_j \to \bar{X}$ of degree d by [22, Lemma 2.23]. By [22, Lemma 2.23], $\pi_1(\bar{X}) = \pi_1(X') * F$, where F is a free group generated by $\pi_1(\bar{\delta}_{i*}(\bar{D}_j))$'s. Enlarging Δ again we assume that the images of the $\bar{\delta}_j$ are contained in Δ.
By Proposition 16 there exists an intermediate finite-sheeted pre-covering

$$\bar{X} \to \hat{X} \to X,$$

into which Δ injects. Since Δ injects into $\hat{X}$ we have that $\bar{\delta}_j$ descends to an elevation $\hat{\delta}_j = \bar{\delta}_j$.

Finally, $\hat{X}$ can be extended to a finite sheeted covering $\hat{X}^+$ by Proposition 19.

We have that Δ injects into $\hat{X}^+$. By Proposition 19, $\pi_1(\hat{X}^+)$ is a $\pi_1(\hat{X})$-GICE group. By Proposition 16, $\pi_1(\hat{X})$ is a $\pi_1(\bar{X})$-GICE group. And $\pi_1(\bar{X}) = \pi_1(X') * F$. Therefore, by transitivity, $\pi_1(\hat{X}^+)$ is a $\pi_1(X') * F$-GICE group. Since $H = \pi_1(X')$, the proposition is proved. $\square$

Theorem 1 follows from the proposition (with the empty set $\mathcal{L}$). Since every limit group is a subgroup of an ICE-group by [15], Corollary 2 follows from Theorem 1. If H is non-abelian, then H-GICE groups are fully residually H and subgroups of fully residually H groups that contain H are also fully residually H. Therefore Theorem 3 follows from Theorem 1.

Example 1. Let us illustrate the proof of Theorem 3 with an example when L is just an extension of a centralizer of a free group. Consider the group

$$L = F(a,b) *_{\langle a \rangle} \langle a, t, | [a,t] = 1 \rangle,$$

where $F(a,b)$ is a free group, a subgroup

$$H = \langle a^2, b^2 \rangle *_{\langle a^2 \rangle} \langle a^2 \rangle *_{\langle a^2 \rangle} \langle a^{2t}, b^{2t} \rangle$$

and $g = \Delta = b \notin H$. Let us construct a finite-index subgroup K such that $H \leq K$, $b \notin K$ and K is an H-GICE group.

In Fig. 1 we show the space X such that $L = \pi_1(X)$. Here X is a graph of spaces with one edge and two vertices. The loops labelled by a and t are generating loops of the torus T with a fundamental group $\langle a, t, | [a,t] = 1 \rangle$ and the bouquet of loops labelled by a and b has a fundamental group $F(a,b)$. A pre-cover X' corresponding to H is a pre-cover with the finite graph. It is a graph of spaces with two edges and three vertices, $H = \pi_1(X')$. The space corresponding to the vertex in the middle is the cylinder that is an infinite cover of the torus T. The other two vertex spaces are infinite covers of the bouquet of loops.

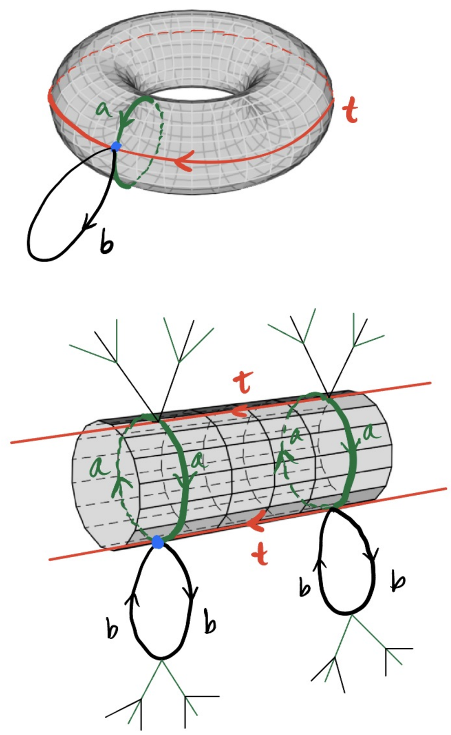

Figure 1: ICE space X and a pre-cover X' with a finite graph

In Fig. 2 we make X' into a finite-sheeted pre-cover $\bar{X}$ as it is done in Proposition 16. The space $\bar{X}$ has the same underlying graph as X', but the vertex spaces are now finite covers of the vertex spaces of X. The torus with the fundamental group generated by a^2, t^2 is a cover of T of degree 4. Two other vertex spaces are graphs that are covers of degree 3 of the bouquet of loops in X. We have

$$\pi_1(\bar{X}) = \langle a^2, b^2, a^{-1}ba, b^{-1}ab \rangle *_{\langle a^2 \rangle} \langle a^2, t^2 \rangle *_{\langle a^2 \rangle} \langle a^{2t}, b^{2t}, t^{-1}a^{-1}bat, t^{-1}b^{-1}abt \rangle.$$

We have that $\pi_1(\bar{X})$ is obtained from H by taking a free product with $\langle a^{-1}ba, b^{-1}ab \rangle$ and $\langle t^{-1}a^{-1}bat, t^{-1}b^{-1}abt \rangle$ and then extending the centralizer of a^2 by t^2. There are two hanging elevations of the loop labelled by a in $\bar{X}$. They both have degree 1.

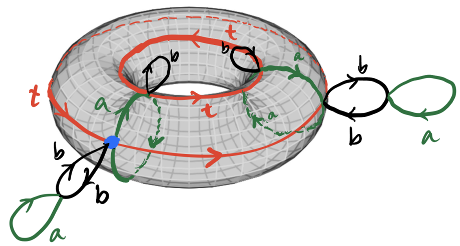

Figure 2: Finite-sheeted pre-cover $\bar{X}$

Figure 3 shows a finite cover $\hat{X}$ of X. It is obtaiinef from $\bar{X}$ by attaching two tori T_1 to the hanging elevations of the loop labelled by a (as in Proposition 19). Then $K = \pi_1(\hat{X})$ is obtained from $\pi_1(\bar{X})$ by extending centralizers of $b^{-1}ab$ (by $b^{-1}tb$) and of $t^{-1}b^{-1}abt$ (by $t^{-1}b^{-1}tbt$). Therefore K is an H-GICE group, $b \notin K$. Notice that $[L : K] = 6$.

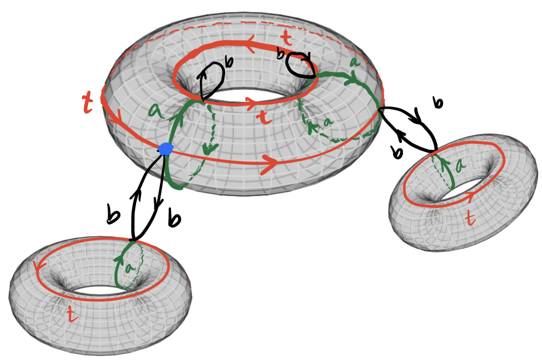

Figure 3: Finite cover $\hat{X}$

Example 2 (Figures 4-6) Now with the same L we take

$$H = \langle a^2, b^2 \rangle *_{\langle a^2 \rangle} \langle a^2 \rangle *_{\langle a^2 \rangle} \langle a^{2t}, b^{2t} \rangle * \langle t^{ba} \rangle$$

and $g = \Delta = b \notin H$. In this example we will have an edge in X' corresponding to an infinite degree elevation of the loop labelled by a. Then $\pi_1(\bar{X})$ is obtained from H by the following chain: $H < H_1$, where

$$H_1 = \langle H, a^{ba}, t^2 | [a^{ba}, t^{ba}] = 1, [a^2, t^2] = 1 \rangle,$$

$H_1 < \pi_1(\bar{X})$, where

$$\pi_1(\bar{X}) = H_1 * \langle a^t, a^{b^{-1}a}, b^{3a}, b^{at}, a^{bt} \rangle,$$

where the last group is freely generated by the five given elements. So, to obtain $pi_1(\bar{X})$ from H we used two centralizer extensions and then a free product with a free group. To obtain $K = \pi_1(\hat{X})$ from $pi_1(\bar{X})$ we make three centralizer extensions, as Figure 6 shows and obtain s finite cover of X of degree 8.

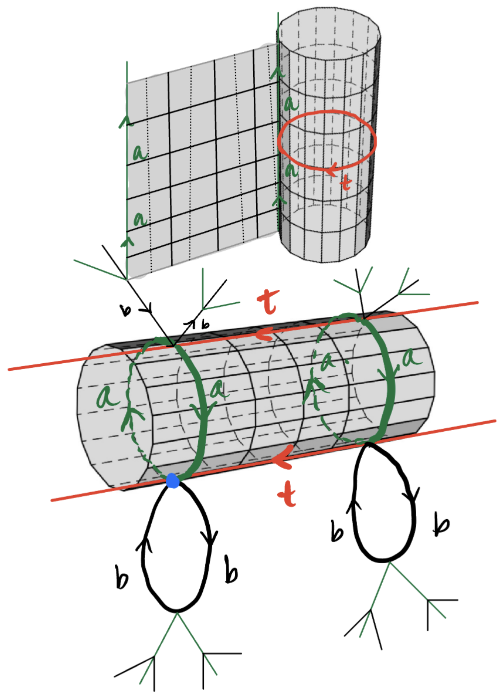

Figure 4: Pre-cover X' with finite graph

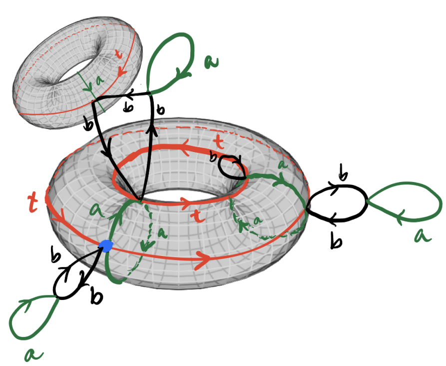

Figure 5: Finite sheeted pre-cover $\bar{X}$

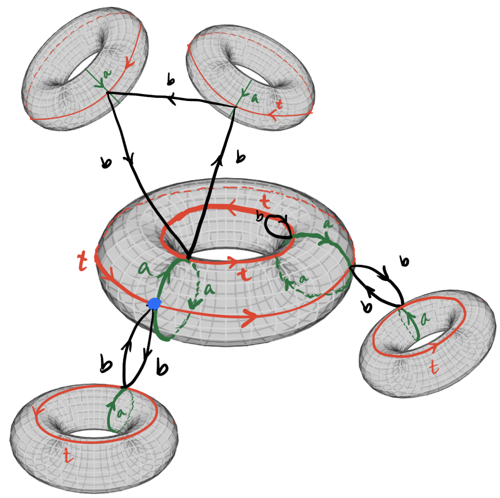

Figure 6: Finite cover $\hat{X}$

Remark 21. Theorem 3 is also true when L is abelian, therefore, free abelian.

Proof. We take a basis $a_1, \dots a_n$ of L such that H has a basis $a_1^{k_1}, \dots, a_r^{k_r}$. Let

$$g = a_1^{m_1} \dots a_n^{m_n}.$$

If m_i is not divisible by k_i for some $i = 1, \dots, r$, then we take K generated by $a_1^{k_1}, \dots, a_r^{k_r}, a_{r+1}, \dots, a_n$. If each m_i is divisible by k_i for $i = 1, \dots, r$, then some of $m_{r+1}, \dots, m_n$ is non-zero, because $g \notin H$. Suppose $m_n \neq 0$. Then take K generated by $a_1^{k_1}, \dots, a_r^{k_r}, a_{r+1}, \dots, a_n^{m_n+1}$.

□

Remark 22. In the case when H is abelian and L is non-abelian a finite-index subgroup of L cannot be fully residually H. In this case there exists $x \in L$ such that $g \notin H_1 = \langle H, x \rangle = H * \langle x \rangle$.

Proof. Take some $x \in L$ such that $[h, x] \neq 1$ for $h \in H$. Then for any $h \in H$, elements h, x generate a free subgroup. Therefore $H_1 = \langle H, x \rangle = H * \langle x \rangle$. If $g \notin H_1$, then we found x. If $g \in H_1$, then g can be uniquely written as

$$g = h_1 x^{k_1} \dots h_r x^{k_r},$$

where $h_1, \dots, h_r$ are elements in H, all, except maybe h_1 non-trivial. Let k be a positive number that is larger than all $|k_1|, \dots, |k_r|$. Then $g \notin H_2 = \langle H, x^k \rangle$ and we can take x^k instead of x. □

4 Proof of Theorem 4

Definition 23. [10] Let G be a finitely generated group and H a finitely generated subgroup of G. For a complex affine algebraic group $\mathbf{G}$ and any representation $\rho_0 \in Hom(G, \mathbf{G})$, we have the closed affine subvariety

$$R_{\rho_0, H}(G, \mathbf{G}) = \{\rho \in Hom(G, \mathbf{G}) : \rho_0(h) = \rho(h) \text{ for all } h \in H\}$$

The representation ρ_0 is said to *strongly distinguish* H in G if there exist representations $\rho, \rho' \in R_{\rho_0, H}(G, \mathbf{G})$ such that $\rho(g) \neq \rho'(g)$ for all $g \in G - H$.

If L is a closed surface group or a free group, then Theorem 4 follows from [10, Theorem1.1]. Suppose L is not a surface group and not an abelian group. Let $\mathbf{G}$ be a complex affine algebraic group. By the following lemma, it is sufficient to construct a faithful representation $\rho \in Hom(L, \mathbf{G})$ that strongly distinguishes H in L.

Lemma 24. *[10, Lemma 3.1] Let G be a finitely generated group, $\mathbf{G}$ a complex algebraic group, and H a finitely generated subgroup of G. If H is strong distinguished by a representation $\rho \in Hom(G, \mathbf{G})$, then there exists a representation $\varrho : G \longrightarrow \mathbf{G} \times \mathbf{G}$ such that $\varrho(G) \cap \overline{\varrho(H)} = \varrho(H)$, where $\overline{\varrho(H)}$ is the Zariski closure of $\varrho(H)$ in $\mathbf{G} \times \mathbf{G}$.*

Proposition 25. *Let L be a limit group and H a non-abelian finitely generated subgroup. There exist a finite-index subgroup $K \leq L$ and a faithful representation $\rho_\omega : K \to \mathbf{G}$ that strongly distinguishes H in K.*

Proof. By Theorem 3, there exists a finite-index subgroup K of L such that K is fully residually H. Let ρ be a faithful representation of H in $\mathbf{G}$. We order balls B_t of radius t in the Cayley graph of K and finite sets $S_t = B_t \cap (K - H)$. Since we have a discriminating family of H-homomorphisms from K to H, we can construct for any $t \in \mathbb{N}$ representations ρ_t and ρ'_t in $Hom(K, \mathbf{G})$ that coincide on H, distinguish all elements in S_t, and map B_t monomorphically. Selecting a non-principal ultrafilter $\omega \in \mathbf{N}$, we have two associated ultraproduct representations $\rho_\omega, {\rho'}_\omega : K \to \mathbf{G}$ (see [10, Proof of Lemma 3.2]). These representations are faithful because each B_t is mapped monomorphically on a co-finite set of $j \in \mathbb{N}$ and for any $g \in K - H$, $\rho_\omega(g) \neq {\rho'}_\omega(g)$. $\square$

Let us prove the first statement of Theorem 4. The proof of [10, Theorem 1.1] shows that it is sufficient to have a representation of K that strongly distinguishes H. Indeed, like in [10, Corollary 3.3], we can construct a representation $\Phi : K \to GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$ such that $\Phi(g) \in Diag(GL(2, \mathbb{C}))$ if and only if $g \in H$. Setting $d_H = [G : K]$, we have the induced representation

$$Ind_K{}^G(\Phi) : G \to GL(2d_H, \mathbb{C}) \times GL(2d_H, \mathbb{C}).$$

Recall, that when Φ is represented by the action on the vector space V and $G = \cup_{i=0}^t g_i K$, then the induced representation acts on the disjoint union $\sqcup_{i=0}^t g_i V$ as follows

$$g \Sigma g_i v_i = \Sigma g_{j(i)} \Phi(k_i) v_i,$$

where $g g_i = g_{j(i)} k_i$, for $k_i \in K$. Taking $\rho = Ind_K{}^G(\Phi)$, it follows from the construction of ρ and definition of induction that $\rho(g) \in \overline{(\rho(H))}$ if and only if $g \in H$. If we set $\rho = \rho_H$, then Theorem 4 is proved.

5 Proof of Corollary 5

Given a complex algebraic group $\mathbf{G} < GL(n, \mathbb{C})$, there exist polynomials $P_1, \ldots, P_r \in \mathbb{C}[X_{i,j}]$ such that

$$\mathbf{G} = \mathbf{G}(\mathbb{C}) = V(P_1, \ldots, P_r) = \left\{ X \in \mathbb{C}^{n^2} \mid P_k(X) = 0, k = 1, \ldots, r \right\}$$

We refer to the polynomials $P_1, \ldots, P_r$ as *defining polynomials* for $\mathbf{G}$. We will say that $\mathbf{G}$ is K–defined for a subfield $K \subset \mathbb{C}$ if there exists defining polynomials $P_1, \ldots, P_r \in K[X_{i,j}]$ for $\mathbf{G}$. For a complex affine algebraic subgroup $\mathbf{H} < \mathbf{G} < GL(n, \mathbb{C})$, we will pick the defining polynomials for $\mathbf{H}$ to contain a defining set for $\mathbf{G}$ as a subset. Specifically, we have polynomials $P_1, ..., P_{r_{\mathbf{G}}}, P_{r_{\mathbf{G}}+1}, ..., P_{r_{\mathbf{H}}}$ such that

$$\mathbf{G} = V(P_1, \ldots, P_{r_{\mathbf{G}}}) \text{ and } \mathbf{H} = V(P_1, \ldots, P_{r_{\mathbf{H}}}) \qquad (3)$$

If $\mathbf{G}$ is defined over a number field K with associated ring of integers $\mathcal{O}_K$, we can find polynomials $P_1, \dots, P_r \in \mathcal{O}_K[X_{i,j}]$ as a defining set by clearing denominators. For instance, in the case when $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$, these are multivariable integer polynomials.

For a fixed finite set $X = \{x_1, \dots, x_t\}$ with associated free group $F(X)$ and any group G, the set of homomorphisms from $F(X)$ to G, denoted by $Hom\,(F\,(X)\,, G)\,,$ can be identified with $G^t = G_1 \times \ldots \times G_t$. For any point $(g_1, \dots, g_t) \in G^t$, we have an associated homomorphism $\varphi_{(g_1,\dots,g_t)} : F\,(X) \longrightarrow G$ given by $\varphi_{(g_1,\dots,g_t)}\,(x_i) = g_i$. For any word $w \in F(X)$, we have a function $\mathrm{Eval}_w : Hom(F(X), G) \longrightarrow G$ defined by $\mathrm{Eval}_w(\varphi_{(g_1,\dots,g_t)})(w) = w(g_1, \dots, g_t)$. For a finitely presented group Γ, we fix a finite presentation $\langle \gamma_1, \dots, \gamma_t \mid r_1, \dots, r_{t'} \rangle$, where $X = \{\gamma_1, \dots, \gamma_t\}$ generates Γ as a monoid and $\{r_1, \dots, r_{t'}\}$ is a finite set of relations. If $\mathbf{G}$ is a complex affine algebraic subgroup of $Gl_n(n, \mathbb{C})$, the set $Hom(\Gamma, \mathbf{G})$ of homomorphisms $\rho : \Gamma \longrightarrow \mathbf{G}$ can be identified with an affine subvariety of G^t. Specifically,

$$Hom(\Gamma, \mathbf{G}) = \left\{(g_1, \dots, g_t) \in \mathbf{G}^t \mid r_j\,(g_1, \dots, g_t) = I_n \text{ for all } j\right\} \tag{4}$$

If Γ is finitely generated, $Hom(\Gamma, \mathbf{G})$ is an affine algebraic variety by the Hilbert Basis Theorem.

The set $Hom(\Gamma, \mathbf{G})$ also has a topology induced by the analytic topology on G^t. There is a Zariski open subset of $Hom(\Gamma, \mathbf{G})$ that is smooth in the this topology called the smooth locus, and the functions $\mathrm{Eval}_w : Hom(\Gamma, \mathbf{G}) \longrightarrow \mathbf{G}$ are analytic on the smooth locus. For any subset $S \in G$ and representation $\rho \in Hom(\Gamma, \mathbf{G})$, $\overline{\rho(S)}$ will denote the Zariski closure of $\rho(S)$ in $\mathbf{G}$.

Lemma 26. *([10, Lemma 5.1]) Let $\mathbf{G} \leq GL\,(n, \mathbb{C})$ be a $\overline{\mathbb{Q}}$-algebraic group, $L \leq \mathbf{G}$ be a finitely generated subgroup, and $\boldsymbol{A} \leq \mathbf{G}$ be a $\overline{\mathbb{Q}}$-algebraic subgroup. Then, $H = L \cap \boldsymbol{A}$ is closed in the profinite topology.*

Proof. Given $g \in L - H$, we need a homomorphism $\varphi : L \longrightarrow Q$ such that $|Q| < \infty$ and $\varphi\,(g) \notin \varphi\,(H)$. We first select polynomials $P_1, ..., P_{r_{\mathbf{G}}}, ..., P_{r_{\mathbf{A}}} \in \mathbb{C}\,[X_{i,j}]$ satisfying (3). Since $\mathbf{G}$ and $\mathbf{A}$ are $\overline{\mathbb{Q}}$-defined, we can select $P_j \in \mathcal{O}_{K_0}\,[X_{i,j}]$ for some number field $K_0/\mathbb{Q}$. We fix a finite set $\{l_1, \dots, l_{r_L}\}$ that generates L as a monoid. In order to distinguish between elements of L as an abstract group and the explicit elements in $\mathbf{G}$, we set $l = M_l \in \mathbf{G}$ for each $l \in L$. In particular, we have a representation given by $\rho_0 : L \longrightarrow \mathbf{G}$ given by $\rho_0(l_t) = M_{l_t}$. We set K_L to be the field generated over K_0 by the set of matrix entries $\left\{(M_t)_{i,j}\right\}_{t,i,j}$. It is straightforward to see that K_L is independent of the choice of the generating set for L. Since L is finitely generated, the field K_L has finite transcendence degree over $\mathbb{Q}$ and so K_L is isomorphic to a field of the form $K(T)$ where $K/\mathbb{Q}$ is a number field and $T = \{T_1, \dots, T_d\}$ is a transcendental basis (See [10]). For each, M_{l_t}, we have $(M_{l_t})_{i,j} = F_{i,j,t}(T) \in K_L$. In particular, we can view the (i,j)–entry of the matrix M_{l_t} as a rational function in d variables with coefficients in some number field K. Taking the ring generated over $\mathcal{O}_{K_0}$ by the set $\left\{(M_{l_t})_{i,j}\right\}_{t,i,j}$, R_L is obtained from $\mathcal{O}_{K_0}\,[T_1, \dots, T_d]$

by inverting a finite number of integers and polynomials. Any ring homomorphism $R_L \longrightarrow R$ induces a group homomorphism $GL(n, R_L) \longrightarrow GL(n, R)$, and since $L \leq GL(n, R_L)$, we obtain $L \longrightarrow GL(n, R)$. If $g \in L - H$ then there exists $r_{\mathbf{G}} < j_g \leq r_{\mathbf{A}}$ such that $Q_g = P_{j_g}\left((M_l)_{1,1}, \ldots, (M_l)_{n,n}\right) \neq 0$. Using Lemma 2.1 in [6], we have a ring homomorphism $\psi_R : R_L \longrightarrow R$ with $|R| < \infty$ such that $\psi_R(Q_g) \neq 0$. Setting, $\rho_R : GL(n, R_L) \longrightarrow GL(n, R)$ we assert that $\rho_R(g) \notin \rho_R(H)$. To see this, set $\overline{M}_\eta = \rho_R(\eta)$ for each $\eta \in L$, and note that $\psi_R(P_j((M_\eta)_{1,1}, \ldots, M_\eta)_{n,n})) = P_j((\overline{M}_\eta)_{1,1}, \ldots, (\overline{M}_\eta)_{n,n})$. For each $h \in H$, we know that $P_{j_l}((M_h)_{i,j}) = 0$ and so $P_j((\overline{M}_\eta)_{1,1}, \ldots, (\overline{M}_\eta)_{n,n}) = 0$. However, by selection of ψ_R, we know that $\psi_R(Q_g) \neq 0$ and so $\rho_R(g) \notin \rho_R(H)$. $\square$

Theorem 4 and Lemma 26 imply Corollary 5.

Proof. Since $H \leq L$ is finitely generated, by Theorem 4, there is a faithful representation

$$\rho_H : L \longrightarrow GL\,(n, \mathbb{C})$$

such that $\overline{\rho_H(H)} \cap \rho_H(L) = \rho_H(H)$. We can construct the representation in Theorem 4 so that $\mathbf{G} = \overline{\rho_H(L)}$ and $\mathbf{A} = \overline{\rho_H(H)}$ are both $\overline{\mathbb{Q}}$-defined. So, by Lemma 26, we can separate H in L. Next, we quantify the separability of H in L. Toward that end, we need to bound the order of the ring R in the proof of Lemma 26 in terms of the word length of the element g. Lemma 2.1 from [6] bounds the size of R in terms of the coefficient size and degree of the polynomial Q_g. It follows from a discussion on pp 412-413 of [6] that the coefficients and degree can be bounded in terms of the word length of g, and that the coefficients and degrees of the polynomials P_j. Because the P_j are independent of the word g, there exists a constant N_0 such that $|R| \leq ||g||^{N_0}$. By construction, the group Q we seek is a subgroup of $GL(n, R)$. Thus, $|Q| \leq |R|^{n^2} \leq ||g||^{N_0 n^2}$. Taking $N = N_0 n^2$ completes the proof.

$\square$

6 The Hanna Neumann conjecture for hyperbolic limit groups

Y. Antolin and A. Jaikin-Zapirain proved in [1] the geometric Hanna Neumann conjecture for surface groups and formulated the Geometric Hanna Neumann conjecture for limit groups [1, Conjecture 1]as follows. Let G be a limit group. Then for every two finitely generated subgroups U and W of G

$$\Sigma_{x \in U \backslash G / W} \bar{\chi}(U \cap xWx^{-1}) \leq \bar{\chi}(U)\bar{\chi}(W)$$

Here for a virtually FL-group Γ we define its Euler characteristic as

$$\chi(\Gamma) = \frac{1}{[\Gamma : \Gamma_0]} \Sigma_{i=0}^{\infty} (-1)^i dim_{\mathbb{Q}} H_i(\Gamma_0, \mathbb{Q}),$$

where Γ_0 is an FL-subgroup of Γ of finite index. And $\bar{\chi}(\Gamma) = \max\{0, -\chi(\Gamma)\}$. Observe that for a non-trivial finitely generated free group Γ, $\bar{\chi}(\Gamma) = d(\Gamma) - 1$, where $d(\Gamma)$ is the number of generators, for a surface group Γ we have $\bar{\chi}(\Gamma) = d(\Gamma) - 2$. By a surface group we mean the fundamental group of a compact closed surface of negative Euler characteristic. Notice that by [1] limit groups are FL-groups. Notice also that for hyperbolic limit groups $dim_{\mathbb{Q}} H_i(\Gamma_0, \mathbb{Q}) = 0$ for $i > 2$.

In this section we will prove the conjecture for hyperbolic limit groups.

The notion of L^2-independence was introduced in [1]. The group G is L^2-Hall, if for every finitely generated subgroup H of G, there exists a subgroup K of G of finite index containing H such that H is L^2-independent in K. Let G be a hyperbolic limit group. By [1, Theorem 1.3], if G satisfies the L^2-Hall property, then the geometric Hanna Neumann conjecture holds for G.

As explained in [1, Lemma 4.1] and the comment after the lemma, since the limit groups satisfy the strong Atiyah conjecture, if G is a limit group and $H \leq K$ subgroups in G, then H is L^2-independent in K if the correstriction map

$$cor : H_1(H; \mathcal{D}_{\mathbb{Q}[G]}) \to H_1(K; \mathcal{D}_{\mathbb{Q}[G]})$$

is injective. Here $\mathcal{D}_{\mathbb{Q}[G]}$ denote the Linnell division ring.

Lemma 27. *Let G be a limit group and $H \leq K$ subgroups of G. Assume that there an abelian subgroup B of G such that $K = \langle H, B\rangle = H *_A B$, where $A = H \cap B$. Then the correstriction map $cor : H_1(H; \mathcal{D}_{\mathbb{Q}[G]}) \to H_1(K; \mathcal{D}_{\mathbb{Q}[G]})$ is injective.*

Proof. By [8, Theorem2(2)], we obtain the exact sequence

$$H_1(A; \mathcal{D}_{\mathbb{Q}[G]}) \to^{(cor,-cor)} H_1(H; \mathcal{D}_{\mathbb{Q}[G]}) \oplus H_1(B; \mathcal{D}_{\mathbb{Q}[G]}) \to^{(cor,cor)} H_1(K; \mathcal{D}_{\mathbb{Q}[G]}).$$

Since A is abelian, $H_1(A; D_{Q[G]}) = 0$. Indeed, the division ring generated by $Q[A]$ inside $D_{Q[G]}$ is isomorphic to the field of fractions R of $Q[A]$, and so $D_{Q[G]}$ is also an R-vector space. Thus, $D_{Q[G]}$ is flat as a $Q[A]$-module. In particular, $H_1(A; D_{Q[G]}) = 0$.

So the correstriction map

$$H_1(H; \mathcal{D}_{\mathbb{Q}[G]}) \to H_1(K; \mathcal{D}_{\mathbb{Q}[G]})$$

is injective.

□

Corollary 28. *A limit group is L^2-Hall.*

Proof. Let G be an ICE- group and H a finitely generated subgroup of G. Then by Theorem 1 there exists a finite chain of groups $H = K_0 < \ldots < K_n = K$ with K of finite index in G, where K_{i+1} is either $K_i * \mathbb{Z}$ or K_{i+1} is an extension of a centralizer of K_i. By Lemma 27, the correstriction maps

$$H_1(K_i; \mathcal{D}_{\mathbb{Q}[G]}) \to H_1(K_{i+1}; \mathcal{D}_{\mathbb{Q}[G]})$$

are injective. Hence H is L^2-independent in K. Now, let $H < L < G$, then H will be L^2-independent in $L \cap K$ because the composition of correstrictions maps is correstriction. Thus L is L^2-Hall. $\square$

Therefore we obtain the following theorem.

Theorem 29. *The geometric Hanna Neumann conjecture is true for hyperbolic limit groups.*

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