STATIONARY RANDOM SUBGROUPS IN NEGATIVE CURVATURE

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ABSTRACT. We show that discrete stationary random subgroups of isometry groups of Gromov hyperbolic spaces have full limit sets as well as critical exponents bounded from below. This information is used to answer a question of Gelander and show that a rank one locally symmetric space for which the bottom of the spectrum of the Laplace–Beltrami operator is the same as that of its universal cover has unbounded injectivity radius.

1. INTRODUCTION

Let G be a connected center-free simple rank one Lie group with associated symmetric space X. In other words X is a real, complex or quaternionic hyperbolic space or the Cayley plane and its group of isometries is $G = \text{Is}(X)$.

A discrete subgroup $\Gamma \leq G$ is called *confined* if there exists a compact subset $Q \subset G$ such that $\Gamma^g \cap Q \neq \{e\}$ for all elements $g \in G$. If the discrete subgroup Γ is torsion-free then being confined is equivalent to the geometric condition saying that the locally symmetric manifold $M_\Gamma = \Gamma \backslash X$ has an upper bound on its injectivity radius at all points. This says that the subgroup Γ is qualitatively "large".

The *critical exponent* of a discrete subgroup $\Gamma \leq \text{Is}(X)$ is a quantitative measure of the exponential growth rate of Γ-orbits in the space X. This is a real number $\delta(\Gamma)$ taking values in the range $[0, d(X)]$ where $d(X) = \dim_{\text{Haus}}(\partial X)$ is the Hausdorff dimension of the visual boundary at infinity. For instance $d(\mathbb{H}^n) = n - 1$ for the real hyperbolic space $\mathbb{H}^n$ of dimension n. See §9 for the precise definitions.

Our main result relates these qualitative and quantitative notions.

Theorem 1.1. *Let Γ be a discrete subgroup of the simple rank one Lie group G. If the subgroup Γ satisfies $\delta(\Gamma) \leq \frac{1}{2}d(X)$ then Γ is not confined.*

This can be seen as a conditional *rank one* analogue of the recent breakthrough by Fraczyk and Gelander [FG23] who showed that any infinite covolume discrete subgroup of a *higher rank* simple Lie group is not confined. Indeed Theorem 1.1 has been conjectured and introduced to us by Gelander.

Assuming the discrete subgroup Γ is torsion-free its critical exponent $\delta(\Gamma)$ is related to the *bottom of the spectrum $\lambda_0(M_\Gamma)$* of the Laplace–Beltrami operator on the locally symmetric manifold M_Γ. The Elstrodt formula [Els74, Pat76, Sul87, Cor90] states that

$$(1.1) \qquad \lambda_0(M_\Gamma) = \begin{cases} \frac{1}{4}d(X)^2 & \text{if} \quad \delta(\Gamma) \in \left[0, \frac{1}{2}d(X)\right], \\ \delta(\Gamma)(d(X) - \delta(\Gamma)) & \text{if} \quad \delta(\Gamma) \in \left[\frac{1}{2}d(X), d(X)\right]. \end{cases}$$

In particular the assumption $\delta(\Gamma) \leq \frac{1}{2}d(X)$ is equivalent to $\lambda_0(M_\Gamma) = \lambda_0(X)$. Note that $\delta(\Gamma) = d(X)$ if and only if the discrete subgroup Γ is co-amenable in the Lie group G. For example every lattice subgroup Γ satisfies $\delta(\Gamma) = d(X)$.

Our result is sharp, at least for the two-dimensional real hyperbolic plane $\mathbb{H}^2$. Indeed consider any uniform lattice Γ in the group of isometries $\mathrm{Is}(\mathbb{H}^2) \cong \mathrm{PSL}_2(\mathbb{R})$. It turns out that Γ admits a sequence of infinite index normal subgroups N_i satisfying $\delta(N_i) \to \frac{1}{2}\delta(\Gamma) = \frac{1}{2}$ [BTMT12, Theorem 3.8]. However every non-trivial normal subgroup of the lattice Γ is confined [FG23, Example 1.2].

Interestingly the condition $\delta(\Gamma) \le \frac{1}{2}d(X)$ for a discrete subgroup Γ of the simple rank one Lie group G is equivalent to saying that the unitary representation $L^2(G/\Gamma)$ is tempered [EO23, Theorem 1.2]. Furthermore, if the discrete subgroup Γ is not confined then the unitary representation $L^2(G)$ is weakly contained in the unitary representation $L^2(G\backslash\Gamma)$ [EO23, Proposition 8.4]. We obtain the following.

Corollary 1.2. *Let Γ be a discrete subgroup of the simple rank one Lie group G. If the unitary representation $L^2(G\backslash\Gamma)$ is tempered then the two unitary representations $L^2(G)$ and $L^2(G\backslash\Gamma)$ are weakly equivalent.*

Stationary random subgroups and critical exponents. While the statement of Theorem 1.1 is purely geometric, the methods we use are probabilistic. Along the way we prove a number of probabilistic results of independent interest. These are considered in the more general setting of Gromov hyperbolic spaces.

Let X be a proper Gromov hyperbolic geodesic metric space with non-elementary group of isometries $\mathrm{Is}(X)$. For example X could be a rank one symmetric space or a Gromov hyperbolic group equipped with the word metric. Let $\mathrm{Sub}(\mathrm{Is}(X))$ denote the Chabauty space of all closed subgroups of the group $\mathrm{Is}(X)$. The group $\mathrm{Is}(X)$ admits a continuous action on this space by conjugation.

Fix a Borel probability measure μ on the group $\mathrm{Is}(X)$ whose support generates as a semigroup a discrete subgroup of $\mathrm{Is}(X)$ acting cocompactly on the space X. Associated to the measure μ are the classical parameters *drift* $l(\mu)$ and *entropy* $h(\mu)$ discussed in some detail in §8. The assumption that $\mathrm{Is}(X)$ is non-elementary implies that $l(\mu) > 0$ as well as $h(\mu) > 0$. It is convenient to denote $\delta(\mu) = \frac{h(\mu)}{l(\mu)}$.

A μ-*stationary random subgroup* of the group $\mathrm{Is}(X)$ is a Borel probability measure ν on the space $\mathrm{Sub}(\mathrm{Is}(X))$ satisfying $\mu * \nu = \nu$ with respect to the action by conjugation. We establish a lower bound on the critical exponents of discrete μ-stationary random subgroups in terms of the parameter $\delta(\mu)$.

Theorem 1.3. *Assume that the probability measure μ has finite first moment. Let ν be a μ-stationary random subgroup of $\mathrm{Is}(X)$ such that ν-almost every subgroup Γ is discrete and not contained in the elliptic radical[1]. Then*

- *(1) $\delta(\Gamma) > \frac{\delta(\mu)}{2}$ holds ν-almost surely, and*
- *(2) if ν-almost every subgroup Γ is of divergence type[2] then $\delta(\Gamma) \ge \delta(\mu)$ holds ν-almost surely.*

If X is a rank one symmetric space then it is possible to choose a probability measure μ as above with finite first moment and with $\delta(\mu) = d(X)$. This fact is discussed in §9. If X is the Cayley graph of a rank k free group then the uniform probability measure μ_k on the standard symmetric generating set satisfies

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[1]The *elliptic radical* of the group $\mathrm{Is}(X)$ is the maximal normal subgroup $\mathrm{E}(\mathrm{Is}(X))$ consisting of elliptic elements. If X is rank one symmetric space or more generally a geodesically complete $\mathrm{CAT}(-1)$-space then the elliptic radical of $\mathrm{Is}(X)$ is trivial.

[2]Critical exponents as well as the notion of divergence type subgroups are discussed in §9.

$\delta(\mu_k) = d(X) = \log(2k-1)$. This provides a new proof of [GL19, Theorem 1.1] for invariant random subgroups in those two special cases[3].

More generally, if the group $\mathrm{Is}(X)$ is acting properly and cocompactly on the space X and L is a uniform lattice in the group $\mathrm{Is}(X)$ then there is a sequence of probability measures μ_i supported on finite generating sets of the lattice L satisfying $\delta(\mu_i) \to \delta(L) = \dim_{\mathrm{Haus}}(\partial X)$.

The study of invariant random subgroups of discrete groups and the spectral radius of random walks on the associated Schreier graphs was initiated by Abert, Glasner and Virag [AGV14]. Their work serves as the inspiration to ours.

On non-confined subgroups. Consider a rank one simple Lie group G with a fixed Borel probability measure μ. Given any discrete subgroup $\Gamma \leq G$ let D_Γ denote the Dirac mass supported on the point $\Gamma \in \mathrm{Sub}(G)$. Then any accumulation point ν of the Cesáro averages $\frac{1}{n} \sum_{i=1}^{n} \mu^{*i} * \mathrm{D}_\Gamma$ is a μ-stationary random subgroup of G. At this point, the lower semi-continuity of the critical exponent with respect to the Chabauty topology would in principle allow us to deduce Theorem 1.1 from Theorem 1.3 along the lines of the strategy introduced in [FG23]. However, there is an additional missing ingredient, namely we need to know that ν-almost every subgroup is discrete. The actual proof is more involved. In particular we use the *key inequality* for the injectivity radius function from [GLM22].

The above strategy showing that a discrete subgroup of the isometry group of some Gromov hyperbolic space with a small critical exponent is not confined works more generally. See Theorem 10.7 for a precise formulation of a general principle. For example, in the case of free groups we recover [Fra20, Corollary 3.2]:

Theorem 1.4 (Fraczyk)**.** *Let F_k be a free group of rank $k \in \mathbb{N}$ and X_k be its Cayley graph with respect to a standard basis. If H is any subgroup of F_k with $\lambda_0(H\backslash X_k) = \lambda_0(X_k)$ then H is not confined.*

Here λ_0 denotes the bottom of the spectrum of the combinatorial Laplacian defined on $2k$-regular graphs [Lub94, §4.2]. In concrete terms, the conclusion of Theorem 1.4 says that for any $R > 0$ there is some element $g \in F_k$ such that the conjugate H^g contains no non-trivial elements of word length less than R.

Limit sets. Let X be a proper Gromov hyperbolic geodesic metric space. Lattices in the group of isometries $\mathrm{Is}(X)$ as well as normal subgroups of such lattices have full limit sets. The same is true for invariant random subgroups of the group $\mathrm{Is}(X)$ as was shown in [ABB$^+$20, Proposition 11.3] and [Osi17]. It turns out that this fact extends to *discrete* stationary random subgroups.

Theorem 1.5. *Let μ be a Borel probability measure on the group $\mathrm{Is}(X)$. Assume that the closed semigroup G_μ generated by $\mathrm{supp}(\mu)$ is a group whose action on the space X has general type.*

Let ν be a discrete μ-stationary random subgroup of $\mathrm{Is}(X)$. If ν-almost every subgroup is not contained in the elliptic radical $\mathrm{E}(\mathrm{Is}(X))$ then the action of ν-almost every subgroup Γ has general type and its limit set $\Lambda(\Gamma)$ contains the limit set $\Lambda(G_\mu)$.

In particular, if the group generated by the support of the measure μ has full limit set $\Lambda(G_\mu) = \partial X$ then so does ν-almost every subgroup.

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[3]More generally, this remark applies to any CAT(-1)-space admitting a uniform lattice, see §9.

The discreteness assumption in Theorem 1.5 is necessary in general, as can be seen by considering the stabilizer of a random point with respect to the G-action on the boundary at infinity ∂X equipped with a μ-stationary probability measure.

Rich and detailed structural information on amenable non-elementary Gromov hyperbolic locally compact groups was obtained in the work [CdCMT15]. These are precisely the groups admitting a proper and cocompact *focal* action on a proper Gromov hyperbolic space, see §3. Taking this work into account and relying on our methods we study stationary random subgroups of such groups.

Theorem 1.6. *Let G be an amenable non-elementary Gromov hyperbolic locally compact group and μ a Borel probability measure on the group G. Assume that the measure μ has finite first moment and is spread out[4]. Then every discrete μ-stationary random subgroup of G is contained in the elliptic radical $\mathrm{E}(G)$.*

Theorem 1.6 applies for example in the case where the group G is the stabilizer of a boundary point at infinity in a rank one symmetric space or a regular tree. The general case is much broader, as is manifested in [CdCMT15].

An immediate consequence of Theorem 1.6 is that amenable non-elementary Gromov hyperbolic groups admit no discrete invariant random subgroups. This is a natural generalization of the fact such groups admit no lattices as they are never unimodular.

CAT(0)-spaces. Let X be a proper geodesically complete CAT(0)-space such that the group of isometries $\mathrm{Is}(X)$ is acting cocompactly on X. A subgroup H of the group $\mathrm{Is}(X)$ is called *geometrically dense* if H preserves no proper closed convex subset of the space X and fixes no point of the visual boundary ∂X.

Let μ be a Borel probability measure on the group $\mathrm{Is}(X)$ such that the closed semigroup generated by $\mathrm{supp}(\mu)$ is a geometrically dense subgroup. Some of our methods are applicable to study μ-stationary random subgroups of $\mathrm{Is}(X)$.

Proposition 1.7. *Let ν be a μ-stationary random subgroup of $\mathrm{Is}(X)$. If ν-almost every subgroup fixes no point of the boundary ∂X then ν-almost every subgroup is geometrically dense.*

This can be seen as a partial analogue of the work [DGLL15] for stationary (rather than invariant) random subgroups, conditional on the extra assumption of having no fixed points on the boundary. We expect that in future work this assumption can be replaced by a discreteness assumption (as was successfully done in the Gromov hyperbolic case in Theorem 1.5).

At present, we are able to combine our methods for Gromov hyperbolic spaces and CAT(0)-spaces to study stationary random subgroups of direct products of finitely many CAT(−1)-spaces.

Theorem 1.8. *Let $X_1, \ldots, X_k$ be a family of proper geodesically complete CAT(−1)-spaces. Let μ_i be a Borel probability measure on the group $\mathrm{Is}(X_i)$ such that $\mathrm{supp}(\mu_i)$ generates $\mathrm{Is}(X_i)$ as a semigroup for each i. Consider the product CAT(0)-space $X = X_1 \times \cdots \times X_k$ and the measure $\mu = \mu_1 \otimes \cdots \otimes \mu_k$ on the group $\mathrm{Is}(X)$.*

Let ν be a discrete μ-stationary random subgroup of $\prod_{i=1}^{k} \mathrm{Is}(X_i)$. If ν-almost every subgroup projects non-trivially to each factor $\mathrm{Is}(X_i)$ then ν-almost every subgroup is geometrically dense.

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[4]A probability measure μ on a locally compact group G is called *spread out* is some convolution power μ^{*n} is non singular with respect to the Haar measure on the group G.

Much more precise information in the special case of the action of a semisimple real Lie group on its symmetric space is given in [FG23, Theorem 6.5].

Lower bounds on critical exponents. As an illustration, we outline a proof of the non-strict variant of Theorem 1.3 asserting that a μ-stationary random subgroup satisfies the non-strict inequality $\delta(\Gamma) \geq \frac{\delta(\mu)}{2}$ almost surely. An analogue of Theorem 1.3 for *invariant* rather than *stationary* random subgroups with the bound $\frac{\delta(\mu)}{2}$ replaced by $d(X) = \frac{\dim_{\mathrm{Haus}}(\partial X)}{2}$ was the main result of [GL19]. In turn, this is a generalization of a result of Matsuzaki, Yabuki and Jaerisch [MYJ20, Theorem 1.4] about the exponential growth rate of orbits for a normal subgroup N of some countable hyperbolic group Γ acting properly and cocompactly on a Gromov hyperbolic space X.

The key idea of [MYJ20] is to bound the critical exponent $\delta(N)$ from below in terms of the exponential growth rate of a single conjugacy class

$$(1.2) \qquad \mathrm{Cl}_\Gamma(\gamma) = \{\gamma^g \ : \ g \in \Gamma\}$$

of some fixed arbitrary hyperbolic element $\gamma \in N$. Fix a basepoint $x_0 \in X$ and denote $\|g\| = d_X(gx_0, x_0)$ for every element $g \in \Gamma$. The triangle inequality implies $\|g\gamma g^{-1}\| \leq 2\|g\| + \|\gamma\|$ for every element $g \in \Gamma$. So

$$(1.3) \qquad \{g \in \Gamma \ : \ \|g\| \leq \frac{R - \|\gamma\|}{2}\} \subset \{g \in \Gamma \ : \ \|g\gamma g^{-1}\| \leq R\} \quad \forall R > 0.$$

On the other hand, the hyperbolic element γ satisfies the following two properties: its centralizer is virtually cyclic and $\lim \frac{1}{n}\|\gamma^n\| > 0$. This implies that

$$(1.4) \qquad |\{h \in \mathrm{Cl}_\Gamma(\gamma) \ : \ \|h\| \leq R\}| \geq \frac{c}{R}|\{g \in \Gamma \ : \ \|g\| \leq \frac{R - \|\gamma\|}{2}\}\| \quad \forall R > 0$$

for some constant $c > 0$. As linear factors do not affect the exponential growth rate we conclude from Equation (1.4) that $\delta(N) \geq \delta(\mathrm{Cl}_\Gamma(\gamma)) \geq \frac{\delta(\Gamma)}{2}$.

Consider a more general situation where ν is a discrete invariant random subgroup of the group of isometries $\mathrm{Is}(X)$ of some Gromov hyperbolic space X. Unlike normal subgroups of lattices, there is a priori no reason for ν-generic subgroups to contain an entire conjugacy class of any particular hyperbolic element. Instead, we show in [GL19] that ν-almost every subgroup contains a *positive proportion* of some hyperbolic "approximate conjugacy class". More precisely, assume that the group $\mathrm{Is}(X)$ admits some auxiliary discrete subgroup Γ acting cocompactly on the space X. Then for any $\varepsilon > 0$ there exists an open relatively compact subset $V \subset \mathrm{Is}(X)$ consisting of hyperbolic elements such that ν-almost every subgroup $\Delta \in \mathrm{Sub}(\mathrm{Is}(X))$ satisfies

$$(1.5) \qquad \liminf_{R \to \infty} \frac{|\{g \in \Gamma \ : \ \|g\| \leq R \quad \text{and} \quad \Delta \cap gVg^{-1} \neq \emptyset\}|}{|\{g \in \Gamma \ : \ \|g\| \leq R\}|} \geq 1 - \varepsilon.$$

This statistical information is sufficient to conclude that $\delta(\Delta) \geq \frac{\delta(\Gamma)}{2}$ along the same lines as in [MYJ20].

To find such a neighborhood $V \subset \mathrm{Is}(X)$ with $\nu(\{\Delta : \Delta \cap V \neq \emptyset\}) > 1 - \varepsilon$ we first show that ν-almost every subgroup admits hyperbolic elements, as a consequence of the fact that it has a full limit set on the Gromov boundary. The maximal ergodic theorem of Bowen and Nevo [BN15] applied to the probability measure preserving action of the auxiliary lattice Γ on the Borel probability space $(\mathrm{Sub}(\mathrm{Is}(X)), \nu)$ implies Equation (1.5) directly.

On our methods. Several important aspects of this strategy need to be modified when ν is a μ-stationary rather than an invariant discrete random subgroup. First, the fact that ν-almost every subgroup has full limit set and hence contains hyperbolic elements now follows from our Theorem 1.5. Second, the maximal ergodic theorem of Bowen and Nevo does not apply to stationary actions. Instead, we rely on *Kakutani's random ergodic theorem.* Roughly speaking, it says that typical trajectories of the μ-random walk become ν-equidistributed. This implies that $\mu^{\otimes\mathbb{N}}$-almost every sequence of random elements $(g_i) \in \Gamma^{\mathbb{N}}$ and ν-almost every subgroup $\Delta \in \mathrm{Sub}(\mathrm{Is}(X))$ satisfy

$$(1.6) \qquad \lim_{n\to\infty} \frac{|\{i \in \{1,\ldots,n\} \,:\, \Delta \cap V^{g_i\cdots g_1} \neq \emptyset\}|}{n} = \nu(\{\Lambda \,:\, \Lambda \cap V \neq \emptyset\}).$$

To get a lower bound on the critical exponent $\delta(\Delta)$ for ν-almost every subgroup Δ we need to estimate the ratios appearing in Equation (1.5) for large R. In other words, we need to show that the subset $\{g \in \Gamma : H \cap V^g \neq \emptyset\}$ is large from the point of view of taking intersections with balls rather than from the point of view of random walks. The connection between these two notions of "size" of subsets of the group Γ is rather complicated (see for instance [Tan17, Theorem 6.2] for an example of a "paradoxical" behaviour in this respect).

In the setting of finitely supported random walks and word metrics on hyperbolic groups, Tanaka showed in [Tan17, §6] that a subset $A \subset \Gamma$ satisfying

$$(1.7) \qquad\qquad \liminf_n \frac{1}{n}|\{i \in \{1,\ldots,n\} : g_i \ldots g_1 \in A\}| > 0$$

for $\mu^{\otimes\mathbb{N}}$-almost surely sequence $(g_i) \in \Gamma^{\mathbb{N}}$ has critical exponent $\delta(A) \geq \delta(\mu) = \frac{h(\mu)}{l(\mu)}$. We prove and make use of Theorem 8.5 which is a vast generalization of Tanaka's result. For us the group Γ need not be hyperbolic, but merely non-amenable. The random walk need not be finitely supported, but only to have finite first moment. This improvement allows us to work with measures having $\delta(\mu) = d(X)$, such as the discretization of Brownian motion. The set A in question is assumed to contain a *positive proportion* of each random walk trajectory, i.e. sample paths visit it statistically rather deterministically. Finally, we conclude that the partial Poincare series associated to the set A diverges at the exponent $\delta(\mu)$.

We expect our results concerning critical exponents can be extended beyond isometry groups of hyperbolic spaces, for instance to rank-one CAT(0)-spaces, Cayley graphs of relatively hyperbolic groups, and products of rank one symmetric spaces. The probabilistic framework for these generalizations is laid out in this paper; one needs to modify a few geometric techniques from [GL19].

Organization of the paper. This work consists of two largely independent parts.

The first part §2–§7 deals with the softer questions of limit sets and fixed points on the boundary. In §2 we introduce the basics of stationary measures and random walks and state Kakutani's ergodic theorem which plays a fundamental role in this work. In §3 we recall the classification of group actions on Gromov hyperbolic spaces, including the focal and the general type cases. In §4 we use boundary hitting measures to study stationary closed subsets. This information is applied to study limit sets. In §5 we show that discrete stationary subgroups cannot have a single fixed point on the boundary, and apply this to study amenable hyperbolic groups. Lastly §6 is dedicated to stationary random subgroups of products and §7 deals with CAT(0)-spaces and geometric density.

The second part §8–§10 deals with critical exponents and confined subgroups. The core probabilistic argument takes place in §8. This is where we discuss entropy, drift, Green's functions and prove our variant of Tanaka's argument. Next in §9 we use this probabilistic machinery to study the Poincare series and the critical exponents of stationary random subgroups. Finally in §10 we consider confined subgroups and implement our strategy towards Theorem 1.1 by observing that the critical exponent is Chabauty semi-continuous.

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Notations.

- We denote $[n] = \{1, \ldots, n\}$ for every natural number $n \in \mathbb{N}$.
- We will often fix a base point $x_0 \in X$. With the base point being implicit in the notation, it will be convenient to write
$$\|g\| = d_X(gx_0, x_0) \quad \forall g \in \mathrm{Is}(X).$$
- To avoid an overuse of the Greek letter δ we will use D_x to denote the Dirac probability measure supported on the point x.
- For a pair of functions f and g the notation $f \approx g$ means that
$$C^{-1}f \le g \le Cf$$
for some constant $C > 1$.
- All probability measures are assumed to be Borel.

2. Stationary measures and Kakutani's ergodic theorem

We introduce some general definitions concerning stationary measures and random walks, which are the main workhorse of this note. In addition we state Kakutani's ergodic theorem and summarize some of its implications.

Sample paths. Let G be a locally compact group. The product space $G^{\mathbb{N}}$ equipped with the Tychonoff topology will be regarded as describing *increments*. Given a sequence of increments $(g_n) \in G^{\mathbb{N}}$ the associated *sample path* is the sequence $\omega = (\omega_n) \in G^{\mathbb{N}}$ defined by

$$(2.1) \qquad \omega_n = g_1 g_2 \cdots g_n \quad \forall n \in \mathbb{N}.$$

Let μ be a probability measure on the group G. The corresponding product measure on the space of increments $G^{\mathbb{N}}$ is denoted $\mu^{\otimes \mathbb{N}}$. Let P_μ be the pushforward of the probability measure $\mu^{\otimes \mathbb{N}}$ via the mapping $(g_n) \mapsto (\omega_n)$. We will call $(G^{\mathbb{N}}, \mathrm{P}_\mu)$ the *space of sample paths*.

On certain occasions it will be useful to consider the probability measure $\check{\mu}$ on the group G given by $\check{\mu}(A) = \mu(A^{-1})$ for every Borel subset $A \subset G$. We will use the notation $\check{w}$ for sample paths distributed with respect to the measure $\mathrm{P}_{\check{\mu}}$. Note that

$$(2.2) \qquad \check{\omega}_n = g_1^{-1} g_2^{-1} \cdots g_n^{-1} = (g_n g_{n-1} \cdots g_1)^{-1} \quad \forall n \in \mathbb{N}.$$

The probability measure μ is *symmetric* if $\mu = \check{\mu}$.

Stationary measures. Let (Z, ν) be a standard Borel probability space admitting a Borel action of the group G. Assume that the measure ν is μ-*stationary*, i.e. $\nu = \mu * \nu$ where

$$(2.3) \qquad \mu * \nu = \int_G g_* \nu \, \mathrm{d}\mu(g).$$

Assume further that ν is *ergodic*, i.e. ν is an extreme point in the simplex of μ-stationary probability measures.

The following ergodic theorem plays a central part in most of our arguments.

Theorem 2.1 (Kakutani's ergodic theorem [Kak51]). *Let $f \in L^1(Z, \nu)$. Then $\mu^{\otimes \mathbb{N}}$-almost every sequence of increments $(g_n) \in G^{\mathbb{N}}$ and ν-almost every point $z \in Z$ satisfy*

$$(2.4) \qquad \frac{1}{N} \sum_{n=1}^{N} f(g_n g_{n-1} \cdots g_1 z) \xrightarrow{N \to \infty} \int f \, \mathrm{d}\nu.$$

We will have occasion to apply Kakutani's ergodic theorem in two ways, which we record here for convenience. First, it can be used to estimate the proportion of time a generic random walk spends inside some fixed subset of the space Y.

Corollary 2.2. *Let $Y \subset Z$ be a ν-measurable subset. Then $\mu^{\otimes \mathbb{N}}$-almost every sequence of increments $(g_n) \in G^{\mathbb{N}}$ and ν-almost every point $z \in Z$ satisfy*

$$(2.5) \qquad \lim_{n \to \infty} \frac{|\{i \in [n] \,:\, g_i g_{n-1} \cdots g_1 z \in Y\}|}{n} = \nu(Y).$$

Proof. The conclusion follows by applying Kakutani's ergodic theorem (Theorem 2.1) to the characteristic function $1_Y \in L^1(Z, \nu)$ of the subset Y. $\square$

Second, Kakutani's theorem implies that a generic random walk "does not escape to infinity", in the following sense.

Corollary 2.3. *Let $F : Z \to [0, \infty)$ be a ν-measurable function. Then $\mu^{\otimes \mathbb{N}}$-almost every sequence of increments $(g_n) \in G^{\mathbb{N}}$ and ν-almost every point $z \in Z$ satisfy*

$$(2.6) \qquad \liminf_{n \to \infty} F(g_n g_{n-1} \cdots g_1 z) < \infty.$$

Proof. Take $D > 0$ to be sufficiently large so that

$$(2.7) \qquad \nu(Z_D) > 0 \quad \text{where} \quad Z_D = \{z \in Z \,:\, F(z) \le D\}.$$

The desired conclusion follows at once by applying Corollary 2.2 with respect to the ν-measurable subset D. $\square$

Stationary measures are quasi-invariant. Let H denote the closure in G of the semigroup generated by the support of μ. It will be useful to keep in mind the following well-known elementary fact. We include a brief proof for completeness.

Lemma 2.4. *Assume that the topological group G is second countable. Then any μ-stationary measure ν is H-quasi-invariant, i.e. if a Borel subset $E \subset Z$ has $\nu(E) = 0$ then $\nu(h^{-1}E) = 0$ for any element $h \in H$.*

Proof. Consider any μ-stationary measure ν on the space Z. By induction $\mu^{*n} * \nu = \nu$ for all $n \in \mathbb{N}$. Let $E \subset Z$ be any ν-null subset. Then

$$(2.8) \qquad 0 = \nu(E) = \int_G \nu(g^{-1}E) \, \mathrm{d}\mu^{*n}(g) \quad \forall n \in \mathbb{N}.$$

As ν is probability measure it must satisfy $\nu(h^{-1}E) = 0$ for μ^{*n}-almost every element $h \in G$ where $n \in \mathbb{N}$ is arbitrary. The second countability assumption implies that for every $n \in \mathbb{N}$ there is a dense subset $S_n \subset \mathrm{supp}(\mu^{*n})$ such that $\nu(h^{-1}E) = 0$ for all elements $h \in S_n$. Denote $H_0 = \bigcup_n S_n$ so that $\overline{H}_0 = H$. The conclusion follows from that fact that the subset of H consisting of all elements h with $\nu(h^{-1}E) = 0$ is closed by [Zim13, Theorem B.3] applied to the Koopman representation. $\qquad\square$

In particular, if Z is a topological space and ν is a μ-stationary Borel measure then $\mathrm{supp}(\nu)$ is an H-invariant closed subset of Z.

3. Group actions on hyperbolic spaces

Let X be a Gromov hyperbolic geodesic metric space[5]. We recall various features of groups acting on X. In addition we discuss several relevant notions, namely quasi-stabilizers and the elliptic radical.

Classification of actions. The *Gromov boundary* ∂X is the collection of geodesic rays in the space X up to the equivalence relation of being within bounded Hausdorff distance. The group $\mathrm{Is}(X)$ is acting on the boundary ∂X by homeomorphisms. The space $X \cup \partial X$ is a natural compactification of the space X.

Let G be a group admitting a continuous action on the space X by isometries. Equivalently we are given a continuous homomorphism $f : G \to \mathrm{Is}(X)$. The *limit set* $\Lambda(G)$ is the set $\partial X \cap \overline{Gx_0}$ for any choice of base point $x_0 \in X$.

Recall our standing notation $\|g\| = d_X(gx_0, x_0)$ where $x_0 \in X$ is any fixed base point. The action of any single element $g \in G$ on the space X can be classified as

- *elliptic* if g has bounded orbits.
- *parabolic* if g has unbounded orbits and $\lim_{n \to \infty} \frac{1}{n}\|g^n\| = 0$.
- *hyperbolic* if $\lim_{n \to \infty} \frac{1}{n}\|g^n\| > 0$.

In terms of the limit set of the cyclic group generated by an element $g \in G$ we can say that

- g is elliptic if and only if $\Lambda(\langle g \rangle) = \emptyset$.
- g is parabolic if and only if $|\Lambda(\langle g \rangle)| = 1$.
- g is hyperbolic if and only if $|\Lambda(\langle g \rangle)| = 2$.

In addition parabolic and hyperbolic elements satisfy $\Lambda(\langle g \rangle) = \mathrm{Fix}_{\partial X}(g)$.

Any action of a group on the space X can be classified as follows [Gro87, §8].

- The action is called *elementary* if it is
  - *bounded:* orbits are bounded,
  - *horocyclic:* it is not bounded but has no hyperbolic isometries, or
  - *lineal:* it has an hyperbolic isometry and any two hyperbolic isometries have the same limit set.
- The action is called *non-elementary* if it is
  - *focal:* it has a hyperbolic isometry, is not lineal and the limit sets of any two hyperbolic isometries are not disjoint.
  - of *general type:* it has two hyperbolic isometries with disjoint limit sets.

---

[5]In this section we do not require the metric space X to be proper (unless explicitly stated otherwise).

We refer to [CdCMT15, Proposition 3.1] for a list of equivalent conditions for each of the above types of actions in terms of the limit set. For instance, a locally compact group is focal hyperbolic with respect to the word metric coming from some compact generating set if and only if it is amenable and non-elementary hyperbolic [CdCMT15, Theorem 7.3].

The elliptic radical. Let G be a group admitting a continuous non-elementary action on the space X by isometries. The *elliptic radical* $\mathrm{E}(G)$ of the group G is the kernel of its action on its limit set, namely it is the closed normal subgroup

$$(3.1) \qquad\qquad \mathrm{E}(G) = \{g \in G \,:\, g\zeta = \zeta \quad \forall \zeta \in \Lambda(G)\}.$$

Alternatively, the elliptic radical $\mathrm{E}(G)$ can be characterized as the maximal normal subgroup of G whose action on the space X is bounded [CdCMT15, Lemma 3.6]. See also [Osi17, Proposition 3.4].

Quasi-stabilizers. Let Γ be a discrete group admitting an action by isometries on the Gromov hyperbolic space X.

Definition 3.1. Given a point $x \in X$ and a radius $R > 0$ the corresponding *quasi-stabilizer* is given by

$$(3.2) \qquad\qquad \Gamma_{x,R} = \{\gamma \in \Gamma \,:\, d_X(\gamma x, x) \le R\}.$$

Note that the actual stabilizer $\Gamma_x = \{\gamma \in \Gamma \,:\, \gamma x = x\}$ is contained in the quasi-stabilizer $\Gamma_{x,R}$ for all $R > 0$. The quasi-stabilizer is not a subgroup in general. In the situation where the metric space X is proper, we know that the action of the group Γ on the metric space X is proper if and only if all quasi-stabilizers are finite.

Lemma 3.2. *There is some sufficiently large $R > 0$ such that if $\gamma \in \Gamma$ is any elliptic or parabolic element then $\Gamma_{x,R} \ne \emptyset$ for some point $x \in X$.*

The lemma is certainly well-known, we include it for the sake of completeness. It is immediate from the definitions if the space X is $\mathrm{CAT}(-1)$ rather than δ-hyperbolic.

Proof of Lemma 3.2. If the element γ under consideration is parabolic then this fact is discussed in [Gro87, 8.1.C]. If the element γ is elliptic we argue as follows. Let O be any orbit of the isometry γ in the space X. The orbit O is bounded. Denote

$$(3.3) \qquad\qquad r_O = \inf\{r > 0 \,:\, O \subset B_x(r) \text{ for some } x \in X\}$$

and

$$(3.4) \qquad\qquad C_1(O) = \{x \in X \,:\, O \subset B_x(r_O + 1)\}.$$

The points of $C_1(O)$ are called the *quasi-centers* of the subset O. The subset $C_1(O)$ is clearly γ-invariant and has diameter less than $4\delta + 2$, see [BH13, III.Γ.3.3]. The claim follows by taking an arbitrary point $x \in C_1(O)$. $\qquad\qquad\square$

4. Stationary closed subsets

Let X be a complete separable Gromov hyperbolic geodesic metric space with fixed base point $x_0 \in X$ and equipped with the metric d_X. For example X could be a proper Gromov hyperbolic length space (see [BH13, §I.3]). Generally speaking, and with some applications in mind, we do not want to assume X is proper.

Boundary measures for random walks. Let G be a locally compact group admitting a continuous action by isometries on the space X. Fix a probability measure μ on the group G. Assume that the closed semigroup G_μ generated by the support of the measure μ is in fact a group. Consider the associated space of sample paths $(G^{\mathbb{N}}, \mathrm{P}_\mu)$ discussed in §2. The following fundamental object plays a key role in our work.

Theorem 4.1 (The boundary measure). *Assume that the action of the group G_μ on the space X is non-elementary. In the focal case assume in addition that*

- *the measure μ has finite first moment[6] and is spread out[7],*
- *the Poisson boundary of the pair (G, μ) is non trivial, and*
- *the space X is proper and the group G is acting cocompactly on X.*

In the general type case no additional assumption is required. Then

(1) *the trajectory $\omega_n x_0$ converges to some point $\zeta_\omega \in \partial X$ for P_μ-almost every sample path $\omega \in G^{\mathbb{N}}$, and*

(2) *the pushforward ν_X of the measure P_μ via the mapping $\omega \mapsto \zeta_\omega$ is the unique μ-stationary probability measure on the boundary ∂X satisfying $\nu_X(\mathrm{Fix}_{\partial X}(G_\mu)) = 0$.*

Note that the limit of a sequence such as $\omega_n x_0$ is independent of the base point. The condition saying that $\nu_X(\mathrm{Fix}_{\partial X}(G_\mu)) = 0$ is clearly relevant only in the focal case, for a general type action fixes no points on the boundary at infinity.

Proof of Theorem 4.1. Assume to begin with that the action of the group G_μ on the metric space X has general type. The case of a random walk on a Gromov hyperbolic group is [Kai94]. The general case for a proper metric space X is treated in [BQ16a]. The case where the group G is countable but the space X may be non-proper is [MT18]. However, as noted by Dussaule and Gouezel, the countability assumption may be dropped. Indeed, the only place where it was used in [MT18] was in their Proposition 4.4, which Dussaule and Gouezel generalized to the uncountable setting, see [Dus, Appendix].

In the focal case, the extended set of assumptions implies that the Poisson boundary of the pair (G, μ) can be identified with the Gromov boundary ∂X equipped with the μ-stationary hitting measure ν_X, see [FT22, Theorem 1.4]. See [Kai96] for a closely related survey of the identification problem for Poisson boundaries. The argument of [BQ16a, Proposition 3.1] shows that ν_X is the unique *non-atomic* μ-stationary probability measure on the boundary ∂X. On the other hand, the only atomic such measure is the Dirac measure D_ζ where $\mathrm{Fix}_{\partial X}(G_\mu) = \{\zeta\} \subset \Lambda(G_\mu)$. $\square$

We will assume throughout §4 and §5 that the assumptions of Theorem 4.1 are satisfied so that the μ-stationary boundary measure ν_X satisfying both statements (1) and (2) of Theorem 4.1 does exist.

For the sake of completeness we include the proof of the following elementary consequence of Theorem 4.1.

---

[6]The probability measure μ has *finite first moment* if $\sum_{g \in G} \mu(g)\|g\| < \infty$.

[7]A measure μ is called *spread out* if some convolution power μ^{*n} is non singular with respect to the Haar measure on the group G.

Lemma 4.2. *The probability measures $\mu^{*n} * D_{x_0}$ converge to the μ-stationary boundary measure ν_X in the weak-$*$ topology of probability measures on the compactification $\overline{X} = X \cup \partial X$.*

Proof. It follows from Theorem 4.1 that

$$(4.1) \qquad \nu_X(E) = P_\mu(\{\omega \,:\, \lim_n \omega_n x_0 \in E\})$$

holds true for any ν_X-measurable subset $E \subset \partial X$. Take U to be any open subset of the compactification $\overline{X}$. If $\lim_n \omega_n x_0 \in U \cap \partial X$ then $\omega_n x_0 \in U$ for all indices n sufficiently large. Fatou's lemma gives

$$(4.2) \qquad \begin{aligned} \nu_X(U) = \nu_X(U \cap \partial X) &\leq \int_{G^{\mathbb{N}}} \liminf_n 1_U(\omega_n x_0) \, dP_\mu(\omega) \leq \\ &\leq \liminf_n \int_X 1_U \, d(\mu^{*n} * D_{x_0}) = \liminf_n \mu^{*n} * D_{x_0}(U). \end{aligned}$$

The desired conclusion follows from the Portmanteau theorem. $\qquad\square$

Proposition 4.3. *The boundary measure ν_X has $\mathrm{supp}(\nu_X) = \Lambda(G_\mu)$.*

Proof. The construction of the boundary measure ν_X in Theorem 4.1 shows that $\mathrm{supp}(\nu_X) \subset \Lambda(G_\mu)$. The subset $\mathrm{supp}(\nu_X)$ is closed and G_μ-invariant by Lemma 2.4. This concludes the proof since $\Lambda(G_\mu)$ is the minimal G_μ-invariant closed subset of the boundary ∂X not contained in $\mathrm{Fix}_{\partial X}(G_\mu)$ [DSU17, Corollary 7.4.3]. $\qquad\square$

Effros space of closed subsets. Let $\mathrm{Cl}(X)$ denote the space of all closed subsets of the space X. The space $\mathrm{Cl}(X)$ is a standard Borel space with respect to the *Effros Borel structure* given by the σ-algebra generated by all subsets of the form

$$(4.3) \qquad \mathcal{O}(U) = \{F \in \mathrm{Cl}(X) \,:\, F \cap U = \emptyset\}$$

for some open subset $U \subset X$ [Kec12, Theorem 12.6]. In the special case where the metric space X is proper the Effros Borel structure coincides with the Borel structure for the Chabauty topology on $\mathrm{Cl}(X)$. For our purposes in §4 it will suffice to regard $\mathrm{Cl}(X)$ as a Borel rather than a topological space.

The action of the group $\mathrm{Is}(X)$ on the space X by isometries is continuous. It determines a Borel action of the group G on the Effros space $\mathrm{Cl}(X)$.

Proposition 4.4. *Let $Y \in \mathrm{Cl}(X)$ be a closed subset satisfying $\Lambda(G_\mu) \setminus \partial Y \neq \emptyset$. Then there is a subset of sample paths $\Omega \subset G^{\mathbb{N}}$ with $P_\mu(\Omega) > 0$ so that*

$$(4.4) \qquad \liminf_n d_X(\omega_n x_0, Y) = \infty$$

holds true for every sample path $\omega \in \Omega$.

Proof. The open subset $U = \Lambda(G_\mu) \setminus \partial Y$ satisfies $\nu_X(U) > 0$ by Proposition 4.3. Let $\Omega = \{\omega \in G^{\mathbb{N}} \,:\, \zeta_\omega \in U\}$ so that $P_\mu(\Omega) > 0$. Note that P_μ-almost every sample path $\omega \in \Omega$ satisfies $\omega_n x_0 \to \zeta_\omega$ for some point $\zeta_\omega \notin \partial Y$. Every such sample path ω must in paticular satisfy Equation (4.4). $\qquad\square$

Stationary closed subsets. Denote $\Lambda_\mu = \Lambda(G_\mu)$. We use the above information to deduce that μ-stationary random closed subsets of the space X have large limit sets.

Proposition 4.5. *Let ν be a μ-stationary probability measure on $\mathrm{Cl}(X)$. Then ν-almost every non-empty closed subset $Y \in \mathrm{Cl}(X)$ satisfies $\Lambda_\mu \subset \partial Y$.*

Proof. Consider the Borel function

$$(4.5) \qquad Y \mapsto d_X(x_0, Y), \quad \mathrm{Cl}(X) \setminus \{\emptyset\} \to [0, \infty)$$

defined on the space of closed non-empty subsets of X. Corollary 2.3 says that

$$(4.6) \qquad \liminf_n d_X(x_0, g_n \cdots g_1 Y) < \infty$$

for $\mu^{\otimes \mathbb{N}}$-almost every sample sequence of increments $(g_n) \in G^{\otimes \mathbb{N}}$ and ν-almost every closed non-empty subset $Y \subset X$. The desired conclusion follows from Equation (4.6) combined with Proposition 4.4 applied with respect to the probability measure $\check{\mu}$ given by $\check{\mu}(A) = \mu(A^{-1})$. $\qquad \square$

Limit sets and fixed points on the boundary. Given an arbitrary closed subset $A \subset \partial X$ we let $\mathrm{WH}(A)$ denote the union of all bi-infinite geodesics in X with both endpoints belonging to A. In particular $\mathrm{WH}(A) \in \mathrm{Cl}(X)$. Note that $\mathrm{WH}(A) \neq \emptyset$ if and only if $|A| \geq 2$.

Proposition 4.6. *Let ν be a μ-stationary random subgroup of G. If ν-almost every subgroup H of G satisfies $|\Lambda(H)| \geq 2$ then $\Lambda_\mu \subset \Lambda(H)$ holds ν-almost surely.*

The assumption that a subgroup $H \leq G$ satisfies $|\Lambda(H)| \geq 2$ is equivalent to saying that H contains an element acting on the space X via a loxodromic isometry.

Proof of Proposition 4.6. Consider the G-equivariant map $F_1 : \mathrm{Sub}(G) \to \mathrm{Cl}(X)$ given by

$$(4.7) \qquad F_1(H) = \mathrm{WH}(\Lambda(H)).$$

The assumption $|\Lambda(H)| \geq 2$ implies that $F_1(H) \neq \emptyset$. Note that if $\Lambda(H) \cap \Lambda_\mu \subsetneq \Lambda_\mu$ then $\partial \mathrm{WH}(H) \cap \Lambda_\mu \subsetneq \Lambda_\mu$. The result follows from Proposition 4.5. $\qquad \square$

Proposition 4.7. *Let ν be a μ-stationary random subgroup of G. If ν-almost every subgroup H is not contained in the elliptic radical $\mathrm{E}(G_\mu)$ then $|\mathrm{Fix}_{\Lambda_\mu}(H)| \leq 1$ holds true ν-almost surely.*

Proof. Consider the G-equivariant map $F_2 : \mathrm{Sub}(G) \to \mathrm{Cl}(X)$ given by

$$(4.8) \qquad F_2(H) = \mathrm{WH}(\mathrm{Fix}_{\Lambda_\mu}(H)).$$

If the subgroup H is ν-almost surely not contained in the elliptic radical $\mathrm{E}(G_\mu)$ then $\mathrm{Fix}_{\Lambda_\mu}(H)$ is ν-almost surely a proper subset of the limit set Λ_μ. In that case it follows from the contrapositive direction of Proposition 4.5 that $F_2(H) = \emptyset$ must hold ν-almost surely. In other words $|\mathrm{Fix}_{\Lambda_\mu}(H)| \leq 1$ holds ν-almost surely. $\qquad \square$

Proposition 4.8. *Let ν be a μ-stationary random subgroup of G. If ν-almost every subgroup H is not contained in the elliptic radical $\mathrm{E}(G_\mu)$ then $\Lambda(H) \neq \emptyset$ holds true ν-almost surely.*

Proof. Assume towards contradiction that the condition $\Lambda(H) = \emptyset$ is not ν-null. Every subgroup H with $\Lambda(H) = \emptyset$ has bounded orbits [CdCMT15, Proposition 3.1]. Fix some radius $R > 0$ and consider the function $F_3 : \mathrm{Sub}(G) \to \mathrm{Cl}(X)$ given by

$$(4.9) \qquad F_3(H) = \{x \in X \ : \ \sup_{h \in H} d_X(hx, h) \leq R\}.$$

Provided that the radius R is taken to be sufficiently large we have

$$(4.10) \qquad \nu(\{H \in \mathrm{Sub}(G) \ : \ F_3(H) \neq \emptyset\}) > 0$$

According to Proposition 4.5 we deduce that ν-almost every subgroup H with $F_3(H) \neq \emptyset$ satisfies $\Lambda_\mu \subset \partial F_3(H)$. Every such subgroup must be contained in the elliptic radical $\mathrm{E}(G_\mu)$. We arrive at a contradiction. $\qquad\square$

Putting together the above information we obtain the following.

Corollary 4.9. *Let ν be a μ-stationary random subgroup of G. If ν-almost every subgroup H is not contained in the elliptic radical $E(G_\mu)$ then either $\Lambda_\mu \subset \Lambda(H)$ or $|\Lambda(H)| = 1$ holds true ν-almost surely.*

Essential freeness of the boundary action. We conclude this section with a nice application to be used in §6 below. Recall that $\Lambda_\mu = \Lambda(G_\mu)$ is the limit set and $\mathrm{E}(G_\mu)$ is the elliptic radical of the group G.

Corollary 4.10. *Let ν be a μ-stationary random subgroup of G. Assume that $|\mathrm{Fix}_{\Lambda_\mu}(H)| = 1$ holds true for ν-almost every subgroup $H \in \mathrm{Sub}(G)$. Then for ν-almost every subgroup $H \in \mathrm{Sub}(G)$ the quotient group $H/(H \cap \mathrm{E}(G_\mu))$ acts essentially freely on the measure space (Λ_μ, ν_X).*

Proof. Let $(\mathrm{Sub}(G) \times \Lambda_\mu, \eta)$ be the μ-join[8] of the two μ-stationary spaces $(\mathrm{Sub}(G), \nu)$ and (Λ_μ, ν_X). This means that the probability measure η is μ-stationary and its marginals on the two factors $\mathrm{Sub}(G)$ and Λ_μ are ν and ν_X, respectively. There is a G-equivariant η-measurable map $\Psi : \mathrm{Sub}(G) \times \Lambda_\mu \to \mathrm{Sub}(G)$ given by $\Psi(H, \zeta) = H_\zeta$ where $H_\zeta = \mathrm{stab}_H(\zeta)$ for all subgroups $H \in \mathrm{Sub}(G)$ and boundary points $\zeta \in \partial X$. The pushforward measure $\Psi_* \eta$ is therefore a μ-stationary random subgroup of G. Note that $|\mathrm{Fix}_{\Lambda_\mu}(H)| \geq 2$ is true $\Psi_* \eta$-almost surely. So $\Psi_* \eta$-almost every stabilizer subgroup H_ζ belongs to the elliptic radical $\mathrm{E}(G_\mu)$ according to Proposition 4.7. The desired conclusion follows. $\qquad\square$

5. Geometric density for discrete stationary random subgroups

We have seen in the previous §4 that the limit set of a μ-stationary random subgroup is either the entire boundary ∂X or a singleton. It is easy to see that both possibilities may occur in general. Our current goal is to exclude the singleton case for *discrete* μ-stationary random subgroups.

We maintain the notations and assumptions of §4. In particular G is a locally compact group admitting a continuous action by isometries on a proper Gromov hyperbolic space X and μ is a probability measure on the group G. The semigroup G_μ generated by $\mathrm{supp}(\mu)$ is a non-elementary group and there exists a μ-stationary boundary measure as in Theorem 4.1. In addition we assume that the group G_μ has full limit set, namely $\Lambda(G_\mu) = \partial X$.

Theorem 5.1. *Let ν be a μ-stationary random subgroup of G. Assume that ν-almost every subgroup Γ acts* properly *on X and is not contained in the elliptic radical $\mathrm{E}(G_\mu)$. Then $\Lambda(\Gamma) = \partial X$ holds true ν-almost surely.*

The proof of Theorem 5.1 will be given towards the end of §5 below, after we develop some more machinery.

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[8]The reason to use the notion of μ-join is that the direct product of two μ-stationary spaces may not be μ-stationary with respect to the diagonal action. See [FG10, §3] for details.

Quasi-stabilizers and fixed points on the boundary. The notion of quasi-stabilizers $\Gamma_{x,R}$ was introduced in Definition 3.1. For every radius $R > 0$ and every $B \in \mathbb{N}$ we denote

$$(5.1) \qquad\qquad Q(\Gamma; R, B) = \overline{\{x \in X \ : \ |\Gamma_{x,R}| > B\}}$$

so that $Q(\Gamma; R, B) \in \mathrm{Cl}(X)$. Roughly speaking, if the group Γ fixes a point on the boundary at infinity then the sets $Q(\Gamma; R, B)$ are coarse analogues of horoballs.

Lemma 5.2. *Let Γ be a subgroup of G acting on the space X without loxodromic isometries. There is some constant $R > 0$ such that*

$$(5.2) \qquad\qquad \mathrm{Fix}_{\partial X}(\Gamma) \subset \partial Q(\Gamma; R, B)$$

for every $B \in \mathbb{N}$ with $B < |\Gamma|$.

The constant R in Lemma 5.2 depends only on the space X and is independent of the particular subgroup Γ.

Proof of Lemma 5.2. Consider an arbitrary boundary point $\zeta \in \mathrm{Fix}_{\partial X}(\Gamma)$ fixed by the action of the group Γ. Let R_0 be the constant provided by Lemma 3.2.

Fix some $B \in \mathbb{N}$ with $B < |\Gamma|$ as well as an arbitrary subset $\{\gamma_1, \ldots, \gamma_{B+1}\} \subset \Gamma$. Since every element of the group Γ is acting on the space X via a parabolic or an elliptic isometry there are points $x_1, \ldots, x_{B+1} \in X$ satisfying $\gamma_i \in \Gamma_{x_i, R_0}$ for all $i \in \{1, \ldots, B+1\}$. Take a geodesic ray l_i from the point x_i to the boundary point ζ for each $i \in \{1, \ldots, B+1\}$. There is some constant $R_1 = R_1(R_0, \delta)$ such that any point $y \in l_i$ satisfies $\gamma_i \in \Gamma_{y, R_1}$.

There is some point $z_B \in X$ within a distance of 5δ from all of the geodesic rays $l_1, \ldots, l_{B+1}$ [BH13, III.H.3.3.(2)]. The previous paragraph implies that $\gamma_i \in \Gamma_{z_B, R}$ for all indices $i \in \{1, \ldots, B+1\}$ provided that R is a constant satisfying $R > 10\delta + R_1$. In other words $z_B \in Q(\Gamma; R, B)$. In particular the subset in question $Q(\Gamma; R, B)$ is non-empty. Lastly note that the point z_B can be taken in an arbitrary small neighborhood of the boundary point ζ. It follows that $\zeta \in \partial Q(\Gamma; R, B)$ as required. $\square$

Lemma 5.3. *Let Γ be a subgroup of G acting on the space X without loxodromic isometries and fixing the point $\zeta_0 \in \partial X$. Let $x_n \in X$ be a sequence of points converging to some boundary point $\zeta \in \partial X$ distinct from ζ_0. Then for each $R > 0$ there is a point $z \in X$ and some $S > 0$ such that $\Gamma_{x_n, R} \subset \Gamma_{z, S}$ for all $n \in \mathbb{N}$.*

Proof. Let $R > 0$ be arbitrary. For each $n \in \mathbb{N}$ take a geodesic ray l_n from the point x_n to the fixed boundary point ζ_0. The sequence of geodesic rays l_n converges uniformly on compact sets to a bi-infinite geodesic line l from the point ζ_0 to the point ζ. Let $z \in l$ be an arbitrary point on the bi-infinite geodesic l. There is a sufficiently large constant $S > 0$ depending on the sequence x_n, on the constant R and on the hyperbolicity constant of the space X such that $\Gamma_{x_n, R} \subset \Gamma_{z, S}$ for all $n \in \mathbb{N}$. $\square$

Lemma 5.4. *Let Γ be a subgroup of G acting on the space X without loxodromic isometries and satisfying $\mathrm{Fix}_{\partial X}(\Gamma) \neq \emptyset$. Fix some constants $R > 0$ as well as $B \in \mathbb{N}$ with $B < |\Gamma|$. Then*

$$(5.3) \qquad\qquad \partial Q(\Gamma; R, B) \subset \{\zeta \in \partial X \ : \ |\Gamma_\zeta| > B\}.$$

Proof. Consider a point $\zeta \in \partial Q(\Gamma; R, B)$. We wish to show that $|\Gamma_\zeta| > B$. Let $\zeta_0 \in \text{Fix}_{\partial X}(\Gamma)$ be an arbitrary boundary point fixed by the action of Γ. We may assume without loss of generality that $\zeta \neq \zeta_0$ for otherwise $|\Gamma_\zeta| = |\Gamma| > B$ in which case we are done.

Take a sequence of points $x_n \in Q(\Gamma; R, B)$ converging to the boundary point ζ. According to Lemma 5.3 there is some point $z \in X$ and some constant $S > 0$ such that $\Gamma_{x_n,R} \subset \Gamma_{z,S}$ for all $n \in \mathbb{N}$. In other words, the subsets $\Gamma_{x_n,R}$ are all contained in the finite subset $\Gamma_{z,S}$ and satisfy $|\Gamma_{x_n,R}| > B$. Therefore

$$(5.4) \qquad |\limsup_n \Gamma_{x_n,S}| > B \quad \text{and} \quad \limsup_n \Gamma_{x_n,S} \subset \Gamma_\zeta.$$

We conclude that $|\Gamma_\zeta| > B$ as required. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Note that in the three previous lemmas (unlike the next lemma) the action of the subgroup Γ on the space X was *not* assumed to be proper.

Lemma 5.5. *Let Γ be a subgroup of G acting on the space X properly and without loxodromic isometries. If $\zeta_0 \in \text{Fix}_{\partial X}(\Gamma)$ then $|\Gamma_\zeta| < \infty$ for every $\zeta \in \partial X \setminus \{\zeta_0\}$.*

Proof. This statement follows immediately from [CdCMT15, Lemma 3.5]. $\qquad\square$

Stationary random subgroups acting properly. We have previously established in §4 that a μ-stationary random subgroup ν not contained in the elliptic radical $\text{E}(G_\mu)$ satisfies $|\text{Fix}_{\partial X}(H)| \leq 1$ for ν-almost every subgroup H. We now refine this result under the additional assumption of discreteness.

Proposition 5.6. *Let ν be a μ-stationary random subgroup of G. If ν-almost every subgroup Γ is acting properly on the space X and is not contained in the elliptic radical $\text{E}(G_\mu)$ then $|\text{Fix}_{\partial X}(\Gamma)| \neq 1$ holds true ν-almost surely.*

Proof. Throughout the proof it will be easier to assume without loss of generality that the measure ν is ergodic.

Assume towards contradiction that ν-almost every subgroup Γ acts properly on X and satisfies $\text{Fix}_{\partial X}(\Gamma) = \{\zeta_\Gamma\}$ for some boundary point $\zeta_\Gamma \in \partial X$. The point ζ_Γ depends on the subgroup Γ in a ν-measurable manner.

We know from Proposition 4.8 that $\Lambda(\Gamma) \neq \emptyset$ holds ν-almost surely. In particular ν-almost every subgroup Γ is infinite. By Gromov's classification of actions on hyperbolic spaces, the action of ν-almost every subgroup Γ is either horocyclic or focal (see §3). In the horocyclic case the action of the group Γ has no loxodromic isometries, and we may proceed with the proof as is.

In the focal case we first argue as follows. Consider the map $\theta : \text{Sub}(G) \to \text{Sub}(G)$ given by $\theta(\Gamma) = \overline{[\Gamma, \Gamma]}$. The push-forward measure $\nu_1 = \theta_*\nu$ is a μ-stationary subgroup of G. The action of ν'-almost every subgroup is proper, horocyclic and fixes at least one point on the boundary (see [CdCMT15, Corollary 3.9]). Proposition 4.7 says that ν'-almost every subgroup $\Gamma' = \theta(\Gamma)$ is either contained in the elliptic radical $\text{E}(G_\mu)$ or satisfies $\text{Fix}_{\partial X}(\Gamma') = \text{Fix}_{\partial X}(\Gamma) = \{\zeta_\Gamma\}$. The first possibility is eliminated by the fact that a focal action admits a pair of loxodoromic isometries γ_1, γ_2 with $\gamma_1^+ = \gamma_2^+ = \zeta_\Gamma$ but $\gamma_1^- \neq \gamma_2^-$. Up to replacing the μ-stationary random subgroup ν by $\nu_1 = \theta_*\nu$ in the focal case, we may assume for the remainder of the proof that ν-almost every subgroup Γ acts without loxodromic isometries.

For every subgroup $\Gamma \in \text{Sub}(G)$ consider the stabilizer sizes $|\Gamma_\zeta|$ as a function from the boundary ∂X to the countable set $\mathbb{N} \cup \{\infty\}$. Clearly $|\Gamma_{\zeta_\Gamma}| = \infty$ holds true

ν-almost surely. On the other hand $|\Gamma_\zeta| < \infty$ for every $\zeta \in \partial X$ with $\zeta \neq \zeta_\Gamma$ and for ν-almost every subgroup Γ, see Lemma 5.5. The Baire category theorem implies that there is some minimal $B \in \mathbb{N}$ such that the subset $\{\zeta \in \partial X \,:\, |\Gamma_\zeta| = B\}$ of the boundary ∂X has non-empty interior. As the measure ν was assumed to be ergodic the same B works for ν-almost every subgroup Γ. In fact, it is $\nu \times \nu_X$-almost surely the case that the pair (Γ, ζ) satisfies $\overline{\{\zeta \in \partial X \,:\, |\Gamma_\zeta| = B\}} = \partial X$, for otherwise the convex hull of the complementary subset of the boundary can be used to reach a contradiction to Proposition 4.5.

Consider the ν-measurable mapping

$$(5.5) \qquad \Phi : \Gamma \mapsto Q(\Gamma; R, B), \quad \Phi : \mathrm{Sub}(G) \to \mathrm{Cl}(X)$$

where $B \in \mathbb{N}$ is the constant determined in the previous paragraph and $R > 0$ is sufficiently large as in Lemma 5.2. On the one hand $\zeta_\Gamma \in \partial Q(\Gamma; R, B)$ holds ν-almost surely according to Lemma 5.2. On the other hand $\partial Q(\Gamma; R, B) \subsetneq \partial X$ occurs ν-almost surely because of the particular choice of B and of Lemma 5.4. Therefore $\Phi_* \nu$-almost every closed subset is non-empty and has proper boundary at infinity. We arrive at a contradiction to Proposition 4.5. $\qquad \square$

We are ready to conclude the following proof as a consequence of Proposition 5.6.

Proof of Theorem 5.1. Let ν be a μ-stationary random subgroup of G such that ν-almost every subgroup Γ acts properly on X and is not contained in the elliptic radical $\mathrm{E}(G_\mu)$. We know that $\Lambda(\Gamma) = \partial X$ or $|\Lambda(\Gamma)| = 1$ holds ν-almost surely, see Corollary 4.9. In the first case we are done. In the second case $\Lambda(\Gamma) \subset \mathrm{Fix}_{\partial X}(\Gamma)$ so that $|\mathrm{Fix}_{\partial X}(\Gamma)| \geq 1$ holds ν-almost surely. On the other hand $|\mathrm{Fix}_{\partial X}(\Gamma)| \leq 1$ according to Proposition 4.7. Putting everything together implies that $|\mathrm{Fix}_{\partial X}(\Gamma)| = 1$ holds ν-almost surely. This contradicts Proposition 5.6. $\qquad \square$

Corollary 5.7. *Assume that the action of the subgroup G_μ on the space X is of general type. Let ν be a μ-stationary random subgroup of G. If ν-almost every subgroup acts properly on the space X and is not contained in the elliptic radical $\mathrm{E}(G_\mu)$ then the action of ν-almost every subgroup is of general type.*

Amenable hyperbolic groups. Caprace, de Cornulier, Monod and Tessera [CdCMT15] developed a rich and detailed structure theory for amenable non-elementary Gromov hyperbolic locally compact groups. The action of a hyperbolic group on itself is focal if and only if the group is amenable and non-elementary hyperbolic. Every such group can be written as $\mathbb{R} \ltimes_\alpha H$ or $\mathbb{Z} \ltimes_\alpha H$ where $\alpha = \alpha(1)$ is a *compacting* automorphism of the locally compact group H, namely there is some compact subset $V \subset H$ such that every element $g \in H$ satisfies $\alpha^n(g) \in V$ for all sufficiently large n. Moreover such a group acts properly and cocompactly by isometries on some proper geodesically complete CAT(-1)-space.

Theorem 5.8. *Let G be an amenable non-elementary hyperbolic locally compact group and μ a probability measure on the group G. Assume that the measure μ has finite first moment and is spread out. Then every discrete μ-stationary random subgroup of G is contained in the elliptic radical $\mathrm{E}(G_\mu)$.*

Proof. First consider the case where the Poisson boundary of the pair (G, μ) is not trivial. This means that the action of the group G on itself by isometries gives rise to a μ-stationary boundary measure ν_G supported on the limit set $\Lambda(G_\mu) \subset \partial G$ by the focal case of Theorem 4.1. Unless ν-almost every discrete subgroup Γ is contained

in the elliptic radical $\mathrm{E}(G)$, we get that $\mathrm{Fix}_{\partial G}(\Gamma) = \emptyset$ for ν-almost every discrete subgroup Γ according to the two Propositions 4.7 and 5.6. This is a contradiction to the fact that this action is focal.

Alternatively, consider the case where the Poisson boundary of the pair (G, μ) is trivial. This means that any μ-stationary probability measure is in fact G-invariant, see e.g. [Fur02]. In particular the probability measure ν is a discrete invariant random subgroup. We may find some compactly supported spread out probability measure μ' on the group G such that the Poisson boundary of the pair (G, μ') is *non* trivial. This is possible by [Jaw95, Theorem 3.16] and by taking into account the fact that the locally compact group G is non-elementary Gromov hyperbolic and as such has exponential growth. The invariant random subgroup ν can be regarded as a μ'-stationary random subgroup. At this point we may conclude the proof exactly as in the previous paragraph. $\qquad\square$

Remark 5.9. *The stabilizer of a point at infinity for a general type action of a locally compact group is a special case of an amenable non-elementary hyperbolic group. For such groups Theorem 5.8 can be proved directly by inducing stationary random subgroups. The proof given above is much more general however.*

Remark 5.10. *If G is any locally compact group with modular function Δ_G and ν is a discrete invariant random subgroup of G then ν-almost every subgroup is contained in $\ker \Delta_G$ [BT17, Corollary 1.2]. In the context of amenable hyperbolic groups $\ker \Delta_G = H$. This can be used to get a weaker conclusion.*

6. Discrete stationary random subgroups of products

Let X be a proper Gromov hyperbolic geodesic metric space for which the action of $\mathrm{Is}(X)$ is of general type. Let Y be any proper metric space. Fix a pair of arbitrary base points $x_0 \in X$ and $y_0 \in Y$. Regard the product $X \times Y$ as a proper metric space with the L^2-product metric. Consider the group G of isometries of the space $X \times Y$ preserving each factor, namely

$$(6.1) \qquad\qquad G = \mathrm{Is}(X) \times \mathrm{Is}(Y).$$

Let p_X and p_Y denote the projection homomorphisms from the group G to its coordinates $\mathrm{Is}(X)$ and $\mathrm{Is}(Y)$ respectively. The group G is acting by isometries on each factor X and Y via the projections p_X and p_Y.

Let $\mathrm{DSub}(G)$ denote the Chabauty space of all discrete subgroups of the group G. Given a discrete subgroup $\Gamma \in \mathrm{DSub}(G)$ it will be convenient to introduce the shorthand notation

$$(6.2) \qquad\qquad F_X(\Gamma) = \mathrm{Fix}_{\partial X}(\Gamma)$$

where the group Γ is understood to act on the space X via the projection p_X.

Fix a probability measure μ_X on the group $\mathrm{Is}(X)$. Assume that $\mathrm{supp}(\mu_X)$ generates as a semigroup a dense subgroup of $\mathrm{Is}(X)$.

Theorem 6.1. *Let ν be a μ_X-stationary probability measure on the space $\mathrm{DSub}(G)$. Then either $F_X(\Gamma) = \emptyset$ or $p_X(\Gamma)$ is contained in the elliptic radical $\mathrm{E}(\mathrm{Is}(X))$ for ν-almost every subgroup Γ.*

Proof. We assume for the sake of convenience and without loss of generality that the measure ν is ergodic.

Assume that the projection $p_X(\Gamma)$ is ν-almost surely *not* contained in the elliptic radical $\mathrm{E}(\mathrm{Is}(X))$. In this case we know by Proposition 4.7 that $|F_X(\Gamma)| \leq 1$ holds true ν-almost surely. Our goal is to show that in fact $F_X(\Gamma) = \emptyset$ holds ν-almost surely.

There are two cases to consider, depending on whether the action of ν-almost every subgroup Γ on the factor X is proper or not. In the proper case $F_X(\Gamma) = \emptyset$ holds true ν-almost surely by Proposition 5.6 and we are done. For the remainder of the proof assume that we are in the non-proper case, namely the projection $p_X(\Gamma)$ is ν-almost surely a non-discrete subgroup of $\mathrm{Is}(X)$, and that $F_X(\Gamma) = \{\zeta_\Gamma\}$ for some point $\zeta_\Gamma \in \partial X$ (depending measurably on the subgroup Γ). We will arrive at a contradiction.

To begin with, and up to replacing the random subgroup ν by its pushforward with respect to the commutator closure map $\Gamma \mapsto \overline{[\Gamma, \Gamma]}$, we may assume that ν-almost every subgroup admits no loxodromic elements in its action on the factor X via the projection p_X. See the proof of Proposition 5.6 for more details concerning this argument.

Fix an arbitrary radius $R > 0$. For each discrete subgroup $\Gamma \in \mathrm{DSub}(G)$ consider the quasi-stabilizer

$$(6.3) \qquad \Gamma_{x_0,R} = \{\gamma \in \Gamma \; : \; d_X(p_X(\gamma)x_0, x_0) \leq R\}$$

corresponding to the action of the subgroup Γ on the factor X via the projection map p_X. This quasi-stabilizer is ν-almost surely infinite as the projection $p_X(\Gamma)$ is non-discrete.

Let $\omega \in \mathrm{Is}(X)^{\mathbb{N}}$ be a P_{μ_X}-random sample path. We claim that

$$(6.4) \qquad \limsup_n p_X(\Gamma_{\omega_n x_0, R}) \subset \mathrm{E}(\mathrm{Is}(X)).$$

Indeed the sequence of points $\omega_n x_0$ converges P_{μ_X}-almost surely to the boundary point $\zeta_\omega \in \partial X$ satisfying moreover $\zeta_\omega \neq \zeta_\Gamma$. We rely on the fact that any element $\gamma \in \Gamma$ such that $p_X(\gamma)$ is not contained in the elliptic radical $\mathrm{E}(\mathrm{Is}(X))$ is acting essentially freely on $(\partial X, \nu_X)$, see Corollary 4.10. Therefore

$$(6.5) \qquad \liminf_n d_X\left(p_X(\gamma)\omega_n x_0, \omega_n x_0\right) = \infty$$

P_{μ_X}-almost surely and for any element $\gamma \in \Gamma$ with $p_X(\gamma) \notin \mathrm{E}(\mathrm{Is}(X))$. In particular any such element γ must satisfy

$$(6.6) \qquad \gamma \notin \limsup_n \Gamma_{\omega_n x_0, R}.$$

The claim follows.

The discreteness of the subgroup Γ implies that $p_Y(\Gamma_{x,R})$ is ν-almost surely a discrete subset of $\mathrm{Is}(Y)$ for any point $x \in X$. We define the Borel function

$$(6.7) \qquad f : \mathrm{DSub}(G) \to [0, \infty), \quad f(\Gamma) = \inf_{\substack{\gamma \in \Gamma_{x_0,R} \\ \gamma \notin \mathrm{E}(G)}} d_Y(p_Y(\gamma)y_0, y_0).$$

Given an element $g \in \mathrm{Is}(X) \leq G$ we obviously have that $p_Y(g)y_0 = y_0$ and so

$$(6.8) \qquad f(\Gamma^g) = \inf_{\substack{\gamma \in \Gamma^g_{x_0,R} \\ \gamma \notin \mathrm{E}(G)}} d_Y(p_Y(\gamma)y_0, y_0) = \inf_{\substack{\gamma \in \Gamma_{g^{-1}x_0,R} \\ \gamma \notin \mathrm{E}(G)}} d_Y(p_Y(\gamma)y_0, y_0).$$

We remark that for P_μ-almost every sample path ω there is some point $z \in X$ and some constant $S > 0$ such that the subsets $p_Y(\Gamma_{\omega_n x_0, R})$ are all contained in the

discrete subset $p_Y(\Gamma_{z,S}) \subset \mathrm{Is}(Y)$, see Lemma 5.3. This remark, combined with the preceding claim applied for each fixed value $R > 0$ and with Equation (6.8), show that

$$\liminf_n f(\Gamma^{g_n g_{n-1} \cdots g_1}) = \infty \tag{6.9}$$

for ν-almost every subgroup Γ and $\mu_X^{\otimes \mathbb{N}}$-almost every sequence of increments $(g_n) \in \mathrm{Is}(X)^{\mathbb{N}}$. We emphasize that the sample path positions ω_n all belong to the factor $\mathrm{Is}(X)$. We arrive at a contradiction in light of Kakutani's ergodic theorem, see Corollary 2.3 specifically.

We conclude that $F_X(\Gamma) = \emptyset$ holds ν-almost surely, as required. $\qquad\square$

Products of hyperbolic spaces. Fix some $k \in \mathbb{N}$. Let $X_1, \ldots, X_k$ be a collection of proper Gromov hyperbolic geodesic metric spaces. Assume that the action of the group $\mathrm{Is}(X_i)$ on the space X_i has general type for each i. Let X denote the product space

$$X = X_1 \times \cdots \times X_n \tag{6.10}$$

endowed with the product metric.

Let G denote the finite index subgroup of the full group of isometries $\mathrm{Is}(X)$ consisting of these isometries that preserve each individual factor X_i for $i \in \{1, \ldots, k\}$. In other words

$$G = \mathrm{Is}(X_1) \times \cdots \times \mathrm{Is}(X_k). \tag{6.11}$$

Let p_i denote the projection homomorphism from the group G to the coordinate $\mathrm{Is}(X_i)$ for each $i \in \{1, \ldots, k\}$. The group G is acting on each factor X_i via the projection p_i.

Fix a probability measure μ_i on the group $\mathrm{Is}(X_i)$ for each $i \in \{1, \ldots, k\}$. Assume that $\mathrm{supp}(\mu_i)$ generates as a semigroup a dense subgroup of $\mathrm{Is}(X_i)$. Finally denote

$$\mu = \mu_1 \otimes \cdots \otimes \mu_k. \tag{6.12}$$

Corollary 6.2. *Let ν be a discrete μ-stationary random subgroup of G. If the projection $p_i(\Gamma)$ is ν-almost surely not contained in the elliptic radical $\mathrm{E}(\mathrm{Is}(X_i))$ for each i then the action of ν-almost every subgroup Γ on the factor X_i has general type.*

Proof. Note that the measure ν is μ_i-stationary for each i by [BS06, Lemma 3.1]. Applying Theorem 6.1 with respect to each factor X_i individually implies that $F_i(\Gamma) = \mathrm{Fix}_{\partial X_i}(\Gamma) = \emptyset$ for each index $i \in \{1, \ldots, k\}$. We may now argue exactly as in the proof of Theorem 5.1 with respect to the action of ν-almost every subgroup Γ on each factor X_i via the projection map p_i. The only difference is that Proposition 5.6 preventing the possibility of admitting a single fixed point on the boundary in the case of a single hyperbolic space is replaced by the information obtained from Theorem 6.1. $\qquad\square$

Product of CAT(-1)**-spaces.** Fix some $k \in \mathbb{N}$. Let $X_1, \ldots, X_k$ be a collection of proper CAT(-1)-spaces. Let X denote the product space $X = X_1 \times \cdots \times X_n$ regarded with the product CAT(0) metric [BH13, p. 168].

The boundary at infinity ∂X is a CAT(1)-space with respect to the angular metric [BH13, Theorem II.9.13]. It is isometric to the spherical join of the individual

boundaries [BH13, §II.8.11], i.e.

$$\partial X = \partial X_1 * \partial X_2 * \cdots * \partial X_k. \tag{6.13}$$

The boundary ∂X is realized as follows. Let Σ be a topological realization of the combinatorial $(k-1)$-simplex with vertex set $V = \{1, \ldots, k\}$. Then

$$\partial X = \partial X_1 \times \partial X_2 \times \cdots \times \partial X_k \times \Sigma / \sim \tag{6.14}$$

where $\sim$ is the equivalence relation defined in terms of the quotient map taking points lying over some face F of the simplex Σ to the sub-product of these factors ∂X_i's for which $i \in F$. The fiber of the natural continuous map $\sigma : \partial X \to \Sigma$ over every interior point of the simplex Σ is homeomorphic to the direct product of the individual boundaries ∂X_i's. More generally, the fiber of the map σ over a point lying on the face F of the simplex Σ is homeomorphic to the product of these factors X_i for which $i \in F$. A boundary point $\zeta \in \partial X$ is called *regular* if $\sigma(\zeta) \in \mathring{\Delta}$. Otherwise the boundary point ζ is called *singular*.

With the additional standing assumption that the spaces X_i are CAT(-1) rather than Gromov hyperbolic, it is possible to extend Theorem 6.1 with respect to all points at infinity of the product CAT(0)-space X, regular as well as singular ones.

Corollary 6.3. *Let ν be a discrete μ-stationary random subgroup of the product $\prod_{i=1}^{k} \mathrm{Is}(X_i)$. If ν-almost every subgroup projects non-trivially to each factor $\mathrm{Is}(X_i)$ then ν-almost every subgroup Γ has $\mathrm{Fix}_{\partial X}(\Gamma) = \emptyset$.*

Proof. Note that the elliptic radical of $\mathrm{Is}(Z)$ is trivial for any CAT(-1)-space Z. For each discrete subgroup $\Gamma \in \mathrm{DSub}(G)$ denote

$$F_i(\Gamma) = \mathrm{Fix}_{\partial X_i}(\Gamma) \subset \partial X_i \quad \text{for each} \quad i \in \{1, \ldots, k\}. \tag{6.15}$$

The spherical join

$$F(\Gamma) = F_1(\Gamma) * \cdots * F_k(\Gamma) \tag{6.16}$$

can be naturally identified with a closed subset of the boundary ∂X. It can be seen from the above explicit realization of the boundary ∂X that $F(\Gamma) = \mathrm{Fix}_{\partial X}(\Gamma)$. In particular $F(\Gamma) = \emptyset$ if and only if $F_i(\Gamma) = \emptyset$ for each $i \in \{1, \ldots, k\}$. With this information at hand the corollary follows immediately from Theorem 6.1 applied with respect to each factor X individually. We recall that the measure ν is in fact μ_i-stationary for each i by [BS06, Lemma 3.1]. $\square$

7. CAT(0)-SPACES AND GEOMETRIC DENSITY

Let X be a proper geodesically complete CAT(0)-space with a fixed base point $x_0 \in X$. We assume that the group of isometries $\mathrm{Is}(X)$ is acting on the space X cocompactly and without fixed points at infinity.

Definition 7.1. A subgroup Γ of the isometry group $\mathrm{Is}(X)$ is acting

- *minimally* if there are no proper closed convex Γ-invariant subsets, and
- *geometrically densely* if it acts minimally on X and without fixed points on the boundary ∂X.

Our standing assumptions imply that the space X is *boundary minimal*, i.e every closed convex subset $Y \subsetneq X$ has $\partial Y \subsetneq \partial X$ [CM09b, Proposition 1.5]. In particular a subgroup Γ of $\mathrm{Is}(X)$ acts geometrically densely provided it admits no proper closed invariant subsets on the boundary ∂X. Some information about the converse direction of this statement is provided by the following result.

Lemma 7.2. *Let G be a subgroup of $\mathrm{Is}(X)$ acting geometrically densely on the space X and $\emptyset \subsetneq B \subset \partial X$ be a G-invariant closed subset. Then no proper closed convex subset $Y \subsetneq X$ has $B \subset \partial Y$.*

Proof. Assume towards contradiction that the family

$$(7.1) \qquad \mathcal{F} = \{Y \subsetneq X \text{ is a closed convex subset such that } B \subset \partial Y\}$$

if non-empty. Consider the intersection

$$(7.2) \qquad\qquad Z = \bigcap_{Y \in \mathcal{F}} Y$$

so that Z is proper G-invariant closed convex subset of X. The minimality of the G-action implies that $Z = \emptyset$. This means that the boundary subset $C = \bigcap_{Y \in \mathcal{F}} \partial Y$ has an intrinsic circumradius of at most $\pi/2$ with respect to the Tits angle[9] on the boundary ∂X [CM09b, Proposition 3.2]. Additionally $B \subset C$ so that $C \neq \emptyset$. Therefore the subset C has a unique circumcentre $z \in \partial Y$ [CM09b, Proposition 3.2]. The G-action fixes the point z. This is a contradiction. $\qquad\square$

Random walks and boundary measures. Let μ be a probability measure on the group $\mathrm{Is}(X)$. Assume that the semigroup G_μ generated by the support of μ is a group acting geometrically densely on the space X. We further assume that the measure μ has positive drift, in the sense that the parameter A given by

$$(7.3) \qquad\qquad A = \lim_{n \to \infty} \frac{1}{n}\|\omega_n\| = \lim_{n \to \infty} \frac{1}{n}d_X(\omega_n x_0, x_0)$$

for P_μ-almost every sample path $\omega \in G^{\mathbb{N}}$ satisfies $A > 0$, see [KM99, Theorem 2.1].

Theorem 7.3 (Karlsson–Margulis [KM99])**.** *For P_μ-almost every sample path $\omega \in G^{\mathbb{N}}$ the sequence $\omega_n x_0$ converges to a boundary point $\zeta_\omega \in \partial X$ depending on ω.*

Let ν_X be the probability measure on the boundary ∂X obtained by pushing forward the measure P_μ via the map $\omega \mapsto \zeta_\omega$. The measure ν_X is μ-stationary.

The following result in the context of general CAT(0)-spaces is to be compared with e.g. [BQ16b, Proposition 9.1.b] in the context of symmetric spaces.

Proposition 7.4. *Any proper closed convex subset $Y \subsetneq X$ has $\mathrm{supp}(\nu_X) \not\subset \partial Y$.*

Proof. The probability measure ν_X is quasi-invariant by Lemma 2.4. In particular $\mathrm{supp}(\nu_X)$ is a G-invariant subset of the boundary ∂X. The desired statement follows immediately from Lemma 7.2. $\qquad\square$

Stationary random convex subsets. Assume that the group $\mathrm{Is}(X)$ is acting geometrically densely on the space X. We are ready to formulate a CAT(0)-space variant of Proposition 4.5.

Proposition 7.5. *Let ν be a μ-stationary probability measure on $\mathrm{Cl}(X)$ such that ν-almost every closed subset is convex. Then $\mathrm{supp}(\nu) \subset \{\emptyset, X\}$.*

Proof. It will be convenient to assume that the measure ν is ergodic. Say towards contradiction that ν-almost every closed convex subset Y has $\emptyset \subsetneq Y \subsetneq X$. Consider the ν-measurable function

$$(7.4) \qquad\qquad \Phi : \mathrm{Cl}(X) \setminus \{\emptyset\} \to [0, \infty), \quad \Phi : Y \mapsto d_X(x_0, Y).$$

---

[9]The Tits angle is called angular metric in [BH13, II.9]

We know that ν-almost every closed convex subset $Y \subset X$ satisfies $\operatorname{supp}(\nu_X) \subsetneq \partial Y$, see Proposition 7.4. This means that a sample path $(\omega_n) \in G^{\mathbb{N}}$ with $\omega_n = g_1...g_n$ satisfies $\omega_n x_0 \to \zeta$ for some $\zeta \notin \partial Y$ with positive P_μ-probability. In particular

$$(7.5) \qquad \Phi(g_n^{-1} \cdots g_1^{-1} Y) = d_X(\omega_n x_0, Y) \xrightarrow{n \to \infty} \infty.$$

This is a contradiction to Corollary 2.3 applied with respect to the function Φ defined in Equation (7.4) and the probability measure $\check{\mu}$ given by $\check{\mu}(A) = \mu(A^{-1})$. $\qquad\square$

Corollary 7.6. *Let ν be a μ-stationary random subgroup of $\operatorname{Is}(X)$. If ν-almost every subgroup fixes no point of the boundary ∂X then ν-almost every subgroup is acting minimally, and is in particular geometrically dense.*

Proof. Our standing assumptions on the space X imply that its Tits boundary ∂X is finite dimensional [Kle99]. This fact combined with the assumption that μ-almost every subgroup has no fixed points on the boundary rules out case (A) of the dichotomy presented in [CM09b, Theorem 4.3]. We deduce that ν-almost every subgroup H admits a *canonical* minimal closed convex non-empty invariant subset $\mathcal{M}(H) \in \operatorname{Cl}(X)$, see case (B.iii) of [CM09b, Theorem 4.3]. Consider the μ-stationary pushforward probability measure $\mathcal{M}_*\nu$ on the space of closed subsets $\operatorname{Cl}(X)$. According to Proposition 7.5 it must be the case that $\mathcal{M}_*\nu = \mathrm{D}_X$. In other words ν-almost every subgroup is acting minimally. $\qquad\square$

As an application of our results on ν-stationary subgroups of the isometry group $\operatorname{Is}(X)$ we obtain the following. This can be seen as a stationary random subgroup variant of the geometric Borel density theorem, see [CM09a, Theorem 2.4] as well as the invariant random subgroup version [DGLL15].

Theorem 7.7. *Assume that the $\operatorname{CAT}(0)$-space X is given by $X_1 \times \cdots \times X_k$ where each X_i is a proper $\operatorname{CAT}(-1)$-space with $\operatorname{Is}(X_i)$ acting geometrically densely. Let ν be a discrete μ-stationary random subgroup of the product $\prod_{i=1}^{k} \operatorname{Is}(X_i)$. If ν-almost every subgroup projects non-trivially to each factor then ν-almost every subgroup is acting geometrically densely.*

Proof. We know that ν-almost every subgroup has no fixed points on the boundary ∂X by Corollary 6.3. Therefore ν-almost every subgroup is geometrically dense by Corollary 7.6. $\qquad\square$

Finally we have the following $\operatorname{CAT}(0)$-space variant of Proposition 4.8.

Proposition 7.8. *Let ν be a μ-stationary random subgroup of $\operatorname{Is}(X)$ such that ν-almost every subgroup is non-trivial. Then ν-almost every subgroup has unbounded orbits in X.*

Proof. Let $\Phi : \operatorname{Sub}(G) \to \operatorname{Cl}(X)$ be the G-equivariant ν-measurable function taking a closed subgroup $H \in \operatorname{Sub}(G)$ to its closed convex fixed point set $\operatorname{Fix}(H) \in \operatorname{Cl}(X)$. Consider the μ-stationary pushforward probability measure $\Phi_*\nu$ on the space of closed subsets $\operatorname{Cl}(X)$. It follows from Proposition 7.5 that $\Phi_*\nu = \alpha \mathrm{D}_\emptyset + (1-\alpha)\mathrm{D}_X$ for some real number $\alpha \in [0,1]$. The only subgroup $H \in \operatorname{Sub}(G)$ with $\Phi(H) = X$ is the trivial subgroup. On the other hand, recall that any bounded subset of a $\operatorname{CAT}(0)$-space has a unique circumcentre [BH13, II.2]. Therefore any subgroup $H \in \operatorname{Sub}(G)$ having bounded orbits in X satisfies $\Phi(H) \neq \emptyset$. The desired conclusion follows by combining these facts. $\qquad\square$

8. Random walks and Poincare series

The main result of this section is Theorem 8.8. It is stated below after we introduce some relevant notions concerning random walks.

Let X be a proper unbounded metric space with a fixed base point $x_0 \in X$. Let Γ be a discrete group acting properly and cocompactly by isometries on the metric space X. Following our standing notation, we will write

$$(8.1) \qquad \|g\| = d_X(gx_0, x_0)$$

for every element $g \in \Gamma$, with the chosen base point x_0 implicit in the notation.

We emphasize that the space X or (what is equivalent) the group Γ are not required to be Gromov hyperbolic.

Green's function. Let μ be a probability measure on the group Γ. Assume that

- the support of the measure μ generates the group Γ as a semigroup,
- the random walk determined by the measure μ is transient, and
- the measure μ has *finite first moment*, namely $\sum_{g \in \Gamma} \mu(g)\|g\| < \infty$.

Note that the random walk determined by μ will be transient provided the group Γ is not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$. Consider the *Green's function* associated to the probability measure μ

$$(8.2) \qquad \mathcal{G}(g, h) = \sum_{n=0}^{\infty} \mu^{*n}(g^{-1}h)$$

as well as the *first return Green's function*

$$(8.3) \qquad \mathcal{F}(g, h) = \frac{\mathcal{G}(g, h)}{\mathcal{G}(e, e)}$$

defined for all elements $g, h \in \Gamma$. The transience assumption implies that the quantities $\mathcal{F}(g, h)$ and $\mathcal{G}(g, h)$ are finite for any pair of elements $g, h \in \Gamma$. The first return Green's function satisfies

$$(8.4) \qquad \mathcal{F}(g, h) = \mathrm{P}_\mu(\{\omega \ : \ g^{-1}h = \omega_n \text{ for some } n \in \mathbb{N}\})$$

for all elements $g, h \in \Gamma$.

The expression $d_\mu = -\log \mathcal{F}$ defines a (possibly asymmetric) distance function on the group Γ called the *Green metric*. The Green metric d_μ is quasi-isometric to a word metric on the group Γ (see [BHM11, Proposition 3.6] for symmetric measures and [GT20, Proposition 7.8] in general).

Proposition 8.1. *If the group Γ has exponential growth then the expression*

$$(8.5) \qquad \sum_{\substack{g \in \Gamma \\ \|g\| \leq n}} \mathcal{F}(e, g)$$

grows subexponentially in n.

Proof. By the Harnack inequality there are constants $k > 0$ and $0 < t < \infty$ depending the group Γ and the measure μ so that

$$(8.6) \qquad \mathcal{G}(g, h) \geq kt^{\|g^{-1}h\|}$$

for all elements $g, h \in \Gamma$, see e.g. [Woe00, (25.1)]. In particular there is a constant $c > 0$ so that $d_\mu(e, g) \leq c\|g\|$ for all elements $g \in \Gamma$. Therefore

$$(8.7) \qquad \sum_{\substack{g \in \Gamma \\ \|g\| \leq n}} \mathcal{F}(e, g) \leq \sum_{\substack{g \in \Gamma \\ d_\mu(e,g) \leq cn}} e^{-d_\mu(e,g)} \leq b \sum_{i=1}^{\lceil cn \rceil} e^{-i} |\{g \in \Gamma : d_\mu(e, g) \leq i\}|$$

for some constant $b > 0$ which depends on the group Γ and the measure μ. The exponential growth condition implies that $|\{g \in \Gamma : d_\mu(e, g) \leq i\}| \leq \varphi(i)e^i$ for some subexponentially growing function $\varphi : \mathbb{N} \to \mathbb{N}$ [BHM08, Proposition 3.1]. This completes the proof. $\qquad \square$

Entropy and drift. Recall that μ is a probability measure on the group Γ such that $\text{supp}(\mu)$ generates Γ as a semigroup.

The *drift* of the measure μ is

$$(8.8) \qquad l(\mu) = \lim_{n \to \infty} \frac{1}{n}\|\omega_n\|.$$

The first moment assumption implies that this limit exists for P_μ-almost every sample path ω and is independent of ω. We have $l(\mu) > 0$ whenever the group Γ is non-amenable, see e.g. [Woe00, Corollary 8.15].

The *entropy* of the measure μ is

$$(8.9) \qquad h(\mu) = \lim_{n \to \infty} \frac{1}{n}\text{H}(\mu^{*n})$$

where $\text{H}(\cdot)$ denotes Shannon entropy. This limit exists and is finite by the finite first moment condition. Again, we have $h(\mu) > 0$ whenever the group Γ is non-amenable [KV83]. The entropy $h(\mu)$ coincides with the drift with respect to the Green metric [BHM08, Theorem 1.1].

The fundamental inequality of Guivarch relates entropy and drift [Gui80]. It states that

$$(8.10) \qquad h(\mu) \leq l(\mu)\delta(\Gamma)$$

where $\delta(\Gamma)$ is the critical exponent[10] of the group Γ with respect to its action on the metric space X. See also [Ver00, BHM08].

From now on we will assume that the group Γ is *non-amenable* (so that in particular $h(\mu) > 0$ and $l(\mu) > 0$). For our purposes it will be convenient to introduce the notation

$$(8.11) \qquad \delta(\mu) = \frac{h(\mu)}{l(\mu)}.$$

Annuli and hitting measures. For each $i \in \mathbb{N}$ consider the *annulus* given by

$$(8.12) \qquad A_i = \{g \in \Gamma : i - 1 \leq \|g\| < i\}.$$

Assume that μ is a probability measure on the infinite group Γ with finite first moment. Given a sample path $\omega \in \Gamma^{\mathbb{N}}$ and for each $i \in \mathbb{N}$ there is a *first hitting time*

$$(8.13) \qquad \tau_i(\omega) = \inf\{n \in \mathbb{N} : \omega_n \in A_i\}.$$

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[10]The notion of critical exponent is discussed in §9 below.

The first hitting time $\tau_i(\omega)$ may be infinite[11] in the situation where $\omega_n \notin A_i$ for all $n \in \mathbb{N}$. We define
$$\theta(\omega) = \{i \in \mathbb{N} \: : \: \tau_i(\omega) < \infty\} \subset \mathbb{N}.$$
The positivity of the drift $l(\mu)$ implies that the set $\theta(\omega)$ is infinite for P_μ-almost every sample path $\omega \in \Gamma^{\mathbb{N}}$.

Let ν_i be the *first hitting measure* on the annulus A_i determined at the first hitting time τ_i for each $i \in \mathbb{N}$. Formally this means

$$(8.14) \qquad \nu_i(E) = \mathrm{P}_\mu(\{\omega \: : \: i \in \theta(\omega) \quad \text{and} \quad \omega_{\tau_i(\omega)} \in E\})$$

for each subset $E \subset A_i$. Formally ν_i is a positive measure on the annulus A_i satisfying $0 \leq \nu_i(A_i) \leq 1$.

Lemma 8.2. *There is a constant $\zeta(\mu) > 0$ such that P_μ-almost every sample path $\omega \in \Gamma^{\mathbb{N}}$ satisfies*

$$(8.15) \qquad \liminf_{n \to \infty} \frac{|\theta(\omega) \cap [n]\,|}{n} > \zeta(\mu).$$

Proof. For each given sample path $\omega \in \Gamma^{\mathbb{N}}$ and each $n \in \mathbb{N}$ consider the closed interval

$$(8.16) \qquad I_n(\omega) = \left[0, \max_{i \in [n]} \|\omega_i\|\right] \subset \mathbb{R}.$$

Further, for each $K > 0$, let $L_{n,K}(\omega)$ denote the total length of the open subintervals of $I_n(\omega)$ of length $\geq K$ not containing any of the n-many points $\{\|\omega_1\|, \ldots, \|\omega_n\|\}$. By the triangle inequality

$$(8.17) \qquad |\|\omega_i\| - \|\omega_{i-1}\|| \leq \|\omega_{i-1}^{-1}\omega_i\| = \|g_i\|$$

where $(g_i) \in \Gamma^{\mathbb{N}}$ is the sequence of increments corresponding to the sample path ω. It follows that $L_{n,K}(\omega)$ is upper bounded by the quantity $M_{n,K}(\omega)$ given by

$$(8.18) \qquad M_{n,K}(\omega) = \sum_{\{i \in [n] \,:\, \|g_i\| \geq K\}} \|g_i\|.$$

The strong law of large numbers implies for P_μ-almost every sample path $\omega \in \Gamma^{\mathbb{N}}$ that

$$(8.19) \qquad \lim_{n \to \infty} \frac{1}{n} M_{n,K}(\omega) = \sum_{\substack{g \in \Gamma \\ \|g\| \geq K}} \mu(g)\|g\|.$$

Finally, the finite first moment assumption means that the right hand side in Equation (8.19) tends to 0 as K tends to infinity. In follows that

$$(8.20) \qquad \lim_{K \to \infty} \limsup_{n \to \infty} \frac{1}{n} L_{n,K}(\omega) = 0.$$

On the other hand, recall that by definition $\lim_{n \to \infty} \frac{\|\omega_n\|}{n} = l(\mu) > 0$. Therefore, provided the parameter $K > 0$ is sufficiently large, a positive proposition of the points in the interval $I_n(\omega)$ are within distance at most K from some point of the form $\|\omega_i\|$. This is equivalent to the desired conclusion, for a suitable choice of the constant $\zeta(\mu) > 0$. $\qquad \square$

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[11]This is precisely the technical challenge that comes up when working with infinitely supported probability measures.

Notation 8.3. $\lim_{i \in \theta(\omega)}$ is a limit taken over all indices i from the set $\theta(\omega)$ in ascending order. This makes sense for P_μ-almost every sample path ω where the subset $\theta(\omega)$ is infinite.

The following is stated inside the proof of [Tan17, Theorem 6.1] for finitely supported random walks on hyperbolic groups. For the sake of the completeness we include the proof, communicated to us by Tanaka. It works identically in our setting.

Lemma 8.4. P_μ-*almost every sample path* $\omega \in \Gamma^{\mathbb{N}}$ *satisfies*

$$(8.21) \qquad \lim_{i \in \theta(\omega)} \frac{\tau_i(\omega)}{i} \to \frac{1}{l(\mu)}.$$

Proof. Clearly P_μ-almost every sample path ω is such that the times $\tau_i(\omega)$ with $i \in \theta(\omega)$ are pairwise distinct. In particular $\tau_i(\omega) \to \infty$ as $i \to \infty$. Moreover, P_μ-almost every sample path ω has

$$(8.22) \qquad \lim_{i \in \theta(\omega)} \frac{\|\omega_{\tau_i(\omega)}\|}{\tau_i(\omega)} \to l(\mu)$$

by definition of the drift $l(\mu)$. Furthermore $\|\omega_{\tau_i(\omega)}\| \in [i-1, i)$ for all $i \in \mathbb{N}$ by definition of the first hitting time τ_i. The result follows. $\qquad \square$

An estimate on first hitting measures. Roughly speaking, we show that the first hitting measure ν_i of "most points" in the i-th annulus A_i decays as $e^{-i\delta(\mu)}$. Here is the precise statement.

Theorem 8.5. *For every* $a > 0$ *we have*

$$(8.23) \qquad \lim_{i \to \infty} \nu_i \left(\{ g \in A_i : e^{-\delta(\mu)(i+a)} \leq \nu_i(\{g\}) \leq e^{-\delta(\mu)(i-a)} \} \right) = 1.$$

The special case of Theorem 8.5 where the measure μ is finitely supported and the group in question Γ is hyperbolic is Tanaka's [Tan17, Theorem 6.1.2]. We provide a detailed proof in the infinitely supported non-amenable case by using our Proposition 8.1. In any case, Theorem 8.5 follows immediately by integrating the statement of the following Proposition 8.6 with respect to the measure P_μ on the space of sample paths.

Proposition 8.6. P_μ-*almost every sample path* ω *satisfies*

$$(8.24) \qquad \lim_{i \in \theta(\omega)} -\frac{\log \nu_i(\{\omega_{\tau_i(\omega)}\})}{i} = \delta(\mu).$$

Proof. We follow Tanaka's argument in [Tan17, Theorem 6.1] very closely. Observe that using first hitting measures which are not probability measures as well as the fact that the sum of the Green function over an annulus grows subexponentially instead of being bounded does not complicate things much.

We argue by establishing an inequality in both directions of Equation (8.24). We have already mentioned the fact that the entropy $h(\mu)$ is equal to the drift with respect to the Green metric [BHM08, Theorem 1.1]. By definition, this means that P_μ-almost every sample path ω satisfies

$$(8.25) \qquad \lim_{n \to \infty} -\frac{\log \mathcal{F}(e, \omega_n)}{n} = h(\mu).$$

By the definition of the first return Green's function and the first hitting time measure it is immediate that

$$\nu_i(\omega_{\tau_i(\omega)}) \leq \mathcal{F}(e, \omega_{\tau_i(\omega)}) \tag{8.26}$$

P_μ-almost surely and for all $i \in \theta(\omega)$. Moreover $\tau_i(\omega)/i \to 1/l(\mu)$ for $i \in \theta(\omega)$ by Lemma 8.4. Equations (8.25) and (8.26) together imply that P_μ-almost every sample path ω has

$$\liminf_{i \in \theta(\omega)} -\frac{\log \nu_i(\{\omega_{\tau_i(\omega)}\})}{i} \geq \delta(\mu). \tag{8.27}$$

We now establish the inequality in the other direction. For every choice of the two parameters $s > \delta(\mu)$ and $0 < t < (s - \delta(\mu))/2$ consider the set

$$P_i = \{g \in A_i : \nu_i(g) \leq e^{-i(s-t)} \quad \text{and} \quad \mathcal{F}(e, g) \geq e^{-i(\delta(\mu)+t)}\}. \tag{8.28}$$

Proposition 8.1 implies that

$$\sum_{g \in A_i} \mathcal{F}(e, g) \leq \varphi(i) \tag{8.29}$$

for some subexponentially growing function $\varphi : \mathbb{N} \to \mathbb{N}$. We deduce that

$$|P_i| \leq \varphi(i) e^{i(\delta(\mu)+t)} \tag{8.30}$$

and so

$$\nu_i(P_i) \leq \varphi(i) e^{i(\delta(\mu)+2t-s)} \tag{8.31}$$

for all $i \in \mathbb{N}$. The Borel–Cantelli lemma implies that for P_μ-almost every sample path ω there exists an $N_\omega \in \mathbb{N}$ such that $\omega_{\tau_i(\omega)} \notin P_i$ for all $i \geq N_\omega$ with $i \in \theta(\omega)$. On the other hand, Equation (8.25) combined with the fact that entropy is the drift with respect to the Green metric implies that P_μ-almost every sample path ω satisfies

$$\mathcal{F}(e, \omega_{\tau_i(\omega)}) \geq e^{-h(\mu)\tau_i(\omega) - \frac{t}{2}} \geq e^{-i(\delta(\mu)+t)} \tag{8.32}$$

for all sufficiently large $i \in \theta(\omega)$. The above facts combined with the definition of the sets P_i imply that

$$\limsup_{i \in \theta(\omega)} -\frac{\log \nu_i(\{\omega_{\tau_i(\omega)}\})}{i} < s - t \tag{8.33}$$

for P_μ-almost every sample path ω. Taking the parameter s to be arbitrary close to $\delta(\mu)$ and the parameter t to be arbitrary small completes the proof. $\square$

The following immediate corollary of Theorem 8.5 can be regarded as a sharpened form of [Tan17, Theorem 6.1.3].

Corollary 8.7. *For every $\varepsilon > 0$ there are constants $C > 0$ and $N > 0$ such that provided $i > N$ any subset $E \subset A_i$ with $\nu_i(E) \geq \varepsilon$ has cardinality $|E| \geq C e^{i\delta(\mu)}$.*

Divergence of Poincare series. The following theorem is the main result of §8. Roughly speaking, it says that if a certain subset W of the group Γ is "large" in the statistical sense of being frequently visited by the random walk, then it must also be "large" in the sense that the partial Poincare series evaluated over W diverges.

Theorem 8.8. *Assume that the group Γ is non-amenable and that the probability measure μ whose supported generates Γ as a semigroup has finite first moment, i.e.*

$$(8.34) \qquad \sum_{g \in \Gamma} \mu(g) \|g\| < \infty.$$

Let $W \subset \Gamma$ be a subset. If P_μ-almost every sample path ω satisfies

$$(8.35) \qquad \liminf_{n \to \infty} \frac{|\{i \in [n] \ : \ \omega_i \in W\}|}{n} > 1 - \zeta(\mu)$$

where $\zeta(\mu) > 0$ is as in Lemma 8.2 then

$$(8.36) \qquad \sum_{\gamma \in W} e^{-\delta(\mu)\|\gamma\|} = \infty.$$

While the arguments of §8 are influenced by Tanaka's work [Tan17], we differ from Tanaka in several aspects. First, we do not require the group Γ (or the space X it acts on) to be hyperbolic, and instead work with arbitrarily cocompact actions of non-amenable groups. Furthermore, we assume that the measure μ has finite first moment rather than finite support. In addition, we assume that sample paths enter the subset W statistically, rather than deterministically. Lastly, the divergence of the series in Equation (8.36) with exponent $\delta(\mu)$ is a stronger statement than the rate of growth estimate in [Tan17].

Proof of Theorem 8.8. We know By Lemma 8.2 that P_μ-almost every sample path $\omega \in \Gamma^{\mathbb{N}}$ satisfies

$$(8.37) \qquad \liminf_{n \to \infty} \frac{|\{i \in [n] \ : \ i = \tau_j(\omega) \text{ for some } j\}|}{n} \geq \zeta > 0.$$

Let $W \subset \Gamma$ be a subset taken so that Equation (8.35) holds. This assumption together with Corollary 2.2 of Kakutani's ergodic theorem give that

$$(8.38) \qquad \varepsilon = \liminf_{n \to \infty} \frac{1}{n} \sum_{i \in \theta(\omega) \cap [n]} 1_W(\omega_{\tau_i(\omega)}) > 0$$

for P_μ-almost every sample path ω. The definition of the first hitting measures ν_i gives that

$$(8.39) \qquad \liminf_{n \to \infty} \frac{|\{i \in [n] \ : \ \nu_i(W) > \varepsilon\}|}{n} > 0$$

for P_μ-almost every sample path ω. By Corollary 8.7 this implies

$$(8.40) \qquad \liminf_{n \to \infty} \frac{|\{i \in [n] \ : \ |W \cap A_i| > Ce^{i\delta(\mu)}\}|}{n} > 0$$

for some constant $C > 0$. Therefore

$$(8.41) \qquad \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} e^{-i\delta(\mu)} |W \cap A_i| > 0.$$

So

$$(8.42) \qquad \sum_{i=0}^{\infty} e^{-i\delta(\mu)} |W \cap A_i| = \infty.$$

This equation is equivalent to the desired conclusion. $\qquad\square$

Divergence of Poincare series weighted by a function. Lastly we consider a certain modification of Theorem 8.8 allowing for a weight function. It will be used below to study discrete stationary random subgroups of divergence type, see e.g. Proposition 9.6 and the discussion surrounding it.

Proposition 8.9. *Assume that the group Γ is non-amenable. Let*

- *μ be a probability measure on the group Γ with finite first moment whose support generates Γ as a semigroup.*
- *$S : \Gamma \to \mathbb{R}_{>0}$ be a function satisfying $\lim_{n\to\infty} S(\omega_n)^{\frac{1}{n}} \to 1$ for P_μ-almost every sample path ω.*
- *$W \subset \Gamma$ be a subset such that P_μ-almost every sample path ω has*

$$(8.43) \qquad \liminf_{n\to\infty} \frac{|\{i \in [n] \,:\, \omega_i \in W\}|}{n} > 1 - \zeta$$

where $\zeta > 0$ is the constant as in Lemma 8.2.

Then for any $a > 0$ we have that

$$(8.44) \qquad \limsup_{i\to\infty} e^{-(\delta(\mu)-a)i} \sum_{g \in W \cap A_i} S(g) > 0.$$

Proof. It follows from Lemma 8.4 that P_μ-almost every sample path ω satisfies

$$(8.45) \qquad \lim_{i\in\theta(\omega)} S(\omega_{\tau_i(\omega)})^{1/i} = \lim_{i\in\theta(\omega)} \left(S(\omega_{\tau_i(\omega)})^{\frac{1}{\tau_i(\omega)}} \right)^{\frac{\tau_i(\omega)}{i}} = 1.$$

Together with Theorem 8.5 this implies that for every $a > 0$ we have
$$(8.46)$$
$$\lim_{i\to\infty} \nu_i \left(\{g \in A_i \,:\, e^{\delta(\mu)(i-a)} \le \nu_i(\{g\}) \le e^{\delta(\mu)(i+a)} \text{ and } e^{-ai} \le S(g) \le e^{ai} \} \right) = 1.$$

Consequently for every choice of $a > 0$ and $\varepsilon > 0$ there are some constants $C > 0$ and $N \in \mathbb{N}$ such that whenever $i > N$ any subset $W \subset A_i$ with $\nu_i(W) \ge \varepsilon$ has

$$(8.47) \qquad \sum_{g \in W \cap A_i} S(g) \ge ce^{i(\delta(\mu)-a)}.$$

By Equation (8.39) appearing in the proof of Theorem 8.8 we have

$$(8.48) \qquad \liminf_{n\to\infty} \frac{|\{i \in [n] \,:\, \nu_i(W) > \varepsilon\}|}{n} > 0$$

for P_μ-almost every sample path ω. The desired conclusion follows by putting together Equations and (8.47) and (8.48). $\qquad\square$

9. Critical exponents of stationary random subgroups

Let X be a proper Gromov hyperbolic geodesic metric space with a fixed base point $x_0 \in X$. Assume that the group $\mathrm{Is}(X)$ is non-elementary. For every element $g \in \mathrm{Is}(X)$ denote

$$(9.1) \qquad \|g\| = d_X(gx_0, x_0)$$

with the chosen base point x_0 implicit in the notation.

Definition 9.1. Given a discrete subgroup Γ of the group of isometries $\mathrm{Is}(X)$:

- The *Poincare series* of the subgroup Γ at the exponent $s \geq 0$ is

$$\mathcal{P}_\Gamma(s) = \sum_{\gamma \in \Gamma} e^{-s\|\gamma\|}.$$

- The *critical exponent* of the subgroup Γ is the number

$$\delta(\Gamma) = \sup\{s \geq 0 \ : \ \mathcal{P}_\Gamma(s) < \infty\} = \inf\{s \geq 0 \ : \ \mathcal{P}_\Gamma(s) = \infty\}.$$

- The subgroup Γ is of *divergence type* if $\mathcal{P}_\Gamma(\delta(\Gamma)) = \infty$.
- The subgroup Γ is of *convergence type* if $\mathcal{P}_\Gamma(\delta(\Gamma)) < \infty$.

The following is the main result of this work.

Theorem 9.2. *Let μ be a probability measure on the group $\mathrm{Is}(X)$. Assume that*

- *the measure μ has finite first moment, and*
- *the semigroup generated by the support of μ is a discrete subgroup acting properly and cocompactly on the metric space X.*

Let ν be a discrete μ-stationary random subgroup of $\mathrm{Is}(X)$. If ν-almost every subgroup is not contained in the elliptic radical $\mathrm{E}(\mathrm{Is}(X))$ then the strict lower bound

$$(9.2) \qquad \delta(\Delta) > \frac{\delta(\mu)}{2}$$

holds for ν-almost every discrete subgroup Δ. In particular

$$(9.3) \qquad \mathcal{P}_\Delta\left(\frac{\delta(\mu)}{2}\right) = \infty$$

holds ν-almost surely. Moreover if ν-almost every subgroup is of divergence type then $\delta(\Delta) \geq \delta(\mu)$ holds true ν-almost surely.

The remainder of this section is dedicated to giving a proof of Theorem 9.2. Note that the assumption that the group $\mathrm{Is}(X)$ contains a uniform lattice implies that it is of general type.

Divergence of Poincare series. The following uses the core arguments of [GL19] adapted to work in the context of random walks. The idea is to relate the critical exponents of ν-almost every discrete subgroup with the quantity $\frac{\delta(\mu)}{2}$.

Proposition 9.3. *Let V be an open and relatively compact subset of $\mathrm{Is}(X)$ consisting of loxodromic elements. Let $\Delta \in \mathrm{DSub}(\mathrm{Is}(X))$ be a discrete subgroup. If*

$$(9.4) \qquad \liminf_{n \to \infty} \frac{|\{i \in [n] \ : \ \Delta^{\omega_i} \cap V \neq \emptyset\}|}{n} > 1 - \zeta(\mu)$$

for P_μ-almost every sample path ω where $\zeta(\mu) > 0$ is as provided by Lemma 8.2 then $\mathcal{P}_\Delta(\frac{\delta(\mu)}{2}) = \infty$.

Proof. To simplify notations write $\delta = \delta(\mu)$. Let Γ be the uniform lattice in the group $\mathrm{Is}(X)$ generated by the support of the measure μ. Write

$$(9.5) \qquad \mathcal{P}_\Delta(\delta/2) = \sum_{h \in \Delta} e^{-\frac{\delta}{2}\|h\|} \quad \text{so that} \quad \mathcal{P}_\Delta(\delta/2) \geq \sum_{\substack{h \in \Delta \\ h^\Gamma \cap V \neq \emptyset}} e^{-\frac{\delta}{2}\|h\|}.$$

It was shown in [GL19, Lemma 3.3] that there is a constant $\beta > 0$ depending only on the space X and the subset V such that

$$(9.6) \qquad e^{-\frac{\delta}{2}\|h\|} \geq \frac{1}{\beta} e^{-\delta\|\gamma\|}$$

for any element $\gamma \in \Gamma$ with $h^\gamma \in V$. Likewise, according to [GL19, Proposition 3.5] there is a constant $\alpha > 0$ depending only on the space X and the subset V such that, if $h^\Gamma \cap V \neq \emptyset$ then there is an element $\gamma_h \in \Gamma$ with

$$(9.7) \qquad e^{-\delta\|\gamma_h\|} \geq \frac{1}{\alpha} \sum_{\substack{\gamma \in \Gamma \\ h^\gamma \in V}} e^{-\delta\|\gamma\|}.$$

Putting the above three equations together gives

$$(9.8) \quad \mathcal{P}_\Delta(\delta/2) \geq \frac{1}{\beta} \sum_{\substack{h \in \Delta \\ h^\Gamma \cap V \neq \emptyset}} e^{-\delta\|\gamma_h\|} \geq \frac{1}{\alpha\beta} \sum_{h \in \Delta} \sum_{\substack{\gamma \in \Gamma \\ h^\gamma \in V}} e^{-\delta\|\gamma\|} \geq \frac{1}{\alpha\beta} \sum_{\substack{\gamma \in \Gamma \\ \Delta^\gamma \cap V \neq \emptyset}} e^{-\delta\|\gamma\|}.$$

To conclude apply Theorem 8.8 with the subset $W = \{\gamma \in \Gamma \ : \ \Delta^\gamma \cap V \neq \emptyset\}$. $\square$

Our strategy towards Theorem 9.2 is to start by showing divergence at the exponent $\frac{\delta(\mu)}{2}$.

Proof of $\mathcal{P}_\Delta(\delta(\mu)/2) = \infty$ part in Theorem 9.2. Let ν be a discrete μ-stationary subgroup not contained in the elliptic radical $\mathrm{E}(\mathrm{Is}(X))$. The action of the subgroup generated by the support of the measure μ has general type. We know from Corollary 5.7 that ν-almost every subgroup is of general type and in particular admits loxodromic elements. By exhaustion it is possible to choose a sufficiently large open and relatively compact subset V of $\mathrm{Is}(X)$ consisting of loxodromic elements so that

$$(9.9) \qquad \nu(\{\Delta \in \mathrm{DSub}(\mathrm{Is}(X)) \ : \ \Delta \cap V \neq \emptyset\}) > 1 - \zeta(\mu)$$

where $\zeta(\mu) > 0$ is as provided by Lemma 8.2.

We know from Corollary 2.2 of Kakutani's ergodic theorem that P_μ-almost every sample path ω and ν-almost every discrete subgroup Δ satisfy

$$(9.10) \qquad \liminf_{n \to \infty} \frac{|\{i \in [n] \ : \ \Delta^{\omega_i} \cap V \neq \emptyset\}|}{n} > 1 - \zeta(\mu).$$

At this point we conclude that $\mathcal{P}_\Delta(\delta(\mu)/2) = \infty$ by Proposition 9.3. $\square$

By regarding an invariant random subgroup as a stationary one it is now possible to recover the non-strict inequality in the main result of [GL19].

Corollary 9.4. *Assume that the group $\mathrm{Is}(X)$ admits a discrete subgroup acting cocompactly on the space X. Let ν be a discrete invariant random subgroup of $\mathrm{Is}(X)$ not contained in the elliptic radical $\mathrm{E}(\mathrm{Is}(X))$. Then ν-almost every discrete subgroup Δ has $\delta(\Delta) \geq \frac{1}{2}\dim(\partial X)$.*

Proof. Fix any uniform lattice Γ in the group $\mathrm{Is}(X)$. The construction of [GMM18, Theorem 1.4] exhibits a sequence μ_i of probability measures supported on finite subsets of the lattice Γ and satisfying $\delta(\mu_i) \to \frac{1}{2} \dim_{\mathrm{Haus}}(\partial X)$. The desired conclusion now follows from Theorem 9.2. $\qquad\square$

Stationary random subgroups of divergence type. Let Γ be a discrete subgroup of $\mathrm{Is}(X)$ having divergence type. There exists a Γ-*quasi-conformal density* η^Γ of dimension $\delta(\Gamma)$. By definition, this is a map assigning to every point $x \in X$ a positive measure η_x^Γ on the boundary ∂X satisfying certain axioms. Such a map is unique up to a multiplicative constant. See e.g. [Nic89], [Coo93] or [GL19, Definition 6.6] for more detailed information.

Normalize $\eta_{x_0}^\Gamma$ to be a probability measure. For discrete subgroups of divergence type, the total measure $\|\eta_x^\Gamma\| = \eta_x^\Gamma(\partial X)$ can be characterized in terms of a certain Poincare series. Namely, for each pair of points $x, y \in X$ denote

$$(9.11) \qquad \mathcal{P}_\Gamma(s; x, y) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma y)}.$$

We get as a special case $\mathcal{P}_\Gamma(s) = \mathcal{P}_\Gamma(s; x_0, x_0)$. Note that $\mathcal{P}_\Gamma(s; x, y) = \mathcal{P}_\Gamma(s; y, x)$. For any pair of points $x, y \in X$ we have according to [MYJ20, Lemmas 6.1 and 6.2] that

$$(9.12) \qquad \|\eta_x^\Gamma\| \approx \lim_{s \searrow \delta(\Gamma)} \frac{\mathcal{P}_\Gamma(s; x, y)}{\mathcal{P}_\Gamma(s; x_0, y)}.$$

We emphasize at this point that the multiplicative constant implicit in all our formulas involving the notation $\approx$ depends only on the metric space X, and not on anything else.

Let $\mathcal{M}(\partial X)$ denote the space of all positive Borel measures on the boundary at infinity ∂X. In fact, if ν is a discrete μ-stationary random subgroup of divergence type then there is a ν-measurable mapping

$$(9.13) \qquad \eta : \mathrm{DSub}(\mathrm{Is}(X)) \times X \to \mathcal{M}(\partial X), \quad \eta : (\Gamma, x) \mapsto \eta_x^\Gamma$$

defined for all discrete subgroups Γ and all points $x \in X$ in such a way that the assignment $x \mapsto \eta_x^\Gamma$ is a Γ-quasi-conformal density of dimension $\delta(\Gamma)$. Consider the expression

$$(9.14) \qquad \pi_\nu : \mathrm{Is}(X) \times \mathrm{DSub}(\mathrm{Is}(X)) \to \mathbb{R}_{>0}, \quad \pi_\nu(g, \Gamma) = \|\eta_{g^{-1}x_0}^\Gamma\|$$

for all elements $g \in \mathrm{Is}(X)$ and all subgroups $\Gamma \in \mathrm{DSub}(\mathrm{Is}(X))$. The function π_ν is a *multiplicative quasi-cocycle*, i.e.

$$(9.15) \qquad \pi_\nu(gh, \Gamma) \approx \pi_\nu(g, \Gamma^h)\pi_\nu(h, \Gamma).$$

For all this see [GL19, §5 and §6].

By combining the definition of the quasi-cocycle π_ν in Equation (9.14) with the characterization of the total mass of the conformal density in Equation (9.12) we obtain the formula

$$(9.16) \qquad \pi_\nu(g, \Gamma) \approx \lim_{s \searrow \delta(\nu)} \frac{\mathcal{P}_\Gamma(s; g^{-1}x_0, g^{-1}x_0)}{\mathcal{P}_\Gamma(s; g^{-1}x_0, x_0)} \approx \lim_{s \searrow \delta(\nu)} \frac{\mathcal{P}_\Gamma(s; g^{-1}x_0, x_0)}{\mathcal{P}_\Gamma(s; x_0, x_0)}.$$

One consequence of the above is that

$$(9.17) \qquad \pi_\nu(g, \Gamma)^2 \approx \lim_{s \searrow \delta(\nu)} \frac{\mathcal{P}_\Gamma(s; g^{-1}x_0, g^{-1}x_0)}{\mathcal{P}_\Gamma(s; x_0, x_0)} = \lim_{s \searrow \delta(\nu)} \frac{\mathcal{P}_{\Gamma^g}(s)}{\mathcal{P}_\Gamma(s)}.$$

The triangle inequality gives for each $s > \delta(\Gamma)$ that

$$(9.18) \qquad -2s\|g\| \le \log \frac{\mathcal{P}_\Gamma(s; g^{-1}x_0, g^{-1}x_0)}{\mathcal{P}_\Gamma(s; g^{-1}x_0, x_0)} \le 2s\|g\|.$$

In particular $|\pi_\nu(g, \Gamma)| \le 2\delta(\Gamma)\|g\|$ for all $g \in \mathrm{Is}(X)$ and $\Gamma \in \mathrm{DSub}(\mathrm{Is}(X))$.

Remark 9.5. *We have allowed for a slight abuse of notation; the above limits as $s \searrow \delta(\Gamma)$ might not exist in the strict sense. The correct statement is that the limits only exist along some subsequence, and the limit is independent of the choice of this subsequence up to a uniform multiplicative constant.*

Proposition 9.6. *Let μ be a probability measure on the group $\mathrm{Is}(X)$ satisfying the requirements stated in Theorem 9.2. Let ν be a discrete μ-stationary random subgroup of $\mathrm{Is}(X)$ such that ν-almost every subgroup has divergence type. Then P_μ-almost every sample path $\omega \in \Gamma^\mathbb{N}$ and ν-almost every subgroup $\Delta \in \mathrm{DSub}(\mathrm{Is}(X))$ satisfy*

$$(9.19) \qquad \lim_{n \to \infty} \pi_\nu(\omega_n, \Delta)^{\frac{1}{n}} = 1.$$

Proof. To simplify notation write $G = \mathrm{Is}(X)$. For each $s > \delta(\Gamma)$ consider the following double integral

$$(9.20) \qquad \int_{G \times \mathrm{DSub}(G)} \log \frac{\mathcal{P}_{\Gamma^g}(s)}{\mathcal{P}_\Gamma(s)} \, \mathrm{d}(\mu \otimes \nu)(g, \Gamma).$$

The function in question is L^1 by the finite first moment assumption together with Equation (9.18). We may apply Fubini's theorem and use the fact that $\mu * \nu = \nu$ to evaluate the double integral in Equation (9.20) as being equal to

$$(9.21) \qquad \int_{\mathrm{DSub}(G)} \left(\int_G \log \mathcal{P}_{\Gamma^g}(s) \, \mu(g) - \log \mathcal{P}_\Gamma(s) \right) \, \mathrm{d}\nu(\Gamma) = 0.$$

Thus, the dominated convergence theorem combined with the limit in Equation 9.17 imply that the function $\log \pi_\nu$ is $(\mu \times \nu)$-integrable. More precisely, it satisfies

$$(9.22) \qquad \left| \int_{G \times \mathrm{DSub}(G)} \log \pi_\nu(g, \Gamma) \, \mathrm{d}(\mu \otimes \nu)(g, \Gamma) \right| \le C_0$$

for some constant $C_0 > 0$ depending only on the metric space X. In particular, the constant C_0 is independent of the measure μ, so that Equation 9.22 continues to hold, say, if μ is replaced by any of its convolution powers μ^{*n}.

We proceed with establishing Equation (9.19). Consider the sequence of functions $f_n : G^\mathbb{N} \times \mathrm{DSub}(G) \to \mathbb{R}_{>0}$ given by

$$(9.23) \qquad f_n((g_i), \Delta) = \log \pi_\nu(g_1 \ldots g_n, \Delta) \quad \forall (g_i) \in G^\mathbb{N}, \Delta \in \mathrm{DSub}(G)$$

and for all indices $n \in \mathbb{N}$. Define the transformation

$$(9.24) \qquad T : G^\mathbb{N} \times \mathrm{DSub}(G) \to G^\mathbb{N} \times \mathrm{DSub}(G)$$

by

$$(9.25) \qquad T((g_i)_{i=1}^\infty, \Delta) = ((g_{i+1})_{i=1}^\infty, \Delta^{g_1}).$$

The transformation T preserves the product probability measure $\mu^\mathbb{N} \times \nu$ and acts ergodically. Since π_μ is a multiplicative quasi-cocycle we get

$$(9.26) \qquad f_n \circ T^m + f_m - C_1 \le f_{n+m} \le f_n \circ T^m + f_m + C_1$$

for all $n, m \in \mathbb{N}$ where $C_1 > 0$ is some constant depending only on the metric space X. Denote $f'_n = f_n + C_1$. Since $\log \pi_v$ belongs to $L^1(\mu \times \nu)$, so do the functions f_n and f'_n for all $n \in \mathbb{N}$. The sequence of functions f'_n satisfies the inequalities

$$(9.27) \qquad f'_{n+m} \leq f'_n \circ T^m + f'_m$$

for all $n, m \in \mathbb{N}$. Kingman's subadditive ergodic theorem implies that

$$(9.28) \qquad \lim_{n \to \infty} \frac{f_n}{n} = \lim_{n \to \infty} \frac{f'_n}{n} = \lim_{n \to \infty} \int \frac{f'_n}{n} \, \mathrm{d}(\mu^{\mathbb{N}} \times \nu) = \lim_{n \to \infty} \int \frac{f_n}{n} \, \mathrm{d}(\mu^{\mathbb{N}} \times \nu)$$

in the sense of pointwise convergence at $(\mu^{\mathbb{N}} \times \nu)$-almost all points. Note that

$$(9.29) \qquad \left| \int_{G \times \mathrm{DSub}(G)} f_n \, \mathrm{d}(\mu^{*n} \times \nu) \right| = \left| \int_{G \times \mathrm{DSub}(G)} \log \pi_\nu \, \mathrm{d}(\mu^{*n} \times \nu) \right| \leq C_1$$

independently of n, as we have explained above, Therefore, we conclude that $\frac{f_n}{n} \to 0$ pointwise at $(\mu^{\mathbb{N}} \times \nu)$-almost all points as $n \to \infty$. This is equivalent to the desired conclusion in Equation (9.19). □

Proof of the divergence type case in Theorem 9.2. Write $G = \mathrm{Is}(X)$ to simplify notations. Let ν be a discrete divergence type μ-stationary random subgroup of the group G. Consider the multiplicative cocycle $\pi_\nu : G \times \mathrm{DSub}(G) \to \mathbb{R}_{>0}$ associated to the random subgroup ν.

Assume without loss of generality that the measure ν is ergodic. Therefore the critical exponent regarded as a ν-measurable function $\delta : \mathrm{DSub}(G) \to \mathbb{R}_{\geq 0}$ is ν-almost everywhere constant [GL19, p. 424]. Denote this constant by $\delta(\nu)$. So $\delta(\Delta) = \delta(\nu)$ for ν-almost every subgroup Δ. In this terminology, our goal is to prove that $\delta(\nu) \geq \delta(\mu)$.

Take a Borel subset $Y \subset \mathrm{DSub}(G)$ with $\nu(Y) > 1 - \zeta(\mu)$ with the properties provided by [GL19, Proposition 7.2]. Here $\zeta(\mu) > 0$ is the constant provided by Lemma 8.2. For each discrete subgroup Δ denote

$$(9.30) \qquad W_\Delta = \{\gamma \in \Gamma : \Delta^\gamma \in Y\}.$$

There is some $k \in \mathbb{N}$ such that ν-almost every discrete subgroup $\Delta \in Y$ satisfies

$$(9.31) \qquad \sum_{\substack{\gamma \in W_\Delta \\ i-k \leq \|\gamma\| \leq i}} \pi_\nu(\gamma, \Delta) \leq c_1 e^{\delta(\nu)i}$$

for some constant $c_1 > 0$ and for all $i \in \mathbb{N}$, see [GL19, Proposition 7.3]. We remark that these two propositions from [GL19] were proved in the context of invariant random subgroups but their proofs don't use the invariance of the measure ν in any way. They are valid for any probability measure on the space $\mathrm{DSub}(G)$ whatsoever. Changing summation from annuli to spheres gives

$$(9.32) \qquad \sum_{\substack{\gamma \in W_\Delta \\ \|\gamma\| \leq j}} \pi_\nu(\gamma, \Delta) \leq \sum_{i=1}^{j} \sum_{\substack{\gamma \in W_\Delta \\ i-k \leq \|\gamma\| \leq i}} \pi_\nu(\gamma, \Delta) \leq c_1 \sum_{i=1}^{j} e^{\delta(\nu)i} \leq c_2 e^{\delta(\nu)j}$$

for all indices $j \in \mathbb{N}$ where $c_2 = \frac{c_1}{1 - e^{-\delta(\nu)}}$.

We contrast the previous paragraph with the fact that $\lim_n \pi_\nu(\omega_n, \Delta)^{\frac{1}{n}} = 1$ holds true for P_ν-almost every sample path ω and ν-almost every discrete subgroup Δ. Fix

an arbitrary parameter $a > 0$. Proposition 8.9 shows that ν-almost every subgroup Δ satisfies

$$(9.33) \qquad \limsup_{j \to \infty} e^{-(\delta(\mu)-a)j} \sum_{\substack{\gamma \in W_\Delta \\ \|\gamma\| \leq j}} \pi_\nu(\gamma, \Delta) > 0.$$

Equation (9.33) is only possible if $\delta(\mu) - a \leq \delta(\nu)$. Taking the parameter a to be arbitrary small we get $\delta(\mu) \leq \delta(\nu)$ as required. $\qquad\square$

The following strategy is taken from [GL19].

Proof of the strict inequality in Theorem 9.2. Let be ν be a discrete μ-stationary random subgroup on $\mathrm{Is}(X)$. We have already proved that $\mathcal{P}_\Delta\left(\frac{\delta(\mu)}{2}\right) = \infty$ so that in particular $\delta(\nu) \geq \frac{\delta(\mu)}{2}$. Assume towards contradiction that $\delta(\nu) = \frac{\delta(\mu)}{2}$. This means that the μ-stationary random subgroup ν is almost surely of divergence type. It follows from the above proof that $\delta(\nu) \geq \delta(\mu)$. This is a contradiction as $\delta(\mu) > 0$. $\qquad\square$

The above paragraphs complete the proof of Theorem 9.2.

A probability measure μ maximizing $\delta(\mu)$. The exact lower bound on the critical exponent of discrete stationary random subgroups obtained in Theorem 9.2 depends on the parameter $\delta(\mu) = h(\mu)/l(\mu)$ of the probability measure μ. In this sense, our results are stronger, the greater is the value of $\delta(\mu)$. We now discuss a particular construction maximizing the parameter $\delta(\mu)$ following Connell and Muchnik [CM07].

Recall that X is a proper Gromov hyperbolic geodesic metric space with a non-elementary group of isometries $\mathrm{Is}(X)$ and a fixed arbitrary base point $x_0 \in X$. Assume that $\mathrm{Is}(X)$ admits a uniform lattice denoted by Γ.

Let η^Γ be a Γ-quasi-conformal density of dimension $\delta(\Gamma)$. Normalize so that $\eta^\Gamma_{x_0}(\partial X) = 1$. The main result of [CM07] asserts that there is a probability measure μ_Γ supported[12] on the lattice Γ such that the probability measure $\eta^\Gamma_{x_0}$ on the boundary ∂X is μ-stationary. Now, on the one hand, the Hausdorff dimension of a μ-stationary probability measure on the boundary equals $h(\mu)/l(\mu)$ [Tan19, Theorem 1.1]. On the other hand, the Hausdorff dimension of $\delta(\Gamma)$-dimensional Γ-quasi-conformal measures equals $\delta(\Gamma) = \dim_{\mathrm{Haus}}(\partial X)$ [Coo93, Theorem 7.7]. We deduce that the measure μ_Γ constructed by the Connell and Muchnik satisfies

$$(9.34) \qquad \delta(\mu_\Gamma) = \delta(\Gamma) = \dim_{\mathrm{Haus}}(\partial X).$$

Recall that $\delta(\Gamma)$ is the maximal possible such value for $\delta(\mu)$ by the fundamental inequality of Guivarch [Gui80].

In the special case where the space X is assumed to be $\mathrm{CAT}(-1)$, the measure μ_Γ constructed in [CM07] can be chosen to have finite first moment. The analogous fact for symmetric spaces of rank one (or more generally, universal covers of closed pinched negatively curved Riemannian manifolds) follows by the method of discretization of Brownian motion, see [Fur71, LS84, Led90, Mar91, BL, BHM11]. For symmetric spaces, the μ_Γ-stationary probability measure on the boundary ∂X coincides with the unique K-invariant one, where $K = \mathrm{stab}_{\mathrm{Is}(X)}(x_0)$.

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[12]In fact μ_Γ can be taken so that $\mathrm{supp}(\mu_\Gamma) = \Gamma$, see [CM07, Remark 0.6].

For general Gromov hyperbolic spaces which are not CAT(-1), it is not known whether one can find such a measure μ_Γ supported on the lattice Γ with $\delta(\mu_\Gamma) = \delta(\Gamma)$ and with finite first moment. However, the construction of Gouezel–Matheus–Maucourant [GMM18, Theorem 1.4] exhibits a sequence of probability measures μ_i supported on finite subsets of the lattice Γ and satisfying $\delta(\mu_i) \to \delta(\Gamma)$. In fact, each such μ_i can be taken to be the uniform probability measure supported on the annulus given by $\{\gamma \in \Gamma \,:\, k \le \|\gamma\| < k+1\}$.

It is interesting to specialize Theorem 9.2 to the situation when we know of an explicit measure μ_Γ satisfying $\delta(\mu_\Gamma) = \dim_{\mathrm{Haus}}(\partial X)$. In that case, we immediately obtain the following.

Corollary 9.7. *Let X be a* CAT(-1) *space. Assume that* Is(X) *admits a uniform lattice Γ. Fix*[13] *a probability measure μ_Γ whose support generates Γ as a semigroup, of finite first moment and satisfying $\delta(\mu_\Gamma) = \dim_{\mathrm{Haus}}(\partial X)$. Let ν be a non-trivial discrete μ-stationary random subgroup on* Is(X). *Then*

- $\delta(\Delta) > \frac{1}{2} \dim_{\mathrm{Haus}}(\partial X)$ *for ν-almost every subgroup Δ, and*
- *if ν-almost every subgroup Δ has divergence type then $\delta(\Delta) = \dim_{\mathrm{Haus}}(\partial X)$.*

Corollary 9.7 certainly applies to all rank one symmetric spaces and Bruhat–Tits buildings. Lastly we consider stationary random subgroups of free groups.

Corollary 9.8. *Let F_k be the free group on k generators. Let μ_k be the uniform measure on the standard symmetric generating set. Then any μ_k-stationary random subgroup of F_k has*

- $\delta(H) > \frac{\log(2k-1)}{2}$ *for ν-almost every subgroup Δ, and*
- *if ν-almost every subgroup H has divergence type then $\delta(H) = \log(2k-1)$.*

Proof. The finitely supported measure μ_k satisfies

$$(9.35) \qquad\qquad \delta(\mu) = \frac{l(\mu)}{h(\mu)} = \dim_{\mathrm{Haus}}(\partial X) = \log(2k-1)$$

by [Led01]. The desired conclusion follows from this and from Theorem 9.2. $\square$

10. RANDOM WALKS AND CONFINED SUBGROUPS

In this section, we apply our results on critical exponents of stationary random subgroups to show that subgroups of small critical exponent are not confined.

Confined subgroups. Let G be a second countable locally compact group.

Definition 10.1. A subgroup H of the group G is called *confined* if there is a compact subset $K \subset G$ so that $(H^g \cap K) \setminus \{e\} \ne \emptyset$ for every element $g \in G$.

The notion of confined subgroups admits an equivalent characterization in terms of the Chabauty topology.

Proposition 10.2. *If the trivial subgroup $\{e\}$ does not belong to the Chabauty closure $\overline{\{H^g \,:\, g \in G\}}$ then the subgroup $H \le G$ is confined. The converse direction holds provided that the group G has no small subgroups*[14].

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[13]Such a probability measure μ_Γ always exists for any CAT(-1)-space admitting a uniform lattice; see the preceding discussion relying on the work of [CM07].

[14]A topological group G has *no small subgroups* if it admits a neighborhood of the identity containing no non-trivial subgroups. For example, Lie groups as well as discrete groups have no small subgroups.

Proof. Assume that the trivial subgroup is not in the Chabauty closure of the family of all subgroups conjugate to H. So, there is some Chabauty neighborhood Ω with $\{e\} \in \Omega$ but $H^g \notin \Omega$ for all elements $g \in G$. The description of the standard sub-basis for the Chabauty topology given in [Gel15, §1] shows that Ω contains a subset of the form $\{L \in \mathrm{Sub}(G) \,:\, L \cap K = \emptyset\}$ for some compact subset $K \subset G$. The desired conclusion follows.

For the converse direction assume that the group G has no small subgroups and that H is a confined subgroup. In particular, there is some compact subset $K \subset G$ so that $(H^g \cap K) \setminus \{e\} \neq \emptyset$ for every element $g \in G$. Let V be a symmetric relatively compact identity neighborhood in the group G containing no non-trivial subgroups. Then the compact subset K can be replaced with $K_1 = (K \cup \overline{V}^2) \setminus V$ to exhibit that the subgroup H is confined. This means that any subgroup in the Chabauty closure $\overline{\{H^g \,:\, g \in G\}}$ intersects the compact set K_1 non-trivially and is in particular non-trivial. $\qquad\square$

Example 10.3. *Let G be a rank one simple Lie group with associated symmetric space X. A discrete torsion-free subgroup Γ of the Lie group G is confined if and only if there is an upper bound on the injectivity radius at all points of the locally symmetric space $M_\Gamma = \Gamma \backslash X$.*

Continuity of the critical exponent. Our method relies on the fact that the critical exponent is lower semi-continuous with respect to the Chabauty topology.

Proposition 10.4. *Let $\Delta_n \to \Delta$ be a Chabauty converging sequence of discrete subgroups. Assume that the subgroup Δ is discrete. Then*

$$(10.1) \qquad\qquad \liminf_n \delta(\Delta_n) \geq \delta(\Delta).$$

This statement is well-known to experts. See for instance [McM99, Theorem 7.7] for the case of Kleinian groups or [MYJ20, Proposition 7.2] for general Gromov hyperbolic spaces (but with an additional assumption that the groups Γ_n are of divergence type). As we could not locate a reference for the general statement (without this additional assumption) we provide a proof for the readers' convenience.

Proof of Proposition 10.4. Fix an arbitrary base point $x_0 \in X$. Denote $\delta_0 = \liminf \delta(\Delta_n)$. Up to passing to a subsequence we may assume that $\lim_n \delta(\Delta_n) = \delta_0$. Let $\mathcal{M}(\partial X)$ denote the space of all positive measures on the boundary ∂X. For each n let $\eta_n : X \to \mathcal{M}(\partial X)$ be some Δ_n-quasi-conformal density of exponent $\delta(\Delta_n)$ normalized so that $(\eta_n)_{x_0}(\partial X) = 1$, in other words so that $(\eta_n)_{x_0}$ is a probability measure. Let η be any weak-$*$ accumulation point of the probability measures $(\eta_n)_{x_0}$ so that η is a probability measure on the boundary ∂X. Enrich η to a quasi-conformal density by setting

$$(10.2) \qquad\qquad \eta_x(\zeta) = e^{\delta_0 \beta_X(x_0, x)} \eta(\zeta) \quad \forall x \in X, \zeta \in \partial X.$$

Then η satisfies the axioms of a Δ-quasi-conformal density of exponent δ_0. Note that a quasi-conformal measure is not required to have its support coincide with the limit set. Since Δ admits such a quasi-conformal measure we infer that $\delta(\Delta) \leq \delta_0$, see [Coo93, Corollaire 6.6] or [MYJ20, Theorem 2.6]. $\qquad\square$

Subgroups of small critical exponent are not confined. We consider discrete subgroups of simple rank one analytic groups and show that small critical exponent implies that the subgroup in question is not confined. The Archimedean and the non-Archimedean cases will be treated in parallel.

In the real analytic case let G be a rank one simple Lie group with associated symmetric space X. Fix an arbitrary base point $x \in X$.

In the non-Archimedean case let k be a non-Archimedean local field. Let $\mathbb{G}$ be a connected simply-connected semisimple k-algebraic linear group with $\mathrm{rank}_k(\mathbb{G}) = 1$ and without k-anisotropic factors. Take $G = \mathbb{G}(k)$ so that G is a k-analytic group. Let X be the Bruhat–Tits building associated to the group G. This means that X is a $(p^{e_1} + 1, p^{e_2} + 1)$-biregular tree for some $e_1, e_2 \in \mathbb{N}$ [Car01] where p is the characteristic of the residue field of k. Fix an arbitrary vertex $x \in X$.

In either case the metric space X is Gromov hyperbolic and the analytic group G is acting on the space X by isometries. To simplify the situation we will find it convenient to pass to the quotient by the finite kernel $\mathrm{Z}(G)$ of the action of the analytic group G on the metric space X and to assume in effect that $G \leq \mathrm{Is}(X)$.

Denote $K = \mathrm{stab}_G(x)$ so that K is a maximal compact subgroup of the analytic group G. The compact subgroup K is acting transitively on the visual boundary ∂X of the Gromov hyperbolic space X. Let m denote the unique K-invariant probability measure[15] on the visual boundary ∂X. Note that the measure m depends on the the choice of the point $x \in X$.

Let μ_0 be the explicit bi-K-invariant probability measure on the analytic group G constructed in the work [GLM22] for the Archimedean as well as for the non-Archimedean case. The measure μ_0 has the following useful property — for every discrete group Γ of the analytic group G, any weak-$*$ accumulation point of the sequence of probability measures $\frac{1}{n} \sum_{i=1}^n \mu_0^{*i} * \mathrm{D}_\Gamma$ is a *discrete* μ_0-stationary random subgroup [FG23, Theorem 1.6].

We remark that in the zero characteristic non-Archimedean case there is no need to consider the measure μ_0 at all, since the analytic group G has no small discrete subgroups [Ser09, Part II, Chapter V.9, Theorem 5].

Let ν_0 be the μ_0-stationary boundary measure ν_0 on the visual boundary ∂X provided by Theorem 4.1. The measure ν_0 is K-invariant. This follows from the equation $\nu_0 = \mu_0 * \nu_0$ and using the fact that the measure μ_0 is bi-K-invariant. Since the visual boundary ∂X admits a *unique* K-invariant probability measure m it must be the case that $\nu_0 = m$.

Consider the probability measure μ_1 on the group $\mathrm{Is}(X)$ as follows. Recall that we are regarding the analytic group G as a subgroup of $\mathrm{Is}(X)$.

(1) *The real case.* Any connected non-compact semisimple Lie group admits uniform (as well as non-uniform) lattices [Bor63]. Fix an arbitrary uniform lattice Γ in the Lie group G. Take μ_1 to be either the probability measure μ_Γ constructed by Connell–Muchnik [CM07] or the discretization of Brownian motion on the Lie group G, supported on the lattice Γ and with respect to the base point x. It is a property of these classes of measures (discussed in §9) that the μ_1-stationary boundary measure ν_1 on the visual boundary ∂X is the unique K-invariant one.

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[15]The probability measure m can be identified with the Haar measure on the coset space K/M where the closed subgroup M is the stabilizer of some boundary point $\infty \in \partial X$.

(2) *The non-Archimedean case.* The fundamental domain for the action of the full group of isometries $\mathrm{Is}(X)$ on the Bruhat–Tits tree X is a single edge. In particular, the group $\mathrm{Is}(X)$ admits a uniform lattice Γ of the form $\Gamma = A * B$ where A and B respectively are finite groups acting simply transitively on the link of two adjacent vertices v_A and v_B of the tree X. Let μ_A and μ_B be the uniform probability measures on the two sets $A \setminus \{e\}$ and $B \setminus \{e\}$ respectively. Let μ_1 be the probability measure supported on the uniform lattice Γ given by

$$(10.3) \qquad \mu = \frac{|A| - 1}{|A| - 1 + \rho}\mu_A + \frac{|B| - 1}{|B| - 1 + \rho}\mu_B$$

where $\rho > 0$ is the unique positive real number satisfying

$$(10.4) \qquad \frac{|A| - 1}{|A| - 1 + \rho} + \frac{|B| - 1}{|B| - 1 + \rho} = 1.$$

The probability measure μ_1 satisfies $\delta(\mu_1) = \frac{h(\mu_1)}{l(\mu_1)} = \delta(\Gamma)$. Furthermore the μ_1-stationary boundary stationary measure ν_1 on the visual boundary ∂X is K-invariant. See [MM07, Proposition 5.2] for both facts. Alternatively, for a less explicit but more uniform treatment, one may take μ_1 to be the measure μ_Γ constructed in [CM07] with respect to this (or any other) uniform lattice.

In either case μ_1 is a probability measure supported on a uniform lattice Γ in the group $\mathrm{Is}(X)$ and the μ_1-stationary boundary measure ν_1 satisfies $\nu_1 = m$.

We conclude that the stationary boundary measures ν_0 and ν_1 for the two probability measures μ_0 and μ_1 on the analytic group G coincide. These stationary boundary measures on the visual boundary ∂X provide a topological model for the Poisson boundary. It follows that a given random subgroup ν of the analytic group G is μ_0-stationary if and only if it is μ_1-stationary (see e.g. the discussion on [GLM22, p. 428]).

Theorem 10.5. *Let G be a simple analytic group[16] of rank one. Let X be the associated symmetric space or Bruhat–Tits tree. Let Δ be a discrete subgroup of the analytic group G. If $\delta(\Delta) \leq \frac{\dim_{\mathrm{Haus}}(\partial X)}{2}$ then $\{e\} \in \overline{\{\Delta^g \,:\, g \in G\}}$.*

Recall that if G is a Lie group then the statement of Theorem 10.5 can be reformulated to say that Δ is not confined, see Proposition 10.2 for details.

Proof of Theorem 10.5. Let μ_1 be the probability measure supported on some fixed uniform lattice in the analytic group G as constructed above, e.g. the one constructed in [CM07]. It satisfies $\delta(\mu_1) = \delta(\Gamma) = \frac{\dim_{\mathrm{Haus}}(\partial X)}{2}$.

Consider a discrete subgroup Δ of the Lie group G and assume that $\delta(\Delta) \leq \frac{\delta(\mu_1)}{2}$. Let ν be any weak-$*$ accumulation point of the sequence of probability measures $\nu_n = \frac{1}{n} \sum_{i=1}^n \mu_0^{*i} * \mathrm{D}_\Delta$. The key property of the probability measure μ_0 implies that the resulting random subgroup ν is *discrete* and μ_0-stationary [FG23, Theorem 1.6]. The last paragraph of the preceding discussion shows that the random subgroup ν is at the same time μ_1-stationary.

We remark that in the zero characteristic non-Archimedean case there is no need to consider the measure μ_0 at all and we may work with the measure μ_1 directly.

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[16]Here G is understood to be either a simple rank one Lie group or a non-Archimedean analytic group of the form $\mathbb{G}(k)$ considered above.

We know from Theorem 9.2 that ν-almost every *non-trivial* subgroup Λ satisfies $\delta(\Lambda) > \frac{\delta(\mu_1)}{2}$. One the other hand, the critical exponent is a conjugation invariant and a lower semi-continuous function on the Chabauty space of discrete subgroups by Proposition 10.4. So $\delta(\Lambda) \leq \delta(\Delta)$ must hold ν-almost surely. We conclude that $\nu = \mathrm{D}_{\{e\}}$ where $\{e\}$ is the trivial subgroup of the Lie group G. In the particular the trivial subgroup $\{e\}$ lies is the Chabauty closure of the conjugacy class of the subgroup Δ by the Portmanteau theorem. $\square$

General Gromov hyperbolic groups. In the case of free groups we recover the result of [Fra20, Corollary 3.2].

Theorem 10.6. *Let F_k be the free group of rank $k \in \mathbb{N}$ and X_k be the Cayley graph of F_k with respect to the standard generating set. Let H be a subgroup of F_k with $\delta(H) \leq \frac{\log(2k-1)}{2}$. Then H is not confined.*

Proof. The proof follows the same strategy as the proof of Theorem 10.5. This time we use the uniform probability measure μ_k supported on the standard symmetric generating set for the group F_k. It is shown in [MM07] that the measure μ_k satisfies

$$(10.5) \qquad \delta(\mu_k) = \delta(\Gamma) = \dim_{\mathrm{Haus}}(\partial X_k) = \log(2k - 1).$$

The proof is simpler in this case as there is no need to worry about discreteness of the limiting stationary random subgroup (for the group F_k itself is discrete). $\square$

Finally we state a weaker analogue of Theorems 10.5 and 10.6 that applies more generally to all Gromov hyperbolic metric spaces.

Theorem 10.7. *Let X be a proper Gromov hyperbolic geodesic metric space with isometry group $\mathrm{Is}(X)$. Assume that the group $\mathrm{Is}(X)$ admits a discrete subgroup Γ which acts properly and cocompactly and which intersects the elliptic radical $\mathrm{E}(\mathrm{Is}(X))$ trivially. If Δ is any subgroup of Γ with $\delta(\Delta) < \frac{\delta(\Gamma)}{2}$ then Δ is not confined.*

The above result applies in particular in the case where the group $\mathrm{Is}(X)$ itself is discrete.

Proof of Theorem 10.7. Let μ_k be the uniform probability measure supported on the finite set

$$(10.6) \qquad \{\gamma \in \Gamma \ : \ k \leq \|\gamma\| < k + 1\}$$

for every $k \in \mathbb{N}$. The sequence of probability measures μ_k satisfies $\delta(\mu_k) \xrightarrow{k\to\infty} \delta(\Gamma)$ [GMM18, Theorem 1.4]. In particular $\delta(\Delta) \leq \frac{\delta(\mu_{k_0})}{2}$ holds for some fixed sufficiently large $k_0 \in \mathbb{N}$. Let ν be any weak-$*$ accumulation point of the sequence of probability measures $\nu_n = \frac{1}{n} \sum_{i=1}^{n} \mu_{k_0}^{*i} * \mathrm{D}_\Delta$. Since the convolution powers $\mu_{k_0}^{*i}$ are all supported on the discrete subgroup Γ and as $\Delta \leq \Gamma$, the resulting μ_{k_0}-stationary random subgroup ν is almost surely contained in the discrete subgroup Γ. From this point we may conclude exactly as in the proof of Theorem 10.5. $\square$

Lastly, if the proper space X in Theorem 10.7 is CAT(-1) rather than simply Gromov hyperbolic, then the strict inequality can be replaced by a non-strict inequality. Namely, any subgroup $\Delta \leq \Gamma$ with $\delta(\Delta) \leq \frac{\delta(\Gamma)}{2}$ is not confined.

References

[ABB⁺20]   Miklos Abert, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet. On the growth of L^2-invariants of locally symmetric spaces, II: Exotic invariant random subgroups in rank one. *International Mathematics Research Notices*, 2020(9):2588–2625, 2020.

[AGV14]   Miklós Abért, Yair Glasner, and Bálint Virág. Kesten's theorem for invariant random subgroups. *Duke Mathematical Journal*, 163(3):465–488, 2014.

[BH13]   Martin Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319. Springer Science & Business Media, 2013.

[BHM08]   Sébastien Blachère, Peter Haïssinsky, and Pierre Mathieu. Asymptotic entropy and Green speed for random walks on countable groups. In *Annals of Probability*, volume 36, pages 1134–1152, 2008.

[BHM11]   Sébastien Blachère, Peter Haïssinsky, and Pierre Mathieu. Harmonic measures versus quasiconformal measures for hyperbolic groups. In *Annales scientifiques de l'Ecole normale supérieure*, volume 44, pages 683–721, 2011.

[BL]   Werner Ballmann and François Ledrappier. *Discretization of positive harmonic functions on Riemannian manifolds and Martin boundary*.

[BN15]   Lewis Bowen and Amos Nevo. Von Neumann and Birkhoff ergodic theorems for negatively curved groups. In *Annales scientifiques de lecole normale superieure*, volume 48, pages 1113–1147, 2015.

[Bor63]   Armand Borel. Compact Clifford-Klein forms of symmetric spaces. *Topology*, 2(1-2):111–122, 1963.

[BQ16a]   Yves Benoist and Jean-François Quint. Central limit theorem on hyperbolic groups. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 80(1):5–26, 2016.

[BQ16b]   Yves Benoist and Jean-François Quint. *Random walks on reductive groups*. Springer, 2016.

[BS06]   Uri Bader and Yehuda Shalom. Factor and normal subgroup theorems for lattices in products of groups. *Inventiones mathematicae*, 163(2):415–454, 2006.

[BT17]   Ian Biringer and Omer Tamuz. Unimodularity of invariant random subgroups. *Transactions of the American Mathematical Society*, 369(6):4043–4061, 2017.

[BTMT12]   Petra Bonfert-Taylor, Katsuhiko Matsuzaki, and Edwars Taylor. Large and small covers of a hyperbolic manifold. *J. Geometric Analysis*, 22:455–470, 2012.

[Car01]   Lisa Carbone. On the classification of rank one groups over non-Archimedean local fields. In *https://sites.math.rutgers.edu/ carbonel/pdfs/Classification.pdf*. Citeseer, 2001.

[CdCMT15] Pierre-Emmanuel Caprace, Yves de Cornulier, Nicolas Monod, and Romain Tessera. Amenable hyperbolic groups. *Journal of the european Mathematical Society*, 17(11):2903–2947, 2015.

[CM07]   Christopher Connell and Roman Muchnik. Harmonicity of quasiconformal measures and Poisson boundaries of hyperbolic spaces. *Geometric and Functional Analysis*, 17:707–769, 2007.

[CM09a]   Pierre-Emmanuel Caprace and Nicolas Monod. Isometry groups of non-positively curved spaces: discrete subgroups. *Journal of Topology*, 2(4):701–746, 2009.

[CM09b]   Pierre-Emmanuel Caprace and Nicolas Monod. Isometry groups of non-positively curved spaces: structure theory. *Journal of topology*, 2(4):661–700, 2009.

[Coo93]   Michel Coornaert. Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov. *Pacific Journal of Mathematics*, 159(2):241–270, 1993.

[Cor90]   Kevin Corlette. Hausdorff dimensions of limit sets I. *Inventiones mathematicae*, 102(1):521–541, 1990.

[DGLL15]   Bruno Duchesne, Yair Glasner, Nir Lazarovich, and Jean Lécureux. Geometric density for invariant random subgroups of groups acting on CAT(0) spaces. *Geometriae Dedicata*, 175(1):249–256, 2015.

[DSU17]   Tushar Das, David Simmons, and Mariusz Urbański. *Geometry and dynamics in Gromov hyperbolic metric spaces*, volume 218. American Mathematical Soc., 2017.

[Dus]   Matthieu Dussaule. Groupes d'isométries et marches aléatoires dans les espaces hyperboliques: Rapport de stage de M2, ENS Lyon.

[Els74] Jürgen Elstrodt. Die resolvente zum eigenwertproblem der automorphen formen in der hyperbolischen ebene. Teil III. *Mathematische Annalen*, 208:99–132, 1974.

[EO23] Sam Edwards and Hee Oh. Temperedness of $L^2(\Gamma \backslash G)$ and positive eigenfunctions in higher rank, 2023.

[FG10] Hillel Furstenberg and Eli Glasner. Stationary dynamical systems. *Dynamical numbers—interplay between dynamical systems and number theory*, 532:1–28, 2010.

[FG23] Mikolaj Fraczyk and Tsachik Gelander. Infinite volume and infinite injectivity radius. *Annals of Mathematics*, 197(1):389–421, 2023.

[Fra20] Mikolaj Fraczyk. Kesten's theorem for uniformly recurrent subgroups. *Ergodic Theory and Dynamical Systems*, 40(10):2778–2787, 2020.

[FT22] Behrang Forghani and Giulio Tiozzo. Shannon's theorem for locally compact groups. *The Annals of Probability*, 50(1):61–89, 2022.

[Fur71] Harry Furstenberg. Random walks and discrete subgroups of Lie groups. *Advances in probability and related topics*, 1:1–63, 1971.

[Fur02] Alex Furman. Random walks on groups and random transformations. In *Handbook of dynamical systems*, volume 1, pages 931–1014. Elsevier, 2002.

[Gel15] Tsachik Gelander. A lecture on invariant random subgroups. *arXiv preprint arXiv:1503.08402*, 2015.

[GL19] Ilya Gekhtman and Arie Levit. Critical exponents of invariant random subgroups in negative curvature. *Geometric and Functional Analysis*, 29(2):411–439, 2019.

[GLM22] Tsachik Gelander, Arie Levit, and Gregori A. Margulis. Effective discreteness radius of stabilizers for stationary actions. *Michigan Mathematical Journal*, 72(1):389–438, 2022.

[GMM18] Sébastien Gouëzel, Frédéric Mathéus, and François Maucourant. Entropy and drift in word hyperbolic groups. *Inventiones mathematicae*, 211(3):1201–1255, 2018.

[Gro87] Mikhael Gromov. Hyperbolic groups. In *Essays in group theory*, pages 75–263. Springer, 1987.

[GT20] Ilya Gekhtman and Giulio Tiozzo. Entropy and drift for Gibbs measures on geometrically finite manifolds. *Transactions of the American Mathematical Society*, 373:2949–2980, 2020.

[Gui80] Yves Guivarch. Sur la loi des grands nombres et le rayon spectral d'une marche aleatoire. *Asterisque*, 74:47–98, 1980.

[Jaw95] Wojciech Jaworski. Strong approximate transitivity, polynomial growth, and spread out random walks on locally compact groups. *Pacific Journal of Mathematics*, 170(2):517–533, 1995.

[Kai94] Vadim A Kaimanovich. The Poisson boundary of hyperbolic groups. *Comptes Rendus de l'Academie des Sciences-Serie I-Mathematique*, 318(1):59–64, 1994.

[Kai96] Vadim A Kaimanovich. Boundaries of invariant Markov operators: the identification problem. *London Mathematical Society lecture note series*, pages 127–176, 1996.

[Kak51] Shizuo Kakutani. Random ergodic theorems and Markoff processes with a stable distribution. Technical report, Yale University, New Haven United States, 1951.

[Kec12] Alexander Kechris. *Classical descriptive set theory*, volume 156. Springer Science & Business Media, 2012.

[Kle99] Bruce Kleiner. The local structure of length spaces with curvature bounded above. *Mathematische Zeitschrift*, 231:409–456, 1999.

[KM99] Anders Karlsson and Gregory A Margulis. A multiplicative ergodic theorem and nonpositively curved spaces. *Communications in mathematical physics*, 208(1):107–123, 1999.

[KV83] Vadim A Kaimanovich and Anatoly M Vershik. Random walks on discrete groups: boundary and entropy. *The annals of probability*, 11(3):457–490, 1983.

[Led90] Francois Ledrappier. Harmonic measures and Bowen-Margulis measures. *Israel J. Math.*, 71(3):275–287, 1990.

[Led01] François Ledrappier. Some asymptotic properties of random walks on free groups. *Topics in probability and Lie groups: boundary theory*, 28(117-152):7, 2001.

[LS84] Terry Lyons and Dennis Sullivan. Function theory, random paths and covering spaces. *Journal of differential geometry*, 19(2):299–323, 1984.

[Lub94] Alex Lubotzky. *Discrete groups, expanding graphs and invariant measures*, volume 125. Springer Science & Business Media, 1994.

[Mar91]   Gregori A Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17. Springer
          Science & Business Media, 1991.
[McM99]   Curtis McMullen. Hausdorff dimension and conformal dynamics. I. Strong convergence
          of Kleinian groups. *Journal of Differential Geometry*, 51(3):471–515, 1999.
[MM07]   Jean Mairesse and Frédéric Mathéus. Random walks on free products of cyclic groups.
          Journal of the London Mathematical Society, 75(1):47–66, 2007.
[MT18]   Joseph Maher and Giulio Tiozzo. Random walks on weakly hyperbolic groups. *Journal
          für die reine und angewandte Mathematik (Crelles Journal)*, 2018(742):187–239, 2018.
[MYJ20]   Katsuhiko Matsuzaki, Yasuhiro Yabuki, and Johannes Jaerisch. Normalizer, divergence
          type, and Patterson measure for discrete groups of the Gromov hyperbolic space.
          Groups, Geometry, and Dynamics, 14(2):369–411, 2020.
[Nic89]   Peter Nicholls. *The ergodic theory of discrete groups*, volume 143. Cambridge Univer-
          sity Press, 1989.
[Osi17]   D Osin. Invariant random subgroups of groups acting on hyperbolic spaces. *Proceedings
          of the American Mathematical Society*, 145(8):3279–3288, 2017.
[Pat76]   Samuel James Patterson. The limit set of a Fuchsian group. *Acta Mathematica*,
          136:241–273, 1976.
[Ser09]   Jean-Pierre Serre. *Lie algebras and Lie groups: 1964 lectures given at Harvard
          University*. Springer, 2009.
[Sul87]   Dennis Sullivan. Related aspects of positivity in Riemannian geometry. *Journal of
          differential geometry*, 25(3):327–351, 1987.
[Tan17]   Ryokichi Tanaka. Hausdorff spectrum of harmonic measure. *Ergodic Theory and
          Dynamical Systems*, 37(1):277–307, 2017.
[Tan19]   Ryokichi Tanaka. Dimension of harmonic measures in hyperbolic spaces,. *Ergodic
          Theory and Dynamical Systems*, 39(2):474–499, 2019.
[Ver00]   Anatoly Vershik. Dynamic theory of growth in groups: Entropy, boundaries, examples.
          Uspekhi Mat. Nauk, 55:59–128, 2000.
[Woe00]   Wolfgang Woess. *Random walks on infinite graphs and groups*, volume 138. Cambridge
          Tracts in Mathematics, 2000.
[Zim13]   Robert J Zimmer. *Ergodic theory and semisimple groups*, volume 81. Springer Science
          & Business Media, 2013.