TWO CURIOUS q-SUPERCONGRUENCES AND THEIR EXTENSIONS

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Abstract. We prove two single-parameter q-supercongruences which were recently conjectured by Guo, and establish their further extensions with one more parameter. Crucial ingredients in the proof are the terminating form of q-binomial theorem and a Karlsson-Minton type summation formula due to Gasper. Incidentally, an assertion of Wang, Li and Tang is also verified by establishing its q-analogue.

1. Introduction

Except for investigating hypergeometric families of Calabi-Yau manifolds, Rodriguez-Villegas [14] also observed (numerically) many possible supercongruences, including the following one: for any odd prime p,

$$\sum_{k=0}^{p-1}\frac{(\frac{1}{2})_k^2}{k!^2}\equiv(-1)^{(p-1)/2}\pmod{p^2}. \tag{1.1}$$

Here and in what follows, the the Pochhammer symbol is defined as and $(x)_0=1$ and $(x)_n=x(x+1)\cdots(x+n-1)$ with n a positive integer. The congruence (1.1) was first proved by Mortenson [13]. Later, Deines et al. [3] gave the following generalization of (1.1): for any integer $d>1$ and prime $p\equiv 1 \pmod d$,

$$\sum_{k=0}^{p-1}\frac{(\frac{d-1}{d})_k^d}{k!^d}\equiv-\Gamma_p\big(\tfrac{1}{d}\big)^d\pmod{p^2}, \tag{1.2}$$

where $\Gamma_p(x)$ denotes the p-adic Gamma function (cf. [2, §1.12]).

For any complex variable x and integer n, define the *q-shifted factorial* as

$$(x;q)_\infty=\prod_{k=0}^{\infty}(1-xq^k)\quad\text{and}\quad(x;q)_n=\frac{(x;q)_\infty}{(xq^n;q)_\infty}.$$

For simplicity, we also adopt the compact expression

$$(x_1,x_2,\ldots,x_m;q)_n=(x_1;q)_n(x_2;q)_n\cdots(x_m;q)_n.$$

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Moreover, $[n] := [n]_q = 1+q+\cdots+q^{n-1}$ denotes the *q-integer*. A q-analogue of (1.1) can be found in Guo and Zeng [10, Corollary 2.4]. Recently, Guo [6, Theorem 1.1] established a q-extension of (1.2): for any integers $d, n > 1$ with $n \equiv 1 \pmod{d}$,

$$\sum_{k=0}^{n-1} \frac{(q^{d-1};q^d)_k^d q^{dk}}{(q^d;q^d)_k^d} \equiv \frac{(q^d;q^d)_{(d-1)(n-1)/d} q^{(d-1)(n-1)(d+n-1)/(2d)}}{(q^d;q^d)_{(n-1)/d}^{d-1}(-1)^{(d-1)(n-1)/d}} \pmod{\Phi_n(q)^2}. \tag{1.3}$$

Here and in what follows, the n-th *cyclotomic polynomial* $\Phi_n(q)$ is given by

$$\Phi_n(q) = \prod_{\substack{1\le k\le n\\ \gcd(k,n)=1}} (q-\zeta^k),$$

where ζ is an n-th primitive root of unity. In the same paper, Guo also presented two analogous q-supercongruences as follows.
(i) For any odd integer $d \ge 3$ and integer n with $n \equiv -1 \pmod{d}$ and $n \ge 2d-1$,

$$\begin{aligned}&\sum_{k=0}^{n-1} \frac{(q^{d+1};q^d)_k^{d-1}(q^{1-d};q^d)_k q^{dk}}{(q^d;q^d)_k^d}\\ &\equiv \frac{(1-q)(1-q^{d-1})(q^d;q^d)_{n-1-(n+1)/d}}{-(q^d;q^d)_{(n+1)/d}^{d-1}} q^{(d(d+n)(n+1)-(n+1)^2)/(2d)-1} \pmod{\Phi_n(q)^2}.\end{aligned} \tag{1.4}$$

(ii) For any even integer $d \ge 4$ and positive integer n with $n \equiv -1 \pmod{d}$,

$$\begin{aligned}&\sum_{k=0}^{n-1} \frac{(q^{d+1};q^d)_k^{d-2}(q;q^d)_k^2 q^{dk}}{(q^d;q^d)_k^d}\\ &\equiv \frac{(1-q)^2(q^d;q^d)_{n-1-(n+1)/d}}{(-1)^{(n+1)/d}(q^d;q^d)_{(n+1)/d}^{d-1}} q^{(d(d+n)(n+1)-(n+1)^2)/(2d)-2} \pmod{\Phi_n(q)^2}.\end{aligned} \tag{1.5}$$

For recent progress on congruences and q-congruences, we recommend the literatures [5, 7–9, 12, 16–21] to readers.

As the complements of (1.4) and (1.5), Guo [6, Conjectures 5.2 and 5.3] proposed two q-supercongruences lined as the following two theorems respectively. Moreover, we shall establish the common generalization of (1.4) and (1.6) and also that of (1.5) and (1.7) with two parameters.

Theorem 1.1 (6, Conjecture 5.2)**.** *Let $d \ge 4$ be an even integer and n an integer with $n \equiv -1 \pmod{d}$ and $n \ge 2d-1$. Then,*

$$\begin{aligned}&\sum_{k=0}^{n-1} \frac{(q^{d+1};q^d)_k^{d-1}(q^{1-d};q^d)_k q^{dk}}{(q^d;q^d)_k^d}\\ &\equiv \frac{(1-q)(1-q^{d-1})(q^d;q^d)_{n-1-(n+1)/d}}{-(-1)^{(n+1)/d}(q^d;q^d)_{(n+1)/d}^{d-1}} q^{(d(d+n)(n+1)-(n+1)^2)/(2d)-1} \pmod{\Phi_n(q)^2}.\end{aligned} \tag{1.6}$$

Theorem 1.2 (6, Conjecture 5.3). *Let $d \geq 3$ be an odd integer and n a positive integer with $n \equiv -1 \pmod{d}$. Then,*

$$\sum_{k=0}^{n-1} \frac{(q^{d+1};q^d)_k^{d-2}(q;q^d)_k^2 q^{dk}}{(q^d;q^d)_k^d} \equiv \frac{(1-q)^2(q^d;q^d)_{n-1-(n+1)/d}}{(q^d;q^d)_{(n+1)/d}^{d-1}} q^{(d(d+n)(n+1)-(n+1)^2)/(2d)-2} \pmod{\Phi_n(q)^2}. \tag{1.7}$$

Particularly, letting $n = p$ be an odd prime and $q \to 1$ in Theorems 1.1 and 1.2, we arrive at Guo's two congruences [6, (5.4) and (5.5)], which have been successfully proved by Wang, Li and Tang [15]. Besides, the authors [15] conjectured that: for any integers $d \geq 2$, n with $n \equiv -1 \pmod{d}$ and $n \geq 2d-1$,

$$\frac{(n-1)!^d d^{dn-d}}{n^2} \sum_{k=0}^{n-1} \frac{(\frac{d+1}{d})_k^{d-2}(\frac{1}{d})_k(\frac{1-d}{d})_k}{(1)_k^d} \in \mathbb{Z}. \tag{1.8}$$

The assertion (1.8) shall be proved by determining the following q-congruence.

Theorem 1.3. *Let $d \geq 2$ and n be integers with $n \equiv -1 \pmod{d}$ and $n \geq 2d-1$. Then,*

$$\frac{(q^d;q^d)_{n-1}^d}{(1-q)^{dn-d}} \sum_{k=0}^{n-1} \frac{(q^{d+1};q^d)_k^{d-2}(q,q^{1-d};q^d)_k q^{dk}}{(q^d;q^d)_k^d} \equiv 0 \pmod{[n]^2}.$$

Clearly, (1.8) follows by letting $q \to 1$ in Theorem 1.3.

The rest of this paper is organized as follows. We shall employ the terminating form of q-binomial theorem to confirm Theorems 1.1 and 1.3 in Section 2. Section 3 is devoted to proving Theorem 1.2 by means of Gasper's Karlsson-Minton type summation formula. In Section 4, we shall present two-parametric extensions of (1.4) and (1.6), together with (1.4) and (1.7). The 'creative microscoping' method introduced by Guo and Zudilin [11] will also be employed in the proofs.

2. Proofs of Theorems 1.1 and 1.3

To prove Theorems 1.1 and 1.3, we require the following key result.

Lemma 2.1. *Let d, r be positive integers with $d \geq 3 + r$ and $\gcd(d, r) = 1$. Let n be an integer such that $n \equiv -r \pmod{d}$ and $n \geq 2d - r$. Then,*

$$\sum_{k=0}^{n-1} \frac{(q^{d+r};q^d)_k^{d-r-1}(q^r;q^d)_k^r(q^{r-d};q^d)_k q^{dk}}{(q^d;q^d)_k^d} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{2.1}$$

Especially, for integers $d \geq 4$ and $n \geq 2d-1$ with $n \equiv -1 \pmod{d}$, we have,

$$\sum_{k=0}^{n-1} \frac{(q^{d+1};q^d)_k^{d-2}(q,q^{1-d};q^d)_k q^{dk}}{(q^d;q^d)_k^d} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{2.2}$$

Proof. Obviously, the ratio $(q^{d+r};q^d)_k^{d-r-1}/(q^d;q^d)_k^d$ contains the factor $(1-q^{(d-1)n})^2$ for $n-(n+r)/d\le k\le n-1$. Besides, it is easy to find that for $k\ge 0$,

$$(q^r;q^d)_k^r(q^{r-d};q^d)_k=(1-q^{r-d})(1-q^r)^{r+1}(q^{d+r};q^d)_{k-2}^{r+1}(1-q^{dk-d+r})^r.$$

Equivalently, we just prove the truth of the subsequent congruence,

$$\sum_{k=0}^{n-1-(n+r)/d}\frac{(q^{d+r};q^d)_k^{d-r-1}(q^{d+r};q^d)_{k-2}^{r+1}(1-q^{dk-d+r})^r q^{dk}}{(q^d;q^d)_k^d}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{2.3}$$

Before that, we confirm the following parametric generalizations of (2.3), which are both symmetric about a and a^{-1}:

$$\begin{aligned}
&\sum_{k=0}^{n-1-(n+r)/d}\frac{(a^{d-1}q^{d+r},a^{d-3}q^{d+r},\cdots,a^{r+2}q^{d+r},a^{-r-2}q^{d+r},\cdots,a^{3-d}q^{d+r},a^{1-d}q^{d+r};q^d)_k q^{dk}}{(q^d,a^{d-2}q^d,a^{d-4}q^d,\cdots,a^{r+3}q^d,a^{-r-1}q^d,\cdots,a^{4-d}q^d,a^{2-d}q^d;q^d)_k}\\
&\qquad\times\frac{(a^rq^{d+r},a^{r-2}q^{d+r},\cdots,a^{-r+2}q^{d+r},a^{-r}q^{d+r};q^d)_{k-2}(1-q^{dk-d+r})^r}{(a^{r+1}q^d,a^{r-1}q^d,\cdots,a^{-r+3}q^d,a^{-r+1}q^d;q^d)_k}\\
&\quad\equiv 0 \pmod{(1-aq^n)(a-q^n)}
\end{aligned}\tag{2.4}$$

for $d+r\equiv 1 \pmod 2$, and

$$\begin{aligned}
&\sum_{k=0}^{n-1-(n+r)/d}\frac{(a^{d-1}q^{d+r},a^{d-3}q^{d+r},\cdots,a^{r+3}q^{d+r},q^{d+r},a^{-r-3}q^{d+r},\cdots,a^{3-d}q^{d+r},a^{1-d}q^{d+r};q^d)_k}{(q^d,a^{d-2}q^d,a^{d-4}q^d,\cdots,a^{r+4}q^d,aq^d,a^{-r-2}q^d,\cdots,a^{4-d}q^d,a^{2-d}q^d;q^d)_k q^{-dk}}\\
&\qquad\times\frac{(a^{r+1}q^{d+r},a^{r-1}q^{d+r},\cdots,a^2q^{d+r},a^{-2}q^{d+r},\cdots,a^{-r+1}q^{d+r},a^{-r-1}q^{d+r};q^d)_{k-2}}{(a^{r+2}q^d,a^rq^d,\cdots,a^3q^d,a^{-1}q^d,\cdots,a^{-r+2}q^d,a^{-r}q^d;q^d)_k(1-q^{dk-d+r})^{-r}}\\
&\quad\equiv 0 \pmod{(1-aq^n)(a-q^n)}
\end{aligned}\tag{2.5}$$

for $d\equiv r\equiv 1 \pmod 2$.

Next, we concentrate on the proof of (2.4). Since $\gcd(d,r)=\gcd(d,n)=1$, the denominator of the left-hand side of (2.4) does not contain the factor $1-aq^n$ or $a-q^n$. Thus, for $a=q^{-n}$ or $a=q^n$, noticing that $(q^{d+r-(d-1)n};q^d)_k=0$ for $k>n-1-(n+r)/d$, the left-hand side of (2.4) equals

$$\begin{aligned}
&\sum_{k=0}^{n-1-(n+r)/d}\frac{(q^{d+r-(d-1)n},q^{d+r-(d-3)n},\cdots,q^{d+r-(r+2)n},q^{d+r+(r+2)n},\cdots,q^{d+r+(d-1)n};q^d)_k q^{dk}}{(q^d,q^{d-(d-2)n},\cdots,q^{d-(r+3)n},q^{d+(r+1)n},\cdots,q^{d+(d-2)n};q^d)_k}\\
&\qquad\times\frac{(q^{d+r-rn},q^{d+r-(r-2)n},\cdots,q^{d+r+(r-2)n},q^{d+r+rn};q^d)_{k-2}(1-q^{dk-d+r})^r}{(q^{d-(r+1)n},q^{d-(r-1)n},\cdots,q^{d+(r-3)n},q^{d+(r-1)n};q^d)_k}\\
&=\sum_{k=0}^{n-1-(n+r)/d}(-1)^k q^{d\binom{n-1-(n+r)/d-k}{2}}\begin{bmatrix}n-1-(n+r)/d\\ k\end{bmatrix}_{q^d}P(q^{dk}).
\end{aligned}\tag{2.6}$$

Here $P(q^{dk})$ is a polynomial in q^{dk} of degree $n-2-(n+r)/d$ and

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

denotes the *q-binomial coefficient*. In the derivation of (2.6), we have utilized the following formulas, which can be certified readily:

$$\frac{(q^{d+r-(d-1)n};q^d)_k q^{dk}}{(q^d;q^d)_k} = (-1)^k \begin{bmatrix} n-1-(n+r)/d \\ k \end{bmatrix}_{q^d} q^{d\binom{k}{2}+(n+2d+r-dn)k},$$

$$d\binom{k}{2} + (n+2d+r-dn)k = d\binom{n-1-(n+r)/d-k}{2} - d\binom{n-1-(n+r)/d}{2},$$

for $1 \le j \le d-1$ and $j \ne (d-r-1)/2, \cdots, (d+r-1)/2$

$$\frac{(q^{d+r-(d-2j-1)n};q^d)_k}{(q^{d-(d-2j)n};q^d)_k} = \frac{(q^{d-(d-2j)n+dk};q^d)_{(n+r)/d}}{(q^{d-(d-2j)n};q^d)_{(n+r)/d}}, \tag{2.7}$$

and for $(d-r-1)/2 \le j \le (d+r-1)/2$

$$\frac{(q^{d+r-(d-2j-1)n};q^d)_{k-2}}{(q^{d-(d-2j)n};q^d)_k} = \frac{(q^{d-(d-2j)n+dk};q^d)_{(n+r)/d-2}}{(q^{d-(d-2j)n};q^d)_{(n+r)/d}}. \tag{2.8}$$

We see that the right-hand sides of (2.7) and (2.8) are polynomials in q^{dk} of degree $(n+r)/d$ and $(n+r)/d-2$ respectively. Then, by the terminating form of q-binomial theorem (see, for example, [1, p. 36]):

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}+jk} = 0 \quad \text{for } 0 \le j \le n-1,$$

we deduce that the right-hand side of (2.6) equals zero. This implies that the congruence (2.4) holds modulo $(1-aq^n)(a-q^n)$. Similarly, we are able to prove (2.5) and we omit the details here.

Apparently, the limit $(1-aq^n)(a-q^n)$ has the factor $\Phi_n(q)^2$ as $a \to 1$. Moreover, the denominators on the left-hand side of (2.4) and (2.5) are both coprime with $\Phi_n(q)$ as $a \to 1$. Hence, letting $a \to 1$ in (2.4) and (2.5), we are led to (2.3). Thus, we verify the correction of (2.1). □

Now, we are ready to prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1. It is routine to check that: for any integer $d \ge 2$,

$$\sum_{k=0}^{n-1} \frac{(q^{d+1};q^d)_k^{d-1}(q^{1-d};q^d)_k q^{dk}}{(q^d;q^d)_k^d}$$
$$= [d]\sum_{k=0}^{n-1} \frac{(q^{d+1};q^d)_k^{d-2}(q,q^{1-d};q^d)_k q^{dk}}{(q^d;q^d)_k^d} - q[d-1]\sum_{k=0}^{n-1} \frac{(q^{d+1};q^d)_k^{d-2}(q;q^d)_k^2 q^{dk}}{(q^d;q^d)_k^d}, \tag{2.9}$$

where we have applied the following identity:

$$(q^{d+1}, q^{1-d}; q^d)_k = -q[d-1]\left(1+\frac{1-q^d}{q^d-q^{dk+1}}\right)(q;q^d)_k^2.$$

Hence, from (1.5), (2.2) and (2.9), we arrive at Theorem 1.1. □

Proof of Theorem 1.3. At first, we claim that the following two q-congruences hold, which are the $d=2,3$ cases of (2.2): for any integer n, modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1}\frac{(q,q^{-1};q^2)_k q^{2k}}{(q^2;q^2)_k^2} \equiv 0 \quad \text{if } n\geq 3 \text{ and } n\equiv 1 \pmod 2, \tag{2.10}$$

$$\sum_{k=0}^{n-1}\frac{(q^4,q,q^{-2},;q^3)_k q^{3k}}{(q^3;q^3)_k^3} \equiv 0 \quad \text{if } n\geq 5 \text{ and } n\equiv 2 \pmod 3. \tag{2.11}$$

In fact, taking $d=2$ in (2.9), we have, modulo $\Phi_n(q)^2$,

$$\begin{aligned}[2]\sum_{k=0}^{n-1}\frac{(q,q^{-1};q^2)_k q^{2k}}{(q^2;q^2)_k^2} &= \sum_{k=0}^{n-1}\frac{(q^3,q^{-1};q^2)_k q^{2k}}{(q^2;q^2)_k^2} + q\sum_{k=0}^{n-1}\frac{(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k^2}\\ &\equiv (-1)^{(n+1)/2}q^{(n^2+3)/4} + q(-1)^{(n-1)/2}q^{(n^2-1)/4},\end{aligned} \tag{2.12}$$

where we have used a q-supercongruence from the work [6, p. 9] and the $d=2$ case of (1.3). Analogously, letting $d=3$ in (2.9), we have, modulo $\Phi_n(q)^2$,

$$\begin{aligned}[3]\sum_{k=0}^{n-1}\frac{(q^4,q,q^{-2};q^3)_k q^{3k}}{(q^3;q^3)_k^3} &= \sum_{k=0}^{n-1}\frac{(q^4,q^4,q^{-2};q^3)_k q^{3k}}{(q^3;q^3)_k^3} + q[2]\sum_{k=0}^{n-1}\frac{(q^4,q,q;q^3)_k q^{3k}}{(q^3;q^3)_k^3}\\ &\equiv \frac{(1-q)(1-q^2)(q^3;q^3)_{(2n-1)/3-1}}{-(q^3;q^3)_{(n+1)/3}^2}q^{(3(3+n)(n+1)-(n+1)^2)/6-1}\\ &\quad + q[2]\frac{(1-q)^2(q^3;q^3)_{(2n-1)/3-1}}{(q^3;q^3)_{(n+1)/3}^2}q^{(3(3+n)(n+1)-(n+1)^2)/6-2},\end{aligned} \tag{2.13}$$

where we have utilized (1.4) and (1.7) with $d=3$. Therefore, the q-results (2.10) and (2.11), respectively, follow from (2.12) and (2.13). In addition, we notice that: for any integer $d\geq 2$,

$$\frac{(q^d;q^d)_{n-1}^d}{(1-q)^{dn-d}} = \prod_{1\leq m<n}\left(\frac{1-q^{md}}{1-q}\right)^d \equiv 0 \pmod{\prod_{\substack{1<m<n\\ m|n}}\Phi_m(q)^2}. \tag{2.14}$$

Combining the following property

$$[n] = \Phi_n(q)\prod_{\substack{1<m<n\\ m|n}}\Phi_m(q)$$

with the congruences (2.2), (2.10), (2.11) and (2.14), we obtain Theorem 1.3. □

3. Proof of Theorem 1.2

Proof. We shall prove Theorem 1.2 by using Gasper's Karlsson-Minton type summation formula (see [4, (1.9.9)] for more general form): for nonnegative integers $N, n_1, \cdots, n_m$ with $N = n_1 + \cdots + n_m$,

$$\sum_{k=0}^{N} \frac{(q^{-N}, b_1 q^{n_1}, \cdots, b_m q^{n_m}; q)_k}{(q, b_1, \cdots, b_m; q)_k} q^k = (-1)^N \frac{(q;q)_N b_1^{n_1} \cdots b_m^{n_m}}{(b_1;q)_{n_1} \cdots (b_m;q)_{n_m}} q^{\binom{n_1}{2} + \cdots + \binom{n_m}{2}}. \tag{3.1}$$

Below, the two conditions $d > 3$ and $d = 3$ of Theorem 1.2 shall be discussed separately.
(i) For $d > 3$, we consider the following generalization of (1.7): modulo $(1 - aq^n)(a - q^n)$,

$$\begin{aligned} &\sum_{k=0}^{n-1} \frac{(a^{d-1}q^{d+1}, a^{d-3}q^{d+1}, \cdots, a^4 q^{d+1}, a^2 q, q^{d+1}, a^{-2}q, a^{-4}q^{d+1}, \cdots, a^{3-d}q^{d+1}, a^{1-d}q^{d+1}; q^d)_k}{(a^{d-2}q^d, a^{d-4}q^d, \cdots, a^{4-d}q^d, a^{2-d}q^d; q^d)_k (q^d;q^d)_k q^{-dk}} \\ &\equiv \frac{(1-a^2q)(1-q/a^2)(q^d;q^d)_{n-1-(n+1)/d}}{(a^{d-2}q^d, a^{d-4}q^d, \cdots, a^{4-d}q^d, a^{2-d}q^d; q^d)_{(n+1)/d}} q^{(d(d+n)(n+1)-(n+1)^2)/(2d)-2}. \end{aligned} \tag{3.2}$$

In fact, we take the substitutions $q \to q^d$, $m = d-1$, $N = n-1-(n+1)/d$, $b_j = q^{d-(d-2j)n}$ $(1 \le j \le d-1)$, $n_{(d-3)/2} = n_{(d+1)/2} = (n+r)/d - 1$ and $n_j = (n+r)/d$ $(1 \le j \le d-1$ and $j \ne (d-3)/2, (d+1)/2)$ in (3.1). Thus, for $a = q^{-n}$ or $a = q^n$, the left-hand side of (3.2) can be simplified as

$$\begin{aligned} &\frac{(-1)^{n-1-(n+1)/d} q^{(d-1)(n+1)-2(d-n)+d(d-3)\binom{(n+1)/d}{2}+2d\binom{(n+1)/d-1}{2}}}{(q^{d-(d-2)n}, q^{d-(d-4)n}, \cdots, q^{d-5n}, q^{d-n}, q^{d+3n}, \cdots, q^{d+(d-4)n}, q^{d+(d-2)n}; q^d)_{(n+1)/d}} \\ &\quad \times \frac{(q^d;q^d)_{n-1-(n+1)/d}}{(q^{d-3n}, q^{d+n}; q^d)_{(n+1)/d-1}} \\ &= \frac{(1-q^{1-2n})(1-q^{1+2n})(q^d;q^d)_{n-1-(n+1)/d} q^{(d(d+n)(n+1)-(n+1)^2)/(2d)-2}}{(q^{d-(d-2)n}, q^{d-(d-4)n}, \cdots, q^{d+(d-4)n}, q^{d+(d-2)n}; q^d)_{(n+1)/d}}. \end{aligned}$$

Namely, the q-congruence (3.2) is true modulo $(1 - aq^n)(a - q^n)$.
(ii) For $d = 3$, we regard the following q-congruence: modulo $(1 - aq^n)(a - q^n)$,

$$\sum_{k=0}^{n-1} \frac{(a^2q, q^4, q/a^2; q^3)_k q^{3k}}{(aq^3, q^3/a, q^3; q^3)_k} \equiv \frac{(1-a^2q)(1-q/a^2)(q^3;q^3)_{n-1-(n+1)/3}}{(aq^3, q^3/a; q^3)_{(n+1)/3}} q^{(n^2+5n-2)/3}. \tag{3.3}$$

Actually, the q-congruence (3.3) follows by choosing $q \to q^3$, $m = 2$, $N = (2n-1)/3$, $b_1 = q^{3-n}$, $b_2 = q^{3+n}$, $n_1 = (n+1)/3$ and $n_2 = (n+1)/3 - 1$ in (3.1). Hence, we finish the proof of Theorem 1.2 by letting $a \to 1$ in (3.2) and (3.3). □

4. The extensions of Theorems 1.1 and 1.2

In this section, we present the common extension of (1.4) and (1.6) and also that of (1.5) and (1.7) with two parameters. Our discoveries are as follows.

Theorem 4.1. *Let d, r be positive integers with $d \geq 3 + r$ and $\gcd(d, r) = 1$. Let n be an integer such that $n \equiv -r \pmod{d}$ and $n \geq 2d - r$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1} \frac{(q^{d+r}; q^d)_k^{d-r} (q^r; q^d)_k^{r-1} (q^{r-d}; q^d)_k q^{dk}}{(q^d; q^d)_k^d} \equiv \frac{(1-q^r)^r (1-q^{d-r}) (q^d; q^d)_{n-1-(n+r)/d}}{-(-1)^{n-1-(n+r)/d} (q^d; q^d)_{(n+r)/d}^{d-1}} q^{A(d,n,r)}, \tag{4.1}$$

where $A(d, n, r) = [d(d + n)(n + r) + dn(r - 1) - (n + r)^2]/(2d) - r(r + 1)/2$. Moreover, the congruence (4.1) also holds for $d = 2, 3$ and $r = 1$.

Theorem 4.2. *Let d, r be positive integers with $d > r$ and $\gcd(d, r) = 1$. Let $n > 1$ be a positive integer such that $n \equiv -r \pmod{d}$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1} \frac{(q^{d+r}; q^d)_k^{d-r-1} (q^r; q^d)_k^{r+1} q^{dk}}{(q^d; q^d)_k^d} \equiv \frac{(1-q^r)^{r+1} (q^d; q^d)_{n-1-(n+r)/d}}{(-1)^{n-1-(n+r)/d} (q^d; q^d)_{(n+r)/d}^{d-1}} q^{A(d,n,r)-r}, \tag{4.2}$$

where $A(d, n, r)$ is stated in Theorem 4.1.

Clearly, taking $r = 1$ in Theorems 4.1 and 4.2, we can easily attain (1.4)-(1.7) on the basis of the parity of d.

In fact, the proof of Theorem 4.1 depends on the relationship between (4.1) and (4.2). So, we first prove Theorem 4.2 via Karlsson-Minton type summation (3.1) again.

Sketch of proof of Theorem 4.2. Totally classified by the parities of d and r, we discuss the following four q-congruences (4.3)–(4.6), which are all symmetric with respect to a and a^{-1}.

(i) For $d + r \equiv 1 \pmod{2}$, modulo $(1 - aq^n)(a - q^n)$,

$$\begin{aligned}&\sum_{k=0}^{n-1} \frac{(a^{d-1}q^{d+r}, a^{d-3}q^{d+r}, \cdots, a^{r+2}q^{d+r}, a^r q^r, a^{r-2}q^r, \cdots, a^{-r}q^r, \cdots, a^{1-d}q^{d+r}; q^d)_k}{(a^{d-2}q^d, a^{d-4}q^d, \cdots, a^{4-d}q^d, a^{2-d}q^d; q^d)_k (q^d; q^d)_k q^{-dk}} \\ &\equiv \frac{(1-a^r q^r)(1-a^{r-2}q^r)\cdots(1-a^{-r}q^r)(q^d; q^d)_{n-1-(n+r)/d}}{(-1)^{n-1-(n+r)/d} (a^{d-2}q^d, a^{d-4}q^d, \cdots, a^{4-d}q^d, a^{2-d}q^d; q^d)_{(n+r)/d}} q^{A(d,n,r)-r}\end{aligned} \tag{4.3}$$

with $d - r \geq 3$, and

$$\sum_{k=0}^{n-1} \frac{(a^{d-1}q^{d-1}, a^{d-3}q^{d-1}, \cdots, a^{3-d}q^{d-1}, a^{1-d}q^{d-1}; q^d)_k q^{dk}}{(q^d, a^{d-2}q^d, a^{d-4}q^d, \cdots, a^{4-d}q^d, a^{2-d}q^d; q^d)_k} \equiv B_q(d, n, d-1) \tag{4.4}$$

with $d - r = 1$. Here $B_q(d, n, r)$ denotes the right-hand side of (4.3).

(ii) For $d \equiv r \equiv 1 \pmod{2}$, modulo $(1 - aq^n)(a - q^n)$,

$$\begin{aligned}&\sum_{k=0}^{n-1} \frac{(a^{d-1}q^{d+r}, a^{d-3}q^{d+r}, \cdots, a^{r+3}q^{d+r}, a^{r+1}q^r, \cdots, a^2q^r, q^{d+r}, a^{-2}q^r, \cdots, a^{1-d}q^{d+r}; q^d)_k}{(a^{d-2}q^d, a^{d-4}q^d, \cdots, a^{4-d}q^d, a^{2-d}q^d; q^d)_k (q^d; q^d)_k q^{-dk}} \\ &\equiv C_q(d, n, r) \qquad \text{with } d - r \geq 4,\end{aligned} \tag{4.5}$$

$$\sum_{k=0}^{n-1}\frac{(a^{d-1}q^{d-2},a^{d-3}q^{d-2},\cdots,a^{2}q^{d-2},q^{2d-2},a^{-2}q^{d-2},\cdots,a^{1-d}q^{d-2};q^d)_kq^{dk}}{(a^{d-2}q^d,a^{d-4}q^d,\cdots,a^{4-d}q^d,a^{2-d}q^d;q^d)_k(q^d;q^d)_k}$$
$$\equiv C_q(d,n,d-2) \qquad \text{with } d-r=2, \tag{4.6}$$

where

$$C_q(d,n,r)=\frac{(q^d;q^d)_{n-1-(n+r)/d}q^{A(d,n,r)-r}}{(a^{d-2}q^d,a^{d-4}q^d,\cdots,a^{2-d}q^d;q^d)_{(n+r)/d}}\prod_{j=1}^{(r+1)/2}(1-a^{2j}q^r)(1-a^{-2j}q^r).$$

Actullay, the proof of per condition needs the help of Karlsson-Minton type summation (3.1). Taking the congruence (4.3) for example, it follows by letting $q\to q^d$, $m=d-1$, $N=n-1-(n+r)/d$, $b_j=q^{d-(d-2j)n}$ $(1\le j\le d-1)$, and

$$n_j=\begin{cases}(n+r)/d-1 & (d-r-1)/2\le j\le (d+r-1)/2\\ (n+r)/d & 1\le j<(d-r-1)/2 \quad\text{and}\quad (d+r-1)/2<j\le d-1\end{cases}$$

in (3.1). Similarly, the other three congruences (4.4)-(4.6) can also be proved and we will not present the details here. Hence, letting $a\to 1$ in (4.3)–(4.6) and making an integration, we are led to Theorem 4.2. □

Proof of Theorem 4.1. In view of

$$(q^{d+r},q^{r-d};q^d)_k=-q^r\frac{[d-r]}{[r]}\left(1+\frac{1-q^d}{q^d-q^{dk+r}}\right)(q^r;q^d)_k^2,$$

we investigate the relation between (4.1) and (4.2): for integers $d>r\ge 1$,

$$\sum_{k=0}^{n-1}\frac{(q^{d+r};q^d)_k^{d-r}(q^r;q^d)_k^{r-1}(q^{r-d};q^d)_kq^{dk}}{(q^d;q^d)_k^d}$$
$$=\frac{[d]}{[r]}\sum_{k=0}^{n-1}\frac{(q^{d+r};q^d)_k^{d-r-1}(q^r;q^d)_k^{r}(q^{r-d};q^d)_kq^{dk}}{(q^d;q^d)_k^d}-\frac{[d-r]}{q^{-r}[r]}\sum_{k=0}^{n-1}\frac{(q^{d+r};q^d)_k^{d-r-1}(q^r;q^d)_k^{r+1}q^{dk}}{(q^d;q^d)_k^d}.$$

Hence, from (2.1), (2.10), (2.11), Theorem 4.2 and the above result, we acquire Theorem 4.1. □

Particularly, letting n be a prime $p\ge 5$ and $q\to 1$ in Theorems 4.1 and 4.2, we gain the following conclusion.

Corollary 4.3. (*i*) *Let d, r be positive integers with $d\ge 3+r$ and $\gcd(d,r)=1$. Then, for any prime $p\ge 2d-r$ with $p\equiv -r \pmod d$,*

$$\sum_{k=0}^{p-1}\frac{(\frac{d+r}{d})_k^{d-r}(\frac{r}{d})_k^{r-1}(\frac{r-d}{d})_k}{k!^d}\equiv \tfrac{d-r}{d}\big(\tfrac{r}{d}\big)^r\Gamma_p\big(-\tfrac{r}{d}\big)^d \pmod{p^2}.$$

(ii) Let d, r be positive integers with $d>r$ and $\gcd(d,r)=1$. Then, for any prime $p\ge 5$ with $p\equiv -r \pmod{d}$,

$$\sum_{k=0}^{p-1}\frac{\big(\frac{d+r}{d}\big)_k^{d-r-1}\big(\frac{r}{d}\big)_k^{r+1}}{k!^d}\equiv -\big(\tfrac{r}{d}\big)^{r+1}\Gamma_p\big(-\tfrac{r}{d}\big)^d \pmod{p^2}.$$

In the derivation of Corollary 4.3, we have used the following formula: for any positive integers d,r with $d>r$ and prime $p\ge 5$ with $p\equiv -r \pmod{d}$,

$$\frac{\big(p-1-\frac{p+r}{d}\big)!}{\big(\frac{p+r}{d}\big)!^{d-1}}\equiv -(-1)^{\frac{p+r}{d}}\Gamma_p\big(-\tfrac{r}{d}\big)^d \pmod{p^2},$$

which can be proved like the formulas [6, (4.4) and (4.8)] by means of some basic properties of the p-adic Gamma function.

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