THE BURNSIDE PROBLEM FOR ODD EXPONENTS

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Abstract. We show that the free Burnside groups $B(n,m)$ are infinite for $n \geqslant 557$ and $m \geqslant 2$. The proof uses iterated small cancellation theory where the induction based on the nesting depth of relators. The main instrument at every step is a new concept of a certification sequence. This decreases the best known lower bound in the Burnside problem for odd exponents from 665 to 557.

1. Introduction

In 1902 Burnside asked whether any finitely generated group of finite exponent is necessarily finite. This question was first answered in the negative in 1964 by Golod and Shafarevitch who constructed an infinite finitely generated torsion group. However, their example has unbounded exponent raising the question whether the so-called free Burnside group

$$B(m,n) = F_m/\langle\langle w^n : w \in F_m\rangle\rangle$$

of exponent n is finite where F_m is the free group in m generators. For exponents $n = 2, 3, 4$ and 6 it is known by work of Burnside [3], Sanov [18], and M. Hall [11] that the free Burnside group is indeed finite for any finite number m of generators. On the other hand, in 1968 Adian and Novikov gave the first proof that the free Burnside group $B(m,n)$ is infinite for odd $n \geqslant 4381$ [2]. Later on, Adian improved the bound to odd $n \geqslant 665$ [1]. The case of even exponent n turned out to be much harder. This case was treated by Ivanov in 1992 [12], he established that $B(m,n)$ is infinite for $n > 2^{48}$ [12]. Then Lysenok in 1996 improved the exponent for the even case to $n \geqslant 8000$ [14]. Together with the work of Adian [1], this yields that $B(m,n)$ is infinite for all $m > 1$ and all $n \geqslant 8000$. The proofs of Adian and Novikov use a very involved induction process with a list of 178 assumptions. So Ol'shanskii's geometric proof based on a deep study of van-Kampen diagrams was an important step. It resulted in the paper [16] for exponents $n > 10^{10}$. The proof is much shorter and more transparent than the one by Adian and Novikov, at the expense of a significantly larger exponent.

Another more geometric approach to free Burnside groups of odd exponent was suggested by Gromov and Delzant in [10]. This has been further developed by Coulon [5]. However, their arguments also require a very large exponent n.

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Note that the restricted version of the Burnside problem asks whether there exist finitely many different finite groups in m generators of exponent n, up to isomorphism. This question was solved in positive by Zelmanov in 1989, [19], [20], for arbitrary exponents.

While arguably the Burnside question has thus long been settled, the precise lower bound for the infiniteness of $B(m,n)$ remains open. Experience shows that decreasing the exponent requires huge efforts even for small steps. We hope that our methods pave the way for further reductions and we believe that an exponent around 300 might be in reach.

We also believe that it is important to provide readable and accessible proofs which give useful lower bounds for the infiniteness of $B(m,n)$ and that the methods developed in this paper are applicable for addressing other Burnside type questions, for instance Engel and quasi-Engel problems, which deal with identities equal to Engel and quasi-Engel words.

Our proof works inductively by choosing a canonical representative for every coset in $B(m,n)$. The induction is based on the rank of a word $w \in F_m$ where we (roughly speaking) define the rank $rk(w)$ to be greater or equal to $k+1$ with respect to our *nesting constant* τ if the word w (cyclically) contains a subword of the form v^τ for some word $v \in F_m$ with $rk(v) \geqslant k$.

We define

$$N_k = \langle\langle w^n \colon rk(w) \leqslant k\rangle\rangle.$$

Thus, we obtain an ascending sequence of normal subgroups

$$N_0 \leqslant N_1 \ldots \leqslant N_i \leqslant \ldots \bigcup N_i = N = \langle\langle w^n \colon w \in F_m\rangle\rangle.$$

We inductively define the canonical form $\mathrm{can}_k(w)$ for a word w as a canonical representative for wN_k. In particular, for all $w_0, w_1 \in F_m$ we have

$$w_0 N_k = w_1 N_k \ \text{ if and only if } \ \mathrm{can}_k(w_0) = \mathrm{can}_k(w_1)$$

and we can define a group operation on the set of canonical forms of rank k making this group isomorphic to F_m/N_k.

In order to define the canonical form $\mathrm{can}_k(w)$ on the basis of $\mathrm{can}_{k-1}(w)$ we use the concept of a *certification sequence*. We think of it as carefully choosing the sides of the relators in a given word. The important point is that for any $w \in F_m$, the canonical form stabilizes, i.e. for any $w \in F_m$ there is some k such that $\mathrm{can}_k(w) = \mathrm{can}_l(w)$ for all $l > k$ and thus $\mathrm{can}_k(w)$ will be the canonical representative for $wN \in B(m,n)$.

In this way we obtain a section $\mathrm{can} : F_m/N \longrightarrow F_m$ i.e. we have

$$\mathrm{can}(w) = \mathrm{can}(w') \text{ if and only if } \ wN = w'N \in B(m,n).$$

The set of canonical forms $\mathrm{can}(F_m)$ with the appropriate multiplication then forms a group isomorphic to $B(m,n)$.

Thus, the main thrust of the paper lies in inductively defining $\mathrm{can}_k(w)$ for any k based on 13 induction hypotheses. We will see that any cube-free element of F_m is already in canonical form and so the infinity of the Burnside group follows immediately from the fact that there are infinitely many cube-free words on two letters.

For our method to give a relatively short and accessible proof, we currently need the exponent n to be at least $n > 36 \cdot 15 + 16 = 556$. However, we expect that this can still be much improved. The proof also yields (the previously known result) that the infinite free Burnside groups are not finitely presented.

2. The set-up

Let $\mathrm{F} = \langle x_1, \ldots, x_m \rangle$ be the free group with free generators $x_1, \ldots, x_m$, $m \geqslant 2$. Then

$$B(m,n) = \mathrm{F}/\langle\langle x_1, \ldots, x_m \mid w^n,\ w \in \mathrm{F}\ \rangle\rangle$$

is called *the free Burnside group of rank m and exponent n.*

In this paper we prove the following

Theorem 2.1. *The free Burnside group $B(m,n)$ is infinite for $m \geqslant 2$ and odd exponents $n \geqslant 557$.*

Throughout the paper n is an odd natural number $\geqslant 557$. Section 3 and Section 4 describe an inductive process for the definition of a canonical form. We apply the results of this induction in Section 8 and show that $B(m,n)$ is infinite for $m \geqslant 2$ and odd exponents $n \geqslant 557$.

The free generators $\{x_1, \ldots, x_m\}$ of F and their inverses $\{x_1^{-1}, \ldots, x_m^{-1}\}$ are called *letters*, sequences of letters are called *words*. A word without cancellations is called a *reduced word*.

We say that a word w *cyclically contains a word A* if A is a subword of a cyclic shift of w.

A *prefix* of a reduced word is any (not necessarily proper) initial segment of this word. Similarly, a *suffix* of a reduced word is any (not necessarily proper) final segment of it.

If N is a normal subgroup of G and $w_1, w_2 \in G$ represent the same element in G/N, we say that w_1 and w_2 *are equivalent* $\mod N$ and we write

$$w_1 \equiv w_2 \mod N.$$

We write $w_1 = w_2$ to denote equality of (reduced) words in the free group.

Let $A, B \in \mathrm{F}$. We denote their product by $A \cdot B$. If we just write AB this implies that $A \cdot B$ has no cancellation. In particular, if we write A^m for some exponent $m \in \mathbb{Z}$, this indicates that A is cyclically reduced.

We will frequently use the following easy observation:

Remark 2.1. Suppose that A and B are reduced words. Then the product $A \cdot B^{-1}$ has cancellation if and only if A and B have a non-trivial common suffix. Similarly, $A^{-1} \cdot B$ has cancellation if and only if A and B have a non-trivial common prefix.

For any word w we denote the number of letters in w by $|w|$ and call it the *length of w*.

Remark 2.2. Note that if $w \neq 1$ is a reduced word in the free group, then $\mathrm{Cen}(w^n) = \langle w \rangle$ if and only if w is not a proper power. In this case we say that w is *primitive.*

3. The list of induction hypotheses

The purpose of the induction is to define *the canonical form of rank i*, $\mathrm{can}_i(A)$, of A, for all words A in the alphabet $\{x_1, \ldots, x_m\} \cup \{x_1^{-1}, \ldots, x_m^{-1}\}$ and for all $i \geqslant 0$. Then Can_i denotes the set of canonical forms of rank i. To start the induction, Can_{-1} is the set of all words (not necessarily reduced) and the canonical form of rank 0 of a word in Can_{-1} is its reduced form, i.e. Can_0 is the set of all reduced words. Then we inductively define the canonical form of rank i for all $A \in \mathrm{Can}_{i-1}$ and extend the definition to all words in Can_{-1} via

$$\mathrm{can}_i(A) = \mathrm{can}_i(\mathrm{can}_{i-1}(\ldots \mathrm{can}_0(A) \ldots)).$$

The elements of Can_i are called *canonical words of rank* i.

Furthermore, we will specify pairwise disjoint sets $\mathrm{Rel}_i \subset \{w^n : w \in \mathrm{F} \text{ primitive}\}$ of relators which are invariant under inverses and cyclic shifts. Note that relators from Rel_i may not belong to Can_{i-1}.

Throughout the paper we fix our *nesting constant* $\tau = 15$.

Definition 3.1 (Fractional powers and Λ_i-measure)**.** *If* u *is a subword of* a^k *for some* $k \in \mathbb{Z}$*, we call* u *a* fractional power of a *and put*

$$\Lambda_a(u) = \frac{|u|}{|a|}.$$

If $a^n \in \mathrm{Rel}_i$*, we call* u *a* fractional power of rank i *and if* $k \geqslant \tau+1$ *we put* $\Lambda_i(u) = \Lambda_a(u)$*. If* $k < \tau + 1$ *we only define its* Λ_i*-measure if it is clear from the context with respect to which relator from* Rel_i *the measure is taken.*

We say that u *has* Λ_i*-measure* at most m *for* $m \geqslant \tau$ *if either* $\Lambda_i(u) \leqslant m$ *or the* Λ_i*-measure of* u *is not defined.*

We show inductively for $i \geqslant 0$ that Can_i is a group with respect to an appropriately defined multiplication.

The induction hypothesis at stage r**:** At stage r we assume inductively that the following statements hold for $i = 0, \ldots, r-1$. Here and in what follows we will refer to Induction Hypothesis 1 as IH 1 etc.

- IH 1. The canonical form of rank i of every word of Can_{i-1} is uniquely defined and
$$\mathrm{Can}_i = \{\mathrm{can}_i(w) \mid \ w \in \mathrm{Can}_{i-1}\}.$$
- IH 2. $\mathrm{Can}_i \subseteq \mathrm{Can}_{i-1}$.
- IH 3. The sets Rel_i, $0 \leqslant i \leqslant r-1$, are closed under cyclic shifts and inverses and pairwise disjoint. We have $\mathrm{Rel}_0 = \{1\}$, and $\mathrm{Rel}_i \subseteq \{w^n \mid w \in \mathrm{F} \textit{ primitive}\}$ for $1 \leqslant i \leqslant r-1$.
- IH 4. If $A \in \mathrm{Can}_{i-1}$ does not contain fractional powers of rank i of Λ_i-measure $> \frac{n}{2} - 5\tau - 2$, then $A \in \mathrm{Can}_i$.

Remark 3.2. Note that by IH 4 we also have $\mathrm{can}_i(1) = 1 \in \mathrm{Can}_i$ where 1 denotes the empty word, and $\mathrm{can}_i(x) = x \in \mathrm{Can}_i$ for every single letter x.

The small cancellation condition is contained in the following induction hypothesis (see Lemma 4.9):

- IH 5. Let $x^n \in \mathrm{Rel}_i$, $y^n \in \mathrm{Rel}_j$, $1 \leqslant i \leqslant j \leqslant r-1$, and let c be their common prefix. If $i < j$, then $|c| < 2|y|$ and if $i = j$ and $|x| \leqslant |y|$, then $|c| < \min\{(\tau+1)|x|, 2|y|\}$.
- IH 6. If $A \in \mathrm{Can}_i$, then $A = \mathrm{can}_i(A)$.
- IH 7. $\mathrm{can}_i(A^{-1}) = (\mathrm{can}_i(A))^{-1}$.

Remark 3.3. By IH 6 we have $\mathrm{can}_i(B) = \mathrm{can}_i(\mathrm{can}_i(B))$ for every $B \in \mathrm{Can}_{-1}$. In other words, can_i is an idempotent operation equivariant with respect to taking inverses by IH 7.

The following axiom states that the canonical form picks unique coset representatives:

- IH 8. Let $A, B \in \mathrm{Can}_{-1}$. Then $A \equiv B \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_i \rangle\rangle$ if and only if $\mathrm{can}_i(A) = \mathrm{can}_i(B)$.

Remark 3.4. Note that for $i \geqslant 0$ the set Can_i is a group with respect to the multiplication defined by

$$A \cdot_i B = \mathrm{can}_i(A \cdot B), \ A, B \in \mathrm{Can}_i,$$

with identity element $1 = \mathrm{can}_i(1)$ and inverses given by inverses in the free group.

In particular, $(\mathrm{Can}_0, \cdot_0)$ is precisely the free group F.

Notice that if $A \equiv B \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_i \rangle\rangle$ and $A \in \mathrm{Can}_i$, then $\mathrm{can}_i(B) = \mathrm{can}_i(A) = A$ by IH 8 and IH 6.

For $A \in \mathrm{Can}_{-1}$ we thus have $A \equiv \mathrm{can}_i(A) \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_i \rangle\rangle$. Furthermore, since $\mathrm{can}_i(1) = 1$, we have $\mathrm{can}_i(v) = 1$ for $v \in \mathrm{Rel}_i$, $1 \leqslant i \leqslant r - 1$.

These previous remarks can be rephrased as:

Corollary 3.5. *Let $A, B \in \mathrm{Can}_{-1}$. Then for $i \geqslant 0$ we have*

$$\begin{aligned}\mathrm{can}_i(A) \cdot_i \mathrm{can}_i(B) &= \mathrm{can}_i(\mathrm{can}_i(A) \cdot \mathrm{can}_i(B)) = \mathrm{can}_i(A \cdot \mathrm{can}_i(B)) \\ &= \mathrm{can}_i(\mathrm{can}_i(A) \cdot B) = \mathrm{can}_i(A \cdot B).\end{aligned}$$

IH 9. Any non-empty subword of a word from Can_i, $i \geqslant 0$, is not equal to 1 in the group $\mathrm{F}/\langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_i \rangle\rangle$.

Definition 3.6. *A reduced word A is α-free modulo rank i if A it does not contain subwords of the form a^α where a is primitive and $a^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_i$.*

A reduced word A is α-free of rank i if it does not contain subwords of the form a^α with $a^n \in \mathrm{Rel}_i$.

We call a triple of words (D_1, D_2, D_3) a canonical triangle *of rank i if they are τ-free modulo rank $i+1$ and $D_1 \cdot D_2 \cdot D_3^{-1} \equiv 1 \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_i \rangle\rangle$.*

The following axiom is crucial:

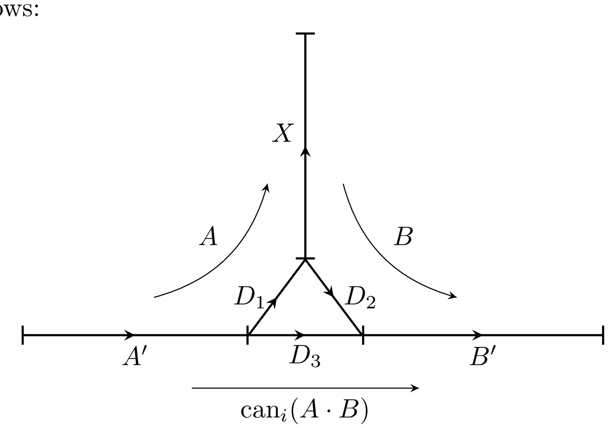
IH 10. **(Canonical triangle hypothesis)** For $A, B \in \mathrm{Can}_i$ there is a canonical triangle (D_1, D_2, D_3) of rank i such that $A = A'D_1X$, $B = X^{-1}D_2B'$ (where $X \cdot X^{-1}$ is the maximal cancellation in $A \cdot B$) such that

$$\mathrm{can}_i(A \cdot B) = A'D_3B'.$$

Furthermore, if $(D_1^{(i)}, D_2^{(i)}, D_3^{(i)})$ is a canonical triangle of rank $i-1$ such that $A = A''D_1^{(i)}X$, $B = X^{-1}D_2^{(i)}B''$ and $\mathrm{can}_{i-1}(A \cdot B) = A''D_3^{(i)}B''$, then A' is a prefix of A'' and B' is a suffix of B'' and if $D_1 = D_1^{(i)}, D_2 = D_2^{(i)}$, then $D_3 = D_3^{(i)}$.

Note that if $D_1 = D_1^{(i)}, D_2 = D_2^{(i)}$, then $A' = A''$ and $B' = B''$ since the maximal cancellation is independent of i.

The multiplication $A \cdot_i B = \mathrm{can}_i(A \cdot B)$ in the group $(\mathrm{Can}_i, \cdot_i)$ can be graphically expressed as follows:



Note that $A{\cdot}B$ and $\mathrm{can}_i(A{\cdot}B)$ represent the same element in $\mathrm{F}/\langle\langle \mathrm{Rel}_0,\dots,\mathrm{Rel}_i\rangle\rangle$ by IH 8. Hence after cancelling A' from the left and B' from the right it follows that $D_1\cdot D_2$ and D_3 represent the same element in $\mathrm{F}/\langle\langle \mathrm{Rel}_0,\dots,\mathrm{Rel}_i\rangle\rangle$. In particular, if two of D_1,D_2,D_3 are equal to 1, then so is the remaining one by IH 9.

The triangles constitute the '*smoothing process*' in the multiplication of canonical words. So IH 10 states that in this smoothing process the perturbation on both sides of the multiplication seam is very limited and, furthermore, in order to obtain higher canonical forms the *smoothing area* given by the canonical triangles may need to increase (but will never shrink).

IH 11. If $L_1A^\tau R_1,\ L_2A^\tau R_2\in \mathrm{Can}_i$ for A primitive, $A^n\notin \mathrm{Rel}_0\cup\ldots\cup\mathrm{Rel}_i$ then $L_1A^NR_2\in\mathrm{Can}_i$ for any $N\geqslant\tau$.

IH 12. If A_1 is a prefix of $A\in\mathrm{Can}_i$, there is a canonical triangle $(D_1,1,D_3)$ such that $A_1=A_1'D_1$ and $\mathrm{can}_i(A_1)=A_1'D_3$.

By taking inverses IH 12 implies also that for a suffix A_2 of $A\in\mathrm{Can}_{r-1}$ there is a canonical triangle $(E_1,E_2,1)$ such that $\mathrm{can}_i(A_2)=E_3A_2'$ and $A_2=E_2A_2'$.

IH 13. If $A\in\mathrm{Can}_{-1}$ and $A^n\notin\langle\langle \mathrm{Rel}_0,\dots,\mathrm{Rel}_i\rangle\rangle$, then there are natural numbers K,M_0 and words W,Z depending only on A and i such that

$$\mathrm{can}_i(\underbrace{A\cdot\ldots\cdot A}_{M\ times})=W\widetilde{A}^{M-K}Z\quad \textit{for all } M\geqslant M_0,$$

and A and $\widetilde{A}$ are conjugate in the group $\mathrm{F}/\langle\langle \mathrm{Rel}_0,\dots,\mathrm{Rel}_i\rangle\rangle$.

We now collect a few immediate consequences of the induction hypotheses which will be widely used throughout:

Corollary 3.7. *Let $La^{N_1}Aa^{N_2}R\in\mathrm{Can}_i$ where A may be empty, a is primitive, $a^n\notin \mathrm{Rel}_0\cup\ldots\cup\mathrm{Rel}_i$ and $N_1,N_2\geqslant 2\tau$. Then*

$$\mathrm{can}_i(La^{N_1})=La^{N_1-\tau}X,$$
$$\mathrm{can}_i(a^{N_2}R)=Ya^{N_2-\tau}R,$$
$$\mathrm{can}_i(a^{N_1}Aa^{N_2})=Ya^{N_1-\tau}Aa^{N_2-\tau}X,$$

where $X\equiv Y\equiv a^\tau \mod \langle\langle \mathrm{Rel}_0,\dots,\mathrm{Rel}_i\rangle\rangle$ and X,Y only depend on a and i.

Proof. By IH 12, there is a canonical triangle $(D_1,1,D_3)$ of rank i such that

$$La^N=La^{N-\gamma}a_1D_1 \text{ and } \mathrm{can}_i(La^N)=La^{N-\gamma}a_1D_3$$

for some $\gamma\leqslant\tau$ and a prefix a_1 of a. Write $X=a^{\tau-\gamma}a_1D_3$, so $La^{N-\gamma}a_1D_3=La^{N-\tau}X$. Since $D_1\equiv D_3 \mod \langle\langle \mathrm{Rel}_0,\dots,\mathrm{Rel}_i\rangle\rangle$, we have

$$a^\tau=a^{\tau-\gamma}a_1D_1\equiv a^{\tau-\gamma}a_1D_3=X \mod \langle\langle \mathrm{Rel}_0,\dots,\mathrm{Rel}_i\rangle\rangle.$$

Since $N\geqslant 2\tau$, by IH 11 we have $L_1a^KX\in\mathrm{Can}_i$ for any $K\geqslant\tau$ and any L_1 such that L_1a^τ is a prefix of a word from Can_i. Now $L_1a^{K+\tau}\equiv L_1a^KX \mod \langle\langle \mathrm{Rel}_0,\dots,\mathrm{Rel}_i\rangle\rangle$, hence by Remark 3.4 we obtain that $\mathrm{can}_i(L_1a^{K+\tau})=L_1a^KX$. So, X depends only on a and i. By taking inverses and applying the previous case on both sides the remaining claims follow. □

For convenience we also note the following:

Corollary 3.8. *Let $La^{N_1+N_2}R\in\mathrm{Can}_i$, $N_1+N_2\geqslant\tau$, where a is primitive and $a^n\notin \mathrm{Rel}_0\cup\ldots\cup\mathrm{Rel}_i$. Let $M\in\mathrm{Can}_{-1}$ be such that $M\equiv a^\alpha \mod \langle\langle \mathrm{Rel}_0,\dots,\mathrm{Rel}_i\rangle\rangle$. Then*

$$\mathrm{can}_i(La^{N_1}\cdot M\cdot a^{N_2}R)=La^{N_1+N_2+\alpha}R.$$

Proof. Since $M \equiv a^\alpha \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_i \rangle\rangle$, we see that

$$La^{N_1} \cdot M \cdot a^{N_2} R \equiv La^{N_1+N_2+\alpha} R \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_i \rangle\rangle.$$

IH 11 implies that $La^{N_1+N_2+\alpha}R \in \mathrm{Can}_i$. Therefore Remark 3.4 implies the result. □

Since canonical triangles are τ-free of rank i, fractional powers of rank i and Λ_i-measure $\geqslant \tau$ block the influence of the smoothing process obtained from the canonical triangles in the computation of the canonical form for subwords and products:

Corollary 3.9. *Let* $A = A'D_1X,\ B_1 = X^{-1}D_2Ma^\tau R \in \mathrm{Can}_i$ *and* $\mathrm{can}_i(A \cdot B_1) = A'D_3Ma^\tau R$ *for some canonical triangle* (D_1, D_2, D_3) *of rank* i *and primitive* a *with* $a^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_i$ *(where* M *may be empty).*
If $B_2 = X^{-1}D_2Ma^\tau R_1 \in \mathrm{Can}_i$*, then* $\mathrm{can}_i(A \cdot B_2) = A'D_3Ma^\tau R_1$*.*

Proof. By IH 11 applied to $A'D_3Ma^\tau$ and $a^\tau R_1$ we have $A'D_3Ma^\tau R_1 \in \mathrm{Can}_i$. Since $D_1 \cdot D_2 \equiv D_3 \mod \langle\langle \mathrm{Rel}_1, \dots, \mathrm{Rel}_i \rangle\rangle$, we see that $A'D_3Ma^\tau R_1 \equiv A \cdot B_2 \mod \langle\langle \mathrm{Rel}_1, \dots, \mathrm{Rel}_i \rangle\rangle$. Thus, Remark 3.4 implies the claim. □

Clearly the corresponding statement for $A_1 \cdot B, A_2 \cdot B$ follows from this by considering inverses. Similarly we have

Corollary 3.10. *Let* $A = La^\tau Mb^\tau WR,\ A_1 = L_1a^\tau Mb^\tau WR_1 \in \mathrm{Can}_i$ *where* a, b *are primitive and* $a^n, b^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_i$ *(where* M, W *may be empty). Then* $\mathrm{can}_i(L_1a^\tau Mb^\tau W)$ *is obtained from* $\mathrm{can}_i(La^\tau Mb^\tau W)$ *by replacing* L *by* L_1*.*

Proof. By IH 12 there is a word D τ-free of rank $i+1$ such that

$$\mathrm{can}_i(La^\tau Mb^\tau W) = La^\tau MXD$$

where X is non-empty and $b^\tau W \equiv XD \mod \langle\langle \mathrm{Rel}_1, \dots, \mathrm{Rel}_i \rangle\rangle$. By IH 11 applied to L_1a^τ and $a^\tau MXD$ we have $L_1a^\tau MXD \in \mathrm{Can}_i$. Since

$$L_1a^\tau MXD \equiv L_1a^\tau Mb^\tau W \mod \langle\langle \mathrm{Rel}_1, \dots, \mathrm{Rel}_i \rangle\rangle,$$

Remark 3.4 implies the claim. □

4. The induction

In this section we start showing that the induction step works. We first establish the **induction basis for** $i = 0$. Note that although we have defined Can_{-1} with index -1, ranks of the canonical form and canonical triangles start from 0.

4.1. **Induction basis.**

Proposition 4.1. *The sets* Rel_0, Can_0*, and* Can_{-1} *satisfy IH 1–13.*

Proof. Since $\mathrm{Rel}_0 = \{1\}$ and the canonical form of rank 0 of a word from Can_{-1} is its reduced form, all the induction hypotheses are easily verified. In particular, all sides of canonical triangles of rank 0 are equal to 1. □

Note that IH 4, IH 5 and IH5 are not defined for $i = 0$, but will be verified for $i \geqslant 1$ inside the proofs.

Now assume that IH 1–13 hold for $\mathrm{Can}_{-1}, \dots, \mathrm{Can}_{r-1}$, $\mathrm{Rel}_0, \dots, \mathrm{Rel}_{r-1}$. In order to prove **the induction step**, we now construct Rel_r and Can_r such that $\mathrm{Can}_{-1}, \dots, \mathrm{Can}_r$, and $\mathrm{Rel}_0, \dots, \mathrm{Rel}_r$ also satisfy IH 1–13.

4.2. **Cyclically canonical words.** The multiplication of canonical words requires the smoothing process given by canonical triangles at the seam between the words (see IH 10). Hence in general $\operatorname{can}_i(A)\cdot\operatorname{can}_i(A)\neq\operatorname{can}_i(A)\cdot_i\operatorname{can}_i(A)=\operatorname{can}_i(A\cdot A)$. We now define *cyclically canonical words* of rank i, $i=0,\dots,r-1$, for which equality holds at least approximately:

Definition 4.2 (cyclically canonical words)**.** *We say that a word A is* cyclically canonical of rank 0 *if it is cyclically reduced. We call a cyclically reduced word A* cyclically canonical of rank i for $i\geqslant 1$ *if A^τ is a subword of a word in Can_i and $A=A_1^K$ for a primitive word $A_1^n\notin\mathrm{Rel}_0\cup\ldots\cup\mathrm{Rel}_i$.*

The set of all cyclically canonical words of rank i, $i\geqslant 0$, is denoted by Cycl_i.

Clearly $\operatorname{can}_0(A^K)=A^K$ for every $A\in\mathrm{Cycl}_0$ and $K\geqslant 0$ and if $A=A_1^K$ is cyclically canonicalof rank i, then so is A_1.

The following are immediate consequences of the induction hypotheses:

Lemma 4.3. *(1)* $\mathrm{Cycl}_i\subseteq\mathrm{Cycl}_{i-1}$.
(2) Cycl_i *is closed under taking cyclic shifts and inverses.*
(3) If $A\in\mathrm{Cycl}_0$ and $A^{N_1}\in\mathrm{Cycl}_i$, then $A^{N_2}\in\mathrm{Cycl}_i$ for all $N_1,N_2\geqslant 1$.
(4) $\mathrm{Cycl}_i\cap(\mathrm{Rel}_0\cup\ldots\cup\mathrm{Rel}_i)=\varnothing$.
(5) If $A\in\mathrm{Cycl}_i$, $i\geqslant 0$ and $K\geqslant 4\tau$, then $\operatorname{can}_i(A^K)=T_1A^{K-2\tau}T_2$ where T_1,T_2 only depend on A and i.

Proof. (1) and (2) are clear and (3) follows from IH 11.
(4) This follows directly from the definition.
(5) is a consequence of IH 11 and Corollary 3.7. □

4.3. **Sets of relators Rel_r and their common parts.** Recall that $\mathrm{Rel}_0=\{1\}$, Cycl_0 is the set of all cyclically reduced words and that throughout the paper we fix the nesting constant $\tau=15$. We put

$$\mathrm{Rel}_1=\{x^n\in\mathrm{Cycl}_0\mid |x|=1\},$$
$$\mathrm{Rel}_2=\{x^n\in\mathrm{Cycl}_1\mid Cen(x)=\langle\langle x\rangle\rangle,|x|>1 \textit{ and } x \textit{ does not cyclically contain } a^\tau \textit{for } a\in\mathrm{Cycl}_0\setminus\{1\}\}.$$

For $r\geqslant 3$ we define:

$$\mathrm{Rel}_r=\big\{x^n\in\mathrm{Cycl}_{r-1}\mid Cen(x)=\langle x\rangle \textit{ and if } x \textit{ cyclically contains } a^\tau \textit{ for } a\in\mathrm{Cycl}_0,\ Cen(a)=\langle a\rangle,\ \textit{then } a^n\in\mathrm{Rel}_1\cup\ldots\cup\mathrm{Rel}_{r-1}\big\}.$$

Remark 4.4. Note that by definition for $r\geqslant 3$, if $x^n\in\mathrm{Cycl}_{r-1}$ and x does *not* cylically contain a subword a^τ with $a^n\in\mathrm{Rel}_{r-1}$, then $x^n\in\mathrm{Rel}_1\cup\ldots\cup\mathrm{Rel}_{r-1}$. In this way the sets of relators Rel_i for $i\geqslant 2$ are defined by the nesting depth of power words that contain at least τ periods (see Corollary 4.8).

After completing the induction process we prove in Corollary 8.7 that by organizing the relators according to their nesting depth, we obtain

$$B(m,n)\cong \mathrm{F}/\left\langle\left\langle\bigcup_i\mathrm{Rel}_i\right\rangle\right\rangle.$$

Since Cycl_{r-1} is closed under inverses and cyclic shifts, if $x^n\in\mathrm{Cycl}_{r-1}$ is such that x cyclically contains some a^τ with $a^n\in\mathrm{Rel}_1\cup\ldots\cup\mathrm{Rel}_{r-1}$, then so does x^{-n} and any cyclic shift of x^n. So with Lemma 4.3(4) and IH 3 for ranks $<r$ we obtain:

Lemma 4.5. IH 3 *holds for* Rel_r.

Corollary 4.6. *If* $x^n, y^n \in \mathrm{Rel}_i, i \geqslant 1$, *then* x^τ *is not cyclically contained in* y.

Proof. This follows directly from $\mathrm{Rel}_i \cap \mathrm{Rel}_j = \varnothing$ for $i \neq j$ and the definition of Rel_i. □

We also note the following:

Lemma 4.7. *If* $x \in \mathrm{Cycl}_{r-1}$ *is primitive, then either* $x^n \in \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$, *or* x *cyclically contains* a^τ *for some* $a^n \in \mathrm{Rel}_r$ *if* $r \geqslant 3$, *(or* $a^n \in \mathrm{Rel}_1 \cup \mathrm{Rel}_2$ *if* $r = 2$*).*

Proof. If x does not cyclically contain any subwords of the form a^τ, then, by definition, $x^n \in \mathrm{Rel}_2$. If x cyclically contains only subwords of the form a^τ with $a^n \in \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_{r-1}$, then, again by definition, $x^n \in \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$. So assume x cyclically contains a subword a^τ where a is primitive and $a^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_{r-1}$. Then $|a| < |x|$, and by Lemma 4.3(4) and induction on $|x|$, we have $a^n \in \mathrm{Rel}_r$ or a (and hence x) cyclically contains b^τ for some $b^n \in \mathrm{Rel}_r$ if $r \geqslant 3$, (or $b^n \in \mathrm{Rel}_1 \cup \mathrm{Rel}_2$ if $r = 2$). By Corollary 4.6 the cases are mutually exclusive. □

Lemma 4.7 and Lemma 4.3(3) with $r-1$ in place of r now imply:

Corollary 4.8. *If* $x^n \in \mathrm{Rel}_r$, *then* x *cyclically contains* a^τ *for some* $a^n \in \mathrm{Rel}_{r-1}$ *if* $r \geqslant 3$, *(or* $a^n \in \mathrm{Rel}_1 \cup \mathrm{Rel}_2$ *if* $r = 2$*).*

The following important statement is proved in [7].

Lemma 4.9. *Let* x^n, y^n *be two reduced words such that* x *and* y *do not centralize each other in* F. *Let* c *be a common prefix of* x^n *and* y^n. *Then* $|c| < |x| + |y| - \gcd(|x|, |y|)$, *where* $\gcd(|x|, |y|)$ *is a greatest common divisor of* $|x|$ *and* $|y|$.

Lemma 4.10. *Let* $x^n, y^n \in \mathrm{Rel}_i$, $i \geqslant 1$, $x \neq y$, *and* c *be a common prefix of* x^n *and* y^n. *Assume* $|x| \leqslant |y|$. *Then* $|c| < \min\{2|y|, (\tau+1)|x|\}$.

Proof. For $r = 1$, by definition of Rel_1, we have $|x| = |y| = 1$, so the claim is obvious. Now let $i \geqslant 2$. Since $x^n \neq y^n \in \mathrm{Rel}_i$ we have $Cen(x) = \langle x \rangle, Cen(y) = \langle y \rangle$ and $\langle x \rangle \cap \langle y \rangle = \{1\}$. So it follows from Lemma 4.9 that $|c| < |x| + |y| \leqslant 2|y|$.

From $|c| < |y| + |x|$ we see that if $|c| > (\tau+1)|x|$, then we must have $|y| > \tau|x|$. Since c is a common prefix of x^n and y^n, this implies that y contains x^τ as a prefix, contradicting Corollary 4.6. □

Corollary 4.11. *Let* x, y *be primitive,* $x \in \mathrm{Cycl}_{r-1}, x^n \notin \mathrm{Rel}_0 \cup \ldots \cup \mathrm{Rel}_i, y^n \in \mathrm{Rel}_i, 1 \leqslant i < r$, *and let* c *be a common prefix of* x^n *and* y^n. *Then* $|c| < 2|x|$.

Proof. If $i = 1$, then y is a single letter and since x is primitive, we have $|c| < |x|$. Thus the claim holds for $i = 1$.

If $i \geqslant 2$, then x cyclically contains a subword a^τ with $a^n \in \mathrm{Rel}_i$ by Corollary 4.8. If $|c| \geqslant 2|x|$, then $|x| < |y|$ by Lemma 4.9 and since any cyclic shift of x is a subword of x^2 and hence of y^2, we see that y also cyclically contains a^τ, contradicting Corollary 4.6. □

Now Lemma 4.10, Corollary 4.11 and $\mathrm{Rel}_i \cap \mathrm{Rel}_j = \varnothing$ for $i \neq j$ directly imply:

Corollary 4.12. *IH 5 and IH 5 hold for* Rel_i, $i = 1, \ldots, r$.

4.4. **Turns of rank r.** If u is a fractional power of a, there exists a cyclic shift $\widehat{a} = a_2a_1$ (a_1, a_2 may be empty) of $a = a_1a_2$ such that u can be written in the form

$$u = \widehat{a}^k a_2 \quad \text{or} \quad u = \widehat{a}^{-k} a_1^{-1},\ k \in \mathbb{N} \cup \{0\}. \tag{1}$$

The set of *fractional powers* of rank $j, 1 \leqslant j \leqslant r$, is defined as

$$\{u \mid u \textit{ is a subword of } R^N, R \in \mathrm{Rel}_j, N \in \mathbb{Z}\}.$$

Note that since Rel_j is closed under cyclic shifts and inverses, this coincides with

$$\{u \mid u \textit{ is a prefix of } R^N, R \in \mathrm{Rel}_j, N \in \mathbb{N}\}.$$

Remark 4.13. If u is a fractional power of rank j, $1 \leqslant j \leqslant r$ of Λ_j-measure $\geqslant \tau + 1$, then by IH 5 (for $1 \leqslant j < r$) and Lemma 4.10 (for $j = r$) there exists a unique relator $a^n \in \mathrm{Rel}_j$ such that u is a prefix of $a^K, K \geqslant 0$. So u can be written uniquely as

$$u = a^k a_1,\ \textit{where } a^n \in \mathrm{Rel}_j,\ a = a_1a_2,\ k \in \mathbb{N} \cup \{0\}. \tag{2}$$

Clearly, any fractional power u of rank j can be represented as in (2). However, without the condition that u contains $\geqslant \tau + 1$ periods of a relator, the relator $a^n \in \mathrm{Rel}_j$ need not be unique, which is why we require in Definition 3.1 that either $k \geqslant \tau + 1$ or that the corresponding relator is clear from the context.

The following simple definition is a crucial concept for everything that follows:

Definition 4.14 (occurrences of rank j, $1 \leqslant j \leqslant r$)**.** *Let U be a subword of $A \in \mathrm{Can}_{-1}$. Then* **the occurrence of U in** *A is determined by its position inside A. We say that an occurrence U is properly contained in A if it is neither prefix nor suffix of A.*

Let $A = LuR \in \mathrm{Can}_{r-1}$ where u is a fractional power of rank j. If u is not properly contained inside an occurrence u_1 in A which is also a fractional power of rank j, then u is called a maximal occurrence *(of rank j) in A.*

I.e. if $A = LUR = L'UR'$ with $L' \neq L, R' \neq R$, then A contains two different occurrences of U.

Note that for any prefix u of a^k with $k \in \mathbb{N}$ and suffix w of a, the word wu is reduced and contained in a^{k+1}. This motivates the following definition:

Definition 4.15 (Prolongation of occurrences of fractional powers)**.** *Let $a^n \in \mathrm{Rel}_r$ and suppose u, w are occurrences in a^K for some $K \in \mathbb{N}$. If u is properly contained in w, we call an occurrence of w in $A \in \mathrm{Can}_{r-1}$ a* prolongation of the occurrence u *in A.*

Remark 4.16. If $u = a^k a_1$ with $a^n \in \mathrm{Rel}_r$, then all prolongations of u with respect to a are fractional powers of a. If $k \geqslant \tau + 1$, then for prolongations of u we do not have to mention a by Remark 4.13 as a is unique (up to cyclic shift). However, if u contains $< \tau + 1$ periods of a, then u may also be a prefix of another relator $b^n \in \mathrm{Rel}_r$. In that case it is possible that u has no proper prolongation in A with respect to a, but u does have a proper prolongation in A with respect to b.

For further reference we can now state the following characterization of maximal occurrences:

Remark 4.17. Suppose $A = LuR \in \mathrm{Can}_{r-1}$ with $u = a^k a_1$, $a^n \in \mathrm{Rel}_j, 1 \leqslant j \leqslant r$, $a = a_1a_2$ (where a_2 can be empty) and $k \in \mathbb{N} \cup \{0\}$. Then by Remark 2.1, u does not have a proper prolongation in A with respect to a if and only if there are no cancellations in the words $L \cdot a^{-1}$ and $a_1^{-1}a_2^{-1} \cdot R$.

In particular we see that if vu and wu are prolongations of u with respect to a, then v is a suffix of w or conversely and the word $v \cdot w^{-1}$ is not reduced.

Corollary 4.18. *Let $A \in \mathrm{Can}_{r-1}$ and let $u = a^k a_1, k \geqslant \tau + 1, a^n \in \mathrm{Rel}_j, a = a_1 a_2$, for some $1 \leqslant j \leqslant r$. Then there exists a unique maximal occurrence of rank j containing u and it coincides with the maximal prolongation of u in A.*

Proof. This follows directly from Lemma 4.10 and the previous remark. □

For further reference we now record the following version of Lemma 4.10. Here and in what follows we say that a word c is an *overlap* of words v, w if c is a suffix of v and prefix of w.

Corollary 4.19. *Let $A \in \mathrm{Can}_{r-1}$, let u_1 be a maximal occurrences of rank r in A, let u_2 be an occurrence in A of rank r not contained in u_1. Write $u_1 = a^k a_1$, $u_2 = b^s b_1$, where $a^n, b^n \in \mathrm{Rel}_r$, $a = a_1 a_2$, $b = b_1 b_2$, and $|a| \leqslant |b|$. If c is the overlap of u_1 and u_2, then $|c| < \min\{(\tau + 1)|a|, 2|b|\}$.*

Proof. By taking inverses if necessary we assume that c is a suffix of u_1 and prefix of u_2. Then we can write c as $c = \widehat{a}^{k_1} \widehat{a}_1$, where $\widehat{a}$ is a cyclic shift of a and $\widehat{a} = \widehat{a}_1 \widehat{a}_2$ and $k_1 \geqslant 0$. Then c is a common prefix of $\widehat{a}^N$ and b^N for some $N \in \mathbb{N}$ and a cyclic shift $\widehat{a}$ of a. Since u_1 is a maximal occurrence and u_2 is not contained in u_1 we see that $\widehat{a} \neq b$ and hence the claim follows from Lemma 4.10. □

If $u = a^k a_1$ is a proper prefix of the relator $a^n \in \mathrm{Rel}_r, a = a_1 a_2$, we call $v = a^{-n} \cdot u = a^{k-n+1} a_2^{-1}$ *the complement of u with respect to the relator a^n*. Clearly, v is the complement of u with respect to a^n if and only if $u \cdot v^{-1} = v^{-1} \cdot u = a^n$.

Remark 4.20. Let $A = LuR \in \mathrm{Can}_{r-1}$ where $u = a^k a_1, k \geqslant 0, a^n \in \mathrm{Rel}_r$, is a maximal occurrence of rank r in A and put $v = a^{-n} \cdot u$. If $v \neq 1$, then we have $L \cdot v \cdot R = LvR$: if $k < n$ there are no cancellations in $L \cdot v \cdot R$ by Remark 4.17 as u is a maximal occurrence and if $k > n$ there are no cancellations in $L \cdot v \cdot R$ because there are no cancellations in the initial word $LuR = La^k a_1 R$. In particular, if $k > n$, then v has no prolongation with respect to a.

Note also that if u contains $\geqslant \tau + 1$ periods of the relator a^n (that is, if $|u| \geqslant (\tau + 1)|a|$), then a^n is the unique relator in Rel_r with prefix u by Remark 4.13. So, the complement of u is defined without referring to the particular relator and in this case we will simply call v *the complement of u*.

The next definition is central to our approach:

Definition 4.21 (turns of rank r)**.** *Let $A = LuR \in \mathrm{Can}_{r-1}$ where $u = a^k a_1, a = a_1 a_2, k \geqslant 0, a^n \in \mathrm{Rel}_r$, is a maximal occurrence of rank r in A. Let $v = a^{-n} \cdot u$. The transformation*

$$A = LuR \longmapsto \mathrm{can}_{r-1}(LvR)$$

is called **a turn of rank r in** *A, or, more specifically,* **the turn of u in** *A.*

Note that we may have $k \geqslant n$ and that the reduced form of v is one of the following:

$$v = \begin{cases} a^{k-n} a_1 & \text{if } k \geqslant n, \\ a^{k-n+1} a_2^{-1} & \text{if } k < n. \end{cases} \tag{3}$$

The following observation will be convenient:

Remark 4.22. (i) Let $a = a_1 a_2$ be a cyclically reduced word and $\widehat{a} = a_2 a_1$ a cyclic shift of a. For any $t, k \in \mathbb{N}$ we have

$$a^{-t} \cdot (a^k a_1) = a^{k-t} \cdot a_1 = a_1 \cdot (a_2 a_1)^{k-t} = a_1 (a_2 a_1)^k \cdot (a_2 a_1)^{-t}$$

$$= a^k a_1 \cdot (a_2 a_1)^{-t} = (a^k a_1) \cdot \widehat{a}^{-t}.$$

Hence if $A = LuR \in \mathrm{Can}_{r-1}$ where $u = a^k a_1$ is a maximal occurrence of rank r in A with complement $v = a^{-n} \cdot u$, then we see that we can move the multiplication with a^{-n} to any position across u by using the appropriate cyclic shift $\widehat{a}^n$ of a^n:

$$LvR = L \cdot a^{-n} \cdot uR = Lu' \cdot \widehat{a}^{-n} \cdot u''R$$

where $u = u'u''$ and u'' is a prefix of $\widehat{a}^N$ for some $N \geqslant 0$.

(ii) Note that this also shows that if u is a not necessarily maximal occurrence of rank r in $A \in \mathrm{Can}_{r-1}$, $u = a^k a_1$ for some $a^n \in \mathrm{Rel}_r$ with $k \geqslant \tau + 1$, then if we multiply u from the left by a^{-n} (and take its canonical form), we automatically turn the maximal prolongation of u with respect to a.

Remark 4.23. Let $A = LuR \in \mathrm{Can}_{r-1}$ where $u = a^k a_1$ is a maximal occurrence of rank r in A with complement $v = a^{-n} \cdot u$. Then by Remark 3.4 and Remark 4.22 the result $\mathrm{can}_{r-1}(LvR)$ of turning u in A satisfies

$$\begin{aligned} \mathrm{can}_{r-1}(LvR) = \mathrm{can}_{r-1}(L \cdot a^{-n} \cdot uR) &\equiv L \cdot a^{-n} \cdot uR \\ &\equiv Lu' \cdot \widehat{a}^{-n} \cdot u''R \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_{r-1} \rangle\rangle, \end{aligned}$$

for any decomposition $u = u'u''$, where $\widehat{a}^n$ is the appropriate cyclic shift of a^n.

In order to describe the resulting word after a turn of rank r we first establish the following general lemma:

Lemma 4.24. *Suppose $a^n \in \mathrm{Rel}_r$, a_2, a_3 are (possibly empty) suffixes of a, La_2a^M and $R^{-1}a_3a^K$ are prefixes of words in Can_{r-1} and assume a_3a^K is a maximal occurrence, $\Lambda_r(a_2a^M) - \tau \geqslant \Lambda_r(a_3a^K) \geqslant 2\tau$. Then there exists a canonical triangle (D_1, D_2, D_3) such that*

$$\mathrm{can}_{r-1}(La_2a^M \cdot a^{-K}a_3^{-1}R) = \widetilde{L}D_3R', \widetilde{L}D_1 = La_2a^M \cdot a^{-K}a_3^{-1}, R = D_2R'.$$

Furthermore, if $\Lambda_r(a_2a^M) - \tau \geqslant \Lambda_r(a_3a^K)$, then $\widetilde{L} = Lw_0$ for a prefix w_0 of a_2a^M with $\Lambda_r(w_0) > \Lambda_r(a_2a^M \cdot a^{-K}a_3^{-1}) - \tau$.

Note that by considering inverses and using the fact that Can_{r-1} and Rel_r are closed under inverses, for the case $2\tau \leqslant \Lambda_r(a_2a^M) \leqslant \Lambda a_3a^K - \tau$ we also obtain

$$\mathrm{can}_{r-1}(La_2a^M \cdot a^{-K}a_3^{-1}R) = L'D_3w_0R, L = L'D_1, D_2w_0R = a_2a^M \cdot a^{-K}a_3^{-1}R$$

for a suffix w_0 of $a^{-K}a_3^{-1}$ and some canonical triangle (D_1, D_2, D_3).

Proof. By Corollary 3.5 we have

$$\mathrm{can}_{r-1}(La_2a^M \cdot a^{-K}a_3^{-1}R) = \mathrm{can}_{r-1}(\mathrm{can}_{r-1}(La_2a^M) \cdot \mathrm{can}_{r-1}(a^{-K}a_3^{-1}R)).$$

By Corollary 3.7 we have

$$\mathrm{can}_{r-1}(La_2a^M) = La^{M-\tau}X \quad \text{and} \quad \mathrm{can}_{r-1}(a^{-K}a_3^{-1}R)) = X^{-1}a^{-K+\tau}a_3^{-1}R.$$

So $\mathrm{can}_{r-1}(La_2a^M \cdot a^{-K}a_3^{-1}R) = \mathrm{can}_{r-1}(La_2a^{M-\tau}X \cdot X^{-1}a^{-K+\tau}a_3^{-1}R)$.

Put $W = a_3a^{K-\tau}X$. Since a_3a^K is maximal, $W \cdot W^{-1}$ is the maximal cancellation in this product. By IH 10 there is a canonical triangle (D_1, D_2, D_3) such that

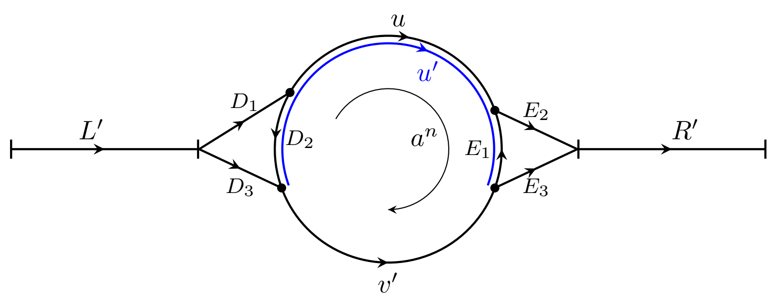
$$La_2a^{M-\tau}X = LwW = \widetilde{L}D_1W \text{ and } X^{-1}a^{-K+\tau}a_3^{-1}R = W^{-1}R = W^{-1}D_2R'$$

for a prefix w of a_2a^{M-K}, prefix $\widetilde{L}$ of Lw and suffix R' of R such that

$$\mathrm{can}_{r-1}(La_2a^{M-K} \cdot a_3^{-1}R) = L'D_3R'.$$

If $\Lambda_r(a_2a^M \cdot a^{-K}a_3^{-1}) \geqslant \tau$, then $\widetilde{L} = Lw_0$ for a nonempty prefix w_0 of a_2a^M with $\Lambda_r(w_0) > \Lambda_r(a_2a^M \cdot a^{-K}a_3^{-1}) - \tau$. □

The following proposition describes the resulting word after a turn of a maximal occurrence of rank r of Λ_r-measure $\geqslant \tau$. Below is an illustration that presents both $A = LuR$ and the result B of the turn of u in A with complement v. Note that the canonical triangles (D_1, D_2, D_3) and (E_1, E_2, E_3) could intersect. In fact, the relative position of these two triangles on the circle corresponds to the Types 2. and 3. in the following proposition.



Lemma 4.25. *Let $A = LuR \in \mathrm{Can}_{r-1}$ where $u = a^k a_1$ is a maximal occurrence of rank r in A, $a^n \in \mathrm{Rel}_r$, $k \geqslant \tau$, and $a = a_1a_2$ (where a_1 can be empty). Let $v = a^{-n} \cdot u$ and consider the turn of rank r in A:*

$$LuR \longmapsto LvR \longmapsto \mathrm{can}_{r-1}(LvR) \text{ if } v \neq 1;$$

$$LuR \longmapsto L \cdot R \longmapsto \mathrm{can}_{r-1}(L \cdot R) \text{ if } v = 1.$$

Put $m = \tau - 1$ if $k < n$ and $m = \tau + k - n$ if $k \geqslant n$.

(i) The result of the turn is of the form

$$L'Q\widetilde{R}$$

where L' is a prefix of L, $\widetilde{R}$ is a proper suffix of uR and we have one of the following three possibilities:

Type 1.

$$\mathrm{can}_{r-1}(LvR) = LvR, \quad L' = L, Q = v \text{ is a fractional power of } a \text{ and } \widetilde{R} = R;$$

Type 2.

$$\mathrm{can}_{r-1}(LvR) = L'D_3v'E_3R', \qquad L = L'D_1,\ R = E_2R',\ v = D_2v'E_1,$$

where the **remainder** v' of v is non empty, and (D_1, D_2, D_3) and (E_1, E_2, E_3) form canonical triangles of rank $r - 1$. Here v is a fractional power of a^{-1} and $Q = D_3v'E_3$.

Type 3.

$$\mathrm{can}_{r-1}(LvR) = L'D_3'E_3\widetilde{R},\ L = L'D_1,\ Q = D_3'E_3,$$

where D_3' is a not-empty prefix of a side of a canonical triangle of rank $r-1$, D_1 and E_3 are a sides of canonical triangles of rank $r - 1$, L' is a prefix of L and $\widetilde{R}$ is a proper suffix of $a^m a_1 R$.

Type 4.

$$\mathrm{can}_{r-1}(LvR) = L'E_3\widetilde{R},\ Q = E_3,$$

where E_3 is a side of a canonical triangle of rank $r-1$, L' is a prefix of L and $\widetilde{R}$ is a proper suffix of $a^m a_1 R$.

(ii) If $k \geqslant n+\tau$, the result is of Type 1 and if $\tau \leqslant \Lambda_r(u) \leqslant n-2\tau$ (or equivalently, $n-\tau \geqslant \Lambda_r(v) \geqslant 2\tau$), the result is of Type 2 with $\Lambda_r(v') > \Lambda_r(v) - 2\tau \geqslant 0$.

(iii) Unless $Q = D_3 v' E_3$ with $|v'| \geqslant (\tau+1)|a|$ or $Q = v$ with $|v| \geqslant (\tau+1)|a|$, Q is $(3\tau+1)$-free of rank r.

Proof. First assume $k-n \geqslant \tau$. Since $LuR = La^k a_1 R \in \mathrm{Can}_{r-1}$ and $k, k-n \geqslant \tau$, it follows from IH 11 that $LvR = La^{k-n}a_1R \in \mathrm{Can}_{r-1}$ as well. Hence $\mathrm{can}_{r-1}(LvR) = LvR$ by IH 6 and so the result is of Type 1.

Now suppose that $\tau \leqslant k < n+\tau$. Then $v = a^{k-n+1} \cdot a_2^{-1}$ and this product is reduced if and only if $k < n$. While we would like to compute $\mathrm{can}_{r-1}(LvR)$ by applying Lemma 4.24 to $La^{-n} \cdot uR$, we do not know that La^{-n} is a prefix of a word in Can_{r-1}. Therefore we appeal to Corollary 3.5 and first write LvR as a product of suitable subwords from Can_{r-1}. To this end we take $N \geqslant 2\tau$ and rewrite LvR as follows:

$$LvR = La^{k-n+1} \cdot a_2^{-1}R = (La^{N+\tau}) \cdot (a^{-N-\tau} \cdot \underbrace{a^{k-n+1} \cdot a_2^{-1}}_{v} a_1^{-1} a^{-N-\tau}) \cdot (a^{N+\tau}a_1R)$$
$$= (La^{N+\tau}) \cdot (a^{-2N-2\tau+k-n}) \cdot (a^{N+\tau}a_1R).$$

Since $A = LuR = La^k a_1 R \in \mathrm{Can}_{r-1}$ and $k \geqslant \tau$, it follows from IH 11 that also $La^K a_1 R \in \mathrm{Can}_{r-1}$ for any $K \geqslant \tau$. Therefore, La^N and $a^N a_1 R$ are prefix and suffix, respectively, of a canonical word of rank $r-1$. Since Can_{r-1} is closed under taking inverses, also $a^{-2N-2\tau+k-n}$ is a subword of a word from Can_{r-1}. Thus, Corollary 3.7 applies to the words La^N, $a^N a_1 R$, $a^{-2N-2\tau+k-n}$ yielding

$$\begin{aligned} Z_1 &= \mathrm{can}_{r-1}(La^{N+\tau}) = La^N X, \\ Z_2 &= \mathrm{can}_{r-1}(a^{-2N-2\tau+k-n}) = X^{-1}a^{-2N+k-n}Y^{-1}, \\ Z_3 &= \mathrm{can}_{r-1}(a^{N+\tau}a_1R) = Ya^N a_1 R. \end{aligned} \tag{4}$$

$$\text{Hence} \quad \mathrm{can}_{r-1}(Z_1 \cdot Z_2) = \mathrm{can}_{r-1}(La^N X \cdot X^{-1} a^{-2N+k-n} Y^{-1})$$
$$= \mathrm{can}_{r-1}(La^N \cdot a^{-2N+k-n}Y^{-1}).$$

By Lemma 4.24 applied to La^N and Ya^{2N-k+n} (see the comment after the lemma) we find a canonical triangle (D_1, D_2, D_3) such that

$$Z = \mathrm{can}_{r-1}(Z_1 \cdot Z_2) = L'D_3v_0Y^{-1} \text{ with } L = L'D_1,\ D_2v_0Y^{-1} = a^{-N-n+k}Y^{-1}.$$

Clearly if v is a fractional power of a^{-1} (i.e. if $k < n$), then v is a prefix of $D_2v_0Y^{-1}$.

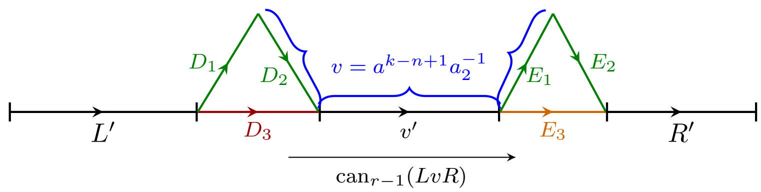
$$\text{Now} \quad \mathrm{can}_{r-1}(LvR) = \mathrm{can}_{r-1}(Z \cdot Z_3) = \mathrm{can}_{r-1}(L'D_3v_0Y^{-1} \cdot Ya^N a_1 R)$$
$$= \mathrm{can}_{r-1}(L'D_3v_0 \cdot a^N a_1 R).$$

Let v_0' be the prefix of v_0 (if any) that is not cancelled in the product $v_0 \cdot a^N a_1$. Note that v_0 may have a proper prolongation $\widetilde{v} = v_1 v_0$ in $L'D_3v_0$ with respect to a^{-1}. We again apply Lemma 4.24 to $L'D_3v_0$, $a^N a_1 R$ or their inverses and obtain another canonical triangle (E_1, E_2, E_3) and the following cases according to the position of this triangle relative to D_3:

Type 2

$$\operatorname{can}_{r-1}(Z \cdot Z_3) = L' D_3 v' E_3 R'$$

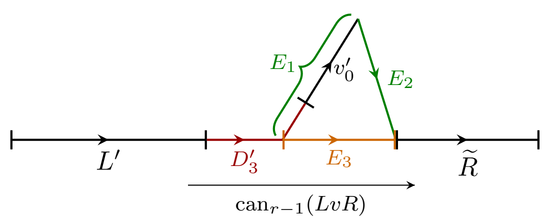
where $L = L'D_1$, $R = E_2R'$, and $v = D_2 v' E_1$. So in particular $\Lambda_r(v') > \Lambda_r(v) - 2\tau$, and this happens exactly if v_0' is not contained inside E_1:



Types 3 and 4 If v_0' is contained in E_1 (in particular if v_0 cancels completely), then

$$\operatorname{can}_{r-1}(Z \cdot Z_3) = L' D_3' E_3 \widetilde{R} \textit{ or } \operatorname{can}_{r-1}(Z \cdot Z_3) = L'' E_3 \widetilde{R},$$

where D_3' is a non-empty prefix of D_3, L'' is a prefix of L' and $\widetilde{R}$ is a suffix of $a^m a_1 R$. Notice that the suffix of $a^N a_1 R$ remaining after cancellation with $L' D_3 v_0$ is a proper suffix of $a^m a_1 R$, so $\widetilde{R}$ is a proper suffix of $a^m a_1 R$. If E_1 is properly contained in $D_3 v_0'$, the we obtain the first formula that gives Type 3. Otherwise we obtain the second formula that gives Type 4.



Now we prove the last part: if $Q = D_3' E_3$ or $Q = E_3$, then Q is 2τ-free of rank r, because D_3' and E_3 are τ-free of rank r.

Let $Q = D_3 v' E_3$ and $|v'| < (\tau+1)|a|$. Assume that Q contains $b^{3\tau+1}$, where $b^n \in \operatorname{Rel}_r$. Since D_3 and E_3 are τ-free of rank r, we obtain that v' contains $b^{\tau+1}$. Hence, it follows from Lemma 4.10 that a^{-1} is a cyclic shift of b. Therefore, $|v| \geqslant (\tau+1)|b| = (\tau+1)|a|$, a contradiction.

Let $Q = v$ and $|v| < (\tau+1)|a|$. Assume that Q contains $b^{3\tau+1}$, where $b \in \operatorname{Rel}_r$. Then it follows from Lemma 4.10 that a is a cyclic shift of b. Therefore, $|v| \geqslant (3\tau+1)|b| > (\tau+1)|a|$, a contradiction.

□

By considering inverses and using IH 7 we also obtain the following "left" version of Type 3 in Lemma 4.25 (instead of the current "right" version). It is important to note that while the description of the canonical form may differ, it is in fact uniquely defined and therefore, these two versions agree.

Remark 4.26. In the situation of Lemma 4.25 (and with the same notation) we obtain the following "left" description of $\operatorname{can}_{r-1}(LvR)$ for Types 3 and 4:

Type 3' and 4'.

$$\mathrm{can}_{r-1}(LvR) = \widetilde{L}F_3G_3'R',$$

where R' is a suffix of R, F_3 is a side of a canonical triangle of rank $r-1$, G_3' is a (possibly empty) suffix of a side of a canonical triangle of rank $r-1$, and $\widetilde{L}$ is a proper prefix of $La^m a_1$.

Convention 4.27. If $A = LuR \in \mathrm{Can}_{r-1}$ for some maximal occurrence u of rank r with $\tau \leqslant \Lambda_r(u) < n$ we say that the turn $A \mapsto \mathrm{can}_{r-1}(LvR) = B$ is of Type 2 provided $B = L'D_3v'E_3R', L = L'D_1, R = D_2R', v = D_2v'E_1$ as in Type 2 of Lemma 4.25.

Corollary 4.28. *Let $A_1 = Lu_1R_1, \quad A_2 = Lu_2R_2 \in \mathrm{Can}_{r-1}$ where $u_1 = a^k a_1, u_2 = a^m a_2$ are maximal occurrences of rank r, u_1 is a prefix of u_2 and $\tau \leqslant \Lambda_r(u_1) \leqslant \Lambda_r(u_2) \leqslant n - 2\tau$. Let $v_i, i = 1, 2$, be the complement of u_i. Then there is a canonical triangle (D_1, D_2, D_3) such that the result B_i of turning u_i in $A_i, i = 1, 2$, is of the form*

$$B_1 = L'D_3v_1'E_3R_1' \quad \text{and } B_2 = L'D_3v_2'F_3R_2'$$

where v_2' and v_1' have a common a prefix of Λ_r-measure $> n - \Lambda_r(u_2) - 2\tau$, R_i' is a suffix of $R_i, i = 1, 2$, and E_3, F_3 are sides of respective canonical triangles of rank $r-1$.

Proof. Consider the decomposition of Lv_1R_1 and Lv_2R_2 into three factors Z_1, Z_2, Z_3 as in the proof of Lemma 4.25. Then the factor $Z_1 = La^NX$ is identical in both cases, the factor Z_2 is of the form $X^{-1}a^{-2N+k-n}Y^{-1}$ and $X^{-1}a^{-2N+m-n}Y^{-1}$, so differs only in the exponent of a by $m-k$, and the third factor is of the form $Ya^Na_iR_i$, $i = 1, 2$. By Corollary 3.9 we see that in either case the product $Z = \mathrm{can}_{r-1}(Z_1 \cdot Z_2)$ is of the form $L'D_3v_{0,1}Y^{-1}$ and $L'D_3v_{0,2}Y^{-1}$, respectively, where $\Lambda_r(v_{0,1}) = \Lambda_r(v_{0,2}) + \Lambda_r(u_2) - \Lambda_r(u_1)$. Thus, looking at the proof of Lemma 4.25 we see that after multiplying either of these results with Z_3 the product will have a prefix of the form $L'D_3v_i'$ for some prefix v_i' of $v_{0,1}$ of Λ_r-measure $> n - \Lambda_r(u_2) - 2\tau$. ☐

Corollary 4.29. *Let $A = LuR \in \mathrm{Can}_{r-1}$ where u is a maximal occurrence of rank r with $\tau \leqslant \Lambda_r(u) \leqslant n - 2\tau$ and let $B = L'D_3v'E_3R'$ be the result of turning u in A. Let w be an occurrence of rank r in R with $\Lambda_r(w) \geqslant \tau$. Then R' contains a non-empty suffix w' of w with $\Lambda_r(w') > \Lambda_r(w) - \tau$.*

Proof. By Lemma 4.25 B is of the given form. By the description of Type 2, we have $R = E_2R'$ where E_2 is τ-free of rank r. So w cannot be entirely contained in E_2, so $\Lambda_r(w') > \Lambda_r(w) - \tau$. ☐

Corollary 4.30. *Let $A_i = LuMb^\tau R_i \in \mathrm{Can}_{r-1}, i = 1, 2$, where u is a maximal occurrence of rank r with $\tau \leqslant \Lambda_r(u) \leqslant n - 2\tau$, $b^n \in \mathrm{Rel}_r$, and assume that the result B_1 of turning u in A_1 is of the form*

$$B_1 = L'D_3v'E_3M'b^\tau R_1,$$

where M' is a suffix of M and $(D_1, D_2, D_3), (E_1, E_2, E_3)$ are canonical triangles of rank $r-1$. Then the result B_2 of turning u in A_2 is of the form

$$B_2 = L'D_3v'E_3M'b^\tau R_2$$

with the same canonical triangles.

Proof. This follows directly from Corollary 4.29, IH 11, IH 6, and IH 8 (see also the proofs of Corollaries 3.9 and 4.28). ☐

The following statement is a useful particular case of Corollary 4.30 with $M = M_1 a^\tau M_2$:

Corollary 4.31. *Let* $A_i = LuM_1 a^\tau M_2 b^\tau R_i \in \mathrm{Can}_{r-1}, i = 1, 2,$ *where* u *is a maximal occurrence of rank* r *with* $\tau \leqslant \Lambda_r(u) \leqslant n - 2\tau$*,* $a^n, b^n \in \mathrm{Rel}_r$*. Then there are canonical triangles* $(D_1, D_2, D_3), (E_1, E_2, E_3)$ *such that the result* $B_i, i = 1, 2,$ *of turning* u *in* A_i *is of the form*

$$B_i = L' D_3 v' E_3 M_2' b^\tau R_i$$

where M_2' *contains a non-empty suffix of* a^τ*, i.e. the canonical triangles do not depend on* R_i*.*

4.5. **Inverse turns.** From now on we will fix the following notational conventions:

Convention 4.32. For $A = LuR \in \mathrm{Can}_{r-1}$ where u is a maximal occurrence of rank r with $\tau \leqslant \Lambda_r(u) \leqslant n - 2\tau$ (i.e. the turn of u is of Type 2) we use the following conventions:

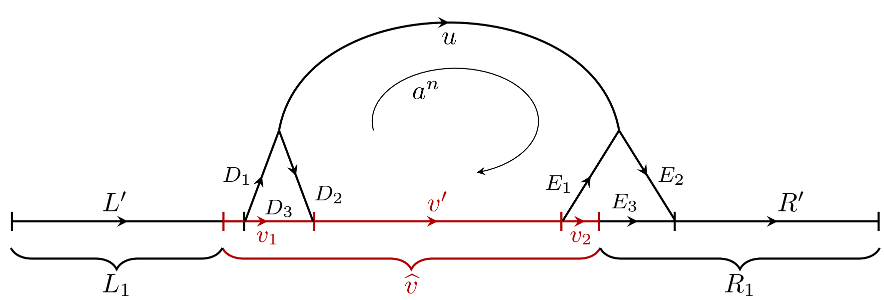
1. v denotes the complement of u;
2. v' denotes the remainder of v after turning u in A;
3. $B = \mathrm{can}_{r-1}(LvR) = L' D_1 v' E_1 R'$ is the result of turning u in A for canonical triangles (D_1, D_2, D_3) and (E_1, E_2, E_3) where $L = L' D_1, R = E_2 R'$ and $v = D_2 v' E_1$;
4. if $\Lambda_r(v') \geqslant \tau + 1$, then $\widehat{v}$ is the maximal prolongation of v' in B (and coincides with the maximal occurrence of rank r in B containing v').

We next prove that a turn of Type 2 of a maximal occurrence u with complement v has a natural inverse turn, namely the turn of the maximal prolongation $\widehat{v}$ of v' in B (provided $\widehat{v}$ is a maximal occurrence).

Lemma 4.33. *Let* $A = LuR \in \mathrm{Can}_{r-1}$*, where* u *is a maximal occurrence of rank* r *in* A *with* $\tau \leqslant \Lambda_r(u) < n - (3\tau + 1)$ *and let* $B = L' D_3 v' E_3 R'$ *be the result of turning* u *in* A*. Then the result of turning* $\widehat{v}$ *in* B *is equal to* A*.*

$$\mathrm{can}_{r-1}(L' D_3 \cdot \widetilde{a}^n \cdot v' E_3 R') = A$$

where u *is a prefix of* $a^n \in \mathrm{Rel}_r$ *and* v' *a prefix of some cyclic shift* $\widetilde{a}^{-n}$ *of* a^{-n}*.*



Thus, if the maximal prolongation $\widehat{v}$ of v' with respect to a^{-1} is a maximal occurrence of rank r in B, then the turn of $\widehat{v}$ in B is defined and by Remark 4.22, (ii), the result of turning $\widehat{v}$ in B is equal to A.

Proof. Suppose that u is a prefix of $a^n \in \mathrm{Rel}_r$ and so v' a prefix of some cyclic shift $\widetilde{a}^{-n}$ of a^{-n}.

We see that $\widetilde{a}^n \cdot v' = D_2^{-1}uE_1^{-1}$ and by the properties of canonical triangles we have $D_1uE_2 \equiv D_3 \cdot D_2^{-1}uE_1^{-1} \cdot E_3 \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_{r-1}\rangle\rangle$. Hence

$$\begin{aligned} A = L'(D_1uE_2)R' &\equiv L'(D_3 \cdot D_2^{-1}uE_1^{-1} \cdot E_3)R' \\ &= L'D_3 \cdot \widetilde{a}^n \cdot v'E_3R' \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_{r-1}\rangle\rangle \end{aligned}$$

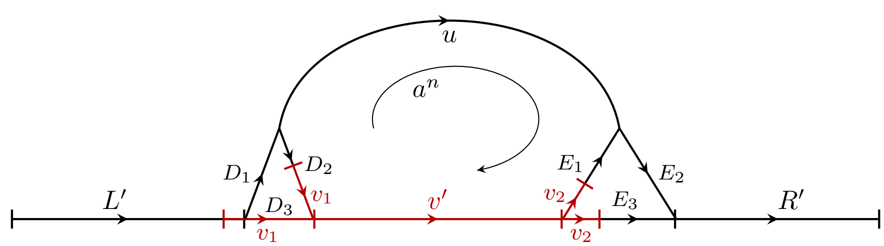
Since $A \in \mathrm{Can}_{r-1}$ by assumption, the claim now follows from IH 8 and IH 6. □

Remark 4.34. Note that turns of Type 3 do not have inverses. Furthermore, if the remainder v' after a turn of Type 2 has Λ_r-measure $< \tau + 1$, then the maximal prolongation $\widehat{v}$ of v' with respect to a^{-1} need not be a maximal occurrence, so again the inverse turn need not exist.

In turns of Type 2, the Λ_r-measure of the maximal prolongation of v' in either direction is bounded by τ:

Lemma 4.35. *Let $A = LuR \in \mathrm{Can}_{r-1}$, where u is a maximal occurrence of rank r in A with $\tau \leqslant \Lambda_r(u) < n$. Let u be a prefix of $a^n \in \mathrm{Rel}_r$ and put $v = a^{-n} \cdot u$. Assume that the result B of turning u in A is of Type 2 and write $B = \mathrm{can}_{r-1}(LvR) = L'D_3v'E_3R'$. Let $\widehat{v} = v_1v'v_2$ be the maximal prolongation of v' in $L'D_3v'E_3R'$ with respect to a^{-1}. Then $|v_1| \leqslant \max\{|D_2|, |D_3|\}$ and $|v_2| \leqslant \max\{|E_1|, |E_3|\}$. Thus, $\Lambda_r(v_i) < \tau, i = 1, 2$ and*

$$n - \Lambda_r(u) - 2\tau < \Lambda_r(v') \leqslant \Lambda_r(\widehat{v}) < \Lambda_r(v') + 2\tau \leqslant n - \Lambda_r(u) + 2\tau.$$



Proof. Assume towards a contradiction that $|v_1| > |D_2|, |D_3|$. Let zD_2 denote the maximal common suffix of $u^{-1}D_2$ and v_1. If $z \neq u^{-1}$, then $zD_2 = v_1$. Otherwise $\Lambda_r(v_1) \geqslant \tau$ since $\Lambda_r(u) \geqslant \tau$. Since D_3 is τ-free of rank r, we see that $|zD_2| > |D_3|$ in either case. Thus, $L'D_3 = L''zD_2 = L''z'D_3$ and hence $z'D_1z^{-1}$ is a subword of A. However, since $D_3 \cdot D_2^{-1} \equiv D_1 \mod \langle\langle \mathrm{Rel}_0 \cup \ldots \cup \mathrm{Rel}_{r-1}\rangle\rangle$ by the definition of a canonical triangle, we have $z'D_1z^{-1} \equiv z'D_3 \cdot D_2^{-1}z^{-1} \equiv 1 \mod \langle\langle \mathrm{Rel}_0 \cup \ldots \cup \mathrm{Rel}_{r-1}\rangle\rangle$, contradicting IH 9. □

For further reference we note the following immediate consequence of the previous lemma:

Corollary 4.36. *Let $A = LuR \in \mathrm{Can}_{r-1}$ where $u = a^ka_1$ is a maximal occurrence of rank r with $\tau \leqslant \Lambda_r(u) \leqslant n - (3\tau + 1)$ and let $B = L'D_3v'E_3R'$ be the result of turning u. Then the maximal occurrence $\widehat{v}$ of rank r in B containing v' is the maximal prolongation of v' with respect to a^{-1} by Corollary 4.18 and*

$$\Lambda_r(\widehat{v}) < \Lambda_r(v') + 2\tau \leqslant \Lambda_r(v) + 2\tau = (n - \Lambda_r(u)) + 2\tau.$$

Hence at least one of u and $\widehat{v}$ has Λ_r-measure $< \frac{n}{2} + \tau$, at least one of u and $\widehat{v}$ has Λ_r-measure $> \frac{n}{2} - \tau$, and the turn $B \mapsto A$ of turning $\widehat{v}$ in B is inverse to the turn $A \mapsto B$.

Proof. Since v' has Λ_r-measure $\geqslant \tau+1$ by Lemma 4.25 (ii), the maximal prolongation $\widehat{v}$ in B with respect to a^{-1} is a maximal occurrence of rank r in B and its turn is inverse to the turn of u in A by Lemma 4.33. The bound on $\Lambda_r(\widehat{v})$ is given in Lemma 4.35. Furthermore if $\Lambda_r(u) \geqslant \frac{n}{2}+\tau$, then $\Lambda_r(\widehat{v}) < n - (\frac{n}{2}+\tau) + 2\tau = \frac{n}{2}+\tau$. □

For convenience we will say that a turn of an occurrence u is a turn of Λ_r-measure $\Lambda_r(u)$.

4.6. **Influence of turns on other maximal occurrences.** In this subsection we describe the effect that a turn of a maximal occurrence has on other maximal occurrences in the original word. We will use the following conventions:

Convention 4.37. We will say that an occurrence u in A is *to the left* (right) of an occurrence w in A if the starting point of u is left (right, resp.) of the starting point of w and between occurrences w and w' if the starting point is.

Convention 4.38. Let $A \in \mathrm{Can}_{r-1}$ and assume that $u_1, \dots, u_t$ is a sequence of maximal occurrences of rank r in A. We use the notation $A = L\ulcorner u_1 \dots u_t \urcorner R$ (thereby, in slight abuse of notation, ignoring overlaps between the occurrences or subwords separating them) where L, R are prefix and suffix of A which may have overlaps with u_1, u_t, respectively, of Λ_r-measure $< \tau+1$.

If the word is clear from the context we may also ignore the prefix and suffix and simply write $A = \ulcorner u_1 \dots u_t \urcorner$, especially in power words.

Convention 4.39. Let $A \in \mathrm{Can}_{r-1}$ and let u_1, u_2 be maximal occurrences of rank r. Let B_1 be the result of turning u_1 in A. Clearly, when turning u_1 the occurrence u_2 might be truncated to a subword u_2'' or even be canceled completely. However, if $\Lambda_r(u_2'') \geqslant \tau+1$, the maximal prolongation $\widetilde{u_2}$ of u_2'' in B_1 is uniquely defined and coincides with the maximal occurrence containing u_2'' by Corollary 4.18. We call $\widetilde{u_2}$ *the occurrence corresponding to* u_2. For ease of notation we may then also write $\Lambda_r(u_2, B_1)$ to refer to the Λ_r-measure of $\widetilde{u_2}$ in B_1.

Turns of occurrences and multiplication of canonical words introduce perturbations on the boundaries of these operations that are captured by the introduction of canonical triangles. Since the sides of these triangles are τ-free in the corresponding ranks, an occurrences u measure $\geqslant \tau$ in the corresponding rank absorbs the effect of the canonincal triangle and protect the remaining word from further perturbation. In other words, we will see that if $A = LuR$ for a maximal occurrence u of Λ_r-measure $\geqslant \tau$, then a turn of rank r of Type 2 inside L will have no effect on R and vice versa. Therefore we introduce the following terminology:

Definition 4.40. *Let $A = Lu_1Wu_2R$ be a reduced word and u_1 and u_2 maximal occurrences of rank r. We say that u_1, u_2 are* isolated *in A if W contains an occurrence a^τ and* strongly isolated from each other *if W contains a subword of the form $a^\tau M_1 b^\tau M_2 c^\tau$ with $a^n, b^n, c^n \in \mathrm{Rel}_r$ (where M_1, M_2 may be empty) and in this case we call W a strong isolation word (in rank r). We say that u_1 and u_2 are* close neighbours *in A if they are not isolated from each other.*

Furthermore, we say that u_1 and u_2 are essentially non-isolated *if there are $f_1 \in \{u_1, v_1\}$ and $f_2 \in \{u_2, v_2\}$ such that turning f_i in $W = L\ulcorner f_1 f_2 \urcorner R'$ does not leave f_j invariant for $\{i,j\} = \{1,2\}$.*

We say that a word W is a strong separation word (in rank r) from the right if in any word $A = LuWR \in \mathrm{Can}_{r-1}$ the maximal occurrence u of rank r is strongly isolated from *any maximal occurrence of rank r in A which has overlap with R (and similarly for the left).*

Examples 4.41. *Words of the following form are strong separation words from the right:*

- *If $A = Lu_1Wu_2R$ is a reduced word such that u_1 and u_2 are essentially non-isolated maximal occurrences with $\tau \leqslant \Lambda_r(u_i) \leqslant n-(3\tau+1)$, then W does not contain a subword of the form $a^\tau M_1 b^\tau M_2 c^\tau$ with $a^n, b^n, c^n \in \mathrm{Rel}_r$ by Lemmas 4.25, 4.33 and 4.35.*
- *$W = a_0^\tau M_1 a_1^\tau M_2 a_2^\tau M_3 a_3^{\tau+1} M_4$, where $a_0^n, a_1^n, a_2^n, a_3^n \in \mathrm{Rel}_r$, M_1, M_2, M_3 can be empty, M_4 is not empty and $a_3^{\tau+1}$ cannot be prolonged to the right.*
- *$W = a_0^\tau M_1 a_1^\tau M_2 a_2^\tau M_3 a_3^{\tau+1}$, where $a_0^n, a_1^n, a_2^n, a_3^n \in \mathrm{Rel}_r$, M_1, M_2, M_3 may be empty, $a_3^{\tau+1}$ cannot be prolonged to the left.*
- *$W = a_0^\tau M_0 a_1^\tau M_1 a_2^\tau M_2 a_3^\tau M_3 a_3^\tau$, where $a_0^n, a_1^n, a_2^n, a_3^n \in \mathrm{Rel}_r$, M_0, M_1, M_2 can be empty, and $M_3 a_3^\tau$ is a primitive word (in particular, M_3 is not empty).*

Proof. Clearly every W is a strong isolation word. We have to show that if $W = W_1 y$ for some fractional power $y = b^k b_1$ of rank r, then W_1 is still a strong isolation word. For the first two cases this follows directly from Lemma 4.19. For the third case, this is immediate if $|b| < |a_3|$ from Corollary 4.19. If $|b| \geqslant |a_3|$, then $|b| < \tau|a_3|$ by Corollary 4.6. Hence by Lemma 4.9 comparing the suffixes of $b^k b_1$ and $a_3^\tau M_3 a_3^\tau$ we see that $|y| < |M_3 a_3^\tau| + |b| < |a_3^\tau M_2 a_3^\tau|$. ☐

If u_1 and u_2 are isolated from each other in A, then u_2 is not affected from turning u_1 and vice versa:

Lemma 4.42. *Let $A = Lu_1Mu_2R \in \mathrm{Can}_{r-1}$, where u_1, u_2 are maximal occurrences of rank r isolated from each other in A and $\tau \leqslant \Lambda_r(u_1) \leqslant n-2\tau$. Let B_1 denote the result of turning u_1. Then*

$$B_1 = L'D_3v_1'E_3M'u_2R \quad \text{for some non-empty suffix } M' \text{ of } M.$$

In particular $\widetilde{u_2} = u_2$ (as words occurring in B_1 and A, respectively, see Convention 4.39).

If u_1, u_2 are strongly isolated, then $\widehat{v_1}$ (if it is defined) is isolated from $\widetilde{u_2}$ in B_1

Proof. Since M contains an occurrence w of rank r with $\Lambda_r(w) \geqslant \tau$, both claims follows from Corollary 4.29 and Lemma 4.35. ☐

In order to consider the influence of a turn on a close neighbour we first note the following:

Lemma 4.43. *Consider Dv', where D is τ-free of rank r and v' is a fractional power or rank r. If z is a maximal occurrence of rank r in Dv' not containing v', then $\Lambda_r(z) < 2\tau+1$.*

Proof. Write $z = z_0z_1$ where z_0 is a suffix of D and z_1 a prefix of v'. Since D is τ-free of rank r, we have $\Lambda_r(z_0) < \tau$ and by Lemma 4.10 we have $\Lambda_r(z_1) < \tau+1$. ☐

Lemma 4.44. *Let $A = LuR \in \mathrm{Can}_{r-1}$ where u, z are distinct maximal occurrences of rank r in A with $\Lambda_r(u) \geqslant \tau+1$ and $\Lambda_r(z) \geqslant 3\tau+2$. Let B be the result of turning u in A. If $B = L'D_3v'E_3R'$ is of Type 2 with $\Lambda_r(v') \geqslant \tau+1$, the occurrence $\widetilde{z}$ corresponding to z in B is well-defined and*

$$\Lambda_r(z) - (2\tau+1) < \Lambda_r(\widetilde{z}) < \Lambda_r(z) + (2\tau+1).$$

Proof. Since z does not contain u, by symmetry we may assume that z is contained in $Lu = L'D_1u$. Thus we may write $z = z'X$ where z' is contained in L' and X is a prefix of D_1u. By Lemma 4.43 we have $\Lambda_r(X) < 2\tau + 1$ and so $\Lambda_r(z') > \tau + 1$. Hence the the corresponding occurrence $\widetilde{z} = z'Y$ in B is well-defined and cannot contain v' by Lemma 4.35 since $\Lambda_r(v') \geqslant \tau + 1$ and $\Lambda_r(z') > \tau$. So $\widetilde{z} = z'Y$ where Y is a proper prefix of D_3v'. Again by Lemma 4.43 we have $\Lambda_r(Y) < 2\tau + 1$ and the result follows. □

We call z' the **remainder** of z after turning u (in analogy to v' in Lemma 4.25).

Remark 4.45. In the previous lemma, both z and $\widetilde{z}$ are prolongations of z'. Since $\Lambda_r(z') \geqslant \tau + 1$, either $z = \widetilde{z}$ or one is a proper prefix of the other by Remark 4.16.

Remark 4.46. If $A = LuR \in \mathrm{Can}_{r-1}$ where u, z are as in Lemma 4.44 and the result $B = L'QR'$ of turning u is of Type 3 or of Type 1 or 2 with $\Lambda_r(v') < \tau + 1$, then $\widetilde{z}$ may contain Q and we may have $\Lambda_r(\widetilde{z}) \geqslant \Lambda_r(z) + (2\tau + 1)$. On the other hand, if the turn is of Type 3 in Lemma 4.25, it is also possible that the occurrence z is completely cancelled and has no trace in the result of the turn.

Corollary 4.47. *Let $A \in \mathrm{Can}_{r-1}$ and let u_1, u_2 be maximal occurrences of rank r in A and $2\tau+1 \leqslant \Lambda_r(u_1), \Lambda_r(u_2) \leqslant n-2\tau$. Write $A = Lu_1R$ and assume that u_2 is contained in u_1R. Let v_1 be the complement of u_1. Then the result of turning u_1 in A is of the form*

$$\mathrm{can}_{r-1}(Lv_1R) = L'D_3v_1'E_3M'u_2'R' \quad \textit{for a suffix } M'u_2'R' \textit{ of } R$$

where u_2' is a non-empty suffix of u_2 with $\Lambda_r(u_2') > \Lambda_r(u_2) - (2\tau + 1)$ and $u_2' = u_2$ if $M' \neq 1$.

Proof. This follows immediately from (the symmetric version of) Lemma 4.44 applied with $u = u_1$ and $z = u_2$. □

Remark 4.48. Let $A \in \mathrm{Can}_{r-1}$ and let u_1, u_2, u_3 be maximal occurrences of rank r in A enumerated from left to right. By Lemma 4.10 the overlap of u_2 with u_1 and u_3 has Λ_r-measure $< \tau + 1$. So if $\Lambda_r(u_2) \geqslant 2\tau + 2$, then there is a subword of u_2 of Λ_r-measure $> \Lambda_r(u_2) - (2\tau + 2)$ not contained in either u_1 or u_2. In particular, if $\Lambda_r(u_2) \geqslant 3\tau + 2$ (or $\geqslant 5\tau + 2$, respectively), then u_1, u_3 are isolated (strongly isolated, respectively) in A (by a subword of u_2).

The previous remark implies the following:

Corollary 4.49. *Let $A \in \mathrm{Can}_{r-1}$ and let $\mathcal{X}$ be a set of maximal occurrences of rank r in A of Λ_r-measure $\geqslant 3\tau + 2$. Then any maximal occurrence in $\mathcal{X}$ to the left of $u \in \mathcal{X}$ is isolated from any maximal occurrence in $\mathcal{X}$ to the right of u, and so any $u \in X$ has at most one close neighbour in $\mathcal{X}$ on either side. Similarly, if all occurrences in $\mathcal{X}$ have Λ_r-measure $\geqslant 5\tau + 2$, then on either side of u there is at most one maximal occurrence in $\mathcal{X}$ which is not strongly isolated from u.*

Lemma 4.50. *Let $A \in \mathrm{Can}_{r-1}$ and let u_1, u_2, u_3 be maximal occurrences of rank r in A enumerated from left to right and of Λ_r-measure $\geqslant 3\tau+2$. Suppose $\Lambda_r(u_2) \leqslant n-k(\tau+1)$ where k is the number of close neighbours of u_2. Let B be the result of turning u_2. Then $\widetilde{u_1}$ and $\widetilde{u_3}$ are isolated in B (witnessed by a subword of v_2') and strongly isolated in B if $\Lambda_r(u_2) \leqslant n - (5\tau + k \cdot (\tau + 1))$*

Proof. By Lemma 4.25 (ii) we have $\Lambda_r(v_2') > n - \Lambda(u_2) - 2\tau$ and by Lemma 4.44 we know that $\Lambda_r(\widetilde{u_1}), \Lambda_r(\widetilde{u_2}) \geqslant \tau + 1$ and so the corresponding occurrences are well-defined. Since $\widetilde{u}_i$, $i = 1, 3$, can have an overlap with $\widehat{v_2}$ only if u_i is a close neighbour of u_2, the claim follows from Corollary 4.19. □

4.7. **Commuting turns of rank r.** Our next aim is to show that the result of turning maximal occurrences of appropriate measures in $A \in \mathrm{Can}_{r-1}$ is independent of the order in which we perform these turns.

Corollary 4.51. *Let $A = Lu_1Mu_2R \in \mathrm{Can}_{r-1}$, where u_1, u_2 are maximal occurrences of rank r in A isolated from each other. If $\tau \leqslant \Lambda_r(u_1), \Lambda_r(u_2) \leqslant n - 2\tau$, then the result C of turning u_1 and u_2 is independent of the order in which we perform these turns and we have*

$$C \equiv L \cdot a_1^{-n} \cdot u_1M \cdot a_2^{-n} \cdot u_2R \qquad \mathrm{mod}\ \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_{r-1} \rangle\rangle.$$

Proof. By Lemma 4.42 we know that the result B_1 of turning u_1 in A satisfies:

$$\begin{aligned} B_1 = \mathrm{can}_{r-1}(Lv_1Mu_2R) &= L'D_3v_1'E_3M'u_2R \\ &\equiv L \cdot a_1^{-n} \cdot u_1Mu_2R \qquad \mathrm{mod}\ \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_{r-1} \rangle\rangle. \end{aligned}$$

It follows from this that for $M = M_0M'$ we have

$$L \cdot a_1^{-n} \cdot u_1M_0 \equiv L'D_3v_1'E_3 \qquad \mathrm{mod}\ \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_{r-1} \rangle\rangle. \tag{5}$$

Similarly, for the result C of turning u_2 in B_1 we obtain

$$\mathrm{can}_{r-1}(L'D_3v_1'E_3M'v_2R) \equiv L'D_3v_1'E_3M' \cdot a_2^{-n} \cdot u_2R \quad \mathrm{mod}\ \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_{r-1} \rangle\rangle.$$

Combining this with (5) we thus obtain

$$C \equiv L \cdot a_1^{-n} \cdot u_1M_0M' \cdot a_2^{-n} \cdot u_2R \qquad \mathrm{mod}\ \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_{r-1} \rangle\rangle.$$

By considering inverses and using Remark 4.23 we see that first turning u_2 and then u_1 yields the same result. So the claim follows from IH 6 and IH 8. ☐

Definition 4.52. *Let $A \in \mathrm{Can}_{r-1}$ and let $u_1, \ldots, u_t$ be maximal occurrences of rank r in A enumerated from left to right with $\tau \leqslant \Lambda_r(u_i) \leqslant n-(3\tau+1)$. Put $\mathcal{Z} = \{u_1, \ldots, u_t\}$. We call an occurrence $u \notin \mathcal{Z}$, solid in A with respect to $\mathcal{Z}$ if after turning any subset of $\mathcal{Z}$ in any order the remainder of u (see the definition after Lemma 4.44) is an occurrence of measure $\geqslant \tau+1$. We call the sequence $u_1, \ldots, u_t$ solid if each $u_i, 1 \leqslant i \leqslant t$, is solid in A with respect to $\mathcal{Z} \setminus \{u_i\}$.*

We say that the sequence $(u_0, \ldots, u_t)$ has a gap *at i if u_i and u_{i+1} are strongly isolated.*

The conditions imply in particular that all $u_i, v_i, i = 1, \ldots, t$, have Λ_r-measure $\geqslant \tau+1$ (and hence their maximal prolongations are unique).

Note that for a solid set of occurrences each turn of one of the occurrences is of Type 2 and has an inverse turn. Clearly any subset of a solid set is again solid.

Lemma 4.53. *Let $A = L\ulcorner u_1, u_2 \urcorner R \in \mathrm{Can}_{r-1}$, where u_1, u_2 is a solid sequence of maximal occurrences of rank r in A. Then the result of turning u_1 and u_2 is independent of the order of the turns.*

Proof. As in the proof of Corollary 4.51 the statement follows directly from the fact that both results of turns are equivalent $\mathrm{mod}\ \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_{r-1} \rangle\rangle$ and IH 8. ☐

We now write $\varepsilon = 2\tau + 1$.

Proposition 4.54. *Let $A \in \mathrm{Can}_{r-1}$ and let $u_1, \ldots, u_t$ be maximal occurrences of rank r in A enumerated from left to right and suppose that u_i is an initial segment of $a_i^n \in \mathrm{Rel}_r$, $i = 1, \ldots, t$. Assume*

(c1) all occurrences $u_i, 1 \leqslant i \leqslant t$, are solid in A with respect to $u_1, \ldots, u_t$;

(c2) $\tau + 1 \leqslant \Lambda_r(u_i) \leqslant n - (4\tau + 1 + k \cdot \varepsilon)$ *if* u_i *has* k *close neighbours*[1] *among* u_j, $j \neq i$.

Then the result of turning (the occurrences corresponding to) the $u_i, i = 1, \ldots, t$, *(in the sense of Remark 4.39) is well-defined and independent of the order of the turns.*

Proof. We know from Corollary 4.51 and Lemma 4.53 that under the given assumptions the turns of any two occurrences $u_i, u_j, i \neq j$, commute. Therefore the result follows once we establish that after turning an occurrence u_i the maximal occurrences corresponding to the remaining occurrences $u_j, j \neq i$, still satisfy the assumptions of this proposition.

By assumption on $\Lambda_r(u_i)$ and Lemma 4.25 Type 2 we have $\Lambda_r(v_i') > \tau + 1 + k \cdot \varepsilon$ where k is the number of close neighbours of u_i among the u_j. Furthermore, if v_i' has overlap with $\widetilde{u_m}$, $m \in \{i-1, i+1\}$, then by Lemma 4.42, u_i was a close neighbour of u_m. Since the overlap of $\widetilde{u_m}$ and v_i' has Λ_r-measure bounded by $\tau + 1$, we see that after turning u_i, the occurrences $\widetilde{u_{i-1}}, \widetilde{u_{i+1}}$ are isolated from each other. Furthermore, $\Lambda_r(\widetilde{u_j}) = \Lambda_r(u_j)$ if u_j and u_i were isolated from each other, and $\Lambda_r(\widetilde{u_j}) < \Lambda_r(u_j) + \varepsilon$ if u_m and u_i were close neigbours for $m \in \{i-1, i+1\}$, in which case the number of close neighbours of $\widetilde{u_m}$ among the $\widetilde{u_j}, j \neq i$, is exactly one less than the number of close neighbours of u_m among the u_j. Since by Condition (c1) we have $\Lambda_r(\widetilde{u_j}) \geqslant \tau + 1$ we see that Condition (c2) holds for $\{\widetilde{u_j}, 1 \leqslant j \neq i \leqslant t\}$.

Since after turning u_i the occurrences $\widetilde{u_{i-1}}, \widetilde{u_{i+1}}$ are isolated from each other, clearly Condition (c1) continues to hold for $\{\widetilde{u_j}, 1 \leqslant j \neq i \leqslant t\}$ by Lemma 4.42. □

Definition 4.55. *We call a sequence of maximal occurrences in* $A \in \mathrm{Can}_{r-1}$ *stable if it satisfies Conditions (c1) and (c2) from Proposition 4.54.*

Remark 4.56. Note that if $A \in \mathrm{Can}_{r-1}$ and $\mathcal{X}$ is a set of maximal occurrences of rank r in A where for each $u \in \mathcal{X}$ we have $5\tau + 3 \leqslant \Lambda_r(u) \leqslant n - 8\tau - 3$, then $\mathcal{X}$ is stable. Furthermore, $u_i, u_j \in \mathcal{X}$ are isolated for $|i - j| \geqslant 2$.

To simplify notation we may now use the following convention:

Convention 4.57. If $A \in \mathrm{Can}_{r-1}$ and $u_1, \ldots, u_t$ is a stable sequence of maximal occurrences of rank r in A with complements $v_1, \ldots, v_t$ and B is the result of turning a subset of $\mathcal{Z} = \{u_1, \ldots, u_t\}$, then by Proposition 4.54 we may simply denote the maximal occurrences $\widetilde{u_i}$ or $\widehat{v_i}$ in B by u_i, v_i, respectively.

Turning an occurrence u_i can be considered as choosing the side v_i in the relator $u_i v_i^{-1}$. Hence for choices $f_i \in \{u_i, v_i\}, 1 \leqslant i \leqslant t$, we will write $B = L' \ulcorner f_1 \ldots f_t \urcorner R'$ for the result of turning members of $\{u_i \in \mathcal{Z} \colon f_i = v_i\}$ in A, extending Convention 4.38.

Note that the proof of Proposition 4.54 shows the following important property:

Corollary 4.58. *Let* $A \in \mathrm{Can}_{r-1}$, *let* $u_1, \ldots, u_t$ *be a stable sequence and let* B *be the result of turning* u_i. *Then the sequence (of occurrences corresponding to)* u_j, $j \neq i$, *is a stable sequence in* B.

Informally speaking, solid occurrences prevent a turn of an occurrence on one side to influence occurrences on the other side:

Lemma 4.59. *Let* $A \in \mathrm{Can}_{r-1}$ *and* $u_1, \ldots, u_t$ *be a stable sequence of maximal occurrences of rank* r *in* A *and let* w *be a solid maximal occurrence of rank* r *in* A *with respect to* $u_1, \ldots, u_t$. *Then there exists a unique maximal occurrence* $\widetilde{w}$ *that corresponds to* w *in*

[1]Note that $0 \leqslant k \leqslant 2$ by Condition (c1).

the result of the turns of $u_1, \dots, u_t$. Furthermore if w is between u_i, u_{i+1} and not isolated from k of them, then $|\Lambda_r(w) - \Lambda_r(\widetilde{w})| < k\varepsilon$ if $k \neq 0$ and $w = \widetilde{w}$ if $k = 0$.

Proof. By Proposition 4.54, we can turn $u_1, \dots, u_t$ in any order. We do induction on $k = 0, 1, 2$. The case $k = 0$ follows from Lemma 4.42. Now assume $k > 0$ and let u_i be a close neighbour of w. Then after turning u_i we have $|\Lambda_r(w) - \Lambda_r(\widetilde{w})| < \varepsilon$ by Lemma 4.44. Note that $\widetilde{w}$ is uniquely defined because w was assumed to be solid with respect to $u_1, \dots, u_t$ and $\widetilde{w}$ isolated from u_{i-1} by Condition (c2) of a stable sequence. Furthermore, $\widetilde{w}$ lies between u_{i-1}, u_{i+1}, is solid with respect to the stable sequence $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_t$ and is not isolated from $k - 1$ of them. Thus the claim follows by induction. □

Remark 4.60. Lemma 4.59 shows that $\widetilde{w}$ only depends on the turns of u_i, u_{i+1}.

Lemma 4.61. *Let $A \in \mathrm{Can}_{r-1}$ and $S = (q_1, \dots, q_t)$ be a stable sequence of maximal occurrences of rank r in A. Let $S_0 = (u_1, \dots, u_s)$ be a subsequence with complements $v_1, \dots, v_s$ such that $\Lambda_r(\widehat{v_i}) \geqslant 5\tau + 3$. Let B be the result of turning $u_1, \dots, u_s$ and assume that all maximal occurrences in B have Λ_r-measure $\leqslant n - 8\tau - 3$. Then the maximal occurrences $\{Q \in S \setminus S_0\} \cup \{\widehat{v_i} \mid i = 1, \dots, s\}$ form a stable sequence of rank r in B.*

Proof. It suffices to verify that these occurrences are solid in B. By Proposition 4.54, Corollary 4.58, Lemma 4.59 and Remark 4.60 it is enough to check that q_i is solid in A with respect to q_{i-1}, q_{i+1} and $\widehat{v}_j$ is solid in the result of turning u_j in A with respect to the occurrences corresponding to q_{j-1}, q_{j+1}. This follows from the initial assumptions. □

Lemma 4.62. *Let $A = LC^N R \in \mathrm{Can}_{r-1}$. Suppose $C = \ulcorner u_1 \dots u_k \urcorner$ where $u_1, \dots, u_k$ is a stable sequence of maximal occurrences $\geqslant 5\tau + 3, k \geqslant 2$. Then for $i = 1, \dots k$, the result B_i of turning all periodic shifts of u_i in C^N is of the form*

$$B_1 = L' D_3 v_1' E_3 \ulcorner u_2 \dots u_k \urcorner (\ulcorner v_1 u_2 \dots u_k \urcorner)^{N-1} R;$$
$$B_i = L(\ulcorner u_1 \dots u_{i-1} v_i u_{i+1} \dots u_k \urcorner)^N R \quad \text{for } i \neq 1, k;$$
$$B_k = L(\ulcorner u_1 \dots u_{k-1} v_k \urcorner)^{N-1} \ulcorner u_1 \dots u_{k-1} \urcorner D_3 v_k' E_3 R'.$$

Furthermore, if $C = \ulcorner u \urcorner$ contains a single maximal occurrence u with $5\tau + 3 \leqslant \Lambda_r(u) \leqslant n - (3\tau + 2)$, i.e. $A = L \ulcorner u \urcorner^N R$, the result B of turning all periodic shifts of u inside C^N is of the form

$$B = L' D_3 v' E_3 (\ulcorner v \urcorner)^{N-2} F_3 v'' G_3 R'$$

where v' and v'' are respective remainders of the complements of maximal prolongations of u.

Proof. This follows from Corollaries 4.28 and 4.31 and their corresponding right version. □

4.8. λ-semicanonical forms of rank r.

Recall that $\varepsilon = 2\tau + 1$.

Definition 4.63 ($\kappa \geqslant \frac{n}{2}$)**.** *A word in Can_{-1} is κ-bounded of rank r if all occurrences of rank r have Λ_j-measure $\leqslant \kappa$. A κ-bounded word from Can_{r-1} is called κ-*semicanonical of rank r *and $\mathrm{SCan}_{\kappa,r}$ denotes the set of all κ-semicanonical words of rank r.*

If $A, A' \in \mathrm{Can}_{r-1}$, $A' \in \mathrm{SCan}_{\kappa,r}$ and A' and A represent the same element of the group $\mathrm{F}/\langle\langle \mathrm{Rel}_1, \dots, \mathrm{Rel}_r \rangle\rangle$, then A' is called a κ-semicanonical form of rank r of A.

We emphasize that κ-semicanonical forms of rank r are not unique and that, by definition, $\mathrm{SCan}_{\kappa,r} \subseteq \mathrm{Can}_{r-1}$. Eventually we will have $\mathrm{Can}_r \subset \mathrm{SCan}_{\frac{n}{2}+3\tau+1,r}$.

Definition 4.64. *Let $A, C \in \mathrm{Can}_{r-1}$ and suppose that either $Z = L'QR' \in \mathrm{Can}_{r-1}$ is the result of a turn of a maximal occurrence u of rank r in $A = LuR$ where Q is $(3\tau+1)$-free or $Z = A \cdot_{r-1} C = A'QC'$ where Q is τ-free of rank r. Suppose that L', R' and A', C' are κ-bounded. If there is a unique maximal occurrence w of Λ_r-measure $\geqslant \kappa + \varepsilon$ in Z, then we call w a* seam occurrence *(with respect to κ). A* seam turn *is a turn of a seam occurrence.*

We collect a number of useful observations:

Lemma 4.65 ($\kappa \geqslant \frac{n}{2} + \tau$)**.** *Let $A = LuR \in \mathrm{Can}_{r-1}$, where L, R are κ-bounded in rank r and u is a maximal occurrence of rank r in A with $\kappa \leqslant \Lambda_r(u) < n$. Let $B = L'QR'$ be the result of turning u in A.*

(i) If $\Lambda_r(v') \geqslant \tau + 1$, then $B \in \mathrm{SCan}_{\kappa+\varepsilon,r}$.

(ii) If $B \notin \mathrm{SCan}_{\kappa+\varepsilon,r}$, then $\Lambda_r(u) > n - (3\tau + 1)$.

(iii) If B contains a maximal occurrence w of Λ_r-measure $\geqslant \kappa+\tau+1$ containing Q, then w is the unique maximal occurrence of Λ_r-measure $\geqslant \kappa + \tau + 1$.

(iv) B contains at most two maximal occurrences w of Λ_r-measure $\geqslant \kappa+\tau+1$, one from the left of Q and one from the right.

(v) If $B \notin \mathrm{SCan}_{\kappa+\varepsilon}$, then B contains a unique occurrence of Λ_r-measure $> \kappa + \varepsilon$.

Proof. We first note that Q is κ-free. This is clear if the turn $A \mapsto B$ is of Type 3 or of Type 2 with $\Lambda_r(v') < \tau + 1$ and follows from $n - \kappa \leqslant \frac{n}{2} - \tau \leqslant \kappa - 2\tau$ and Lemma 4.35 in case it is of Type 2 with $\Lambda_r(v') \geqslant \tau + 1$.

(i) We have $\tau + 1 \leqslant \Lambda_r(v') \leqslant \kappa - 2\tau$ by Lemma 4.25. Thus, $\Lambda_r(\widehat{v}) < \kappa$ by Lemma 4.35 and the Λ_r-measure of any other maximal occurrence in L and R can increase from the turn of u at most by Λ_r-measure $< \varepsilon$ by Lemma 4.43. Hence $B \in \mathrm{SCan}_{\kappa+\varepsilon}$.

(ii) If $B \notin \mathrm{SCan}_{\kappa+\varepsilon,r}$, then $\Lambda_r(v') < \tau + 1$ by part (i). Hence $\Lambda_r(u) > n - (3\tau + 1)$.

(iii) Let w be a maximal occurrence in B of Λ_r-measure $> \kappa + \tau + 1$ containing Q. If $\Lambda_r(Q) \geqslant \tau + 1$, then w is the unique occurrence containing Q by Lemma 4.10. If $\Lambda_r(Q) < \tau + 1$, then w contains a suffix of $L'Q$ and a prefix QR' each of Λ_r-measure $\geqslant \tau + 1$ and hence w is the unique occurrence containing Q by Lemma 4.10. If w' is another maximal occurrence in B of Λ_r-measure $\geqslant \kappa+\tau+1$, then w' is properly contained in $L'Q$ or QR' and the overlap of w' with Q must have Λ_r-measure $\geqslant \tau + 1$. Hence the overlap of w and w' has Λ_r-measure $\geqslant \tau + 1$ again contradicting Lemma 4.10.

(iv) If there are at least two maximal occurrences in B of Λ_r-measure $\geqslant \kappa+\tau+1$, then by part (iii) they must be contained in $L'Q$ or QR'. Since L', R' and Q are κ-bounded, any maximal occurrence of Λ_r-measure $\geqslant \kappa + \tau + 1$ in $L'Q$ contains both a suffix of L' and a prefix of Q of Λ_r-measure $\geqslant \tau + 1$ (and similarly for QR'). Hence such an occurrence is unique by Lemma 4.10.

(v) If $B \notin \mathrm{SCan}_{\kappa+\varepsilon}$, then $\Lambda_r(v') < \tau + 1$-free by part (ii) and so the turn is of Type 1 or 2 with $\Lambda_r(v') < \tau + 1$ or of Type 3. If the turn $A \mapsto B$ is of Type 3, then Q is 2τ-free. Hence any maximal occurrence of Λ_r-measure $\geqslant \kappa + \varepsilon$ contains Q and so is unique by part (ii).

Now assume that the turn $A \mapsto B$ is of Type 1 or 2 and $Q = Dv'E$ where D, E are τ-free and $\Lambda_r(v') < \tau + 1$ so that Q is $3\tau + 1$-free. In this case any maximal occurrence w of Λ_r-measure $\geqslant \kappa + \varepsilon$ must contain a prefix of $Dv'ER'$ and a suffix of $L'Dv'E$ of Λ_r-measure $> \varepsilon$ and hence there can only be one such w by Lemma 4.10. ∎

Lemma 4.66 ($\kappa \geqslant \frac{n}{2} + \tau$)**.** *Suppose $A, B \in \mathrm{SCan}_{\kappa,r}$. Then*

$$Z = \mathrm{can}_{r-1}(A \cdot B) = A' D_3 B' \in \mathrm{SCan}_{\kappa+\varepsilon,r}$$

unless Z contains a seam occurrence.

Proof. Since D_3 is τ-free and A, B are κ-semicanonical, any occurrence of Λ_r-measure $> \kappa + \varepsilon$ in $A'D_3B'$ must contain both a suffix of A' and a prefix of B' of Λ_r-measure $\geqslant \tau + 1$, and hence is unique. □

For a sufficiently big constant μ the natural greedy algorithm of turning occurrences of Λ_r-measure $> \mu$ converges and leads to a μ-semicanonical form of rank r of the word:

Lemma 4.67 ($\mu \geqslant \frac{n}{2} + 9\tau$, $\alpha = 5\tau + 3$)**.** *If $A \in \mathrm{Can}_{r-1}$ and $A = LuR \mapsto B$ is the turn of the maximal occurrence u of rank r in A with $\Lambda_r(u) > \mu$, then $d(B) < d(A)$ where $d(X)$ denotes the sum of the Λ_r-measures of all maximal occurrences of rank r in X of Λ_r-measure $\geqslant \alpha$ for $X \in \mathrm{Can}_{r-1}$.*

Proof. Note that $d(A) - d(B) \geqslant \Lambda_r(u) - S$ where S is the sum of Λ_r-measures of maximal occurrences in $B = L'QR'$ that did not contribute to $d(A)$ but count for $d(B)$. These arise from maximal occurrences in B of Λ_r-measure $\geqslant \alpha$ having nontrivial overlap with Q. Note that if $w = \ell q$ is a maximal occurrence in A that contained in $L'Q$, where ℓ is a suffix of L' with $\Lambda_r(\ell) \geqslant \alpha$, then only $\Lambda_r(q)$ may contribute to S, and similarly for maximal occurrences contained in QR'. So in order to compute an upper bound for S, we may assume in such cases that $\Lambda_r(\ell) < \alpha$ and hence $\Lambda_r(\ell q) < \alpha + \Lambda_r(q)$. Note that by Lemma 4.25 (iii) and Lemma 4.43 we have $\Lambda_r(q) < 3\tau + 1$.

If Q is $3\tau + 1$-free, any maximal occurrence in B that contributes to S must contain a suffix of L' or prefix of R' (or both) and at least one of the suffix or prefix must have Λ_r-measure $\geqslant \tau + 1$. By Lemma 4.10 there can be at most one such occurrence from either side of Q and only one if both overlaps with L' and R' are of Λ_r-measure $\geqslant \tau + 1$. Hence we can estimate the contributions in S by $S < 2(\alpha + 4\tau + 2) = 18\tau + 10 < \mu < \Lambda_r(u)$ (because $n > 18\tau + 20$).

By Lemma 4.25 (iii) it remains to consider the case that the turn is of Type 2, so $Q = D_3 v' E_3$ (where D_3, E_3 may be empty) with $\Lambda_r(v') \geqslant \tau + 1$.

Here contributions to S can arise from $\widehat{v}$ and, as before, from maximal occurrences containing a suffix of L' or a prefix of R'. By Lemma 4.35 we have $\Lambda_r(\widehat{v}) < \Lambda_r(v) + 2\tau$, and so $\Lambda_r(u) - \Lambda_r(\widehat{v}) > \Lambda_r(u) - (n - \Lambda_r(u) + 2\tau) > 2(9\tau) - 2\tau = 16\tau$. Furthermore the occurrences containing a suffix of L' or a prefix of R' may contribute at most $2\alpha + 2(2\tau + 1) = 14\tau + 8$. Hence again $\Lambda_r(u) - S > 16\tau - 14\tau - 8 > 0$, and this finishes the proof. □

The previous lemma immediately implies:

Corollary 4.68 ($\mu = \frac{n}{2} + 9\tau \geqslant n - 7\tau - 3$)**.** *Any $A \in \mathrm{Can}_{r-1}$ can be transformed into a μ-semicanonical form of rank r of A by a sequence of turns of occurrences of rank r of Λ_r-measure $> \mu$, starting from A.*

While the previous algorithm is the most intuitive way to obtain a semicanonical form, the bound $\mu = \frac{n}{2} + 9\tau$ will not be good enough for our purpose. Therefore we will further improve this bound below.

For future reference we record the following observation:

Lemma 4.69 ($\kappa \geqslant \frac{n}{2} + \tau$)**.** *If $A \mapsto B$ with $A, B \in \mathrm{SCan}_{\kappa,r}$ is the turn of a maximal occurrence u in A of Type 2, then $n - \kappa - 2\tau < \Lambda_r(u) \leqslant \kappa$.*

Proof. By Lemma 4.25 (ii) we have $n - \Lambda_r(u) - 2\tau < \Lambda_r(v') \leqslant \kappa$. □

The following lemma will be used in Section 6 to define an auxilliary group structure:

Lemma 4.70 ($\frac{n}{2} + \tau \leqslant \kappa \leqslant n - 7\tau - 3$)**.** *Let* $A, C \in \mathrm{SCan}_{\kappa,r}$ *and* $Z = \mathrm{can}_{r-1}(A \cdot C) = A_1 D_3 C_1$. *Then there is a sequence of seam turns*

$$Z = A_1 D_3 C_1 \mapsto Z_2 = A_2 Q_2 C_2 \mapsto \ldots \mapsto Z_k = A_k Q_k C_k = Z' \in \mathrm{SCan}_{\kappa+(3\tau+1),r}$$

such that A_{i+1}, C_{i+1}, $i \leqslant k-2$, *are proper prefix and suffix of* A_i, C_i, *respectively,* Q_i *is* $(3\tau+1)$*-free of rank* r *for* $i < k$, *the last turn has* Λ_r*-measure* $> \kappa + 3\tau + 1$ *and all other turns have* Λ_r*-measure* $> n - (3\tau + 1)$. *We write* $Z' = \mathrm{prod}_{\kappa+(3\tau+1),r}(A \cdot C)$.

Proof. If $Z = A_1 D_3 C_1 \notin \mathrm{SCan}_{\kappa+\varepsilon,r}$, then, by Lemma 4.66, Z contains a seam occurrence w of Λ_r-measure $> \kappa + \varepsilon$ that properly contains D_3. Hence the result of turning w in Z is of the form $Z_2 = A_2 Q_2 C_2$ where A_2, C_2 are proper prefix and suffix of A_1, C_1, respectively. If $Z_2 \notin \mathrm{SCan}_{\kappa+(3\tau+1),r}$, then $\Lambda_r(w) > n - (3\tau + 1)$ by Lemma 4.65 (ii) and Z_2 contains a unique maximal occurrence w_2 of Λ_r-measure $\geqslant \kappa+(3\tau+1)$ by Lemma 4.65 (v). Since $n - 3\tau - 1 > \kappa + 3\tau + 1)$, w_2 has non-trivial overlap both with A_2 and C_2. Let Z_3 be the result of turning w_2 in Z_2. If $\Lambda_r(w_2) \leqslant n - (3\tau + 1)$, then $Z_3 \in \mathrm{SCan}_{\kappa+\varepsilon,r}$ by Lemma 4.65 (ii), and we are done. Otherwise w_2 is the seam occurrence in Z_2 (of Λ_r-measure $> n - 3\tau - 1$). We continue until $Z_k = Z' \in \mathrm{SCan}_{\kappa+3\tau+1,r}$. Since at each step we obtain a proper prefix of A_i and B_i by the description in Lemma 4.25, this process stops with $Z' = \mathrm{prod}_{\kappa+3\tau+1,r,}(Z)$ after finitely many turns. □

Lemma 4.71 ($\frac{n}{2} + 3\tau + 1 \leqslant \mu_2 \leqslant \mu_1 \leqslant n - 7\tau - 3$)**.** *For* $A \in \mathrm{SCan}_{\mu_1,r}$ *there exists a sequence of turns of rank* r *and* Λ_r*-measure* $> \mu_2 - \varepsilon$

$$A = C_1 \mapsto C_2 \mapsto \ldots \mapsto C_l \in \mathrm{SCan}_{\mu_2,r}$$

such that all C_i *are* $(\mu_1 + \varepsilon)$*-semicanonical of rank* r.

Proof. Let $(u_0, \ldots, u_m)$ be an enumeration of all maximal occurrences in A of Λ_r-measure $> \mu_2 - 2\varepsilon$ enumerated from left to right. Note that this forms a stable sequence (see Remark 4.56) and hence u_i, u_j are isolated for $|i - j| \geqslant 2$. Let $u = u_i$ be the left-most maximal occurrence of rank r in A of Λ_r-measure $> \mu_2 - \varepsilon$ and write $A = LuR$. Then L is $(\mu_2 - \varepsilon)$-bounded of rank r and R is μ_1-bounded. Since $n - (3\tau + 1) \geqslant \mu_1 \geqslant \Lambda_r(u) > \mu_2 - \varepsilon$, the result $B = L'\widehat{v}R'$ of turning u belongs to $\mathrm{SCan}_{\mu_1+\varepsilon}$ by Lemma 4.65(i) and $\Lambda_r(\widehat{v}) < \mu_2 - \varepsilon$ by Lemma 4.25. Furthermore, $L'\widehat{v}$ is μ_2-bounded by Lemma 4.44.

We consider in B the left-most maximal occurrence w in $\widehat{v}R'$ of Λ_r-measure $> \mu_2 - \varepsilon$. Then w corresponds to u_j, $j > i$, in the original sequence. Since $\Lambda_r(u) \leqslant \mu_1 \leqslant n - (3\tau + 1)$, by Lemma 4.65 (i) $B \in \mathrm{SCan}_{\mu_1+\varepsilon}$. Moreover, there exists at most one maximal occurrence of Λ_r-measure $> \mu_1$ in B and if it exists, it is contained in $\widehat{v}R'$ and must agree with w. Write $B = L_1 w R_1$. Then R_1 is μ_1-bounded. Now we turn w and argue as above. Let $L_1'\widehat{z}R'$ be the result of the turn. Although L_1 is not $\mu_2 - \varepsilon$-bounded now, since $\Lambda_r(\widehat{v}) \geqslant 3\tau + 2$, all maximal occurrences in B from the left of $\widehat{v}$ stay unchanged in the result of the turn. Therefore $L_1'\widehat{z}$ is μ_2-bounded. We continue to argue in the same way until we reach the end of the sequence $(u_0, \ldots, u_m)$. □

From now on we fix $\lambda = \frac{n}{2} + 3\tau + 1$**.**

Corollary 4.72. *Every* $A \in \mathrm{Can}_{r-1}$ *has a* λ*-semicanonical form of rank* r.

Proof. By Lemma 4.68 every $A \in \mathrm{Can}_{r-1}$ has a $(\frac{n}{2}+9\tau)$-semicanonical form. Now apply Lemma 4.71 with $\lambda = \mu_2 \leqslant \mu_1 = \frac{n}{2}+9\tau \leqslant n-7\tau-3$. □

While by Proposition 4.54 the result of a number of turns is independent of the order of the turns, the properties of the intermediate results may depend on the order.

Lemma 4.73 ($\kappa \geqslant 5\tau+3$)**.** *Let $A \in \mathrm{SCan}_{\kappa,r}$ and let $u_1,\dots,u_t$ be a stable sequence of maximal occurrences of rank r in A enumerated from left to right. Let B be the result of turning all $u_i, i=1,\dots,t$, and assume $B \in \mathrm{SCan}_{\kappa,r}$.*

If $A = X_0 \mapsto X_1 \mapsto \dots \mapsto X_t = B$ is the sequence of turns from left to right, then $X_i \in \mathrm{SCan}_{\kappa+\varepsilon,r}$ for $i=1,\dots,t$.

Proof. Assume towards a contradiction that $X_i \notin \mathrm{SCan}_{\kappa+\varepsilon,r}$ for some i. Let w be a maximal occurrence of rank r in X_i that does not correspond to u_{i+1}. If there are occurrences to the left of u that are not yet turned in X_i, then w is to the right of $\widetilde{u_{i+1}}$. Since u_{i+1} is solid with respect to $u_1,\dots,u_i$, the occurrence corresponding to w in A is equal to w as a word. So, $\Lambda_r(w) \leqslant \kappa$.

If $\Lambda_r(w) > \kappa+\varepsilon$, then by Lemma 4.59 w does not correspond to any u_j, $j \geqslant i+1$. So all occurrences that are not yet turned in X_i are to the right of w and w is solid with respect to them. By Lemma 4.59 the occurrence corresponding to w in B has Λ_r-measure $> \kappa$, a contradiction. □

Corollary 4.74 ($\mu = n-8\tau-3$)**.** *Let $X_0 \mapsto X_1 \mapsto \dots \mapsto X_l$ be a sequence of turns of rank r and Λ_r-measure $\geqslant 9\tau+5$, where $X_0 \in \mathrm{SCan}_{\mu,r}$ and $X_i \in \mathrm{Can}_{r-1}$. Then there exists a stable sequence of maximal occurrences $u_1,\dots,u_t$ of rank r and Λ_r-measure $\geqslant 5\tau+3$ in X_0 such that the result of the corresponding turns is equal to X_l.*

Proof. The proof is by induction on l. If $l=1$, there is nothing to prove. So consider $l>1$ and assume inductively that there exists a stable sequence $(q_1,\dots,q_s)$ of maximal occurrences of Λ_r-measure $\geqslant 9\tau+5$ in X_0 whose turns results in X_{l-1}. If we turn them from left to right, then by Lemma 4.59 every turn is of Λ_r-measure $< \mu+\varepsilon = n-6\tau-2$. Hence the maximal occurrence that contain the remainder of the complement has Λ_r-measure $> 4\tau+2$. So again by Lemma 4.59 the corresponding maximal occurrence in X_{l-1} is well defined has Λ_r-measure $> 2\tau+1$. Let $z_1,\dots,z_s$ be maximal occurrences in X_{l-1} that correspond to the remainders of the complements of $q_1,\dots,q_s$.

Assume that $X_{l-1} \mapsto X_l$ is the turn of an occurrence $\widetilde{u}$. Then either $\widetilde{u}$ coincides with some z_i, or $\widetilde{u}$ lies between z_i, z_{i+1} for some i. If $\widetilde{u}$ coincides with some z_i, we put $u = q_i$. Now consider the second possibility. By the initial assumptions, $\Lambda_r(\widetilde{u}) \geqslant 9\tau+5$. Hence q_i and q_{i+1} are isolated in X_0 and $X_0 = Lq_iMq_{i+1}R$. When we first turn the q_j, $j \neq i, i+1$, then by Lemma 4.59 the result is of the form $L_1\widetilde{q_i}M\widetilde{q_{i+1}}R_1$, where $\Lambda_r(\widetilde{q_i}), \Lambda_r(\widetilde{q_{i+1}}) < \mu+\varepsilon = n-6\tau-2$. Since $\widetilde{u} \geqslant 9\tau+5$, we obtain that $X_{l-1} = L_2{w'}_iE_3M'F_3{w'}_{i+1}R_2$, where w_i, w_{i+1} are complements of $\widetilde{q_i}, \widetilde{q_{i+1}}$ and ${w'}_i, {w'}_{i+1}$ are their remainders. Hence the common part of M' and $\widetilde{u}$ has Λ_r-measure $> 5\tau+3$. So its maximal prolongation u in X_0 is unique. We denote it by u.

If $u = q_i$ for some $i \in \{1,\dots,s\}$, then we claim that $\{q_1,\dots,q_s\} \setminus \{q_{i_0}\}$ is the required set of occurrences. Clearly, they form a stable sequence. By Proposition 4.54 we may assume that turning q_i is the last turn and hence the turn $X_{l-1} \mapsto X_l$ is its inverse. Therefore, X_l is the result of turning the occurrences in $\{q_1,\dots,q_s\} \setminus \{q_i\}$.

If $u \notin \{q_1,\dots,q_s\}$, then $\{q_1,\dots,q_s\} \cup \{u\}$ is the required set of occurrences. Indeed, since $5\tau+3 \leqslant \Lambda_r(q_i), \Lambda_r(u) \leqslant n-8\tau-3$, they form a stable sequence and clearly X_l is the result of their turns. □

We need the following lemma in Section 6.

Lemma 4.75 ($\mu = n-8\tau-3, \lambda = \frac{n}{2}+3\tau+1$)**.** *Let $A \in \mathrm{SCan}_{\mu,r}$ and $A_1, A_2 \in \mathrm{SCan}_{\lambda+\tau,r}$. Assume that A_1 and A_2 are obtained from A by sequences of turns of Λ_r-measure $\geqslant 9\tau+5$. Then A_1 can be obtained from A_2 by a sequences of turns where all intermediate words are in $\mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$.*

Proof. By Lemma 4.74, there exist stable sets $\mathcal{X}_1, \mathcal{X}_2$ of maximal occurrences of rank r in A of Λ_r-measure $\geqslant 5\tau+3$ such that A_i is the result of turning the occurrences in $\mathcal{X}_i, i = 1, 2$. Since all occurrences in $\mathcal{X}_i, i = 1, 2$ satisfy the restrictions $5\tau + 3 \leqslant \Lambda_r(u) \leqslant \mu$, $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ is a stable set. Therefore Lemma 4.61 implies that the maximal occurrences in A_1 that correspond to $\mathcal{X}$ (and to remainders of complements of turned occurrences) form a stable set $\mathcal{Y}$ in A_1. Clearly turns of some subset of $\mathcal{Y}$ in A_1 give A_2. Using Lemma 4.73, we turn them from left to right and obtain the required sequence of turns. □

We next aim to show that turns commute (under suitable conditions) with the multiplication of canonical words. This will be used in Section 6.

Lemma 4.76. *Let $A = LuX, C \in \mathrm{Can}_{r-1}$ for a maximal occurrence $u = a^k a_1, a^n \in \mathrm{Rel}_r$, and let B be the result of turning u. Assume that $\mathrm{can}_{r-1}(A \cdot C) = A'DC'$ where A', C' are prefix and suffix of A and C, respectively, and D is τ-free of rank r. Let u' be the (possibly empty) common part of u and A'. Then the following holds:*

(1) If $\Lambda_r(u') \geqslant \tau + 1$, and $\widetilde{u}$ is the maximal occurrence containing u' in $A'D_3C'$, then the following diagram commutes:

$$\begin{array}{ccc} A = LuX & \xrightarrow{\text{turn of } u} & B \\ \downarrow & & \downarrow \\ \mathrm{can}_{r-1}(A\cdot C) & \xrightarrow{\text{turn of } \widetilde{u}} & \mathrm{can}_{r-1}(B\cdot C) \end{array}$$

(2) If $\Lambda_r(u) \geqslant \frac{n}{2} - 3\tau - 1$ and $\Lambda_r(u') < 2\tau + 1$, then $C = X^{-1}c^{-1}R$, where $X^{-1}c^{-1}$ is the maximal cancellation in $uX \cdot C$. Then c^{-1} is a fractional power of a^{-1} with $\Lambda_r(c^{-1}) > \frac{n}{2}-(6\tau+2)$. Let $\widehat{w}$ be the maximal occurrence in $\mathrm{can}_{r-1}(B{\cdot}C)$ corresponding to c^{-1} (note that $\widehat{w}$ then also corresponds to $\widehat{v}$, if this is defined). Then $\Lambda_r(\widehat{w}) > n-(4\tau+1)$ and the following diagram commutes:

$$\begin{array}{ccc} A = LuX & \xrightarrow{\text{turn of } u} & B \\ \downarrow & & \downarrow \\ \mathrm{can}_{r-1}(A\cdot C) & \xleftarrow{\text{turn of } \widehat{w}} & \mathrm{can}_{r-1}(B\cdot C) \end{array}$$

Proof. In the first case we can write $\mathrm{can}_{r-1}(A \cdot C) = L\widetilde{u}R_1$ and let Z be the result of the turn of $\widetilde{u}$. Then $Z \equiv La^{-n} \cdot \widetilde{u}R_1 \equiv La^{-n} \cdot uX \cdot C \equiv \mathrm{can}_{r-1}(La^{-n} \cdot uX) \cdot C = B \cdot C$ mod $\langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_{r-1} \rangle\rangle$. So the first part follows from IH 6 and IH 8.

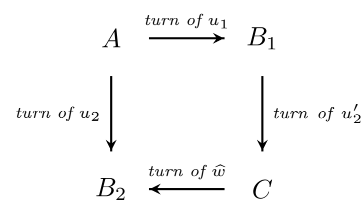
For the second case if $\Lambda_r(u) \geqslant \frac{n}{2}-3\tau-1$ and $\Lambda_r(u') < 2\tau+1$, then $\Lambda_r(c) > \frac{n}{2}-(6\tau+2)$. So $\mathrm{can}_{r-1}(B \cdot C) = \mathrm{can}_{r-1}(La^{-n} \cdot u \cdot c^{-1}R) = \mathrm{can}_{r-1}(LwR)$, where $w = a^{-n} \cdot u \cdot c^{-1}$ is a fractional power of a^{-1} with $\Lambda_r(w) > n - (3\tau + 1)$.

Now write $LwR = La^N \cdot a^{-N}wR$. Since $\Lambda_r(u), \Lambda_r(c) \geqslant \tau$, IH 11 and then Lemma 4.24 are applicable and imply that $\mathrm{can}_{r-1}(LwR) = L'F_3w'R$, where w' is a suffix of w with $\Lambda_r(w') > \Lambda_r(w) - \tau > n - (4\tau + 1)$. So w' has a unique maximal prolongation $\widehat{w}$ and by Remark 4.22 and IH 8 the result of the turn of $\widehat{w}$ is equal to $\mathrm{can}_{r-1}(La^n \cdot wR) = \mathrm{can}_{r-1}(La^n \cdot (a^{-n} \cdot u \cdot c^{-1})R) = \mathrm{can}_{r-1}(A \cdot C)$. □

Note that by symmetry, if both $\widetilde{u}$ and $\widehat{w}$ are defined in $\mathrm{can}_{r-1}(A \cdot C)$ and $\mathrm{can}_{r-1}(B \cdot C)$, respectively, then both diagrams in the above situation commute. Furthermore, if $\widehat{v}$ is defined in Lemma 4.76, then $\widehat{w}$ comes from merging of $\widehat{v}$ and c^{-1} (this effect is described in Remark 4.46).

Similarly to Lemma 4.76 (2) the following extension of Lemma 4.53 holds.

Lemma 4.77. *Let $A = L\ulcorner u_1 u_2 \urcorner R \in \mathrm{Can}_{r-1}$ be such that $\Lambda_r(u_1) \geqslant \frac{n}{2} - \tau$, u_1 is a fractional power of $a^n \in \mathrm{Rel}_r$, u_2 is solid with respect to u_1 and u_1 is not solid with respect to u_2. Then $A = L\ulcorner u_1 u_2 \urcorner X c^{-1} R_1$, where c^{-1} is a fractional power of a^{-1} with $\Lambda_r(c^{-1}) > \frac{n}{2} - (9\tau + 3)$. Let B_i be the result of turning u_i in A and C be the result of turning the remainder of u_2 in B_1. Then there exists $\widehat{w}$ a maximal occurrence of rank r in C that corresponds to c^{-1} (and to $\widehat{v_1}$ if it is defined) such that $\Lambda_r(\widehat{w}) > n - (4\tau + 1)$, and the following diagram commutes:*



Proof. We can write $A = Lu_1' M u_2 R$, where u_1' is a prefix of u_1 with $\Lambda_r(u_1') > \Lambda_r(u_1) - \tau - 1$, $u_1' = u_1$ if $M \neq 1$. Let $Z_1 = \mathrm{can}_{r-1}(Lu_1'M)$, $Z_2 = \mathrm{can}_{r-1}(u_2 R)$, W_1 be the result of turning the occurrence that corresponds to u_1' in Z_1 (which has Λ_r-measure $> \Lambda_r(u_1) - 2\tau - 1 \geqslant \frac{n}{2} - 3\tau - 1$), and W_2 be the result of tuning the occurrence that corresponds to u_2 in Z_2. Then by IH 8 and Remark 4.22 $B_1 = \mathrm{can}_{r-1}(W_1 \cdot Z_2)$, $B_2 = \mathrm{can}_{r-1}(Z_2 \cdot W_2)$ and $C = \mathrm{can}_{r-1}(W_1 \cdot W_2)$. So the result follows from Lemma 4.76 (2) applied to Z_1, W_1, W_2. □

5. Defining the canonical form of rank r

5.1. **Determining winner sides.** Recall that $\tau = 15$, $\varepsilon = 2\tau + 1$ and let $\mu = n - 8\tau - 3 \geqslant \frac{n}{2} + 9\tau$.

In this section we define *the canonical form of rank r* of $A \in \mathrm{SCan}_{\mu,r} \subseteq \mathrm{Can}_{r-1}$. For this we consider all maximal occurrences of rank r in $A \in \mathrm{SCan}_{\mu,r}$ of Λ_r-measure $\geqslant 5\tau + 3$. Since $\mu = n - (4\tau + 1 + 2\varepsilon)$, these occurrences form a stable sequence in A and their complements are defined. Hence the result of turning any subset of these occurrences in A is well-defined by Proposition 4.54 and the canonical form $\mathrm{can}_r(A)$ is the result of turning a specific subset of these occurrences. Roughly speaking, for every maximal occurrence of rank r in A of Λ_r-measure $\geqslant \tau + 1 + 2\varepsilon$ we decide whether or not to turn it using a threshold of Λ_r-measure roughly $\frac{n}{2}$. For every maximal occurrence u in A at least one of u or its complement will be below this threshold (see Corollary 4.36).

5.2. **Rank $\mathbf{r = 1}$.** This case is much simpler than the general case because relators in Rel_1 are of the form x^n, where x is a single letter, so maximal occurrences of rank 1 have no overlaps. Since canonical triangles of rank 0 are trivial (i.e. all sides are equal to 1), a turn of a rank 1 occurrence consists simply of replacing an occurrence by its complement. Furthermore, for a maximal occurrence u of rank 1 we have $\Lambda_1(u) = |u|$. Since the exponent n is odd, either u or its complement has Λ_1-measure $< \frac{n}{2}$.

Now for $A \in \mathrm{SCan}_{\mu,1}$, the canonical form of A of rank 1, denoted by $\mathrm{can}_1(A)$, is defined as the word obtained from A by replacing all maximal occurrences of rank 1 of Λ_1-measure $> \frac{n}{2}$ by their respective complements.

Lemma 5.1. *Let $A \in \mathrm{SCan}_{\mu,1}$, and let $A \mapsto B$ be a turn of rank 1. Assume that $B \in \mathrm{SCan}_{\mu,1}$. Then $\mathrm{can}_1(A) = \mathrm{can}_1(B)$.*

Proof. Since a turn of rank 1 just consist of replacing an occurrence by its complement, it does not change any other maximal occurrences and so this follows directly from the definition of the canonical form of rank 1. □

5.3. **Rank $\mathbf{r \geqslant 2}$.** From now on until the end of Section 5.1 we consider the general case, namely, rank $r \geqslant 2$ and we fix the following set-up:

Let A be in μ-semicanonical form, and let u be a maximal occurrence of rank r in $A = LuR$ of Λ_r-measure $\geqslant \tau + 1 + 2\varepsilon$ where $u = a^t a_1$, for some $a^n \in \mathrm{Rel}_r$, $a = a_1 a_2$ (a_1 can be empty).

We now state conditions whether or not to turn u when we construct $\mathrm{can}_r(A)$. Let λ_1 and λ_2 be two constants with the following properties:

(λ1) $n - (11\tau + 5) \geqslant \lambda_1 > \lambda_2 \geqslant \frac{n}{2} + 5\tau + 2$.

(λ2) $\lambda_1 - \lambda_2 \geqslant \varepsilon$

For $n > 36\tau + 16$ the interval $[\frac{n}{2} + 5\tau + 2, n - (11\tau + 5)]$ has length $\geqslant 2\tau + 1$ and hence such λ_1, λ_2 exist.

We will use the fact that there exist sequences $m : \mathbb{N} \longrightarrow \{1, 2\}$ without subsequences of the form BBb [4] where b is a nontrivial initial segment of B. By Proposition 4.72 we know that we can obtain λ_2-semicanonical forms by making a number of appropriate turns. In the certification process we test whether a given occurrence can be made short enough by appropriate turns without significantly increasing other occurrences. The cubic-free sequence given by m will ensure that we are not creating new power words (of higher rank) in the process.

We first note the following:

Lemma 5.2. *Let $u_1, \dots, u_k$ be maximal occurrences of rank r in a word $W = D^\ulcorner u_1 \dots u_k{}^\urcorner E$ such that u_i, u_{i+1} are not essentially isolated and $\Lambda_r(u_i) \geqslant \tau + 1$ for all i. Suppose that D, E are τ-free of rank r. Let u be a maximal occurrence of rank r in W with $\Lambda_r(u) \geqslant 5\tau + 2$. Then u coincides with one of the u_i.*

Proof. Clearly, if u has nontrivial overlap with D or E, then $u = u_1$ or u_k respectively by Corollary 4.19. If u has a common part with the gap between u_i, u_{i+1} for some $1 \leqslant i \leqslant k - 1$, this common part has Λ_r-measure $< 3\tau$. Since $\Lambda_r(u) \geqslant 5\tau + 2$, the overlap of u with u_i or u_{i+1} has Λ_r-measure $\geqslant \tau + 1$ and hence u coincides with that occurrence. □

Recall that in a stable sequence any turn of a member of the sequence has an inverse turn by Lemma 4.33.

Definition 5.3 (certification sequence)**.** *Let $A \in \mathrm{SCan}_{\mu,r}$. Then a stable sequence $(u = u_0, u_1, u_2, \dots, u_t), t \geqslant 1$, of maximal occurrences of Λ_r-measure $\geqslant 5\tau + 2$ in $A = L^{\ulcorner} u_0 \dots u_t{}^{\urcorner} R$ (enumerated from left to right) with complements $v = v_0, v_1, v_2, \dots, v_t$ is called a* certification sequence *in A to the right of u (with respect to $m : \mathbb{N} \to \{1, 2\}$) if the following holds*

1. *u_1 is essentially non-isolated from u_0;*
2. *there is a choice $f_i \in \{u_i, v_i\}, 0 \leqslant i \leqslant t$, such that in $W = L'^{\ulcorner} f_0 f_1 \dots f_t{}^{\urcorner} R'$ the maximal occurrences (corresponding to) f_i for $i = 1, \dots, t$ satisfy $\Lambda_r(f_i) \leqslant \lambda_{m(i)}$.*
3. *After turning f_0 in W and denoting the occurrence corresponding to f_1 in the result by $\widetilde{f_1}$ we have $\Lambda_r(\widetilde{f_1}) \geqslant \Lambda_r(f_1)$. Moreover if $\Lambda_r(\widetilde{f_1}) = \Lambda_r(f_1)$, then $f_0 = u_0$.*
4. *For $2 \leqslant i \leqslant t$, after turning f_i in W the occurrence corresponding to f_{i-1} has Λ_r-measure $> \lambda_{m(i-1)}$.*
5. *If there is a maximal occurrence w in A of Λ_r-measure $\geqslant 5\tau + 3$ to the right of u_t, then after turning (the occurrence corresponding to) w in W, in the resulting word we still have $\Lambda_r(f_t) \leqslant \lambda_{m(t)}$.*

We say that the sequence certifies *f_1 to the right of u, i.e. either $f_1 = u_1$ or $f_1 = v_1$ is certified by the sequence. W is called the witness for the certification (of u_1 or v_1, respectively), exhibiting the choices $f_i \in \{u_i, v_i\}$.*

We let $\mathcal{Y}_R(u) = \mathcal{Y}_R(u, A)$ denote the set of sides f_1 which are certified by a certification sequence to the right of u. (Note that if $f_1 = v_1$ this is not an occurrence in A.) Similarly we define $\mathcal{Y}_L(u, A)$ as the set of inverses of $\mathcal{Y}_R(u^{-1}, A^{-1})$ and put $\mathcal{Y}(u) = \mathcal{Y}_L(u) \cup \mathcal{Y}_R(u)$. Note that $\mathcal{Y}_L(u), \mathcal{Y}_R(u)$ contain at most two elements and are empty if there are no maximal occurrences of Λ_r-measure $\geqslant 5\tau + 3$ essentially non-isolated from u from the left or right, respectively.

We say that a stable sequence $(u = u_0, u_1, u_2, \dots, u_t), t \geqslant 1$, is an un-certification sequence *if it satisfies 1., 3. and 4. above and in place of 2. and 5. it satisfies the following:*

- 2'. *there is a choice $f_i \in \{u_i, v_i\}, 0 \leqslant i \leqslant t$, such that in $W = L'^{\ulcorner} f_0 f_1 \dots f_t{}^{\urcorner} R'$ the maximal occurrences (corresponding to) f_i for $i = 1, \dots, t - 1$ satisfy $\Lambda_r(f_i) \leqslant \lambda_{m(i)}$ and $\Lambda_r(f_t) > \lambda_{m(t)}$.*
- 5'. *If there is a maximal occurrence w in A of Λ_r-measure $\geqslant 5\tau + 3$ to the right of u_t, then after turning (the occurrence corresponding to) w in W, in the resulting word we still have $\Lambda_r(f_t) > \lambda_{m(t)}$.*

Similarly we define (un-)certification sequences to the left in the obvious way by considering inverses. We then say that a maximal occurrence w or its complement of Λ_r-measure $\geqslant 5\tau+3$ contained in uR is certified (or uncertified, respectively) in A to the left of u by a stable sequence $(u_t, \dots, u = u_0)$ enumerated from right to left if $(u_0^{-1}, \dots, u_t^{-1})$ is an (un-)certification sequence for w^{-1} to the rigth of u^{-1} in A^{-1}.

If there is no maximal occurrence of Λ_r-measure $\geqslant 5\tau + 3$ to the right of u and essentially non-isolated from u, then we say that $(u = u_0)$ is both the certification and uncertification sequence to the right of u.

Remark 5.4. Let $A = LuR \in \mathrm{SCan}_{\mu,r}$ and let $(u = u_0, \dots, u_s)$ be an enumeration of all maximal occurrences of Λ_r-measure $\geqslant 5\tau + 3$ in uR enumerated from left to right. Then u_i, u_{i+2} are strictly isolated, and hence, essentially isolated from each other. Combining Conditions 1, 2, 4, we see that (un-)certification sequences have no gaps. Therefore, any (un-)certification sequence to the right of u is an initial segment of $(u = u_0, \dots, u_s)$.

For an (un-)certification sequence to the right of u it suffices to check Conditions 5 and 5' for the left most occurrence w to the right of u_t with $\Lambda_r(w) \geqslant 5\tau + 3$ because all maximal occurrences of rank r in A to the right of w are strictly isolated from u_t.

We first record the following remarks, which follow directly from Definition 5.3:

Remark 5.5. Let $A = LuR \in \mathrm{SCan}_{\mu,r}$ and let $(u = u_0, \dots, u_t)$ be an (un-) certification sequence to the right of u in A.

(1) By Condition 4, a proper prefix of $(u = u_0, \dots, u_t)$ can be neither a certification nor an un-certification sequence.
(2) For $i = 0, \dots t-1$, the members u_i and u_{i+1} are essentially non-isolated by Condition 4 and $\lambda_2 - 2\varepsilon < \Lambda_r(u_i) < \mu$. In particular, any (un-)certification sequence is stable.
(3) If in A we have $\Lambda_r(u_1) < \lambda_{m(1)} - \varepsilon$ for a maximal occurrence u_1 not essentially isolated from u, then by Condition 3 and Lemma 4.44, u_1 is certified with the sequence (u, u_1) and witness A. Since $\lambda_2 - \varepsilon > \frac{n}{2} + \tau$, at least one of u_1 and v_1 is certified in A by Corollary 4.36 with certification sequence (u, u_1). So for at most one of u_1 and v_1 we have a certification or un-certification sequence that contains > 2 occurrences.
(4) Suppose $(u_0 = u, u_1, u_2, \dots, u_t)$ is an (un-)certification sequence to the right of u with witness $W = L'^{\ulcorner} f_0 \dots f_t{}^{\urcorner} R'$. If in W we have $\Lambda_r(f_i, W) \leqslant \lambda_{m(i)} - \varepsilon$ (or $\Lambda_r(f_i, W) \geqslant \lambda_{m(i)} + \varepsilon$, respectively) for some $1 \leqslant i \leqslant t$, then $i = t$ by Conditions 2, 2' and 4.
(5) If y is certified in A to the right of u, then by Lemma 4.59 $\Lambda_r(y) < \lambda_{m(1)} + k\varepsilon$ where k is the number of close neighbours of y among $u_0, \dots, u_t$.

Lemma 5.6. *Let $A = LuR \in \mathrm{SCan}_{\mu,r}$ where $\Lambda_r(u) \geqslant 5\tau + 3$. Let u_1 be a maximal occurrence of rank r in uR essentially non-isolated from u with $\Lambda_r(u_1) \geqslant 5\tau + 3$. Then for any choice $f_1 \in \{u_1, v_1\}$ either there exists a unique certification sequence or a unique un-certification sequence for f_1. In either case, the witness W is unique.*

Proof. Let $(u_0 = u, u_1, \dots, u_s)$ be an enumeration of all maximal occurrences of Λ_r-measure $\geqslant 5\tau + 3$ in uR enumerated from left to right. Any (un-)certification sequence for u is an initial segment of this sequence by Remark 5.4.

Clearly there exists a unique choice for $f_0 \in \{u_0, v_0\}$ such that (u_0, u_1) (for the choice for f_1) satisfies either Conditions 1–3, or Conditions 1, 2', 3. If it also satisfies one of Conditions 5 and 5', then (u_0, u_1) is a certification or an un-certification sequence, respectively. If it satisfies neither Condition 5, nor Condition 5', then there exists u_2 such that (u_0, u_1, u_2) satisfies either Conditions 1–4, or Conditions 1, 2', 3, 4. Therefore adding u_i one by one, we eventually obtain either a certification, or an un-certification sequence. Moreover, by Conditions 2 and 4 the choice of $f_i \in \{u_i, v_i\}$ for every added occurrence, $i > 1$, is unique. □

Remark 5.7. Lemma 5.6 implies that $f_1 \in \{u_1, v_1\}$ cannot be both certified and "un-certified". So if there exists an un-certification sequence for f_1, then f_1 is not certified to the right of u.

The proof of Lemma 5.6 shows that certification sequences are equivariant under turns in the following sense:

Corollary 5.8. *Let $A = L^{\ulcorner} u_0 \dots u_t{}^{\urcorner} R \in \mathrm{SCan}_{\mu,r}$ where $(u_0 = u, \dots, u_t)$, $t \geqslant 1$, is the (un-) certification sequence in A for $f_1 \in \{u_1, v_1\}$. Let w be a maximal occurrence in*

u_tR of Λ_r-measure $\geqslant 5\tau+3$ with complement y and assume $C \in \mathrm{SCan}_{\mu,r}$ is the result of turning w in C. Then the following holds:

(i) If $\Lambda_r(u_t, C) < 5\tau+3$, then $(u_0 = u, \dots, u_{t-1})$, is the certification sequence for f_1 in C.

(ii) If $\Lambda_r(u_t, C) \geqslant 5\tau+3$ and C does not contain an occurrence between u_t and y of Λ_r-measure $\geqslant 5\tau+3$, then $(u_0 = u, \dots, u_t)$ is the (un-)certification sequence for f_1 in C.

(iii) If $\Lambda_r(u_t, C) \geqslant 5\tau+3$ and C contains an occurrence z between u_t and y of Λ_r-measure $\geqslant 5\tau+3$, then $(u_0, \dots, u_t)$ or $(u_0 = u, \dots, u_t, z)$ are the (un-)certification sequence for f_1 in C.

Furthermore, f_1 is certified in C to the right of u if and only if this holds in A.

(iv) If $t = 0$ and C contains a maximal occurrence z essentially non-isolated from u with $\Lambda_r(z, C) \geqslant 5\tau+3$, then the complement of z is not certified to the right of u in C by Remark 5.5(5).

Proof. (i) and (ii) follow directly from the definition and the proof of Lemma 5.6.

For part (iii) assume that there exists a maximal occurrence z of rank r in C with $\Lambda_r(z, C) \geqslant 5\tau+3$ with $\Lambda_r(z, A) < 5\tau+3$. Then by Lemma 4.44 $\Lambda_r(\widetilde{z}) < 7\tau+4$. If $(u_0, \dots, u_t)$ in C still satisfies Condition 5 or 5', then $(u_0, \dots, u_t)$ is a (un-)certification sequence in C. So assume that $(u_0, \dots, u_t)$ in C violates the corresponding condition (5 for a certification sequence and 5' for an un-certification sequence). This can happen only because of z. Consider the sequence $(u_0, \dots, u_t, z)$ and the choice of $f_{t+1} \in \{z, y\}$, where y is the complement of z, such that this sequence satisfies Condition 2 or 2'. Since $(u_0, \dots, u_t)$ does not satisfy Condition 5 or 5', we see that $(u_0, \dots, u_t, z)$ satisfies Condition 4. If $f_{t+1} = z$, then both $(u_0, \dots, u_t)$ in A and $(u_0, \dots, u_t, \widetilde{z})$ in C satisfy Conditions 2 and 5, so they both are certification sequences.

Assume that $f_{t+1} = y$. Since $(u_0, \dots, u_t)$ in A satisfies Conditions 2' and 5', the occurrence that corresponds to $f_{t+1} = y$ after turning u_i such that $f_i = v_i$, $0 \leqslant i \leqslant t$, has Λ_r-measure $> n - \Lambda_r(z) - 2\tau - \varepsilon > n - (11\tau+5) \geqslant \lambda_1$. Hence $(u_0, \dots, u_t, z)$ in C satisfies Conditions 2'. To see that it satisfies also Condition 5', let A' be the result of turning z. Then $\Lambda_r(y, A') > n - (5\tau+3) - 2\tau \geqslant \lambda_1 + 2\varepsilon$. Let g be the complement of w. Then g is essentially non-isolated in C from z and $\Lambda_r(g) \geqslant 5\tau+3$. So, we check Condition 5' for $(u_0, \dots, u_t, z)$ in C using g. Thus the occurrence that corresponds to $f_{t+1} = y$ after turning u_i such that $f_i = v_i$, $0 \leqslant i \leqslant t$, and the turn of the occurrence (corresponding to) g has Λ_r-measure $> \Lambda_r(y, A') - 2\varepsilon > \lambda_1$. Therefore, $(u_0, \dots, u_t, z)$ in A satisfies Condition 5' as required.

For part (iii) we see that $(u_0, \dots, u_t)$ satisfies Conditions (1) – (4) of Definition 5.3. If it also satisfies (5) or (5'), then $(u_0, \dots, u_t)$ is the (un-)certification sequence for f_1 in W. Now suppose it does not satisfy either (5) or (5') and $(u_0, \dots, u_t)$ is a certification sequence for f_1 in A. Then after turning z in C the occurrence corresponding to f_t has Λ_r-measure $> \lambda_{m(t)}$. Since $\Lambda_r(f_t, C) \leqslant \lambda_{m(t)}$ and $\Lambda_r(z, C) < 7\tau+4$, it follows that $(u_0, \dots, u_t, z)$ is the certification sequence for f_1 in C.

On the other hand, if $(u_0, \dots, u_t)$ is an un-certification sequence for f_1 in A, let x be the complement of z and D be the result of turning z in C. Then $\Lambda_r(f_t, D) < \lambda_{m(t)}$. Since $\Lambda_r(z, C) < 7\tau+4$, it follows that $\Lambda_r(x, D) > n - 9\tau - 4$. Hence by $(u_0, \dots, u_t, z)$ is the certification sequence for f_1 in C. ∎

Corollary 5.9. *Let $A \in \mathrm{SCan}_{\mu,r}$, and let $(u_0, \dots, u_t)$ be an (un-)certification sequence to the right of u_0 in A with witness $W = L^{\ulcorner} f_0 \dots f_t{}^{\urcorner} R$. Let $g_i \in \{u_i, v_i\}, i = 0, \dots, t$, let C be the result of turning all occurrences u_i in A with $g_i = v_i$ and suppose $C \in \mathrm{SCan}_{\mu,r}$ and $\Lambda_r(g_0, C) \geqslant 5\tau + 3$. Then the following hold:*

(i) $\Lambda_r(g_i, C) \geqslant 5\tau + 3$ for $0 \leqslant i \leqslant t - 1$.

(ii) If $\Lambda_r(g_t, C) < 5\tau + 3$ and $t \geqslant 2$, then the sequence (corresponding to) $(g_0, \dots, g_{t-1})$ in C is an (un-)certification sequence to the right of g_0 for the side that corresponds to f_1.

(iii) Assume that $t = 1$ and $\Lambda_r(g_1, C) < 5\tau + 3$. If $g_1 = u_1$, then v_1 is not certified in A to the right of u. If $g_1 = v_1$, then u_1 is not certified in A to the right of u.

(iv) Assume that $\Lambda_r(g_t, C) \geqslant 5\tau + 3$. Then either the sequence (corresponding to) $(g_0, \dots, g_t)$ in C, or $(g_0, \dots, g_t, \widetilde{z})$ is a (un-)certification sequence to the right of g_0 for the side that corresponds to f_1, where $\widetilde{z}$ corresponds to some maximal occurrence z in A with $\Lambda_r(z) < 5\tau + 3$.

Furthermore, the choices for g_i in (un-)certification sequences in (ii) and (iv) agree with the choices f_i in the initial sequence $(u_0, \dots, u_t)$.

Proof. If $t = 1$, then (i) immediately holds. So let $t \geqslant 2$ and assume towards a contradiction that $\Lambda_r(g_i, C) < 5\tau + 3$ for some $1 \leqslant i \leqslant t - 1$. If $g_i = f_i$, then $\Lambda_r(f_i, W) < \Lambda_r(g_i, C) + 2\varepsilon < 9\tau + 5 < \lambda_2 - \varepsilon$, contradicting Remark 5.5 (iii). If $g_i \neq f_i$ (i.e. $g_i \in \{u_i, v_i\} \setminus \{f_i\}$), then $\Lambda_r(f_i, W) > n - \Lambda_r(g_i, C) - 2\tau - 2\varepsilon > \lambda_1$, which contradicts to Condition 2 or 2'.

(ii)–(iv) are proved as in Corollary 5.8. □

Lemma 5.10. *Let $A \in \mathrm{SCan}_{\mu,r}$, let u be a maximal occurrence of Λ_r-measure $\geqslant 5\tau + 3$ and let $(u = u_0, \dots, u_t), t \geqslant 1$, be an (un-)certification sequence for $f_1 \in \{u_1, v_1\}$ in A to the right of u. Let $\kappa = \Lambda_r(f_t, W)$. Write (in the notation of Convention 4.38) $A = L^{\ulcorner} u_0 \dots u_t{}^{\urcorner} MR$ and let $B = L' v_t' EM'R$ be the result of turning u_t in A, where M' is a suffix of M and E is τ-free of rank r, such that*

- *if $\kappa \leqslant \lambda_{m(t)} - \varepsilon$ or $\kappa \geqslant \lambda_{m(t)} + \varepsilon$, then M or EM' contain an occurrence $a^\tau M_0 b^\tau$, $a^n, b^n \in \mathrm{Rel}_r$;*
- *if $\lambda_{m(t)} - \varepsilon < \kappa < \lambda_{m(t)} + \varepsilon$, then M or EM' contain a strong separation word (see Definition 4.40).*

Then for any $A' = L^{\ulcorner} u_0 \dots u_t{}^{\urcorner} MR' \in \mathrm{SCan}_{\mu,r}$ the corresponding sequence $(u_0, \dots, u_t), t \geqslant 1$, in A' is still an (un-)certification sequence for the corresponding f_1 in A' (with the same choices for all $f_i \in \{u_i, v_i\}$).

Proof. This follows directly from the definition, Corollaries 4.30 and 4.31 and Definition 4.40. □

Definition 5.11. *Let $A = LuMR \in \mathrm{SCan}_{\mu,r}$, where u is a maximal occurrence of rank r with $\frac{n}{2} - 5\tau - 2 < \Lambda_r(u) < \frac{n}{2} + 5\tau + 2$. We say that uM is a* right context *for u in A if any (un-)certification sequence on the right of u in A is properly contained in uM and for any word $A' = LuMR' \in \mathrm{SCan}_{\mu,r}$ the sequence of corresponding occurrences is an (un-)certification sequence to the right of u in A' for the same f_1.*

Note that such M might not exist and in this case the right context is not defined.

Let $A = LuR \in \mathrm{SCan}_{\mu.r}$ for a maximal occurrence u of rank r. When we decide for an occurrence u in a word $A \in \mathrm{SCan}_{\mu,r}$ whether to turn it (and thus replace it essentially by the complement) we need to take into account the effect the turn has on the neighbouring occurrences because we want the canonical form to be invariant under certain turns. We therefore make this decision after also considering the neighbours of u and their possible turns. We first note that an occurrence with sufficiently small Λ_r-measure will always be shorter than its complement no matter which neighbours we turn, and, conversely, if the Λ_r-measure of an occurrence is sufficiently large, then no matter which neighbours we turn, it will always be the longer than its complement:

Lemma 5.12. *Let $A = L^\ulcorner y_1, u, y_2{}^\urcorner R \in \mathrm{SCan}_{\mu,r}$, u, y_1, y_2 be maximal occurrences of rank r with complements v, z_1, z_2, respectively, and assume that $\Lambda_r(y_i) \geqslant 3\tau + 2, i = 1,2$. Let B be the result of turning u in A and for choices $f_i, g_i \in \{y_i, z_i\}, i = 1, 2$, let $A' = L'^\ulcorner f_1, u, f_2{}^\urcorner R'$, $B' = L''^\ulcorner g_1, v, g_2{}^\urcorner R''$,*

1. *If $5\tau + 3 \leqslant \Lambda_r(u) \leqslant \frac{n}{2} - 5\tau - 2$, then $\Lambda_r(u, A') < \Lambda_r(v, B')$.*
2. *If $\frac{n}{2} + 5\tau + 2 \leqslant \Lambda_r(u) \leqslant \mu$, then $\Lambda_r(u, A') > \Lambda_r(v, B')$.*

Proof. 1. If $\Lambda_r(u) \leqslant \frac{n}{2} - 2\varepsilon - \tau$, then $\Lambda_r(v) = n - \Lambda_r(u) \geqslant \frac{n}{2} + 2\varepsilon + \tau$. So after possibly turning neighbours of u by Lemma 4.44 the corresponding occurrence u satisfies $\Lambda_r(u, A') < \Lambda_r(u) + 2\varepsilon \leqslant \frac{n}{2} - \tau$ whereas $\Lambda_r(v, B') > \Lambda_r(v, B) - 2\varepsilon > \Lambda_r(v) - 2\tau - 2\varepsilon \geqslant \frac{n}{2} - \tau$.

2. If $\Lambda_r(u) \geqslant \frac{n}{2} + 2\varepsilon + \tau$, then $\Lambda_r(v, B) \leqslant \frac{n}{2} - 2\varepsilon - \tau$. By Lemma 4.44 we have that $\Lambda_r(u, A') > \Lambda_r(u, A) - 2\varepsilon \geqslant \frac{n}{2} + \tau$ and $\Lambda_r(v, B') < \Lambda_r(v, B) + 2\varepsilon < \Lambda_r(v) + 2\tau + 2\varepsilon \leqslant \frac{n}{2} + \tau$. □

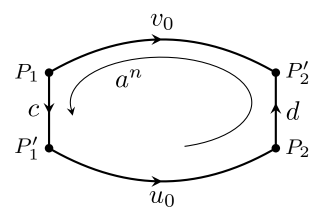
So in these cases, no matter which neighbours we turn, the occurrence corresponding to u remains shorter (or longer, respectively) than the one corresponding to v. Thus, according to our definition we never turn an occurrence u of Λ_r-measure $\leqslant \frac{n}{2} - 5\tau - 2$ and we always turn an occurrence u of Λ_r-measure $\geqslant \frac{n}{2} + 5\tau + 2$. Therefore we can now restrict our attention to occurrences u with $\frac{n}{2} - 5\tau - 2 < \Lambda_r(u) < \frac{n}{2} + 5\tau + 2$. Note that in this situation all occurrences to the left of u are strictly isolated from all occurrences to the right of u. Therefore we can consider the left and right side separately.

We now define $\widetilde{u}$ to be the shortest occurrence among the occurrences corresponding to u when we turn neighbours of u of Λ_r-measure $\geqslant 5\tau + 3$ according to the certified sides in $\mathcal{Y}(u)$. If $\mathcal{Y}(u) = \varnothing$, then $\widetilde{u} = u$. Also we define $\widetilde{v}$ to be the shortest occurrence among the occurrences corresponding to $\widehat{v}$ using the same set $\mathcal{Y}(u)$.

If $|\widetilde{u}| \neq |\widetilde{v}|$, then we choose the shorter occurrence as the winner side. Since the canonical form is equivariant with respect to inversion, we need to make sure that the choice for $A = LuR$ is consistent with the choice for A^{-1}. Therefore we use the following more intricate procedure to determine the winner side in case $|\widetilde{u}| = |\widetilde{v}|$:

Consider $a^n \in \mathrm{Rel}_r$ as a cyclic word. Let I_u be the starting point of u, F_u be the end point of u. Since u contains at least one period of a, I_u and F_u are fixed up to a cyclic shift by some number of periods of a. Following the construction of $\widetilde{u}$ and $\widetilde{v}$ from u, we mark the initial and the final points of $\widetilde{u}$ and $\widetilde{v}$ in a^n with respect to the points I_u and F_u. Denote them by $I_{\widetilde{u}}$, $F_{\widetilde{u}}$, $I_{\widetilde{v}}$ and $F_{\widetilde{v}}$ respectively. Notice that $\widetilde{u}$ and $\widetilde{v}$ may or may not have overlaps in the cyclic word a^n and so the overlaps or gaps between $\widetilde{u}$ and $\widetilde{v}$ have measure $< 3\tau + 1$.

Consider the subword of a^n with endpoints $[I_{\widetilde{u}}, I_{\widetilde{v}}]$ of Λ_r-measure $< 3\tau + 1$ and let c denote the middle letter if the length of this is odd, otherwise let c mark the mid point between the two middle letters. Similarly, consider the subword of a^n with endpoints $[F_{\widetilde{u}}, F_{\widetilde{v}}]$ and define d in the same way. We denote the segment corresponding to c and d, respectively, by the (possibly empty) intervals $[P_1, P_1']$ and $[P_2, P_2']$ (see diagram below). Let u_0 be the subword of a^n starting at P_1' and ending at P_2, and let v_0 be the subword of a^{-n} starting at P_1 and ending at P_2'. So, we have a partition of a^n into four segments: u_0, d, v_0^{-1}, c, where $|c|, |d| \leqslant 1$.



Now we are ready to specify the conditions for turning u in A in order to construct $\mathrm{can}_r(A)$.

Remark 5.13. Note that for $a^n \in \mathrm{Rel}_r$ we have $a^n \neq Z^2$ for all $Z \in \mathrm{Cycl}_0$: if $a^n = Z^2$, then $Z \in Cen(a^n)$. By definition of Rel_r we have $Cen(a^n) = \langle a \rangle$, so $Z = a^k$ for some $k \in \mathbb{Z}$. Then $Z^2 = a^{2k} \neq a^n$ since n is odd.

The following is well-known:

Lemma 5.14. *Let $b \neq 1$ be cyclically reduced. If $b = xy$ where $|x| \geqslant \frac{1}{2}|b|$, then no cyclic shift of b contains x^{-1} as a subword.*

Proof. Suppose otherwise. Then either there exists an occurrence of x^{-1} in b, or $b = x_1^{-1} z x_2^{-1}$, where $x = x_1 x_2$. Since $|x| \geqslant \frac{b}{2}$ and b is a reduced word, in either case we obtain that x and x^{-1} have an overlap which is impossible. $\square$

Recall that $a^n \in \mathrm{Rel}_r$ as a cyclic word is separated into four parts u_0, d, v_0^{-1}, c, where c and d are either empty, or a single letter (independently from each other).

Lemma 5.15. *For $a^n \in \mathrm{Rel}_r, r \geqslant 2$, the sets of words*

$$\{u_0,\ u_0^{-1},\ cu_0,\ u_0 d,\ u_0^{-1} c^{-1},\ d^{-1} u_0^{-1}\},$$
$$\{v_0,\ v_0^{-1},\ c^{-1} v_0,\ v_0 d^{-1},\ v_0^{-1} c,\ d v_0^{-1}\}.$$

are not equal to each other.

Proof. Since Rel_r is invariant under cyclic shifts by IH 3, we may assume $a^n = u_0 d v_0^{-1} c$. Now assume to the contrary that

$$\{u_0,\ u_0^{-1},\ cu_0,\ u_0 d,\ u_0^{-1} c^{-1},\ d^{-1} u_0^{-1}\} = \{v_0,\ v_0^{-1},\ c^{-1} v_0,\ v_0 d^{-1},\ v_0^{-1} c,\ d v_0^{-1}\}.$$

The words u_0 and u_0^{-1} are the shortest in the left-hand set, and similarly v_0 and v_0^{-1} are shortest on the right-hand side. Hence, either $u_0 = v_0$, or $u_0 = v_0^{-1}$. Since v_0 is a subword of a^{-n} and $\Lambda_r(u_0), \Lambda_r(v_0) \geqslant 1$, by construction, we have $u_0 \neq v_0$ by Lemma 5.14 and so $u_0 = v_0^{-1}$. If both c and d are empty, then $a^n = u_0 v_0^{-1} = u_0^2$, contradicting Remark 5.13. So, we may assume that at least one of c and d is not empty. By symmetry assume that $d \neq 1$.

For the sets to be equal, we must have $u_0 d \in \{c^{-1} v_0, v_0 d^{-1}, v_0^{-1} c, d v_0^{-1}\}$. Since $u_0 = v_0^{-1}$, we have $u_0 d \notin \{c^{-1} v_0, v_0 d^{-1}\}$ by Lemma 5.14. If $u_0 d = v_0^{-1} c = u_0 c$, then $c = d$ and hence $a^n = (u_0 d)(v_0^{-1} c) = (u_0 d)^2$, contradicting Lemma 5.13. And finally if $u_0 d = d v_0^{-1} = d u_0$, then $d \in Cen(u_0)$ and hence $u_0 \in \langle\langle d \rangle\rangle$. Since $\Lambda_r(u_0) \geqslant 1$, we also have $a \in \langle\langle d \rangle\rangle$. However, since $r \geqslant 2$ we have $|a| > 1$ by definition. This contradiction proves the lemma. $\square$

Definition 5.16 (deglex order)**.** *Fix an ordering on the set of letters. For reduced words C_1, C_2 we say that $C_1 <_{\text{deglex}} C_2$ if either $|C_1| < |C_2|$, or $|C_1| = |C_2|$ and C_1 is lexicographically smaller than C_2 with respect to the order that we fixed on the letters. For finite sets of words $\mathcal{U} \neq \mathcal{V}$, we put $\mathcal{U} <_{\text{deglex}} \mathcal{V}$ if the minimal element of $\mathcal{U} \cup \mathcal{V} \setminus (\mathcal{U} \cap \mathcal{V})$ belongs to $\mathcal{U}$.*

Now let $\mathcal{U} = \{u_0,\ u_0^{-1},\ cu_0,\ u_0d,\ u_0^{-1}c^{-1},\ d^{-1}u_0^{-1}\}$ and $\mathcal{V} = \{v_0,\ v_0^{-1},\ c^{-1}v_0,\ v_0d^{-1},\ v_0^{-1}c,\ dv_0^{-1}\}$. By Lemma 5.15 we have $\mathcal{U} \neq \mathcal{V}$. If $\mathcal{U} <_{\text{deglex}} \mathcal{V}$, we do not turn u and call u the winncer side, otherwise we turn u and v is called the winner side.

Thus v is the winner side if and only if either $|\widetilde{u}| > |\widetilde{v}|$ (as defined above) or, in case $|\widetilde{u}| = |\widetilde{v}|$, if $\mathcal{V}$ is smaller than $\mathcal{U}$ with respect to the $<_{\text{deglex}}$ order.

Lemma 5.17. *Let $A = LuR \in \mathrm{SCan}_{\mu,r}$ for a maximal occurrence u of rank r and suppose that q is a maximal occurrence of Λ_r-measure $\geqslant 5\tau + 3$ essentially non-isolated from u to the right of u. Then the winner side for q is certified to the right of u.*

Proof. Indeed, if the winner side for q is not certified, then by turning occurrences not essentially isolated from q of Λ_r-measure $\geqslant 5\tau+3$, the maximal occurrence corresponding to the winner side can be made $> \lambda_2 = \frac{n}{2} + 5\tau + 2$ contradicting Lemma 5.12. ☐

Definition 5.18 (canonical form for λ-semicanonical words, $\lambda = \frac{n}{2} + 3\tau + 1$)**.** *For $A \in \mathrm{SCan}_{\lambda,r}$, consider the set of all maximal occurrences of Λ_r-measure $\geqslant \frac{n}{2} - 5\tau - 2$ and turn each one of them according to the decision process described above. The result is denoted by $\mathrm{can}_r(A)$.*

By Proposition 4.54, the result $\mathrm{can}_r(A)$ does not depend on the order in which we perform the necessary turns.

Lemma 5.19. *For $A \in \mathrm{SCan}_{\mu,r}$ we have $\mathrm{can}_r(A) \in \mathrm{SCan}_{\lambda,r}$.*

Proof. Let $A \in \mathrm{SCan}_{\mu,r}$ and u some maximal occurrence of rank r in A, let $f \in \{u, v\}$ be the winner side for u and let q_1, q_2 be maximal occurrences in A of Λ_r-measure $\geqslant 5\tau + 2$ not essentially isolated from u on the left and right, respectively. (If no such q exists, the statement follows from Corollary 4.36 and for only one such q, the proof is essentially the same as here.)

Assume towards a contradiction that $\Lambda_r(f, \mathrm{can}_r(A)) > \lambda = \frac{n}{2} + 3\tau + 1$. Then the shortest occurrence that corresponds to the side f has Λ_r-measure $> \frac{n}{2} + 3\tau + 1 - 2\varepsilon = \frac{n}{2} - \tau - 1$. Since the winner sides for q_1, q_2 are certified by Lemma 5.17, the shortest occurrence that corresponds to the complement of f has Λ_r-measure $< n - \Lambda_r(\widetilde{f}) + 2\tau = \frac{n}{2} - \tau - 1$ contradicting our assumption that f is the winner side. ☐

Proposition 5.20. *Let $A, C \in \mathrm{SCan}_{\mu,r}$ and let $A \mapsto C$ be the turn of a maximal occurrence f in A of rank r and Λ_r-measure $\geqslant \tau + 1$. Then $\mathrm{can}_r(A) = \mathrm{can}_r(C)$.*

Proof. Since $A, C \in \mathrm{SCan}_{\mu,r}$, we have $5\tau + 3 \leqslant \Lambda_r(f) \leqslant \mu$ by Lemma 4.69. Hence the turn $A \mapsto C$ is of Type 2 with inverse turn $C \mapsto A$ of the maximal occurrence $\widehat{g}$ where g is the complement of f. By symmetry we also have $5\tau + 3 \leqslant \Lambda_r(\widehat{g}) \leqslant \mu$. Let u be a maximal occurrence in A with $\Lambda_r(u) \geqslant 5\tau + 3$. Assume that at least one of $\Lambda_r(u, A), \Lambda_r(u, C) \leqslant \frac{n}{2} - 5\tau - 2$ or at least one of $\Lambda_r(u, A), \Lambda_r(u, C) \geqslant \frac{n}{2} + 5\tau + 2$. Then as in Lemma 5.12 we obtain that both in A and C the winner side is u or v, respectively. So we now assume that this does not happen.

Assume that there exists a maximal occurrence q to the right of u essentially non-isolated from u with $\Lambda_r(q) \geqslant 5\tau + 3$ and the complement z. We need to show that in

A and in C the certified sides of q are the same. This is clear if f is to the left of u, so we assume that either $f = u$, or f is to the right of u. Then the result follows from Corollaries 5.8 and 5.9. If such occurrence q does not exist in A but exists in C, then we consider the inverse turn $C \mapsto A$ and the result follows. $\square$

6. An auxilliary group structure

In order to prove IH 8 we will need to show that for $A, B \in \mathrm{Can}_{-1}$ with $A \equiv B \mod \langle\langle \mathrm{Rel}_0 \cup \ldots \cup \mathrm{Rel}_r \rangle\rangle$ we have $\mathrm{can}_r(A) = \mathrm{can}_r(B)$. We begin with showing this for $A, B \in \mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$ by introducing a group structure on equivalence classes on $\mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$ where (as always) $\lambda = \frac{n}{2} + 3\tau + 1$. We will show that the equivalence relation coincides with equality in $\mathrm{F}/\langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r \rangle\rangle$. Using this equivalence relation, we will then define the canonical form of rank r of arbitrary words in Section 6.1.

Definition 6.1. *For $A_1, A_2 \in \mathrm{SCan}_{\kappa+\varepsilon,r}$ we define $A_1 \sim_{\kappa,r} A_2$ if and only if there exists a (possibly empty) sequence of turns of rank r of Λ_r-measure $\geqslant \tau$*

$$A_1 = C_1 \longmapsto C_2 \longmapsto \ldots \longmapsto C_t = A_2$$

with $C_i \in \mathrm{SCan}_{\kappa+\varepsilon,r}$ for $2 \leqslant i \leqslant t-1$. The same $\sim_{\kappa,r}$ is defined also for $A_1, A_2 \in \mathrm{SCan}_{\kappa,r}$.

In this section we will use the relation $\sim_{\kappa,r}$ with $\kappa = \lambda + \tau = \frac{n}{2} + 4\tau + 1$.

Remark 6.2. Since all C_i in this sequence belong to $\mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$, by Lemma 4.69 any occurrence u which is turned in the sequence satisfies $\frac{n}{2} - 8\tau - 2 < \Lambda_r(u) \leqslant \lambda + \tau + \varepsilon$. Since $n - (\lambda + \tau + \varepsilon) = \frac{n}{2} - 6\tau - 2 > 12\tau$, such a turn is of Type 2 by Lemma 4.25 (ii) with remainder v' of Λ_r-measure $> 10\tau$ and hence has an inverse turn of Λ_r-measure $> 10\tau$ by Lemma 4.33. Thus, if

$$A_1 = C_1 \mapsto C_2 \mapsto \ldots \mapsto C_t = A_2$$

is a sequence witnessing $A_1 \sim_{\lambda+\tau,r} A_2$, then $A_2 \sim_{\lambda+\tau,r} A_1$ is witnessed by its inverse sequence

$$A_2 = C_t \mapsto C_{t-1} \mapsto \ldots \mapsto C_1 = A_1.$$

By this observation we obtain:

Corollary 6.3. *The relation $\sim_{\lambda+\tau,r}$ is an equivalence relation on $\mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$ and on $\mathrm{SCan}_{\lambda+\tau,r}$ with finite equivalence classes. Moreover every equivalence class in $\mathrm{SCan}_{\lambda+\tau r}$ has a representative in $\mathrm{SCan}_{\lambda,r}$.*

Proof. By Lemma 4.69 every turned occurrence in the sequence witnessing $A_1 \sim_{\lambda+\tau,r} A_2$ has Λ_r-measure $> \frac{n}{2} - 8\tau - 2 \geqslant 9\tau + 5$. Hence Corollary 4.74 implies that there exists a stable sequence of occurrences in A_1 such that their turns give A_2. Therefore the members of an equivalence class of A_1 correspond to choices of sides in maximal occurrences u in A such that $\Lambda_r(u) \geqslant \tau + 1$, and there are only finitely many of these.

Lemma 4.71 implies that every equivalence class in $\mathrm{SCan}_{\lambda+\tau,r}$ has a representative in $\mathrm{SCan}_{\lambda,r}$. $\square$

The equivalence class of a word $A \in \mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$ is denoted by $[A]$. Recall that in Section 5.1 we defined can_r for μ-semicanonical words, where $\mu = n - (8\tau + 3)$. Since $\lambda + \tau + \varepsilon = \frac{n}{2} + 6\tau + 2 \leqslant \mu$, Proposition 5.20 now implies:

Corollary 6.4. *If $A_1, A_2 \in \mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$ are $\sim_{\lambda+\tau,r}$-equivalent, then $\mathrm{can}_r(A_1) = \mathrm{can}_r(A_2)$.*

We will now define an (auxilliary) group structure on $\mathrm{SCan}_{\lambda+\tau,r}/\sim_{\lambda+\tau,r}$ and establish that for $A_1, A_2 \in \mathrm{SCan}_{\lambda+\tau,r}$ we have $A_1 \sim_{\lambda+\tau,r} A_2$ if and only if A_1 and A_2 represent the same element in $\mathrm{F}/\langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r\rangle\rangle$. Since we were not able to show directly that different $\lambda+\tau$-semicanonical forms of a given word are $\sim_{\lambda+\tau,r}$-equivalent, we need this group structure to show that we obtain a well-defined canonical form of rank r of an arbitrary word using an arbitrary $\lambda+\tau$-semicanonical form for it.

We first define the multiplication $\times_{\lambda+\tau,r}$ on $\mathrm{SCan}_{\lambda+\tau,r}$. For technical reasons we define it on larger set $\mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$.

Definition 6.5. *For $A, C \in \mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$, let $Z' = \mathrm{prod}_{\lambda+6\tau+6,r}(A\cdot C) \in \mathrm{SCan}_{\lambda+6\tau+2,r}$ and let Z'' be a $\lambda+\tau$-semicanonical form of rank r of Z' obtained by turns of Λ_r-measure $\geqslant \frac{n}{2}+2\tau$ (such sequence of turns exists by Lemma 4.71). Define $A \times_{\lambda+\tau,r} C = [Z''] \in \mathrm{SCan}_{\lambda+\tau,r}/\sim_{\lambda+\tau,r}$.*

Remark 6.6. In Definition 6.5 we define the multiplication $\times_{\lambda+\tau,r}$ in two steps: for $A, C \in \mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$ we find $Z = \mathrm{can}_{r-1}(A\cdot C)$, compute a *specific* $(\lambda+6\tau+2)$-semicanonical form Z' of Z and then find a $\lambda+\tau$-semicanonical form Z'' of Z'.

Note that every word in $\mathrm{SCan}_{\lambda+6\tau+2,r}$ has a $\lambda+\tau$-semicanonical form by Corollary 4.72. Since $\lambda+6\tau+2 = \frac{n}{2}+9\tau+3 \leqslant \mu = n-8\tau-3$, Lemma 4.75 implies that the resulting equivalence class does not depend on the particular $\lambda+\tau$-semicanonical form of Z' as long as the descent is obtained from turns of Λ_r-measure $\geqslant 9\tau+5$.

We will just write $\times$ for $\times_{\lambda+\tau,r}$ if the parameters are clear from the context. We emphasis that $A \times_{\lambda+\tau,r} B$ is not a single word, but an equivalence class in $\mathrm{SCan}_{\lambda+\tau,r}/\sim_{\lambda+\tau,r}$.

Remark 6.7. It follows directly from the definition that for $A, C \in \mathrm{SCan}_{\lambda+\tau,r}$ (rather than $\mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$) we have

(1) $A \times_{\lambda+\tau,r} 1 = [A]$;
(2) $A \times_{\lambda+\tau,r} A^{-1} = [1]$; and
(3) $(A \times_{\lambda+\tau,r} C)^{-1} = [C^{-1} \times_{\lambda+\tau,r} A^{-1}]$.

We next show that the multiplication factors through $\sim_{\lambda+\tau,r}$:

Proposition 6.8. *For $[A], [C] \in \mathrm{SCan}_{\lambda+\tau,r}/\sim_{\lambda+\tau,r}$ the multiplication $[A]\times[C] = [A\times C]$ is well-defined.*

The crucial step for the proof is contained in the following lemma:

Lemma 6.9. *Let $A, B \in \mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$. If $A \mapsto B$ is a turn of a maximal occurrence u in A with $\Lambda_r(u) \geqslant \tau$, then for any $C \in \mathrm{SCan}_{\lambda+\tau,r}$ we have $A \times C = B \times C$.*

Proof. By Remark 6.2 the inverse turn $B \mapsto A$ is defined. Therefore by Corollary 4.36 we can assume that $\Lambda(u) > \frac{n}{2} - \tau$. Hence by Lemma 4.76 we may assume that the turn of u commutes with the product with C. That is, we can assume that $\mathrm{can}_{r-1}(A\cdot C) \mapsto \mathrm{can}_{r-1}(B\cdot C)$ is a turn of measure $\geqslant \tau+1$ of maximal occurrence $\widetilde{u}$.

Now consider the sequences of seam turns

$$X_0 = \mathrm{can}_{r-1}(A\cdot C) \mapsto X_1 \mapsto \ldots \mapsto X_m = \mathrm{prod}_{\lambda+6\tau+2,r}(A\cdot C) \quad \text{and}$$
$$Y_0 = \mathrm{can}_{r-1}(B\cdot C) \mapsto Y_1 \mapsto \ldots \mapsto Y_k = \mathrm{prod}_{\lambda+6\tau+2,r}(B\cdot C).$$

First assume that $\widetilde{u}$ is isolated from the seam occurrences in X_i for all $0 \leqslant i \leqslant m$. Then $\Lambda_r(\widetilde{u}, X_m) = \Lambda_r(\widetilde{u}, X_0) > \frac{n}{2} - \tau$. Recall from Lemma 4.70 that all but possibly the last turns in each sequence have Λ_r-measure $> n-(3\tau+1)$ and the last turn has Λ_r-measure $> \frac{n}{2}+8\tau+3$. At each step the turn of $\widetilde{u}$ commutes with the seam turn by Lemma 4.53 and hence we see that $m = k$ and we obtain Y_k from X_k by turning the occurrence corresponding to $\widetilde{u}$ in X_k. Since $X_,Y_k \in \mathrm{SCan}_{\lambda+6\tau+2,r}$ and $\lambda + 6\tau + 2 = \frac{n}{2} + 9\tau + 3 \leqslant n - 8\tau - 3$, by Lemma 4.75 $A \times C = B \times C$.

Now we consider the general case. If $X_0 \mapsto Y_0$ is the seam turn (which is unique by definition), then $X_1 = \mathrm{can}_{r-1}(B \cdot C)$. Since all seam turns are uniquely defined, we obtain that $X_m = Y_k$. So as above the result follows from Lemma 4.75.

Assume that $X_0 \mapsto Y_0$ is not a seam turn. If u or v correspond to the seam occurrences in X_1 or Y_1, respectively, then by Lemma 4.53 we see that $X_2 = Y_1$ or $X_1 = Y_2$. Hence as above $X_m = Y_k$ and the result follows from Lemma 4.75.

If u does not correspond to the seam occurrence in X_1 and its complement v does not correspond to the seam occurrence in Y_1, then $X_1 \mapsto Y_1$ is the turn of the occurrence corresponding to u by Lemma 4.53. By Lemma 4.36 we can assume that it has Λ_r-measure $> \frac{n}{2} - \tau$ (otherwise we switch to the inverse turn) and denote the turning occurrence by u_1. Repeating the above argument for the turn of u_1 until we reach the end of one of the sequences, we obtain the required result.

Assume that the second sequence of seam turns is empty and the first sequence of seam turns is not empty. This means that there are no seam occurrences in Y_0, so the occurrence in Y_0 that corresponds the the seam occurrence in X_0 is not a seam occurrence in Y_0. In particular it has Λ_r-measure $< \lambda+6\tau+2$ and the corresponding seam occurrence in X_1 has Λ_r-measure $< \lambda + 6\tau + 2 + \varepsilon < n - (3\tau + 1)$. So the first sequence is of length one and $X_1 \in \mathrm{SCan}_{\lambda+6\tau+2,r}$. However, by Remark 6.6 we still can turn this occurrence in Y_0 and after that take its $\lambda + \tau$-semicanonical form. As above then $X_1 \mapsto Y_1$ is the turn of the occurrence corresponding to u. Therefore the result follows from Lemma 4.75. ☐

Proof. (of Proposition 6.8) We denote the operation $\times_{\lambda+\tau,r}$ for short by $\times$.

Let $A_1, A_2, C \in \mathrm{SCan}_{\lambda+\tau,r}$ with $A_1 \sim_{\lambda+\tau,r} A_2$. By Remark 6.7 (iii) it suffices to prove that $A_1 \times C = A_2 \times C$. Since $A_1 \sim_{\lambda+\tau,r} A_2$, there exists a sequence of turns of rank r

$$A_1 = X_1 \longmapsto X_2 \longmapsto \ldots \longmapsto X_t = A_2$$

such that all $X_i \in \mathrm{SCan}_{\lambda+\tau+\varepsilon,r}$ for $2 \leqslant i \leqslant t-1$ and all turns have Λ_r-measure $\geqslant \tau$. By Lemma 6.9 we have $X_i \times C = X_{i+1} \times C$ and hence $A_1 \times C = A_2 \times C$. ☐

Remark 6.10. Let $C \in \mathrm{SCan}_{\kappa,r}$.

(i) If C_1 is a prefix of C, then by IH 12 we have $\mathrm{can}_{r-1}(C_1) = C_1' D$ for a prefix C_1' of C_1 and a τ-free side of a canonical triangle D, so $\mathrm{can}_{r-1}(C_1) \in \mathrm{SCan}_{\kappa+\tau,r}$.

(ii) If $z = \mathrm{can}_{r-1}(z)$ is a single letter, then $\mathrm{can}_{r-1}(C \cdot z) = C' D z'$ for a prefix C' of C, a τ-free side D of a canonical triangle and $z' \in \{1, z\}$, so $\mathrm{can}_{r-1}(C \cdot z) \in \mathrm{SCan}_{\kappa+\tau}$.

Lemma 6.11. *Let $C = z_1 \cdots z_t$ be a reduced word such that for every initial segment $C_s = z_1 \cdots z_s$ we have $\mathrm{can}_{r-1}(C_s) \in \mathrm{SCan}_{\lambda+\tau,r}$. Then*

$$(\ldots(z_1 \times z_2) \times z_3) \times \ldots) \times z_t = [\mathrm{can}_{r-1}(C)]$$

where $\times$ is $\times_{\lambda+\tau,r}$ and z_i, $1 \leqslant i \leqslant s$ are single letters.

Note that by Remark 6.10 this applies in particular if $C = z_1 \cdots z_t \in \mathrm{SCan}_{\lambda,r}$.

Proof. We prove $(\ldots(z_1 \times z_2) \times z_3) \times \ldots) \times z_s = [\mathrm{can}_{r-1}(z_1 \cdots z_s)]$ for all $s \leqslant t$ by induction on s.

For $s = 1$ there is nothing to prove, so assume inductively

$$(\ldots(z_1 \times z_2) \times z_3) \times \ldots \times z_{s-1}) \times z_s = \mathrm{can}_{r-1}(z_1 \cdots z_{s-1}) \times z_s.$$

By Corollary 3.5 and Remark 6.10 we have

$$\mathrm{can}_{r-1}(\mathrm{can}_{r-1}(z_1 \cdots z_{s-1}) \cdot z_s) = \mathrm{can}_{r-1}(z_1 \dots z_s) \in \mathrm{SCan}_{\lambda+\tau,r}.$$

So $\mathrm{can}_{r-1}(z_1 \cdots z_{s-1}) \times z_s = [\mathrm{can}_{r-1}(z_1 \cdots z_s)]$ and for $s = t$ we obtain the required result. □

In order to establish that $\mathrm{SCan}_{\lambda+\tau,r}/\sim_{\lambda+\tau,r}$ is a group with respect to $\times_{\lambda+\tau,r}$, we now show that the multiplication is associative.

Lemma 6.12. *Let $A, B, C \in \mathrm{SCan}_{\lambda+\tau,r}$. Then $(A \times B) \times C = A \times (B \times C)$, where $\times$ is $\times_{\lambda+\tau,r}$.*

Proof. First we prove the statement for $C = z$ a single letter. By Corollary 6.3, we can assume that $B \in \mathrm{SCan}_{\lambda,r}$. Then by Remark 6.10, $B \cdot_{r-1} z \in \mathrm{SCan}_{\lambda+\tau,r}$, therefore $B \times C = [B \cdot_{r-1} z]$. By definition $A \times B$ is calculated using a sequence of turns

$$A \cdot_{r-1} B = X_0 \mapsto \ldots \mapsto X_m \in \mathrm{SCan}_{\lambda+\tau,r},$$

and $A \times (B \times z) = A \times (B \cdot_{r-1} z)$ is calculated using a sequence of turns

$$A \cdot_{r-1} (B \cdot_{r-1} z) = Y_0 \mapsto \ldots \mapsto Y_k \in \mathrm{SCan}_{\lambda+\tau,r}.$$

Recall that all turns in these sequences are of Λ_r-measure $> \frac{n}{2} + 2\tau$. When we multiply the first sequence by z, Lemma 4.76 implies that $X_i \cdot_{r-1} z \mapsto X_{i+1} \cdot_{r-1} z$ are turns of rank r and by Remark 6.10 they have Λ_r-measure $> \frac{n}{2} + \tau$.

If in at least one of the sequences there are no seam turns, then all maximal occurrences in X_0, Y_0 are of Λ_r-measure $< \lambda+\tau+3\tau+1+\tau$ by Remark 6.10, so $X_0, Y_0 \in \mathrm{SCan}_{\frac{n}{2}+8\tau+2,r}$. Hence the result follows from Lemma 4.75. Assume that the first sequence has seam turns and let $X_i \mapsto X_{i+1}$ be a seam turn and $X_i \cdot_{r-1} z = Y_i$. Then either the corresponding turn $X_i \cdot_{r-1} z \mapsto X_{i+1} \cdot_{r-1} z$ is the seam turn in Y_i so $X_{i+1} \cdot_{r-1} z = Y_{i+1}$, or the corresponding occurrence in Y_i has Λ_r-measure $< \lambda + \tau + 3\tau + 1$. In the latter case $X_i \in \mathrm{SCan}_{\frac{n}{2}+8\tau+2,r}$ by Remark 6.10 so the result again follows from Lemma 4.75. In the first case we repeat the argument until we exhaust all seam turns in one the sequences.

Now let $C = z_1 \cdots z_s \in \mathrm{SCan}_{\lambda,r}$, C_t be a prefix of C of length t and $Z_t = \mathrm{can}_{r-1}(C_t)$. Then Lemma 6.11 implies that $Z_{t-1} \times z_t = [Z_t]$. Therefore using induction on t we have

$$\begin{aligned}(A \times B) \times Z_t = (A \times B) \times (Z_{t-1} \times z_t) &= ((A \times B) \times Z_{t-1}) \times \ z_t = \\ &= (A \times (B \times Z_{t-1})) \times z_t = A \times ((B \times Z_{t-1}) \times z_t) = \\ &= A \times (B \times (Z_{t-1} \times z_t)) = A \times (B \times Z_t),\end{aligned}$$

and for $t = s$ we obtain the final result. □

By Proposition 6.8 and Lemma 6.12, we now have

Corollary 6.13. $(\mathrm{SCan}_{\lambda+\tau,r}/\sim_{\lambda+\tau,r}, \times_{\lambda+\tau,r})$ *is a group.*

The main statement of this section is

Proposition 6.14. *For $A_1, A_2 \in \mathrm{SCan}_{\lambda+\tau,r}$ the following are equivalent:*

(1) $A_1 \sim_{\lambda+\tau,r} A_2$;
(2) A_1 *and* A_2 *represent the same element in* $\mathrm{F}/\langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r \rangle\rangle$;
(3) $\mathrm{can}_r(A_1) = \mathrm{can}_r(A_2)$.

For the proof we define an epimorphism

$$\varphi : \mathrm{F} = \langle x_1, \ldots, x_m \rangle \longrightarrow \mathrm{SCan}_{\lambda+\tau,r} / \sim_{\lambda+\tau,r} : \quad x_i \mapsto [x_i].$$

By the universal property of free groups φ is well-defined. For a reduced word $C = z_1 \cdots z_t \in \mathrm{Can}_0$ we then have

$$\varphi(C) = \varphi(z_1 \cdots z_t) = \varphi(z_1) \times \ldots \times \varphi(z_t) = z_1 \times \ldots \times z_t.$$

Remark 6.15. By Lemma 6.11 we have $\varphi(C) = [C]$ for $C \in \mathrm{SCan}_{\lambda,r}$. Therefore φ is surjective by Corollary 6.3.

Remark 6.16. We will repeatedly make use of the observation that, by the very definition of a turn, if $A \mapsto B$ is a turn of rank $i \leqslant r$, then $A \equiv B \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_i \rangle\rangle$.

Lemma 6.17. *Let $R = a^n = z_1 \cdots z_t \in \mathrm{Rel}_i$, $i \leqslant r$. Write $R_s = z_1 \cdots z_s$ and let V_s denote its complement. Then for $s \leqslant t$ the following holds:*

1. *If $\Lambda_i(R_s) \geqslant 3\tau + 1$, then $\mathrm{can}_{i-1}(R_s) = DR'_sE \mapsto \mathrm{can}_{i-1}(V_s)$ is a turn of rank i of the maximal prolongation of R'_s.*
2. *If $i < r$, then*

$$\mathrm{can}_{r-1}(R_s) = \mathrm{can}_{r-1}(V_s) = \begin{cases} \mathrm{can}_{i-1}(R_s) & \text{if } \mathrm{can}_{i-1}(R_s) \in \mathrm{Can}_i, \\ \mathrm{can}_{i-1}(V_s) & \text{if } \mathrm{can}_{i-1}(V_s) \in \mathrm{Can}_i. \end{cases}$$

Furthermore $\mathrm{can}_{r-1}(R_s)$ is 6-semicanonical of rank r.

Proof. Part 1: By IH 12 we have $\mathrm{can}_{i-1}(R_s) = DR'_sE$ for an appropriate subword R'_s of R_s and τ-free of rank i words D, E. If $\Lambda_i(R_s) \geqslant 3\tau + 1$, then $\Lambda_i(R'_s) \geqslant \tau + 1$ and so the maximal prolongation of R'_s is a maximal occurrence. By Remark 3.4, IH 8 and Remark 4.22, we have

$$\mathrm{can}_{i-1}(D \cdot \widehat{a}^{-n} \cdot R'_sE) = \mathrm{can}_{i-1}(a^{-n} \cdot R_s) = \mathrm{can}_{i-1}(V_s)$$

where $\widehat{a}$ is the corresponding cyclic shift of a.

Part 2: If $\mathrm{can}_{i-1}(R_s) \notin \mathrm{Can}_i$, then only the maximal prolongation of R'_s can have Λ_i-measure $> 3\tau + 1$. Hence by Part 1, turning R'_s in DR'_sE yields $\mathrm{can}_i(R_s) = \mathrm{can}_{i-1}(V_s)$. The word $\mathrm{can}_{i-1}(V_s)$ is 6-free of rank $> i$ by IH 5 and hence $\mathrm{can}_j(R_s) = \mathrm{can}_i(R_s)$ for $j > i$ by IH 4. □

Corollary 6.18. *Let $R = a^n = z_1 \cdots z_t \in \mathrm{Rel}_i, i \leqslant r, R_s = z_1 \cdots z_s$. Then for $s \leqslant \lceil \frac{t}{2} \rceil$ we have*

$$z_1 \times \ldots \times z_s = [\mathrm{can}_{r-1}(R_s)].$$

Proof. Since $\mathrm{can}_{r-1}(R_s)$ is 6-semicanonical of rank r for $i < r$ by Lemma 6.17 and $\mathrm{can}_{r-1}(R_s)$ is $\frac{n}{2} + 2\tau + 1$-semicanonical of rank r for $i = r$, the result follows immediately from Lemma 6.11. □

For the proof of Proposition 6.14 we need

Lemma 6.19. $\langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r \rangle\rangle \leqslant \ker \varphi$.[2]

Proof. Let $R = a^n = z_1 \cdots z_t \in \mathrm{Rel}_i, i \leqslant r, R_s = z_1 \cdots z_s$ and $T_s = z_{s+1} \cdot \ldots \cdot z_t$. By Corollary 6.18 we have

$$Z_1 = z_1 \times \ldots \times z_{\lceil \frac{t}{2} \rceil} = [\mathrm{can}_{r-1}(R_{\lceil \frac{t}{2} \rceil})]$$

[2]In fact, one can show that $\ker \varphi = \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r \rangle\rangle$.

and similarly, by considering the appropriate cyclic shift,

$$Z_2 = z_{\lceil \frac{t}{2} \rceil+1} \times \ldots \times z_t = [\mathrm{can}_{r-1}(T_{\lceil \frac{t}{2} \rceil})].$$

Then, by Definition 6.5, $Z_1 \times Z_2 = z_1 \times \ldots \times z_t$ is computed from

$$\mathrm{can}_{r-1}(Z_1 \cdot Z_2) = \mathrm{can}_{r-1}(R_{\lceil \frac{t}{2} \rceil} \cdot_{r-1} T_{\lceil \frac{t}{2} \rceil}) = \mathrm{can}_{r-1}(R)$$

by first taking seam turns. However, if $R \in \mathrm{Rel}_i, i < r$, then $\mathrm{can}_{r-1}(R) = 1$ by Remark 3.4 and hence in this case no seam turn is necessary and $\varphi(R) = \mathrm{can}_{r-1}(R) = 1$. In case $R \in \mathrm{Rel}_r$, we have $\mathrm{can}_{r-1}(R) = DR'E$ for some τ-free of rank r D, E, so by Lemma 4.66 the maximal prolongation of R' is a seam occurrence and the result of the seam turn is equal to 1 by Lemma 6.17 part 1. □

Proof. (of Proposition 6.14) Let $A_1, A_2 \in \mathrm{SCan}_{\lambda+\tau,r}$.

1.⇒ 2. and 1.⇒ 3.: If $A_1 \sim_{\lambda+\tau,r} A_2$, there is a sequence of turns of rank r such that $A_1 = X_0 \mapsto \ldots \mapsto X_k = A_2$. Thus we have $A_1 \equiv A_2 \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r \rangle\rangle$ by Remark 6.16 and $\mathrm{can}_r(A_1) = \mathrm{can}_r(A_2)$ by Corollary 6.4.

2.⇒ 1.: Suppose $A_1 \equiv A_2 \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r \rangle\rangle$. By Corollary 6.3 we may assume $A_1, A_2 \in \mathrm{SCan}_{\lambda,r}$. Hence by Remark 6.15 and Lemma 6.19 we have $[A_1] = \varphi(A_1) = \varphi(A_2) = [A_2]$ and hence $A_1 \sim_{\lambda+\tau,r} A_2$.

3.⇒ 2.: Suppose $\mathrm{can}_r(A_1) = \mathrm{can}_r(A_2)$. Since $\mathrm{can}_r(A_i)$ is obtained from A_i, $i = 1, 2$, by turns of rank r, we have

$$A_1 \equiv \mathrm{can}_r(A_1) = \mathrm{can}_r(A_2) \equiv A_2 \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r \rangle\rangle$$

□

Corollary 6.20. *If A, B are λ-semicanonical forms of rank r of some $C \in \mathrm{Can}_{r-1}$, then* $\mathrm{can}_r(A) = \mathrm{can}_r(B)$.

Proof. If A, B arise from C by turns of rank r, then $A \equiv C \equiv B \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r \rangle\rangle$, so the claim follows Proposition 6.14. □

6.1. **Canonical form of rank r of arbitrary words.** Recall that since

$$\mu = n - (8\tau + 3) > \lambda = \frac{n}{2} + 3\tau + 1$$

for our choice of the exponent n, every λ-semicanonical form of rank r is also μ-semicanonical.

Definition 6.21 (canonical form of rank r)**.** *For $A \in \mathrm{Can}_{r-1}$ we define* the canonical form of rank r of A, $\mathrm{can}_r(A)$, *in two steps as follows:*

(1) choose a λ-semicanonical form A' of rank r for A.
(2) put $\mathrm{can}_r(A) = \mathrm{can}_r(A')$ as defined in Section 5.1.

For $A \in \mathrm{Can}_{-1}$ we define $\mathrm{can}_r(A) = \mathrm{can}_r(\mathrm{can}_{r-1}(\ldots \mathrm{can}_0(A) \ldots))$.

Note that $A \equiv A' \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r \rangle\rangle$ by construction and $\mathrm{can}_r(A)$ does not depend on A' by Corollary 6.20.

Lemma 6.22 ($\mu = n - (8\tau + 3)$)**.** *If $A \in \mathrm{SCan}_{\mu-\varepsilon_2,r}$, then Definition 6.21 and choosing winner sides from Section 5.1 give the same* $\mathrm{can}_r(A)$*.*

Proof. By Lemma 4.71 there exists a λ-semicanonical form A' of A that is obtained from A by a sequence of turns with all intermediate words μ-semicanonical. Thus the result follows from Proposition 5.20. □

Lemma 6.23. *Let* $A \in \mathrm{Can}_{-1}$*. Then* $A \equiv \mathrm{can}_r(A) \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_r \rangle\rangle$.

Proof. Let $\mathrm{can}_{r-1}(A) = B$. By definition, $\mathrm{can}_r(A) = \mathrm{can}_r(B)$. By Remark 3.4, we have $B \equiv A \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_{r-1} \rangle\rangle$. By definition, $\mathrm{can}_r(B)$ is obtained from B by a sequence of turns of rank r, therefore, $B \equiv \mathrm{can}_r(B) \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_r \rangle\rangle$. Thus,

$$A \equiv \mathrm{can}_r(B) = \mathrm{can}_r(A) \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_r \rangle\rangle.$$

□

We now verify some further induction hypotheses:

Proposition 6.24. *IH 8 and IH 6 hold for rank* r*: for* $A, B \in \mathrm{Can}_{-1}$ *we have* $\mathrm{can}_r(A) = \mathrm{can}_r(B)$ *if and only if* $A \equiv B \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_r \rangle\rangle$*. In particular,* $A = \mathrm{can}_r(A)$ *for* $A \in \mathrm{Can}_r$*.*

Proof. If $\mathrm{can}_r(A) = \mathrm{can}_r(B) = C$, then Lemma 6.23 implies that $A \equiv C \equiv B \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_r \rangle\rangle$.

Conversely, assume that $A \equiv B \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_r \rangle\rangle$. Let A', B' be λ-semicanonical forms of rank r of A, B, respectively. Then

$$A' \equiv A \equiv B \equiv B' \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_r \rangle\rangle.$$

Hence, Corollary 6.4 implies that $\mathrm{can}_r(A) = \mathrm{can}_r(B)$.

If $A = \mathrm{can}_r(B) \in \mathrm{Can}_r$, then $A \equiv B \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_r \rangle\rangle$ and hence $\mathrm{can}_r(A) = \mathrm{can}_r(B) = A$ by the previous. □

Proposition 6.25. *Inductive Hypotheses 1– 8 hold for* can_r*.*

Proof. IH 1–3 follow trivially from the definition of can_r.

IH 4 follows from Lemma 5.12.

IH 5 and IH 5 are proved in Corollary 4.12.

IH 6 and 8 are proved in Proposition 6.24.

IH 7 follows from IH 7 for rank $r - 1$, and the decision process in Section 5.1.

□

Remark 6.26. If $A \in \mathrm{Can}_r \subseteq \mathrm{SCan}_{\lambda,r}$, then for every maximal occurrence u of rank r in A it follows from IH 6 that u is the winner side in the process from Section 5.1.

Lemma 6.27. *If* $A \in \mathrm{SCan}_{\lambda+\tau,r} \setminus \{1\}$ *for* $\lambda = \frac{n}{2} = 3\tau + 1$*, then* $A \notin \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_r \rangle\rangle$.

Proof. If $A \equiv 1 \mod \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_r \rangle\rangle$, then $A \sim_{\lambda,r} 1$ by Proposition 6.14. However, this is not possible, because the equivalence class of 1 consists only of 1 itself. □

Corollary 6.28. *IH 9 holds for rank* r*.*

Proof. Let U be a subword of $A \in \mathrm{Can}_r$. Then $\mathrm{can}_{r-1}(U) = DU'E$, where D and E are sides of canonical triangles of rank $r - 1$. Since D and E are τ-free of rank r, it follows from Lemma 5.19 that $\mathrm{can}_{r-1}(U)$ is $\left(\frac{n}{2} + 5\tau + 1\right)$-semicanonical. Using Lemma 4.71 we find a λ-semicanonical form U_1 of $\mathrm{can}_{r-1}(U)$. In particular the lemma implies that $U_1 \neq 1$. Then byLemma 6.27 $U_1 \notin \langle\langle \mathrm{Rel}_0, \dots, \mathrm{Rel}_r \rangle\rangle$, and neither is U. □

7. Power subwords in canonical words of rank r

We start with the following natural definition:

Definition 7.1. *Let* $A = LX^K X_1 R \in \mathrm{Can}_{r-1}$ *where* X *is primitive,* $X^n \notin \mathrm{Rel}_r$*, and* X_1 *is a prefix of* X*. If* u *is a maximal occurrence of rank* r *in* $X^K X_1$*, a* periodic shift of u in $X^K X_1$ *is a shift of* u *by* $\pm k|X|$ *in* $X^K X_1$ *contained in* $X^K X_1$*.*

If u is a maximal occurrence properly contained in $X^K X_1$, then clearly u is also a maximal occurrence of rank r in A and so are all periodic shifts of u that are properly contained in $X^K X_1$. However, if a periodic shift of u is a prefix or a suffix of $X^K X_1$, it may have a prolongation in A.

Remark 7.2. Let x, a be primitive, not cyclic shifts of each other and let $K \geqslant 2$. If x^K contains a maximal occurrence u which is a fractional power of a with $\Lambda_a(u) \geqslant 2$, then u is a prefix of a cyclic shift of x^K. Hence, if $|x| \geqslant |u|$, then clearly a cyclic shift of x contains u. Otherwise by Lemma 4.9 we have $|x| < |u| < |x| + |a|$ and hence a cyclic shift of x contains a^m where $m = \lfloor \Lambda_a(u) \rfloor$. In particular, $|u| < 2|x|$ and so u is a proper subword of x^3. Hence there exist $\geqslant K - 2$ different periodic shifts of u in x^K. Note that if (and only if) u is a prefix or suffix of x^K, the periodic shifts of u may have proper prolongations with respect to a in x^K. If there exist precisely $K - 2$ different periodic shifts of u in x^K, then u is not a subword of x^2 and so $u = u_1 x u_2$ where u_1, u_2 are nonempty suffix and prefix, respectively, of x with $0 < \Lambda_a(u_1) + \Lambda_a(u_2) < 1$ and $\Lambda_a(x) > \Lambda_a(u) - 1$. In particular, all periodic shifts are proper subwords of x^K and are maximal occurrences in x^K.

We state these observations for further applications in the following form:

Corollary 7.3. *Let $A = LX^K R \in \mathrm{Can}_{r-1}$, where $K \geqslant 3$ and X is primitive, $X^n \notin \mathrm{Rel}_r$, and let u be a maximal occurrence of rank r in A that is contained in X^K, $\Lambda_r(u) \geqslant 2$. There exist $\geqslant K - 2$ different periodic shifts of u in X^K that are maximal occurrences of rank r in A. Moreover, there exist precisely $K - 2$ such periodic shifts of u in X^K if and only if $u = u_1 X u_2$ and $0 \leqslant \Lambda_r(u_1) + \Lambda_r(u_2) < 1$.*

Further we use the notations from Definition 5.3.

Lemma 7.4. *Let $(u_0, \dots, u_t)$ be an (un-)certification sequence in $A = L^\ulcorner u_0 \dots u_t{}^\urcorner R$ to the right of u_0 and let $i, j \in \{1, \dots, t\}$ with $\Lambda_r(u_i) = \Lambda_r(u_j)$. If $t \in \{i, j\}$ assume that $\lambda_{m(t)} - \varepsilon < \Lambda_r(f_t, W) < \lambda_{m(t)} + \varepsilon$. Then $f_i = u_i$ if and only if $f_j = u_j$.*

Proof. Suppose $f_i = u_i$ and $f_j = v_j$. Then in A by Condition 2 of Definition 5.3 we have

$$\lambda_{m(i)} + 2\varepsilon > \Lambda_r(u_i) = \Lambda_r(u_j) > \lambda_{m(i)} - 2\varepsilon \geqslant \frac{n}{2} + \tau.$$

Write $W_j = L'^\ulcorner f_0 \dots f_j u_{j+1} \dots u_t{}^\urcorner R'$ for the result of turning the necessary occurrences $u_i, i \leqslant j$. Then in W_j we have $\Lambda_r(v_j, W_j) < \frac{n}{2} + 3\tau + 1 \leqslant \lambda_2 - \varepsilon$. Thus $j = t$ by Condition 4, contradicting our assumption on f_t. □

Remark 7.5. Suppose $W = L^\ulcorner f_0 \dots f_t{}^\urcorner R$ is the witness of an (un-)certification sequence to the right of $u = u_0$ in A. Let $i, j \in \{1, \dots, t-1\}$. Then $\Lambda_r(f_i, W) = \Lambda_r(f_j, W)$ implies $m(i) = m(j)$ since, by definition, $\lambda_{m(i)} \geqslant \Lambda_r(f_i, W) = \Lambda_r(f_j, W) > \lambda_{m(i)} - \varepsilon$ and the intervals $[\lambda_1, \lambda_1 - \varepsilon)$ and $[\lambda_2, \lambda_2 - \varepsilon)$ are disjoint. In particular, by the choice of the function m, there are no subsequences of the form BBb in $\Lambda_r(f_1, W), \dots, \Lambda_r(f_{t-1}, W)$.

We will need the following refinement of Lemma 4.9:

Lemma 7.6. *Let u be a fractional power of B with $B^n \in \mathrm{Rel}_r$. Let $C = wM$ be primitive, and assume that $w = a^m a_0 = a_0(a_1 a_0)^m, m \geqslant \tau$, is a maximal occurrence of rank r in $MwM = MC$, where $a^n \in \mathrm{Rel}_r$, $a = a_0 a_1$ and $C^n \notin \mathrm{Rel}_0 \cup \ldots \cup \mathrm{Rel}_r$, $M \neq 1$. Then the following holds:*

1. *If w is a subword of u and a is not a cyclic shift of B, then $m = \tau$ and a cyclic shift of B is of the form $a^{\tau-1} a_2$ for a prefix a_2 of a;*

(2) wMa^2 *is not a subword of* u*;*
(3) $(a_1a_0)^2Mw$ *is not a subword of* u*.*

Proof. 1. If $a^ma_0, m \geqslant \tau$, is contained in u we see from Lemma 4.9 that $|a^ma_0| < |B|+|a|$. Hence by Corollary 4.6, we may assume (after taking a cyclic shift) that $B = a^{\tau-1}a_2$ for a proper prefix a_2 of a. Hence $m = \tau$ by Lemma 4.9 and $|B| < |C|$.

2. If wMa^2 is a subword of u and a is a cyclic shift of B, then w is not a maximal occurrence in wM, contradicting to our assumption. So by Part 1, $B = a^{\tau-1}a_2$, hence for $Ca^2 = wMa^2$ to be a subword of u, C must be a prefix of B^K for some K. Hence we may write $C = B^kM'$ where k is maximal possible and M' is not empty. Then M' is a proper prefix of B and a non-empty suffix of C. Hence M' and M have a common suffix. Notice that occurrences of a^2 in B^K arise only inside the maximal prolongations of a^τ. If $Ca^2 = B^kM'a^2$ is a prefix of B^K, then $M'a^2$ is a prefix of B^{K-k}. However, this implies that M' has a common suffix with a, since $|M'| < |B|$. Then M has a common suffix with a contradicting our assumption that w is maximal in MwM.

3. follows from 2. by considering the inverses w^{-1}, u^{-1} and $M^{-1}w^{-1}M^{-1}$. □

Lemma 7.7. *Let* a, B, C *be primitive and* $B = a^sa_1 \neq C = a^ta_2$ *for* $4 \leqslant s \leqslant t \leqslant s+1$ *and* a_1, a_2 *nontrivial prefixes of* a*. If* D *is a common prefix of* $B^m, C^m, |B| \leqslant |C|$*, then* $|D| < |C| + |a|$*.*

Proof. Suppose $|D| \geqslant |C|+|a|$, then Ca is a prefix of B^2. So Ba is a prefix of C. Therefore Ba is a common prefix of B^K and a^K, and we get a contradiction to Lemma 4.9 applied to B and a. □

Remark 7.8. Let $C \in \mathrm{Cycl}_{r-1}$ is primitive and $C^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$. Then by Corollary 4.7 C cyclically contains $\widehat{a}^\tau$ for some $\widehat{a}^n \in \mathrm{Rel}$. One can see that there exist a and C_0 cyclic shifts of $\widehat{a}$ and C_0, respectively, such that $C_0 = a^ma_0C_1$, $m \geqslant \tau$, $a = a_0a_1$, and either $C_1 = 1$, or a^ma_0 is a maximal occurrence of rank r in $C_1a^ma_0C_1$.

Lemma 7.9. *Let* $A = LuMzR \in \mathrm{Can}_{r-1}$*, where* u, w *are maximal occurrences of rank* r *with* $\Lambda_r(u), \Lambda_r(z) \geqslant \tau+1$*. Let* C^N *be a subword of* uMz *where* $C \in \mathrm{Cycl}_{r-1}$ *is primitive and* $C^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$*. Let* $C_0 = a^ma_0C_1$*,* $m \geqslant \tau$*,* $a^n \in \mathrm{Rel}_r$*, be a cyclic shift of* C *as in Remark 7.8. Then* M *contains* $\geqslant N-4$ *occurrences of* a^ma_0*.*

Proof. Let $a^ma_0 = w$ and assume the contrary. Clearly uMz contains $\geqslant N-1$ occurrences of w. By Corollary 4.11 each of u and z contain at most one occurrence of w, so uMz contains precisely $N-1$ occurrence of w and u and z properly contain occurrences of w. Then wC_1a^2 is a subword of z or $(a_1a_0)^2C_1w$ is a subword of u (since C^N is a subword of uMz and $m > 4$). If $C_1 \neq 1$, this contradicts to Lemma 7.6, otherwise this contradicts to Lemma 7.7. □

Lemma 7.10. *Let* $A = LDu_0\ulcorner u_1 \ldots u_t\urcorner R \in \mathrm{SCan}_{\mu,r}$ *where* $(u_0, \ldots, u_t)$ *is an (un-)certification sequence to the right of* u_0 *in* A *and* D *is* τ*-free of rank* r*. Let* C^N *be a subword of* $Du_0\ulcorner u_1 \ldots u_t\urcorner$ *where* $C \in \mathrm{Cycl}_{r-1}$ *is primitive and* $C^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$*. If* C^N *cyclically contains* $a^{2\tau}$ *with* $a^n \in \mathrm{Rel}_r$*, then* $N \leqslant 5$*, otherwise* $N \leqslant 6$*.*

Proof. First assume that C^N does not contain any of u_i, $0 \leqslant i \leqslant t$. Since there is no gap in the sequence $(u_0, \ldots, u_t)$, it follows from Lemma 7.9 that $N \leqslant 6$. Assume moreover that C^N cyclically contains $a^{2\tau}$ with $a^n \in \mathrm{Rel}_r$. Then also by Lemma 7.9 $N \leqslant 5$.

Hence we may assume that C^N contains some u_i. Let $W = L'E\ulcorner f_0f_1 \ldots f_t\urcorner R'$ be the witness of the certification sequence (where L' is a prefix of LD, E is τ-free of rank r if

$f_0 = v_0$ and $L'E = LD$ if $f_0 = u_0$). Let $i \leqslant t$ be minimal such that u_i is contained in C^N and all its periodic shifts in C^N are maximal occurrences in C^N. Let C_0 be a cyclic shift of C such that u_i is a prefix of C_0^2. So $C_0^{N-1}C_0'$ is a subword of $Du_0\ulcorner u_1 \dots u_t\urcorner$, where C_0' is a prefix of C_0. By Convention 4.38 we write $C_0 = \ulcorner u_i \dots u_{i+k-1}\urcorner$ for some $k \geqslant 1$. Then $C_0' = \ulcorner u_i \dots u_{i+k-2}\urcorner$ for $k \geqslant 2$, here u_{i+k-2} in C_0' is a periodic shift of a prefix of u_{i+k-2} with Λ_r-measure $> 4\tau + 2$. Since $\Lambda_r(u_j) \geqslant 5\tau + 3$ for all $0 \leqslant j \leqslant t$, all periodic shifts of u_s that are maximal occurrences in C^N are equal to some u_j by Lemma 5.2 (except possibly shifts that are a prefix and a suffix of C^N). Thus u_{i+j+sk} are equal to each other for $0 \leqslant s \leqslant N-1$ with a fixed $0 \leqslant j \leqslant k-3$, and are equal to each other for $0 \leqslant s \leqslant N-2$ with a fixed $j \in \{k-2, k-1\}$. Moreover $i + (N-1)k + k - 3 \neq t$ and $i + (N-2)k + k - 1 \neq t$ for $k \geqslant 2$.

From now on we assume additionally that $i \neq 0$. If C^N contains only u_0, then clearly $N \leqslant 2$, so the result follows.

If $k \geqslant 2$, then by Lemma 7.4 the choices in W for f_{i+j+sk} are the same for $0 \leqslant s \leqslant N-1$ with a fixed $0 \leqslant j \leqslant k-3$, and and are the same for $0 \leqslant s \leqslant N-2$ with a fixed $j \in \{k-2, k-1\}$. Hence by Lemma 4.62 $\Lambda_r(f_{i+sk}, W)$ are the same for $1 \leqslant s \leqslant N-2$ and $\Lambda_r(f_{i+j+sk}, W)$ are the same for $0 \leqslant s \leqslant N-3$ with a fixed $1 \leqslant j \leqslant k-1$ (we put these indices in order to consider cases $k = 2$ and $k \geqslant 3$ simultaneously). So by Remark 7.5 $m(i + sk)$ are equal to each other for $1 \leqslant s \leqslant N-2$, and $m(i + j + sk)$ with a fixed $1 \leqslant j \leqslant k-1$ are equal to each other for $0 \leqslant s \leqslant N-3$. Since m is BBb-free, we must have $N - 2 \leqslant 2$, so $N \leqslant 4$.

If $k = 1$, we write $C_0 = \ulcorner u_i\urcorner$, then $u_i = u_{i+1} = \ldots = u_{i+N-3}$, $i + N - 3 \neq t$, and u_{i+N-2} has a prefix equal to u_i. So the choices f_j in the witness W are the same for u_j with $i \leqslant j \leqslant i + N - 3$. Hence by Lemma 4.62 and Remark 7.5 the $m(j)$ are the same for $i + 1 \leqslant j \leqslant i + N - 4$. It follows from Corollary 4.28 that the turn of u_{j+1} have the same influence on u_j for all $i \leqslant j \leqslant i + N - 2$. If $N - 2 \geqslant 4$, then for one of u_j with $i + 1 \leqslant j \leqslant i + N - 3$ any choice for u_{j+1} fits for a witness, since $\lambda_1 - \lambda_2 \geqslant \varepsilon_2$. This contradicts Condition 4 of Definition 5.3, so $N - 2 \leqslant 3$ and $N \leqslant 5$. □

In fact the proof of Lemma 7.10 shows the following

Corollary 7.11. *Let $A = LDu_0\ulcorner u_1 \dots u_t\urcorner R \in \mathrm{SCan}_{\mu,r}$ where $(u_0, \dots, u_t)$ is an (un-)certification sequence to the right of u_0 in A and D is τ-free of rank r. Let C^6 be a subword of $Du_0\ulcorner u_1 \dots u_t\urcorner R$ that properly contains some u_i where $C \in \mathrm{Cycl}_{r-1}$ is primitive and $C^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$. Then $Du_0\ulcorner u_1 \dots u_t\urcorner$ contains $\leqslant 3$ periodic shifts of u_i that are maximal occurrences of rank r in A different from u_0 and u_t.*

Corollary 7.12. *Let $A = LuR \in \mathrm{SCan}_{\mu,r}$, where u is a maximal occurrence of rank r with $\Lambda_r(u) \geqslant 5\tau + 3$. Let $C \in \mathrm{Cycl}_{r-1}$ be primitive and $C^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$. Let C^N be a subword of DuR, where D is τ-free of rank r suffix of L, $N = 6$ if C cyclically contains $a^{2\tau}$ with $a^n \in \mathrm{Rel}_r$, and $N = 7$ otherwise. Then the prefix of R that ends at the end point of C^N is a right context for u.*

Proof. Let $(u = u_0, u_1, \dots, u_t)$ be an (un-)certification sequence from the right of u for side f_1 and $A = L\ulcorner u_0, \dots, u_t\urcorner R_1$. Then C^N is not contained in $D\ulcorner u_0, \dots, u_t\urcorner$ by Lemma 7.10, so u_t ends strictly from the left of the end of C^N. Denote by M a prefix of R that ends at the end point of C^N and let $R = MR_1$ and consider a word $B = LuMR_2 \in \mathrm{SCan}_{\mu,r}$. First notice that $(u = u_0, u_1, \dots, u_t)$ is a certification or an uncertification sequence for the side f_1 in B, because it cannot be extended or shortened by Lemma 7.10 and Conditions 4 and 5 or 5'.

It remains to show that $(u = u_0, u_1, \ldots, u_t)$ in B cannot change its status from certification sequence for f_1 to un-certification sequence and vice versa. Let Q be a suffix of M that starts at the end of u_t. It is sufficient to show that Q contains a subword of the form $a^\tau M_1 b^\tau$, $a^n, b^n \in \mathrm{Rel}_r$ (because the only possible problematic case is when $f_t = v_t$ and $\Lambda_r(f_t)$ is different in the witnesses for A and for B). If C^N does not contain any u_i, this follows from Corollaries 4.7 and 4.11.

If C^N contains only u_t, then the result is clear. So we can assume that C^N properly contains u_t. Then by Corollary 7.11 $D\ulcorner u_0, \ldots, u_t \urcorner$ contains $\leqslant 4$ periodic shifts of u_t different from u_0. Hence Q contains a periodic shift of a suffix of u_t with Λ_r-measure $> 5\tau + 1$. This completes the proof. □

Corollary 7.13. *Let $A = LC^N R \in \mathrm{SCan}_{\mu,r}$, $N \geqslant \tau$, where C is primitive and $C^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$. Let u be a maximal occurrence of rank r in A contained in C^N with $\Lambda_r(u) \geqslant 5\tau + 3$ such that its periodic shifts are maximal occurrences in A (except possibly the first and the last one). Then there exist periodic shifts of u that are contained neither in LC^6, nor in $C^6 R$. Furthermore the left and right contexts together for these the periodic shifts of u are contained in C^{13} and the (un-)certification sequences are periodic shifts of each other (certification sequences are shifted to certification sequences, un-certification sequences are shifted to un-certification sequences).*

Proof. The existence follows from Remark 7.2, since $\tau \geqslant 13$. The second part follows directly from Corollary 7.12. □

Corollary 7.14. *Let $A = LC^N R \in \mathrm{SCan}_{\mu,r}$, where $N \geqslant \tau$ is a sufficiently big positive number, C is primitive and $C^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$. Then $\mathrm{can}_r(A) = \widetilde{L} Y^{N-\gamma} \widetilde{R}$, where C and Y are conjugate in rank r, and γ does not depend on N.*

Proof. By Corollary 7.13 the certification sequences for any maximal occurrence of rank r in A that is contained in C^N and is contained neither in LC^6, nor $C^6 R$ are contained inside C^N and are periodic shifts of each other. Hence the winner choice for periodic shifts is the same and the result follows from Lemma 4.62. □

Remark 7.15. If $A = LC^\tau R \in \mathrm{Can}_r$, then by Corollaries 7.12 and 7.13 we see that the left and the right context together for any maximal occurrence u in A is contained in either LC^{12}, C^{13} or $C^{12} R$. In particular, there is no maximal occurrence in A whose left and right contexts have nontrivial overlap with both L and R.

Corollary 7.16. *IH 11 holds for rank r.*

For the proof we first note the following:

Lemma 7.17 ($\kappa = \mu - \varepsilon = n - 10\tau - 4$)**.** *Let $A = L_1 C^{N_1} R_1, B = L_2 C^{N_2} R_2 \in \mathrm{SCan}_{\kappa,r}$, where C is primitive, $C^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$, and $N_1, N_2 \geqslant \tau$. Then $L_1 C^S R_2 \in \mathrm{SCan}_{\kappa,r}$ for any $S \geqslant \tau$.*

Proof. Since $A, B \in \mathrm{SCan}_{\kappa,r} \subseteq \mathrm{Can}_{r-1}$ and $C^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$, we have $L_1 C^S R_2 \in \mathrm{Can}_{r-1}$ by IH 11 in rank $r-1$. If $L_1 C^S R_2$ contains a maximal occurrence u of rank r of Λ_r-measure $> \kappa$, then u is contained in $L_1 C^2$, in $C^2 R_2$, or in C^3 by Remark 7.2 and Corollary 4.11. This is impossible since $A, B \in \mathrm{SCan}_{\kappa,r}$. □

Proof of Corollary 7.16. For $r = 1$ this follows directly from the definition of can_1 in Section 5.1. So assume $r > 1$. Let $X_1 = L_1 C^\tau R_1, X_2 = L_2 C^\tau R_2 \in \mathrm{Can}_r$. Then $L_1 C^S R_2 \in \mathrm{SCan}_{\kappa,r}$ for any $S \geqslant \tau$ by Lemma 7.17. If all maximal occurrences in

$L_1C^SR_2$ have Λ_r-measure $\leqslant \frac{n}{2} - 5\tau - 2$, the claim follows directly from Lemma 5.12. So let u be a maximal occurrence in $L_1C^SR_2$ with $\Lambda_r(u) > \frac{n}{2} - 5\tau - 2$. By Remark 7.15 we see that the left and right contexts of u are contained in LC^{12}, C^{13} or $C^{12}R$. So they coincide with the corresponding ones in X_1 or in X_2. Since any occurrence u in X_1, X_2 is the winner side (because $X_1, X_2 \in \mathrm{Can}_r$), the same is true for u in $L_1C^SR_2$, proving the claim. □

7.1. **Multiplication and canonical triangles.** We now prove that the multiplication of canonical words of rank r can be described in terms of canonical triangles of rank r.

We say that a word W contains *a gap* if it contains a subword of the form $a^\tau M b^\tau M' c^\tau$ where $a^n, b^n, c^n \in \mathrm{Rel}_r$.

Lemma 7.18. *Let $A = LuW_0FW_1wR \in \mathrm{Can}_{r-1}$ where u, w are maximal occurrences of Λ_r-measure $\geqslant \tau + 1$, and W_0, W_1 do not contain strong separation words (from the right and left, respectively). Assume that at least one of the following conditions holds:*

1. *F is τ-free of rank r;*
2. *W_0, W_1 do not contain gaps and $F = DzE$, where D, E are τ-free of rank r, and z is an occurrence of rank r;*
3. *at least one of W_0, W_1 is τ-free of rank r and $F = DzE$ as above.*
4. *W_0 does not contain a gap, $F = DE$, where D, E are τ-free of rank r;*
5. *W_0 contains a subword $b^{2\tau+1}$, $b^n \in \mathrm{Rel}_r$, and $F = DzE$ as above.*

Let C^N be a subword of uW_0FW_1w where $C \in \mathrm{Can}_0$ is primitive and $C^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$. Then $N < \tau = 15$.

Proof. If $C \notin \mathrm{Cycl}_{r-1}$, then by Definition 4.2 A does not contain C^τ. So we suppose that $C \in \mathrm{Cycl}_{r-1}$. Hence Lemma 7.9 implies that W_0FW_1 contains $\geqslant N - 4$ occurrences of a^τ, $a^n \in \mathrm{Rel}_r$ not overlapping with each other. If one of Conditions 1—4 holds, then the result follows from Example 4.41 and Corollary 4.11 for a common part of C^N and z.

Under Condition 4 either C^N is contained in uW_0, or C cyclically contains $b^{2\tau}$. In the first case the result follows from the above argument. In the second case we count directly periodic shifts of $b^{2\tau}$ and obtain $N < \tau$. □

We now show a preliminary version of IH 10:

Proposition 7.19 ($\lambda = \frac{n}{2} + 3\tau + 1$)**.** *Let $A, B \in \mathrm{Can}_r$. Then $\mathrm{can}_r(A \cdot B) = A_1'M_3B_1'$, $A = A_1'M_1X$, $B = X^{-1}M_2B_1'$, where $X \cdot X^{-1}$ is the maximal cancellation in $A \cdot B$, and M_3 is τ-free modulo r.*

Proof. By IH 10 for rank $r-1$, we know that $\mathrm{can}_{r-1}(A \cdot B) = A''EB''$, where E is τ-free of rank r. Since by Lemma 5.19 we have that $A, B \in \mathrm{SCan}_{\lambda,r}$, we can apply Lemma 4.70 to $A''EB''$ and obtain $C_1 = A_1QB_1 \in \mathrm{SCan}_{\lambda+3\tau+1,r}$ for a prefix A_1 of A'' and suffix B_1 of B'' by iterated turns of Λ_r-measure $> \lambda + (3\tau + 1)$. By Lemma 4.25 $Q = DzE$, where D, E are τ-free of rank r and z is an occurrence of rank r (any part can be empty).

Since $\lambda + 3\tau + 1 = \frac{n}{2} + 6\tau + 2 < n - (8\tau + 3) - 2\tau - 1$, by Lemma 6.22 $C = \mathrm{can}_r(A \cdot B)$ is obtained from $C_1 = A_1QB_1$ by choosing the winner sides in the maximal occurrences of rank r in C_1. Clearly we can write $C = A_1'M_3B_1'$ for some prefix A_1' of A_1, suffix B_1' of B_1 and some word M_3. We need to show that it is possible to take M_3 τ-free modulo r.

To determine the prefix A_1' and the suffix B_1', let u and w be the left- and right-most occurrences, respectively, which are turned in C_1. Let $u = u_0, \ldots, u_s = w$ be an enumeration from left to right of all maximal occurrences in $C_1 = A_1QB_1$ of Λ_r-measure $\geqslant 5\tau + 3$ between u and w. By Remark 5.4 an initial segment of $u_0, \dots, u_s$ is an initial segment of a (un-)certification sequence to the right of u in C (and in A if u is a maximal occurrence in A), a final segment of $u_0, \dots, u_s$ is a final segment of a (un-)certification sequence to the left of w in C (and in B if w is a maximal occurrence in B).

First suppose that u is contained in A_1 and w is contained in B_1. Since the winner side for $u = u_0$ is different in A and in C, A_1 cannot contain a right context for u_0. Hence if a common part of u_k and A_1 has Λ_r-measure $\geqslant \tau + 1$, then consecutive occurrences in $(u_0, \dots, u_k)$ are essentially not isolated. If a common part of u_k and A_1 has Λ_r-measure $\geqslant 5\tau + 3$, then $(u_0, \dots, u_k)$ is an initial segment of an (un-)certification sequence in A and in C_1 by Condition 5 or 5' of Definition 5.3. The corresponding properties hold for $(u_k, \dots, u_s)$.

Let i be the maximal index such that A_1 does not contain u_i as a suffix, and j be the minimal index such that B_1 does not contain u_j as a prefix. Then $0 < j - i \leqslant 4$, since $Q = DzE$.

We now turn all necessary occurrences in C_1 according to the choices of the winner sides. Denote the occurrences corresponding to u_k in C by f_k. Since A_1, B_1 do not contain contexts for u_0 and u_s, respectively, there exist at most 3 maximal occurrences of Λ_r-measure $\geqslant 5\tau + 3$ in C between f_i and f_j.

Corollaries 5.8 and 5.9 imply that either $(f_0, \dots, f_i)$ is an initial segment of an (un-)certification sequence in C from the right of f_0, or $\Lambda_r(f_i) < 5\tau + 3$. Hence in the second case there exist at most 3 maximal occurrences of Λ_r-measure $\geqslant 5\tau + 3$ in C between f_{i-1} and f_j. The symmetric property holds from the other side. So we obtain the following sequence of maximal occurrences of Λ_r-measure $\geqslant 5\tau + 3$ in C between f_0 and f_s: $(f_0, \dots, f_{i_0})$ is an initial segment of an (un-)certification sequence, where either $i_0 = i - 1$, or $i_0 = i$, $(f_{j_0}, \dots, f_s)$ is a final segment of an (un-)certification sequence, where either $j_0 = j$, or $i_0 = j + 1$, and there exist at most 3 maximal occurrences of Λ_r-measure $\geqslant 5\tau + 3$ in C between f_{i_0} and f_{j_0}.

Let V be primitive such that $V^n \notin \mathrm{Rel}_1 \cup \ldots \cup \mathrm{Rel}_r$. If $V \notin \mathrm{Cycl}_{r-1}$, we are done by the definition of Cycl_{r-1}. So assume $V \in \mathrm{Cycl}_r$. Assume that V^N contains some maximal occurrence x with $\Lambda_r(x) \geqslant 5\tau + 3$, and not as a prefix or suffix. Then by Corollary 7.11 and by the previous considerations V^N contains $\leqslant 11$ periodic shifts of x, so $N \leqslant 13$.

Now assume that V^N does not contain any occurrence with Λ_r-measure $\geqslant 5\tau + 3$. Clearly it is sufficient to consider a space between f_{i_0} and f_{j_0}, which is of the form W_1QW_2, where W_1, W_2 do not contain strong separation words. If W_1 does not have a common suffix with A_1 or W_2 does not have a common prefix with B, then the result follows from Lemma 7.18 (3). If $Q = E$, the result follows from Lemma 7.18 (1). Assume that $Q = DzE$ with non-empty z with $\Lambda_r(z) < 5\tau + 3$. Then the last occurrence that is turned in order to obtain C_1 is of Λ_r-measure $> n - (7\tau + 3)$. Hence it has common parts both with A and B of Λ_r-measure $> \frac{n}{2} - 5\tau - 2$. Denote their maximal prolongations by w_1 and w_2, respectively. If W_1 contains a gap, then w_1 is strongly isolated from f_{i_0}. Hence there must exist some x in A with $\Lambda_r(x) \geqslant 5\tau + 3$ between f_{i_0} and w_1 (otherwise the winner side for u_0 is the same in A and C_1). Since the space between f_{i_0} and w_1 is of the form W_1D_1 with D_1 τ-free of rank r, W_1 contains $b^{2\tau+1}$, $b^n \in \mathrm{Rel}_r$. The symmetric property holds for w_2 and W_2. So the result follows from Lemma 7.18 (2) and (5). If $Q = DE$, then we argue in the same way but only from the left side. Then the result follows from Lemma 7.18 (4)

Finally we need to consider the case that u or w are not contained in A_1, B_1, respectively. Suppose that u is not contained in A_1. Then the sequence $(u = u_0, \dots, u_m = w)$ contains at most three maximal occurrences not contained in B_1 and similarly for the other side. Thus we see from the previous arguments that M_3 is τ-free modulo rank r. □

Now we can finish the proof of IH 10 for rank r.

Corollary 7.20. *Let $A, B \in \mathrm{Can}_r$ and $\mathrm{can}_{r-1}(A\cdot B) = A''E_3B''$ by IH 10 for rank $r-1$. There exists a canonical triangle (D_1, D_2, D_3) of rank r such that $\mathrm{can}_r(A\cdot B) = A_1D_3B_1$, $A = A_1D_1X$, $B = X^{-1}D_2B_1$, where $X \cdot X^{-1}$ is the maximal cancellation in $A \cdot B$ and A_1, B_1 are prefix and suffix of A'', B'', respectively. Furthermore if $A_1 = A''$ and $B_1 = B''$, then $\mathrm{can}_r(A \cdot B) = \mathrm{can}_{r-1}(A \cdot B)$.*

Proof. By Proposition 7.19 we have $\mathrm{can}_r(A\cdot B) = A'M_3B'$, $A = A'M_1X$, $B = X^{-1}M_2B'$, where $X \cdot X^{-1}$ is the maximal cancellation in $A \cdot B$. By IH 10 for rank $r-1$ we have that $A = A''E_1X$ and $B = X^{-1}E_2B''$, where (E_1, E_2, E_3) is a canonical triangle of rank $r-1$. By construction we have that A' is a prefix of A'' and B' is a suffix of B''.

First suppose that $A' = A''$ and $B' = B''$. Then by construction all turns are done in E_3. However, since E_3 is τ-free of rank r, it does not contain any occurrences of rank r to turn. So there are no turns in $A''E_3B''$ in order to obtain $\mathrm{can}_r(A \cdot B)$, hence $\mathrm{can}_r(A \cdot B) = \mathrm{can}_{r-1}(A \cdot B)$ and we can put $D_i = E_i$, $i = 1, 2, 3$.

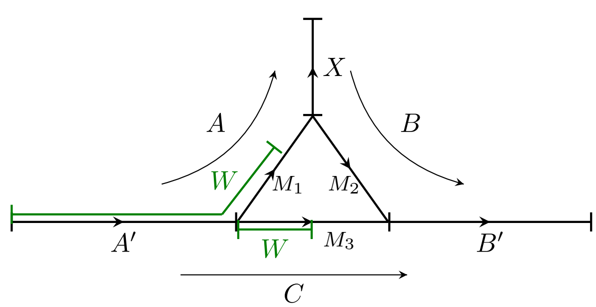
By definition of $\cdot_r$, we can write $A \cdot_r B = \mathrm{can}_r(A \cdot B) = C$. Therefore $A = C \cdot_r B^{-1}$. Now we apply Proposition 7.19 to $A = C \cdot_r B^{-1}$ and obtain $A = C'F_3{B'}^{-1}$, where F_3 is τ-free modulo rank r, C' is a prefix of C, ${B'}^{-1}$ is a suffix of B^{-1}. Since $X \cdot X^{-1}$ is the maximal cancellation in $A \cdot B$, we have that ${B'}^{-1}$ is a suffix of X^{-1}.

If C' is a prefix of A', then M_1 is a subword of F_3, since ${B'}^{-1}$ is a suffix of X^{-1}. So, M_1 is τ-free modulo rank r.

Otherwise A' is a proper prefix of C', so $C' = A'W$. Since C' is left after the maximal cancellations in $C \cdot B^{-1}$, we obtain that W is a prefix of M_3. Then $A'W$ is a prefix of A, because C' is also a prefix of A. If M_1 is contained in W, then M_1 is a subword of M_3, so it is τ-free modulo r. If W is a proper prefix of M_1, then W is a common prefix of M_1 and M_3. In this case we fold W in the sides M_1 and M_3 and obtain a new triangle $\widetilde{M_1}$, $\widetilde{M_2} = M_2$, $\widetilde{M_3}$. Then $\widetilde{M_1}$ is a prefix of F_3 and $\widetilde{M_3}$ is a suffix of M_3. So, $\widetilde{M_1}$ and $\widetilde{M_3}$ are τ-free modulo rank r.

Assume that after the above procedure side E_1 is not a suffix of $\widetilde{M_1}$ anymore. Then instead of complete folding W we fold it until E_1 plus one extra letter. Then $\widetilde{M_1} = xE_1$ for some single letter x. Since E_1 is τ-free modulo rank $r-1$, $\widetilde{M_1}$ is τ-free modulo rank r by Lemma 4.7.

After that we deal similarly with $\widetilde{M_2} = M_2$ in the new triangle and as a result obtain the required canonical triangle (D_1, D_2, D_3).



In this case we can fold W in the sides D_1 and D_3 and obtain a new triangle $\widetilde{D}_1$, $\widetilde{D}_2 = D_2$, $\widetilde{D}_3$, where $\widetilde{D}_1$ is a prefix of F_3 and $\widetilde{D}_3$ is a suffix of D_3. So, $\widetilde{D}_1$ and $\widetilde{D}_3$ are τ-free modulo rank r.

After that we deal similarly with $\widetilde{D}_2 = D_2$ in the new triangle and as a result obtain the required canonical triangle (D_1, D_2, D_3). □

7.2. **Canonical form of power words.**

We start with some preliminary lemmas:

Lemma 7.21. *Let $A = XWX^{-1} \in \mathrm{Can}_{r-1}$ and $\mathrm{can}_{r-1}(A \cdot A) = X_1W_1(X_1)^{-1}$ where W and W_1 are cyclically reduced. If W is τ-free of rank r, then W_1 is 3τ-free of rank r.*

Proof. By IH 10 there exists a canonical triangle (D_1, D_2, D_3) of rank $r-1$ such that $XW = A'D_1$, $WX^{-1} = D_2A''$ and $\mathrm{can}_{r-1}(A \cdot A) = A'D_3A''$. If $A' = XW'$ and $A'' = W''X^{-1}$ (where W' or W'' may be empty), then $X = X'$ and the claim is immediate. Otherwise, either $A' = X'$, $A'' = W_0X_0^{-1}(X')^{-1}$, or symmetrically $A' = X'X_0W_0$, $A'' = (X')^{-1}$, where X' is a prefix of X, X_0 is τ-free subword of X and W_0 is a subword of W. Then again the claim is immediate. □

Lemma 7.22. *Let W', W'' be 3τ-free of rank r and let D, E be τ-free of rank r. If $(EW'DW'')^N$ contains an occurrence u of Λ_r-measure $\geqslant 11\tau + 1$, then $EW'DW'' = a^s$ for some $s \geqslant 0$ and $a^n \in \mathrm{Rel}_r$.*

Proof. Let $u = \widehat{a}^k\widehat{a}_1$ for $\widehat{a}^n \in \mathrm{Rel}_r$ and $k \geqslant 11\tau + 1$. If $|\widehat{a}| \geqslant |EW'DW''|$, then by Lemma 4.9 $EW'DW''$ is a cyclic shift of $\widehat{a}$.

If $|\widehat{a}| < |EW'DW''|$, the assumptions on W', W'', D, E imply that u contains a cyclic shift Y of $EW'DW''$, and Y is 11τ-free. So since $\Lambda_r(u) \geqslant 11\tau + 1$, the common part of u and $(EW'DW'')^N$ has length $\geqslant |Y| + |\widehat{a}|$. Hence by Lemma 4.9 $EW'DW'' = a^s$ for a a cyclic shift of $\widehat{a}$. □

Lemma 7.23. *Let $A = XWX^{-1} \in \mathrm{Can}_{r-1}$ where W is cyclically reduced and contains an occurrence u of Λ_r-measure $\geqslant 3\tau$. Then there is a canonical triangle (D_1, D_2, D_3) of rank $r-1$ such that*

$$Q = \mathrm{can}_{r-1}(\underbrace{A \cdot \ldots \cdot A}_{N\,times}) = XD_2(MD_3)^{N-1}MD_1X^{-1}$$

where $W = D_2MD_1$. In particular, MD_3 is conjugate to W in $\mathrm{F}/\langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_{r-1}\rangle\rangle$. Furthermore, if W is κ-bounded of rank r for some $\kappa \geqslant 3\tau$, then either $(MD_3)^N$ is $2\kappa + \tau + 1$-free of rank r, or $MD_3 = a^s$ for $a^n \in \mathrm{Rel}_r$.

Proof. By IH 10 there is a canonical triangle (D_1, D_2, D_3) of rank $r-1$ such that $\mathrm{can}_{r-1}(A \cdot A) = XW'D_3W''X^{-1}$ for some non-empty prefix W' and suffix W'' of W. Since W contains u, we can write $W = D_2MD_1$ with non-empty M. Then $\widetilde{W} = D_1D_2M$ is a cyclic conjugate of W and we have

$$\widetilde{W} = D_1D_2M \equiv D_3M \mod \langle\langle \mathrm{Rel}_0 \cup \ldots \cup \mathrm{Rel}_{r-1}\rangle\rangle.$$

Since D_1 and D_2 is τ-free of rank r, W' and W'' contain occurrences of Λ_r-measure $\geqslant 2\tau$. Hence by Corollary 3.9 we obtain

$$\mathrm{can}_{r-1}(A \cdot A \cdot A) = XW'D_3MD_3W''X^{-1} \text{ and } W' = D_2M,\ W'' = MD_1.$$

Inductively Corollary 3.9 yields

$$Q = XW'\underbrace{D_3M\dots D_3M}_{N-2\ times}D_3W''X^{-1} = XD_2(MD_3)^{N-1}MD_1X^{-1}.$$

The last sentence follows from Lemma 4.9. □

Lemma 7.24. *Let $A = XWX^{-1} \in \mathrm{Can}_{r-1}$, where W is a cyclically reduced and 3τ-free of rank r and $W^n \notin \langle\langle \mathrm{Rel}_1, \dots, \mathrm{Rel}_r\rangle\rangle$. By IH 13 for rank $r-1$ write*

$$\mathrm{can}_{r-1}(\underbrace{A\cdot\dots\cdot A}_{K\ times}) = T\widetilde{A}^{K-\gamma}S \quad \textit{for all } K \geqslant \gamma,$$

where $\widetilde{A}, T, S, \gamma$ depend only on A and r, and $A, \widetilde{A}$ are conjugate in the group $\mathrm{F}/\langle\langle \mathrm{Rel}_1, \dots, \mathrm{Rel}_{r-1}\rangle\rangle$. Then $\widetilde{A}^N$ is $11\tau+1$-free of rank r for all $N \geqslant 1$ and hence

$$\mathrm{can}_{r}(\underbrace{A\cdot\dots\cdot A}_{K\ times}) = T'\widetilde{A}^{K-\gamma'}S' \quad \textit{for all } K \geqslant \gamma'$$

where T', S' and γ' only depend on A and r. (Note that $\widetilde{A}$ does not change.)

Proof. Let $A_s = \mathrm{can}_{r-1}(\underbrace{A\cdot\dots\cdot A}_{2^s\ times}) = X_sW_sX_s^{-1}$, where W_s is cyclically reduced.

First assume that W_s is 3τ-free for all $s \geqslant 0$. For all $K = 2^s$ we have $T\widetilde{A}^{K-\gamma}S = X_sW_sX_s^{-1}$. Notice that if an overlap of X_s and $\widetilde{A}^{K-\gamma}$ contains a whole period $\widetilde{A}$, then an overlap of X_s^{-1} and $\widetilde{A}^{K-\gamma}$ cannot contain $\widetilde{A}$. So if there exists $s \geqslant \gamma+3$ such that T, S are contained in X_s, X_s^{-1}, respectively, then W_s contains $\widetilde{A}^3$. Otherwise $|X_s| \leqslant \max\{|S|, |T|\}$ for $s \geqslant \gamma+3$ because $|S|, |T|$ do not depend on s. In this case W_s contains $\widetilde{A}^3$ for all sufficiently large s. Thus $\widetilde{A}^3$ is 3τ-free of rank r, and so is $\widetilde{A}^{N_1}$ for all $N_1 \geqslant 1$.

Now let $t \geqslant 1$ be minimal such that W_t is not 3τ-free of rank r so W_{t-1} is not τ-free of rank r by Lemma 7.21. Therefore by IH 10, $W_t = W'_{t-1}EW''_{t-1}$, where E is τ-free of rank r, W'_{t-1}, W''_{t-1} are a non-empty suffix and prefix of W_{t-1}, respectively. Then it follows from Lemma 7.23 that

$$\mathrm{can}_{r-1}(\underbrace{A_t\cdot\dots\cdot A_t}_{N\ times}) = X_tD_2(MD_3)^{N-1}MD_1X_t,$$

where $W_t = D_2MD_1$, and (D_1, D_2, D_3) is a canonical triangle of rank $r-1$. Since W_{t-1} is 3τ-free of rank r, by construction, MD_3 is of the form $W'E'W''D_3$, where E' is τ-free and W', W'' are 3τ-free of rank r (some parts can be empty). Hence Lemma 7.22 implies that either $(MD_3)^{N-1}$ is $11\tau+1$-free of rank r, or $MD_3 = a^k$ for some $a^n \in \mathrm{Rel}_r$.

For sufficiently large N and $K = N\cdot 2^t$, the common part of $(MD_3)^{N-1}$ and $\widetilde{A}^{K-\gamma}$ has length $> |MD_3| + |\widetilde{A}|$. Hence by Lemma 4.9 $MD_3 = Z^{k_1}$, $\widetilde{A} = Z^{k_2}$ for some word Z. So if $(MD_3)^{N-1}$ is $11\tau+1$-free of rank r, $\widetilde{A}^{N_1}$ is also $11\tau+1$-free of rank r for all $N_1 \geqslant 1$. If $MD_3 = a^k$, then $\widetilde{A} = a^{k_2}$, since a is primitive. Hence $\widetilde{A}^n \in \mathrm{Rel}_r$, a contradiction.

For sufficiently large N by IH 12 we have $\mathrm{can}_{r-1}(\widetilde{A}^N) = D\widetilde{A}_1\widetilde{A}^{N-\delta}\widetilde{A}_2E$, where D, E are τ-free of rank r, $\widetilde{A}_1, \widetilde{A}_2$ are a suffix and prefix of $\widetilde{A}$, respectively. By IH 11 $D, E, \widetilde{A}_1, \widetilde{A}_2, \delta$ do not depend on N. By Lemma 7.24 $D\widetilde{A}_1\widetilde{A}^{N-\delta}\widetilde{A}_2E$ is $12\tau+1$-free of rank r for big enough N. Since $12\tau+1 \leqslant \frac{n}{2} - 5\tau - 2$, it follows from Lemma 5.12 that

$\mathrm{can}_{r-1}(\widetilde{A}^N) \in \mathrm{Can}_r$, so $\mathrm{can}_r(\widetilde{A}^N) = \mathrm{can}_{r-1}(\widetilde{A}^N)$. For sufficiently large K we obtain

$$\begin{aligned}\mathrm{can}_r(\underbrace{A\cdot\ldots\cdot A}_{K\ times}) &= \mathrm{can}_r(T)\cdot_r \mathrm{can}_r(\widetilde{A}^{K-\gamma})\cdot_r \mathrm{can}_r(S)\\ &= \mathrm{can}_r(T)\cdot_r \left(D\widetilde{A}_1\widetilde{A}^{K-\gamma-\delta}\widetilde{A}_2 E\right)\cdot_r \mathrm{can}_r(S) \;=\; T'\widetilde{A}^{K-\gamma'}S'.\end{aligned}$$

□

Lemma 7.25 ($\mu = n - 8\tau_1 - 3$)**.** *Let $A \in \mathrm{Cycl}_{r-1}$ such that $A^n \notin \langle\langle \mathrm{Rel}_0 \cup \ldots \cup \mathrm{Rel}_r\rangle\rangle$. Then there exists $\widetilde{B}$ conjugate to A in the group $\mathrm{F}/\langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r\rangle\rangle$ such that one of the following holds:*

- *$\widetilde{B} \in \mathrm{Can}_{r-1}$ and its cyclically reduced part is 3τ-free of rank r.*
- *$\widetilde{B} \in \mathrm{Cycl}_{r-1}$ and $\widetilde{B}^K$ is $\mu-\tau$-bounded for all $K \geqslant 1$.*

Proof. Similarly to the proof of Lemma 4.67, we do induction on $d'(A) = \max\{d(X)\}$, where X runs through all cyclic shifts of A and $d(X)$ is the sum of all Λ_r-measures of all maximal occurrences of rank r in X with Λ_r-measure $\geqslant \beta = 3\tau + 2$.

Let $Y = \mathrm{can}_{r-1}(A^{N_1}) = LA^NR$, $N = N_1 - \delta$, and let u be a maximal occurrence of rank r properly contained in A^N with $\Lambda_r(u) > \mu - \tau$. Since $A^n \notin \langle\langle \mathrm{Rel}_0 \cup \ldots \cup \mathrm{Rel}_r\rangle\rangle$, by Remark 7.2 u is contained in A^3 and there are periodic shifts of u starting in every period of X that are maximal occurrences in Y.

We now turn the occurrence of u in Y starting in the second period of A^N. Then there are the following configurations in the resulting word.

1. There exists a cyclic shift V_2V_1 of $A = V_1V_2$ of the form LuR such that the result of the turn of u in Y is of the form $(LV_1)(L'Q\widetilde{R})V_2A\cdots AR$, where L' is a prefix of V_2 that contains b_1^τ and $\widetilde{R}$ is a suffix of uR that contains b_2^τ for some $b_1^n, b_2^n \in \mathrm{Rel}_r$. Then Corollary 3.9 implies that the result of the turns of all periodic shifts of u in A is equal to $(LV_1)(L'Q\widetilde{R})\cdots(L'Q\widetilde{R})V_2R$, so $L'Q\widetilde{R} \in \mathrm{Cycl}_{r-1}$. Clearly $L'Q\widetilde{R}$ and A are conjugate in the group $\mathrm{F}/\langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r\rangle\rangle$.

We have

$$\begin{aligned}d(L'Q\widetilde{R}) &\leqslant d(V_2V_1) - (\mu-\tau) + (n-\mu+3\tau) + 2\beta + 2\varepsilon =\\ &= d(V_2V_1) - (\mu-\tau) + (n-\mu+3\tau) + 2(3\tau+2) + 2(2\tau+1) =\\ &= d(V_2V_1) - (n - 30\tau - 12).\end{aligned}$$

Hence $d'(L'Q\widetilde{R}) < d(L'Q\widetilde{R}) + 2\beta \leqslant d(V_2V_1) - (n-30\tau-12) + 6\tau + 4 = d(V_2V_1) - (n - 36\tau - 16) < d'(A)$. So the claim holds by the induction hypothesis.

2. Assume that $|u| \leqslant |A|$ and we are not in Case 1. Then for every cyclic shift of A of the from LuR we see that L, R do not contain words of the form $a^\tau M b^\tau$, $a^n, b^n \in \mathrm{Rel}_r$. Consider $\mathrm{can}_{r-1}(LuR) = L_1\widetilde{u}R_1$, where $\Lambda_r(u) - 2\tau < \Lambda_r(\widetilde{u}) < \Lambda_r(u) + 2\tau$. Let us turn $\widetilde{u}$ and let B be the resulting word. If the cyclically reduced part of B is 3τ-free of rank r, then we take it as $\widetilde{B}$ and we are done.

Let $\widetilde{v}$ be the complement of $\widetilde{u}$. Then the turn is of Type 2 and $\Lambda_r(\widehat{\widetilde{v}}, B) < n - (\mu - 3\tau) + 2\tau = 13\tau + 3$. So B is $13\tau+3$-bounded. Then by Lemma 7.23 the periodic part of $\mathrm{can}_{r-1}(B\cdot\ldots\cdot B)$ is $27\tau+7$-bounded. Since $27\tau + 7 < \mu - \tau = n - 9\tau - 3$, we take a period of this periodic part as $\widetilde{B}$.

3. The last case is $|u| > |A|$. Then there exists a cyclic shift of A equal to u_1 a prefix of u with $\Lambda_r(u_1) > \Lambda_r(u) - 1 > \mu - \tau - 1$. Then we take $\mathrm{can}_{r-1}(u_1) = L_1\widetilde{u}_1R_1$ and argue as

above. Using the same notations, we have that $\Lambda_r(\widehat{\widetilde{v}}, B) < n-(\mu-3\tau-1)+2\tau = 13\tau+4$, so B is $13\tau+4$-bounded. Hence the periodic part of $\mathrm{can}_{r-1}(B\cdot\ldots\cdot B)$ is $27\tau+9$-bounded. Since $27\tau+9 < \mu-\tau = n-9\tau-3$, we take a period of this periodic part as $\widetilde{B}$. □

Proposition 7.26. *IH 13 holds for rank r.*

Proof. By IH 8 and IH 13 for rank $r-1$ we can assume that $A \in \mathrm{Cycl}_{r-1}$. Let $\widetilde{B}$ be the conjugate of A given by Lemma 7.25. If $\widetilde{B} \in \mathrm{Can}_{r-1}$ and its cyclically reduced part is 3τ-free of rank r, the result follows from Corollary 7.24.

If the second case of Lemma 7.25 holds, then $\mathrm{can}_{r-1}(\widetilde{B}^K) = Z_1\widetilde{B}_1\widetilde{B}^{K-\delta}\widetilde{B}_1 Z_2 \in \mathrm{SCan}_{\mu,r}$, where $\widetilde{B}_1, \widetilde{B_2}$ are a prefix and suffix of B, respectively, Z_1, Z_2 are τ-free of rank r, and δ does not depend on K. Then Corollary 7.14 implies that $\mathrm{can}_r(\widetilde{B}^K) = \widetilde{L}\widetilde{X}^{K-\gamma}\widetilde{R}$, where $\widetilde{X}$ and $\widetilde{B}$ are conjugate in the group $\mathrm{F}/\langle\langle \mathrm{Rel}_1, \ldots, \mathrm{Rel}_r\rangle\rangle$, and $\widetilde{X}, \gamma$ do not depend on K. Since $A \equiv Y \cdot \widetilde{B} \cdot Y^{-1} \mod \langle\langle \mathrm{Rel}_1, \ldots, \mathrm{Rel}_r\rangle\rangle$, we have

$$\mathrm{can}_r(\underbrace{A\cdot\ldots\cdot A}_{K\ times}) = \mathrm{can}_r(Y)\cdot_r \mathrm{can}_r(\widetilde{B}^K)\cdot_r \mathrm{can}_r(Y^{-1}).$$

So, A satisfies IH 13 with $\widetilde{A} = \widetilde{X}$. □

8. Completion of the proof of Theorem 2.1

It is left to show that the canonical form stabilizes and that our relators $\bigcup_{i\in\mathbb{N}} \mathrm{Rel}_i$ yield a quotient group isomorphic to the free Burnside group $B(m, n)$. We start with the first point:

Lemma 8.1. *Assume that $A, B \in \cap_{i=0}^{\infty}\mathrm{Can}_i$. Then there exists r_0 such that $\mathrm{can}_{r_0}(A\cdot B) \in \cap_{i=0}^{\infty}\mathrm{Can}_i$.*

Proof. Since $A, B \in \mathrm{Can}_i$ for all $i \geqslant 0$, by IH 10 we have

$$\mathrm{can}_i(A\cdot B) = A_i D_3^{(i)} B_i,\ A = A_i D_1^{(i)} X,\ B = X^{-1} D_2^{(i)} B_i,$$

where $X\cdot X^{-1}$ is the maximal cancellation in $A\cdot B$ and $(D_1^{(i)}, D_2^{(i)}, D_3^{(i)})$ is a canonical triangle of rank i, A_{i+1} is a prefix of A_i and B_{i+1} is a suffix of B_i. Let $r_0 \geqslant 0$ be such that for all $i \geqslant r_0$ we have $A_{r_0} = A_i, B_{r_0} = B_i$. Since the maximal cancellation X does not depend on i, this implies $D_1^{(i)} = D_1^{(r_0)}$ and $D_2^{(i)} = D_2^{(r_0)}$ and hence, by IH 10, also $D_3^{(i)} = D_3^{(r_0)}$ for all $i \geqslant r_0$. We obtain $\mathrm{can}_i(A\cdot B) = A_{r_0} D_3^{(r_0)} B_{r_0} = \mathrm{can}_{r_0}(A\cdot B)$ for all $i \geqslant r_0$ and r_0 is as required. □

Proposition 8.2. *For every word $A \in \mathrm{Can}_{-1}$ there exists r_0 such that $\mathrm{can}_{r_0}(A) \in \cap_{i=0}^{\infty}\mathrm{Can}_i$.*

Proof. Clearly we may assume that A is reduced, so $A \in \mathrm{Can}_0$, and do induction on $|A|$. If $|A| = 1$, then it follows from Remark 8.8 that $A \in \cap_{i=0}^{\infty}\mathrm{Can}_i$.

For the induction step assume that $A = A_1 x$, where x is a single letter. By our induction assumption there is some s such that $\mathrm{can}_s(A_1), \mathrm{can}_s(x) = x \in \cap_{i=0}^{\infty}\mathrm{Can}_i$. By Lemma 8.1 and Corollary 3.5, there exists some $r_0 \geqslant s$ such that

$$\mathrm{can}_{r_0}(A) = \mathrm{can}_{r_0}(A_1 x) = \mathrm{can}_{r_0}(\mathrm{can}_{r_0}(A_1)\cdot x) \in \cap_{i=0}^{\infty}\mathrm{Can}_i.$$

□

By Proposition 8.2 the sequence $\mathrm{can}_i(A)$, $i \geqslant 0$, stabilizes after a finite number of steps (depending on A). Therefore we can now define:

Definition 8.3. *For* $A \in \mathrm{Can}_{-1}$*, the canonical form* $\mathrm{can}(A)$ *of* A *is defined as* $\mathrm{can}(A) = \mathrm{can}_i(A)$ *where* i *is such that* $\mathrm{can}_i(A) \in \bigcap_{i=0}^{\infty} \mathrm{Can}_i$*, and* $\mathrm{Can} = \{ \mathrm{can}(A) \mid A \in \mathrm{Can}_{-1}\} = \bigcap_{i=0}^{\infty} \mathrm{Can}_i$.

It follows directly from the definition, IH 6 and Remark 3.4 that we have

Corollary 8.4. $\mathrm{can}(\mathrm{can}(A)) = \mathrm{can}(A) \equiv A \mod \langle\langle\ \mathrm{Rel}_i \mid\ i \geqslant 0\ \rangle\rangle$ *for* $A \in \mathrm{Can}_{-1}$.

Lemma 8.5. *Let* a *be a primitive word and* a^τ *be an occurrence in* $A_i \in \mathrm{Can}_i$ *for every* $i \geqslant 0$*. Then* $a^n \in \mathrm{Rel}_r$ *for some* $r \geqslant 0$.

Proof. By the definition of Cycl_i, $a \in \mathrm{Cycl}_i$ for all $i \geqslant 0$. The proof is by induction on $|a|$. If $|a| = 1$, then by definition $a^n \in \mathrm{Rel}_1$.

If $|a| > 1$ and a cyclically contains b^τ, then $|b| < |a|$. By the induction hypothesis $b^n \in \mathrm{Rel}_j$ for some j. Let r be minimal such that a does not cyclically contain any occurrence of the form b^τ with $b^n \in \mathrm{Rel}_r$. Then we have that $a^n \in \mathrm{Rel}_r$ by the definition of Rel_r. □

Proposition 8.6. *If* $A = XWX^{-1} \in \mathrm{Can}_0$ *with* W *cyclically reduced, then* $W^n \in \langle\langle \mathrm{Rel}_i \mid\ i \in \mathbb{N}\rangle\rangle$.

Proof. Write $\mathcal{H} = \langle\langle \mathrm{Rel}_i \mid\ i \in \mathbb{N}\rangle\rangle$ and suppose $W^n \notin \mathcal{H}$. By Corollary 8.4 we may assume $A = \mathrm{can}(A)$. Let r be minimal such that A does not contain any maximal occurrence of rank r of Λ_r-measure $\geqslant 3\tau$. By IH 13 for all $j \geqslant 0$ we have

$$\mathrm{can}_j(\underbrace{A \cdot \ldots \cdot A}_{K\ times}) = T_j \widetilde{A}_j^{K-\gamma_j} S_j \ \textit{for}\ K \geqslant \gamma_j,$$

where $\widetilde{A}_j, T_j, S_j, \gamma_j$ depend only on A and j, and A and $\widetilde{A}_j$ are conjugate in $\mathrm{F}/\mathcal{H}$. Lemma 7.24 implies that $\widetilde{A}_j = \widetilde{A}_r$ for all ranks $j \geqslant r$. Hence $\widetilde{A}_r^\tau$ is a subword of words from Can_i for all i. Thus by Lemma 8.5 $\widetilde{A}_r^n \in \mathcal{H}$ and hence $W^n \in \mathcal{H}$, a contradiction. □

Since the sets $\mathrm{Rel}_i, i \geqslant 0$, consist of n-th powers, we now obtain:

Corollary 8.7. *The normal subgroup of* F *generated by* $\langle\langle\ \mathrm{Rel}_i \mid\ i \in \mathbb{N}\rangle\rangle$ *coincides with the normal subgroup generated by all* n*-th powers.*

Theorem 8.1. *For every* $A, B \in \mathrm{Can}_{-1}$ *the words* A *and* B *represent the same element of the group* $B(m,n)$ *if and only if* $\mathrm{can}(A) = \mathrm{can}(B)$.

Proof. If $\mathrm{can}(A) = \mathrm{can}(B)$, then $A \equiv B \mod \langle\langle\ \mathrm{Rel}_i \mid\ i \geqslant 0\ \rangle\rangle$ by Corollary 8.4. Thus clearly A and B represent the same element of the group $B(m,n)$.

Converseley, if A and B represent the same element in $B(m,n)$, then, by definition, $A \equiv B \mod \langle\langle w_1^n, \ldots, w_k^n \rangle\rangle$ for some cyclically reduced words w_i. By Corollary 8.7 we have $w_i^n \in \langle\langle \mathrm{Rel}_0, \mathrm{Rel}_1, \ldots, \mathrm{Rel}_r\rangle\rangle$ for some r and so $A \equiv B \mod \langle\langle \mathrm{Rel}_0, \ldots, \mathrm{Rel}_r\rangle\rangle$. Thus $\mathrm{can}_r(A) = \mathrm{can}_r(B)$ by IH 8 and, by construction, $\mathrm{can}(A) = \mathrm{can}(B)$. □

Finally we are ready to prove Theorem 2.1:

Remark 8.8. If $A \in \mathrm{Can}_0$ and all subwords of A of the form $a^k a_1$, $a = a_1 a_2$, satisfy $\Lambda_a(a^k a_1) \leqslant \frac{n}{2} - 5\tau - 2$, then by iterated application of IH 4 we have $A \in \cap_{i=0}^{\infty} \mathrm{Can}_i$.

The proof of Theorem 2.1. By Theorem 8.1 words $A, B \in \mathrm{Can}_{-1}$ represent the same element in $B(m,n)$ if and only if $\mathrm{can}(A) = \mathrm{can}(B)$. Now consider the set of cube free words in Can_0, which is an infinite set by [6]. Clearly for our choice of the exponent n we have $3 \leqslant \frac{n}{2} - 5\tau - 2$ and hence by Remark 8.8 every cube free word is in Can. Thus Can is infinite and hence so is $B(m,n)$. □

References

[1] S.I. Adian (1979) The Burnside problem and identities in groups. Translated from the Russian by John Lennox and James Wiegold. Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], 95. Springer-Verlag, Berlin-New York.

[2] S.I. Adian, P.S. Novikov, "Infinite periodic groups. I–III", Izv. Akad. Nauk SSSR, Ser. Mat. 32(1–3), 212–244, 251–524, 709–731 (1968) [Math. USSR, Izv. 2, 209–236, 241–479, 665–685 (1968)] (in Russian).

[3] W. Burnside, "On an unsettled question in the theory of discontinuous groups", Quart J. Pure and Applied Math. 33(1902), 230–238.

[4] C. Choffrut, J. Karhumäki, "Combinatorics on words", G. Rozenberg, A. Salomaa (Eds.), Handbook of Formal Languages, Vol. I, Springer, Berlin, Heidelberg (1997), pp. 329–438

[5] R. Coulon, "On the geometry of burnside quotients of torsion free hyperbolic groups", International Journal of Algebra and Computation Vol. 24, No. 03, pp. 251–345 (2014).

[6] E.D. Fife, "Binary sequences which contain no BBb", Trans. Amer. Math. Soc. 261 (1980) 115–13.

[7] N. Fine, H. Wilf, Uniqueness theorems for periodic functions. Proc. Amer. Math. Soc.16(1965), 109–114.

[8] E.S. Golod, I.R. Shafarevich, "O bashne poleii klassov" Izv. Akad. Nauk SSSR Ser. Mat.28,2 (1964), 261–272 (in Russian).

[9] W. H. Gottschalk, G.A. Hedlund, "A characterization of the Morse minimal set", Proc. Amer. Math. Soc. 15 (1964), 70–74.

[10] T. Delzant, M. Gromov, "Courbure ḿesoscopique et théorie de la toute petite simplification", J. Topology 1(4) (2008) 804–836.

[11] M. Hall, Jr., "Solution of the Burnside Problem for Exponent Six", Illinois J. of Math. 2 (1958), 764–786.

[12] S.V. Ivanov (1994), "The Free Burnside Groups of Sufficiently Large Exponents", International Journal of Algebra and Computation. 04: 1–308.

[13] R. Lyndon, P. Schupp, "Combinatorial group theory. Reprint of the 1977 edition", Classics in Mathematics. Springer-Verlag, Berlin. (2001).

[14] I.G. Lysenok (1996). "Infinite Burnside groups of even exponent", Izvestiya: Mathematics. 60 (3): 453–654.

[15] I.G. Lysenok (2023). A sample iterated small cancellation theory for groups of Burnside type, https://arxiv.org/abs/2301.05000.

[16] A.Yu. Ol'shanskii (1989) Geometry of defining relations in groups. Translated from the 1989 Russian original by Yu. A. Bakhturin (1991) Mathematics and its Applications (Soviet Series), 70. Dordrecht: Kluwer Academic Publishers Group.

[17] E. Rips, "Generalized small cancellation theory and applications I. The word problem.", Israel J. Math. 41, 1–146 (1982).

[18] I.N. Sanov, "Reshenie problemy Bernsaiida dlya pokazatelya 4", Leningrad State Univ. Ann 10 (1940), 166–170 (in Russian).

[19] E. Zelmanov (1991). "Solution of the Restricted Burnside Problem for Groups of Odd Exponent". Mathematics of the USSR-Izvestiya. 36 (1): 41–60.

[20] E. Zelmanov (1992). "A Solution of the Restricted Burnside Problem for 2-groups". Mathematics of the USS–Sbornik. 72 (2): 543–565.