Three components of stochastic entropy production associated with the quantum Zeno and anti-Zeno effects

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We investigate stochastic entropy production in a two-level quantum system that performs Rabi oscillations while undergoing quantum measurement brought about by continuous random disturbance by an external measuring device or environment. The dynamics produce quantum Zeno and anti-Zeno effects for certain measurement regimes, and the stochastic entropy production is a measure of the irreversibility of the behaviour. When the strength of the measurement disturbance is time-dependent, the stochastic entropy production separates into three components. Two represent relaxational behaviour, one being specific to systems represented by coordinates that are odd under time reversal symmetry, and a third characterises the nonequilibrium stationary state arising from breakage of detailed balance in the dynamics. The study illustrates how the ideas of stochastic thermodynamics may be applied in similar ways to both quantum and classical systems.

I. INTRODUCTION

Entropy quantifies subjective uncertainty in the configuration of a system and it can be argued that similar applications of this concept should apply in both classical and quantum mechanics, where configurations are described by phase space coordinates and by elements of a density matrix, respectively. The effective stochastic dynamics of such variables brought about by coupling the system to a coarse grained environment will increase the configurational uncertainty of the world (the system together with its environment) as time passes. Such a loss of information is often manifested in the dispersal of energy and matter or the loss of correlations: consequences of the chaotic nature of the underlying deterministic dynamics but nevertheless captured by stochastic modelling. This is the content of the second law of thermodynamics [1].

The aim of this paper is to compute the stochastic entropy production and hence loss of information when a simple quantum system undergoing Rabi oscillations is subjected to continuous measurement of two non-commuting observables [2]. The system exhibits Zeno and anti-Zeno effects [3] depending on the relative strengths of the two measurement processes. It is of particular interest to consider the division of the stochastic entropy production into three components when the strength of measurement is time-dependent [4, 5]. Each component describes an aspect of the nonequilibrium, irreversible behaviour of the system.

In Section II we derive Markovian stochastic differential equations, or Itô processes, that describe the evolution of the system. We examine Zeno and anti-Zeno effects where the mean rate of change of a system coordinate is reduced or increased, respectively, when measurement is made more intense. The nature of the three components of stochastic entropy production for timedependent measurement of one of the observables is discussed in Section III. We give our conclusions in Section IV.

II. STOCHASTIC DYNAMICS

The reduced density matrix ρ is a specification of the state of an open quantum system and under certain conditions of coupling to the environment its evolution can be modelled using a stochastic Lindblad equation:

$$d\rho = -i[H,\rho]dt + \sum_{i} \left(c_i \rho c_i^{\dagger} - \frac{1}{2} \rho c_i^{\dagger} c_i - \frac{1}{2} c_i^{\dagger} c_i \rho \right) dt + \left(\rho c_i^{\dagger} + c_i \rho - C_i \rho \right) dW_i,$$
(1)

with $C_i = \text{Tr}\left((c_i + c_i^{\dagger})\rho\right)$, where *H* is the system Hamiltonian. The Lindblad operators c_i represent the modes of interaction between the system and the environment, and the dW_i are a set of independent Wiener increments [6–8]. This framework is a form of quantum state diffusion, where evolution of ρ is continuous, without jumps [9].

We consider a two-level bosonic system represented by $\rho = \frac{1}{2} (\mathbb{I} + \mathbf{r} \cdot \boldsymbol{\sigma})$, where \mathbf{r} is the coherence or Bloch vector and σ_k are the Pauli matrices, with $H = \epsilon \sigma_z$, $c_1 = \alpha_x \sigma_x$ and $c_2 = \alpha_y \sigma_y$. The dynamics describe a system that performs Rabi oscillations in the expectation values $r_x = \text{Tr}(\sigma_x \rho)$ and $r_y = \text{Tr}(\sigma_y \rho)$ when isolated, but which departs stochastically from such regular behaviour when the coupling coefficients α_x and α_y are non-zero. The situation is similar to a two-level system undergoing the measurement of one observable, studied previously [10].

The Lindblads c_1 and c_2 tend to drive the system towards eigenstates of σ_x and σ_y , respectively, and their use in Eq. (1) can be regarded as an implementation of the continuous, simultaneous measurement of these two system observables [7]. The coefficients α_x and α_y are *measurement* strengths, since increasing α_x while α_y is held constant brings about a greater concentration of the pdf in the vicinity of the eigenstates of σ_x , and vice versa.

The dynamics of the components of r can be expressed

as Itô processes:

$$dr_x = -2\left(\epsilon r_y + \alpha_y^2 r_x\right) dt + 2\alpha_x \left(1 - r_x^2\right) dW_x - 2\alpha_y r_x r_y dW_y$$

$$dr_y = 2\left(\epsilon r_x - \alpha_x^2 r_y\right) dt - 2\alpha_x r_x r_y dW_x + 2\alpha_y \left(1 - r_y^2\right) dW_y$$

$$dr_z = -2r_z \left(\alpha_x^2 + \alpha_y^2\right) dt - 2r_z \left(\alpha_x r_x dW_x + \alpha_y r_y dW_y\right),$$

(2)

where dW_x and dW_y are Wiener increments.

We consider a (pure) state denoted by $r_z = 0$, $r_x = \sin \phi$ and $r_y = \cos \phi$. The coherence vector lies in the equatorial plane of the Bloch sphere and its rotation about the r_z axis is specified by an azimuthal angle $\phi = \tan^{-1}(r_x/r_y)$. The stochastic evolution of ϕ can be derived from Eq. (2) using Itô's lemma:

$$d\phi = \left(2\epsilon - \left(\alpha_x^2 - \alpha_y^2\right)\sin 2\phi\right)dt$$
$$- 2\alpha_x \sin\phi \, dW_x + 2\alpha_y \cos\phi \, dW_y$$
$$= \left(2\epsilon - \left(\alpha_x^2 - \alpha_y^2\right)\sin 2\phi\right)dt$$
$$- 2\left(\alpha_x^2 \sin^2\phi + \alpha_y^2 \cos^2\phi\right)^{1/2}dW, \qquad (3)$$

where dW is also a Wiener increment. The dynamics produce a linear increase in ϕ with time when the system is isolated ($\alpha_x = \alpha_y = 0$). This drift is distorted by random disturbances when the system is coupled to the environment, here regarded as a measuring device. The associated Fokker-Planck equation for the probability density function (pdf) $p(\phi, t)$ is

$$\frac{\partial p(\phi, t)}{\partial t} = -\frac{\partial J}{\partial \phi},\tag{4}$$

where the probability current is

$$J = \left(2\epsilon - \left(\alpha_x^2 - \alpha_y^2\right)\sin 2\phi\right)p(\phi, t) - 2\frac{\partial}{\partial\phi}\left(\alpha_x^2\sin^2\phi + \alpha_y^2\cos^2\phi\right)p(\phi, t).$$
(5)

The situation with equal non-zero measurement strengths $\alpha_x = \alpha_y = \alpha \neq 0$ is described by $d\phi = 2\epsilon dt - 2\alpha dW$. The system then evolves towards a stationary state with $p_{\rm st}(\phi) = (2\pi)^{-1}$ and $J = \epsilon/\pi$. When $\alpha_x \neq \alpha_y$, there is also a stationary state with constant J but characterised by a nonuniform pdf. These are nonequilibrium situations with consequent stochastic entropy production, which we investigate in the next section.

We consider the dependence of the mean rate of change of ϕ on the measurement strengths α_x and α_y . The mean of ϕ evolves according to

$$\frac{d\langle\phi\rangle}{dt} = 2\epsilon - \left(\alpha_x^2 - \alpha_y^2\right)\langle\sin 2\phi\rangle,\tag{6}$$

where the angled brackets represent an average over the stochasticity. Results from numerical simulations of Eq. (3) are given in Figure 1 for $\epsilon = 1/2$, $\alpha_x = 1$ and a range of values of α_y . The Zeno effect operates for $\alpha_y > \alpha_x$; a slowing of the average evolution of the system as the

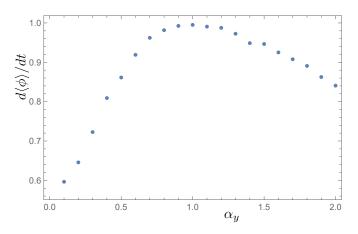


Figure 1. Zeno and anti-Zeno effects are apparent in the mean rate of Rabi oscillations, arising from changing the measurement strength α_y for $\alpha_x = 1$ and $\epsilon = 1/2$. The average is obtained over a time interval $\Delta t = 10^5$ for each value of α_y and the error in the mean is estimated to be less than 0.01.

strength of measurement α_y is increased at a constant α_x [10]. There is also an anti-Zeno effect for $\alpha_y < \alpha_x$, where the mean evolution is speeded up when α_y is increased. For $\alpha_x = \alpha_y$ and $p_{\rm st}(\phi) = (2\pi)^{-1}$ the average of $\sin 2\phi$ vanishes, and $d\langle\phi\rangle/dt = 2\epsilon$: the effects of the two measurement processes on the mean rate of Rabi oscillation then cancel each other out, somewhat counter-intuitively.

III. STOCHASTIC ENTROPY PRODUCTION

We now consider the stochastic thermodynamics associated with the dynamics

$$d\phi = \left(A^{\text{rev}}(\phi) + A^{\text{irr}}(\phi)\right)dt + B(\phi, t)dW, \qquad (7)$$

where the terms involving A^{rev} and A^{irr} represent deterministic rates of change of ϕ that satisfy and violate time reversal symmetry, respectively. The stochastic entropy production is given by [5]

$$d\Delta s_{\text{tot}} = -d\ln p(\phi, t) + \frac{A^{\text{irr}}}{D} d\phi - \frac{A^{\text{rev}}A^{\text{irr}}}{D} dt + \frac{dA^{\text{irr}}}{d\phi} dt$$
$$-\frac{dA^{\text{rev}}}{d\phi} dt - \frac{1}{D} \frac{\partial D}{\partial \phi} d\phi + \frac{(A^{\text{rev}} - A^{\text{irr}})}{D} \frac{\partial D}{\partial \phi} dt$$
$$-\frac{\partial^2 D}{\partial \phi^2} dt + \frac{1}{D} \left(\frac{\partial D}{\partial \phi}\right)^2 dt, \tag{8}$$

where $D(\phi, t) = \frac{1}{2}B(\phi, t)^2$. For dynamics that possess an equilibrium state (a stationary state with vanishing probability current J) characterised by a pdf $p_{\rm st}(\phi)$, Eq. (8) reduces to the simpler expression $d\Delta s_{\rm tot} =$ $-d \ln (p(\phi, t)/p_{\rm st}(\phi))$, showing explicitly how stochastic entropy production can arise from a statistical deviation from equilibrium. The system under consideration here, however, does not possess an equilibrium state in general, but instead a nonequilibrium stationary state with non-zero J. For bosonic systems, the time reversal operation corresponds to taking a complex conjugate of the density matrix. Thus the components r_x and r_z of the coherence vector are even and the component r_y is odd under time reversal symmetry. This means that ϕ is also odd and we deduce that $A^{\text{rev}} = 2\epsilon$ and $A^{\text{irr}} = -(\alpha_x^2 - \alpha_y^2) \sin 2\phi$. The diffusion coefficient is $D = 2(\alpha_x^2 \sin^2 \phi + \alpha_y^2 \cos^2 \phi)$. We take the coefficients α_x and α_y to be time-independent (for now) and write $dA^{\text{irr}}/d\phi = -2(\alpha_x^2 - \alpha_y^2) \cos 2\phi$, $dD/d\phi = 2(\alpha_x^2 - \alpha_y^2) \sin 2\phi$, $d^2D/d\phi^2 = 4(\alpha_x^2 - \alpha_y^2) \cos 2\phi$, and obtain

$$d\Delta s_{\text{tot}} = -d\ln p + \frac{9\left(\alpha_x^2 - \alpha_y^2\right)^2 \sin^2 2\phi}{2\left(\alpha_x^2 \sin^2 \phi + \alpha_y^2 \cos^2 \phi\right)} dt$$
$$-6\left(\alpha_x^2 - \alpha_y^2\right) \cos 2\phi dt + \frac{3\left(\alpha_x^2 - \alpha_y^2\right) \sin 2\phi}{\left(\alpha_x^2 \sin^2 \phi + \alpha_y^2 \cos^2 \phi\right)^{1/2}} dW.$$
(9)

For $\alpha_y = 0$, and hence measurement of σ_x alone, this reduces to

$$d\Delta s_{\text{tot}} = -d\ln p + 6\alpha_x^2 \left(1 + \cos^2 \phi\right) dt + 6\alpha_x \cos \phi \, dW,\tag{10}$$

and we conclude that in a stationary state, for a given value of α_x , the stochastic entropy production increases on average at a constant rate given by

$$\frac{d\langle\Delta s_{\rm tot}\rangle}{dt} = 6\alpha_x^2 \left(1 + \langle\cos^2\phi\rangle\right),\tag{11}$$

since $\langle -d \ln p \rangle = dS_G = 0$ in these circumstances, where $S_G = -\int p \ln p \, d\phi$ is the Gibbs entropy. For larger α_x^2 , the pdf becomes more concentrated in the region of $\phi = 0$ and π , corresponding to the eigenstates of σ_x [10], such that $\langle \cos^2 \phi \rangle$ increases with α_x^2 towards an upper limit of unity. Thus a increase in measurement strength brings about a higher mean rate of production of stochastic entropy, which can be associated intuitively with the increased Zeno slowing down, on average, of the Rabi oscillations.

We now consider a situation where the measurement strength α_x is time-dependent. In such circumstances the stochastic entropy production separates into three identifiable components [5], written

$$d\Delta s_{\rm tot} = d\Delta s_1 + d\Delta s_2 + d\Delta s_3. \tag{12}$$

The rate of change of the mean value of the first component may be written in the form

$$\frac{d\langle\Delta s_1\rangle}{dt} = -\int \frac{\partial p}{\partial t} \ln \frac{p(\phi, t)}{p_{\rm st}^{\alpha_x}(\phi)} d\phi.$$
(13)

Evidently, this is a relaxational entropy production that vanishes when the system is in a stationary state characterised by the pdf $p_{\rm st}^{\alpha_x}(\phi)$ associated with a specified

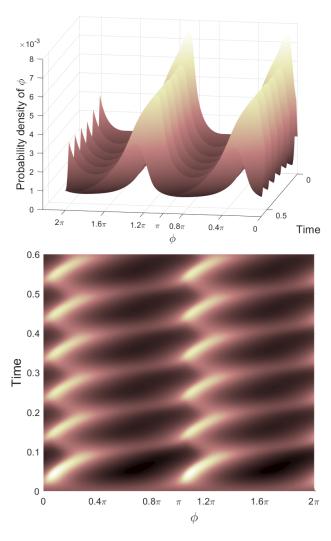


Figure 2. Probability density function for ϕ against time (side and top views) brought about by competition between a timedependent measurement strength $\alpha_x = 2 + \sin 20\pi t$ that draws the system towards eigenstates of σ_x at $\phi = 0$ and π , and Rabi oscillations characterised by $\epsilon = 10$ that favour a positive drift for ϕ .

value of α_x . Esposito and Van den Broeck denoted this component the nonadiabatic entropy production [11]. Its mean rate of change can never be negative.

We solve the Fokker-Planck equation for $\epsilon = 10$ and a range of values of α_x (with $\alpha_y = 0$) to obtain stationary pdfs $p_{\rm st}^{\alpha_x}(\phi)$. We then introduce a time-dependent measurement strength $\alpha_x = 2 + \sin 20\pi t$ to obtain a time-dependent pdf $p(\phi, t)$ that settles into a periodic stationary state, as shown in Figure 2. The principal feature to notice is that the system is periodically attracted, statistically speaking, towards the eigenstates of the σ_x observable at $\phi = 0$ and π , though displaced to higher values by the Rabi rotation.

We have calculated the average of Δs_1 as a function of time for $\epsilon = 10$ and $\alpha_x = 2 + \sin 20\pi t$ using meth-

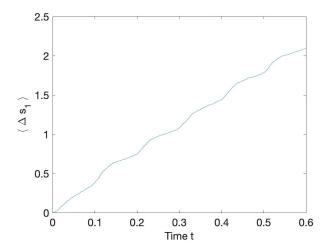


Figure 3. The average of Δs_1 , the nonadiabatic component of stochastic entropy production, evolving with time, for $\alpha_x = 2 + \sin 20\pi t$, $\alpha_y = 0$ and $\epsilon = 10$, adopting a periodic stationary state to accompany the Zeno slowing down of the Rabi oscillations and reflecting the time-dependence of the measurement strength α_x .

ods described in [5] and the results are given in Figure 3. Since the system is prevented from reaching a stationary state through the time-dependence of the measurement strength, the mean rate of change of this component of stochastic entropy production never falls to zero, but instead continues to evolve periodically.

The average of the second component of stochastic entropy production evolves according to [4]

$$\frac{d\langle\Delta s_2\rangle}{dt} = \int \frac{p}{D} \left(\frac{J_{\rm st}^{\rm irr}(\phi^{\rm T})}{p_{\rm st}^{\alpha_x}(\phi^{\rm T})}\right)^2 d\phi, \qquad (14)$$

where ϕ^{T} is the transform of ϕ under time reversal: since ϕ is odd, $\phi^{\mathrm{T}} = -\phi$. Δs_2 is a contribution to stochastic entropy production arising from the breakage of detailed balance, which permits the emergence of a non-zero irreversible probability current in a stationary state, given by

$$J_{\rm st}^{\rm irr}(\phi) = A^{\rm irr} p_{\rm st}^{\alpha_x}(\phi) - \frac{\partial}{\partial \phi} D(\phi, \alpha_x) p_{\rm st}^{\alpha_x}(\phi), \qquad (15)$$

where the diffusion coefficient is specified by the current value of α_x . Esposito and Van den Broeck referred to Δs_2 as the adiabatic entropy production [11] and Spinney and Ford, who included a consideration of dynamical variables that are odd as well as even under time reversal symmetry, denoted it the generalised housekeeping entropy production [4]. Like the nonadiabatic entropy production, its mean rate of change is never negative. The evolution of $\langle \Delta s_2 \rangle$ for $\epsilon = 10$ and $\alpha_x = 2 + \sin 20\pi t$ is illustrated in Figure 4.

The average rate of change of the third contribution to

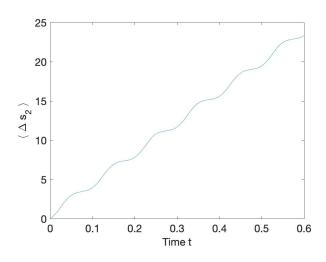


Figure 4. Average of Δs_2 , the adiabatic component of stochastic entropy production, against time, for $\alpha_x = 2 + \sin 20\pi t$, $\alpha_y = 0$ and $\epsilon = 10$.

the stochastic entropy production may be written

$$\frac{d\langle\Delta s_3\rangle}{dt} = -\int \frac{\partial p}{\partial t} \ln \frac{p_{\rm st}^{\alpha_x}(\phi)}{p_{\rm st}^{\alpha_x}(\phi^{\rm T})} d\phi.$$
(16)

 Δs_3 is a contribution associated with relaxation towards a stationary state and in this respect is similar to Δs_1 . It explicitly vanishes when there are no odd variables in the dynamics, but here it does not vanish. It was designated the transient housekeeping entropy production by Spinney and Ford [4]. The evolution of $\langle \Delta s_3 \rangle$ for $\epsilon = 10$ and $\alpha_x = 2 + \sin 20\pi t$ is illustrated in Figure 5. Notice that a negative mean rate of production is permitted, in contrast to the other two contributions. The mean rate of change of Δs_{tot} is, of course, positive for all times, in accordance with the second law [12].

If we were to re-introduce the measurement of σ_y , and hence create conditions for an anti-Zeno effect in the dynamics, the stochastic entropy production would similarly divide into three components and quantify the relative contributions of different sources of irreversibility.

IV. CONCLUSIONS

Employing the framework of quantum state diffusion as a model of the evolution of an open quantum system allows us to investigate the effect of measurement on the intrinsic dynamics of a quantum system. We have previously investigated a Zeno effect in a multilevel bosonic system: a slowing down of Rabi oscillations, on average, when measurements are performed to determine the level currently occupied [10]. Here we extend that study to demonstrate that simultaneous measurement of a second, non-commuting observable can produce a counter-intuitive anti-Zeno effect, specifically that the

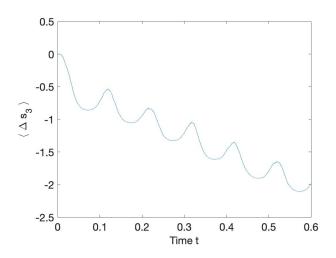


Figure 5. Average of Δs_3 , the transient housekeeping stochastic entropy production, against time, for $\alpha_x = 2 + \sin 20\pi t$, $\alpha_y = 0$ and $\epsilon = 10$.

slowed down evolution can be speeded up when measurement of the second observable is introduced.

The principal aim of the study, however, has been to compute the stochastic entropy production associated with the evolution of a two-level system when a Zeno effect is operating as a result of the continuous measurement of one observable. Our investigation of the division of the stochastic entropy production into its three components is motivated by a wish to demonstrate that the ideas underpinning stochastic thermodynamics can apply

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with equal validity to classical and quantum mechanics. Stochastic entropy production measures the irreversibility of the evolution of a system when subjected to unpredictable external disturbance. This is the extent to which two sequences of events, one the reverse of the other, occur with different probabilities in such circumstances. We take the trajectory followed by the reduced density matrix of an open quantum system, when it is subjected to continuous measurement, to be analogous to the Brownian path of a classical particle under the influence of an unpredictable environment. Irreversibility occurs in both situations and can be quantified.

The division of stochastic entropy production into components demonstrates how irreversibility can be associated with the relaxation of a system towards stationarity (components Δs_1 and Δs_3) and with the breakage of detailed balance and the consequent existence of a nonequilibrium stationary state (component Δs_2). These three contributions emerge in the two-level quantum system when we make the strength of measurement a periodic function of time, such that the reduced density matrix and the mean stochastic entropy production also evolve periodically. We conclude that stochastic entropy production associated with nonequilibrium behaviour, reflecting a continuing loss of information concerning the configuration of the world, can be demonstrated in open quantum systems as well as in classical situations.

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