Unimodality of partition polynomials related to Borwein's conjecture

Janet J. W. Dong[1] and Kathy Q. Ji[2]

[1,2] Center for Applied Mathematics, Tianjin University, Tianjin 300072, P.R. China

[1]dongjinwei@tju.edu.cn and [2]kathyji@tju.edu.cn,

Abstract. The objective of this paper is to prove that the polynomials $\prod_{k=0}^{n}(1+q^{3k+1})(1+q^{3k+2})$ are symmetric and unimodal for $n \geq 0$ by an analytical method.

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1 Introduction

The study of unimodality of polynomials (or combinatorial sequences) has drawn great attention in recent decades. There is a remarkable diversity of applicable tools, ranging from analytic to topological, and from representation theory to probabilistic analysis. In this paper, we establish the unimodality of the polynomials defined in (1.6) by refining the method of Odlyzko-Richmond [13]. Recall that a polynomial

$$a_0 + a_1 q + \cdots + a_N q^n$$

with integer coefficients is called unimodal if for some $0 \leq j \leq N$,

$$a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_N,$$

and is called symmetric if for all $0 \leq j \leq N$,

$$a_j = a_{N-j}.$$

See [20, p. 124, Ex. 50]. It is well-known that the Gaussian polynomials

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}$$

are symmetric and unimodal, as conjectured by Caylay [7] in 1856 and confirmed by Sylvester [22] in 1878 based on semi-invariants of binary forms. For more information, we refer to [6, 12, 14, 16].

R. C. Entringer may be the first to investigate the unimodality of polynomials by an analytical method. By extending the argument of van Lint [11], Entringer [9] showed that the polynomials

$$(1+q)^2(1+q^2)^2\cdots(1+q^n)^2$$

are unimodal for $n \geq 1$. This method was greatly extended by Odlyzko and Richmond [13] to establish the almost unimodality of a class of polynomials of the form

$$(1+q^{a_1})(1+q^{a_2})\cdots(1+q^{a_n})$$

when n is large enough, where $\{a_i\}_{i=1}^{\infty}$ is a non-decreasing sequence of positive integers. More precisely, let

$$\prod_{i=1}^{n}(1+q^{a_i}) = \sum_{m=0}^{N} b_n(m)q^m, \qquad \text{where} \quad N = \sum_{i=1}^{n} a_i, \tag{1.1}$$

Odlyzko and Richmond showed that under suitable conditions (conditions (I) and (II) in Roth and Szekeres [17, p. 241]) on the infinite sequence $\{a_i\}$, the polynomials (1.1) are almost unimodal for n sufficiently large, that is, when $n \to \infty$,

$$b_n(A) \leq b_n(A+1) \leq \cdots \leq b_n(K) \geq b_n(K+1) \geq \cdots \geq b_n(N-A), \tag{1.2}$$

where A is some fixed constant and $K = N/2$ or $K = (N+1)/2$.

When $a_i = i$ for $1 \leq i \leq n$ in (1.1), Odlyzko and Richmond [13] verified that the inequality (1.2) holds for $A = 1$ when $n \geq 60$. It can be checked that inequality (1.2) also holds for $A = 1$ when $n \leq 59$. Hence Odlyzko and Richmond concluded that the polynomials

$$(1+q)(1+q^2)\cdots(1+q^n) \tag{1.3}$$

are unimodal for $n \geq 1$. The first proof of the unimodality of the polynomials (1.3) was given by Hughes [10] with the aid of Lie algebra results. Stanley [19] provided an alternative proof by using the Hard Lefschetz Theorem. Stanley [18] also established the general result of this type based on a result of Dynkin [8].

When $a_i = 2i - 1$ for $1 \leq i \leq n$ in (1.1), Almkvist [1] proved that the inequality (1.2) holds for $A = 3$ when $n \geq 83$. This leads to the polynomials

$$(1+q)(1+q^3)\cdots(1+q^{2n-1}) \tag{1.4}$$

are unimodal for $n \geq 27$, except at the coefficient of q^2 and q^{n^2-2} conjectured by Stanley [19]. Pak and Panova [15] showed that the polynomials (1.4) are strict unimodal by interpreting the differences between numbers of certain partitions as Kronecker coefficients of representations of S_n.

In [1], Almkvist also made the following conjecture.

Conjecture 1.1 (Almkvist) *For even $r \geq 2$ or odd $r \geq 3$ and $n \geq 11$, the polynomials*

$$\prod_{k=1}^{n} \frac{1-q^{rk}}{1-q^k} \tag{1.5}$$

are unimodal.

When $r = 2$, the polynomials (1.5) reduces to the polynomials (1.3). Almkvist [2] first showed that the conjecture is true when $r = 4$ by refining the method of Odlyzko-Richmond [13]. Subsequently, Almkvist [3] showed that the conjecture is true when $3 \leq r \leq 20$, $r = 100$ and 101.

In this paper, we establish the unimodality of the following polynomials.

Theorem 1.2 *For $n \geq 0$, the polynomials*

$$\prod_{k=0}^{n} (1+q^{3k+1})(1+q^{3k+2}) \tag{1.6}$$

are symmetric and unimodal.

It is worth mentioning that Borwein conjectured that the coefficients of the polynomials

$$\prod_{k=0}^{n} (1-q^{3k+1})(1-q^{3k+2})$$

have a repeating sign pattern of $+--$, which has been called as Borwein's conjecture, see Andrews [4]. Recently, Borwein's conjecture has been proved by Wang [23] by an analytical method.

2 Preliminaries

In this section, we collect several identities and inequalities which will be useful in the proof of Theorem 1.2.

$$e^{ix} = \cos(x) + i\sin(x), \tag{2.1}$$

$$\cos(2x) = 2\cos^2(x) - 1 \tag{2.2}$$

$$= 1 - 2\sin^2(x), \tag{2.3}$$

$$\sin(2x) = 2\sin(x)\cos(x), \tag{2.4}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta), \tag{2.5}$$

$$\sin(x) \geq xe^{-x^2/3} \quad \text{for} \quad 0 \leq x \leq 2, \tag{2.6}$$

$$\cos(x) \geq e^{-\gamma x^2} \quad \text{for} \quad |x| \leq 1, \ (\gamma = -\log\cos(1) = 0.615626....), \tag{2.7}$$

$$x - \frac{x^3}{6} \leq \sin(x) \leq x \quad \text{for} \quad x \geq 0, \tag{2.8}$$

$$|\cos(x)| \leq \exp\left(-\frac{1}{2}\sin^2(x) - \frac{1}{4}\sin^4(x)\right) \quad \text{for} \quad x \geq 0, \tag{2.9}$$

$$\left|\frac{\sin(nx)}{\sin(x)}\right| \leq n, \tag{2.10}$$

$$\sum_{k=1}^{n} \sin^2(kx) = \frac{n}{2} - \frac{\sin((2n+1)x)}{4\sin(x)} + \frac{1}{4}, \tag{2.11}$$

$$\sum_{k=1}^{n} \sin^4(kx) = \frac{3n}{8} - \frac{\sin((2n+1)x)}{4\sin(x)} + \frac{\sin((2n+1)2x)}{16\sin(2x)} + \frac{3}{16}. \tag{2.12}$$

The identity (2.1) is Euler's identity, see [21, p. 4]. For the formulas (2.2)–(2.5) of trigonometric functions, please see [5, Chapter 8]. The inequalities (2.6)–(2.10) were proved by Odlyzko and Richmond [13, p. 81].

It remains to show (2.11) and (2.12).

Proofs of (2.11) *and* (2.12). First, by (2.5), we obtain

$$\begin{aligned}
&2\sin(x)\left(\frac{1}{2} + \sum_{k=1}^{n}\cos(2kx)\right) \\
&= \sin(x) + 2\sin(x)\cos(2x) + 2\sin(x)\cos(4x) + \cdots + 2\sin(x)\cos(2nx) \\
&\overset{(2.5)}{=} \sin(x) + (\sin(3x) - \sin(x)) + (\sin(5x) - \sin(3x)) \\
&\quad + \cdots + (\sin((2n+1)x) - \sin((2n-1)x)) \\
&= \sin((2n+1)x).
\end{aligned}$$

Hence, we have

$$\sum_{k=1}^{n}\cos(2kx) = \frac{\sin((2n+1)x)}{2\sin(x)} - \frac{1}{2}. \tag{2.13}$$

Using (2.3) and (2.13), we deduce that

$$\begin{aligned}\sum_{k=1}^{n}\sin^2(kx) &\overset{(2.3)}{=} \frac{n}{2}-\frac{1}{2}\sum_{k=1}^{n}\cos(2kx)\\ &\overset{(2.13)}{=} \frac{n}{2}-\frac{1}{2}\left(\frac{\sin((2n+1)x)}{2\sin(x)}-\frac{1}{2}\right)\\ &= \frac{n}{2}-\frac{\sin((2n+1)x)}{4\sin(x)}+\frac{1}{4},\end{aligned}$$

which is (2.11).

The identity (2.12) can be derived in the same way. To wit,

$$\begin{aligned}\sum_{k=1}^{n}\sin^4(kx) &\overset{(2.3)}{=} \sum_{k=1}^{n}\left(\frac{1-\cos(2kx)}{2}\right)^2\\ &\overset{(2.2)}{=} \frac{3n}{8}-\frac{1}{2}\sum_{k=1}^{n}\cos(2kx)+\frac{1}{8}\sum_{k=1}^{n}\cos(4kx)\\ &\overset{(2.13)}{=} \frac{3n}{8}-\frac{1}{2}\left(\frac{\sin((2n+1)x)}{2\sin(x)}-\frac{1}{2}\right)+\frac{1}{8}\left(\frac{\sin((2n+1)2x)}{2\sin(2x)}-\frac{1}{2}\right)\\ &= \frac{3n}{8}-\frac{\sin((2n+1)x)}{4\sin(x)}+\frac{\sin((2n+1)2x)}{16\sin(2x)}+\frac{3}{16},\end{aligned}$$

in agreement with (2.12). This completes the proof. ∎

3 Proof of Theorem 1.2

Let $d_n = 3(n+1)^2$ and define

$$B_n(q)=\prod_{k=0}^{n}(1+q^{3k+1})(1+q^{3k+2})=\sum_{m=0}^{d_n}a_n(m)q^m. \tag{3.1}$$

In order to prove Theorem 1.2, we first show the following lemma.

Lemma 3.1 *If* $n\ge 1$ *and* $\frac{3n^2}{2}\le m\le\frac{3(n+1)^2}{2}$, *then*

$$a_n(m)-a_n(m-1)\ge 0. \tag{3.2}$$

Proof. We first show that (3.2) holds for $n \geq 168$ and $\frac{3n^2}{2} \leq m \leq \frac{3(n+1)^2}{2}$. Putting $q = e^{2i\theta}$ in (3.1), by (2.1), (2.2) and (2.4), we derive that

$$\begin{aligned}
B_n(e^{2i\theta}) &= \prod_{k=0}^{n}(1+(e^{2i\theta})^{3k+1})(1+(e^{2i\theta})^{3k+2}) \\
&\overset{(2.1)}{=} \prod_{k=0}^{n}(1+\cos(2(3k+1)\theta)+i\sin(2(3k+1)\theta)) \\
&\qquad\qquad \times (1+\cos(2(3k+2)\theta)+i\sin(2(3k+2)\theta)) \\
&\overset{(2.2)\&(2.4)}{=} \prod_{k=0}^{n}\left(2\cos^2((3k+1)\theta)+2i\sin((3k+1)\theta)\cos((3k+1)\theta)\right) \\
&\qquad\qquad \times \left(2\cos^2((3k+2)\theta)+2i\sin((3k+2)\theta)\cos((3k+2)\theta)\right) \\
&\overset{(2.1)}{=} \prod_{k=0}^{n} 4\cos((3k+1)\theta)\cos((3k+2)\theta)\exp(i(3k+1)\theta)\exp(i(3k+2)\theta) \\
&= 4^{n+1}\exp(id_n\theta)\prod_{k=0}^{n}\cos((3k+1)\theta)\cos((3k+2)\theta). \qquad (3.3)
\end{aligned}$$

Using Taylor's theorem [21, p. 47–49], we find that

$$\begin{aligned}
a_n(m) &= \frac{1}{2\pi i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{B_n\left(e^{2i\theta}\right)}{\left(e^{2i\theta}\right)^{m+1}}\mathrm{d}\left(e^{2i\theta}\right) \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} B_n\left(e^{2i\theta}\right)e^{-2im\theta}\mathrm{d}\theta \\
&\overset{(3.3)}{=} \frac{4^{n+1}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp(i(d_n-2m)\theta)\prod_{k=0}^{n}\cos((3k+1)\theta)\cos((3k+2)\theta)\mathrm{d}\theta \\
&\overset{(2.1)}{=} \frac{4^{n+1}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos((d_n-2m)\theta)+i\sin((d_n-2m)\theta))\prod_{k=0}^{n}\cos((3k+1)\theta)\cos((3k+2)\theta)\mathrm{d}\theta.
\end{aligned}$$

Observe that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin((d_n-2m)\theta)\prod_{k=0}^{n}\cos((3k+1)\theta)\cos((3k+2)\theta)\mathrm{d}\theta = 0,$$

we have therefore,

$$a_n(m) = \frac{2^{2n+3}}{\pi}\int_{0}^{\frac{\pi}{2}} \cos((d_n-2m)\theta)\prod_{k=0}^{n}\cos((3k+1)\theta)\cos((3k+2)\theta)\mathrm{d}\theta.$$

We next show that

$$\frac{\partial}{\partial m}a_n(m) \geq 0 \quad \text{for} \quad n \geq 168 \quad \text{and} \quad \frac{3n^2}{2} \leq m \leq \frac{3(n+1)^2}{2}, \qquad (3.4)$$

from which, it follows that (3.2) is valid for $n \geq 168$ and $\frac{3n^2}{2} \leq m \leq \frac{3(n+1)^2}{2}$.

It is easy to see that

$$\frac{\partial}{\partial m} a_n(m) = \frac{2^{2n+4}}{\pi} \int_0^{\frac{\pi}{2}} \theta \sin((d_n - 2m)\theta) \prod_{k=0}^{n} \cos((3k+1)\theta) \cos((3k+2)\theta) \mathrm{d}\theta.$$

Let $d_n - 2m = \mu$, and let

$$I_n(\mu) = \int_0^{\frac{\pi}{2}} \theta \sin(\mu\theta) \prod_{k=0}^{n} \cos((3k+1)\theta) \cos((3k+2)\theta) \mathrm{d}\theta.$$

Under the condition that $\frac{3n^2}{2} \leq m \leq \frac{3(n+1)^2}{2}$, we see that

$$0 \leq \mu = d_n - 2m \leq 6n + 3. \tag{3.5}$$

To prove (3.4), it suffices to show that

$$I_n(\mu) \geq 0 \quad \text{for} \quad n \geq 168 \quad \text{and} \quad 0 \leq \mu \leq 6n+3. \tag{3.6}$$

To this end, we write

$$\begin{aligned} I_n(\mu) &= \left\{ \int_0^{\frac{\pi}{6n+4}} + \int_{\frac{\pi}{6n+4}}^{\frac{\pi}{2}} \right\} \theta \sin(\mu\theta) \prod_{k=0}^{n} \cos((3k+1)\theta) \cos((3k+2)\theta) \mathrm{d}\theta \\ &= I_n^{(1)}(\mu) + I_n^{(2)}(\mu). \end{aligned}$$

We next show that

$$I_n^{(1)}(\mu) \geq |I_n^{(2)}(\mu)| \quad \text{for} \quad n \geq 168 \quad \text{and} \quad 0 \leq \mu \leq 6n+3, \tag{3.7}$$

which implies (3.6).

We first evaluate the value of $I_n^{(1)}(\mu)$, which is defined by

$$I_n^{(1)}(\mu) := \int_0^{\frac{\pi}{6n+4}} \theta \sin(\mu\theta) \prod_{k=0}^{n} \cos((3k+1)\theta) \cos((3k+2)\theta) \mathrm{d}\theta. \tag{3.8}$$

When $0 \leq \theta \leq \frac{1}{3n+2}$, by (3.5), we have

$$0 \leq \mu\theta \leq 2 \;\text{ and }\; 0 \leq (3k+1)\theta \leq (3k+2)\theta \leq 1 \;\text{ for }\; 0 \leq k \leq n,$$

so that

$$\theta \sin(\mu\theta) \prod_{k=0}^{n} \cos((3k+1)\theta) \cos((3k+2)\theta)$$

$$\overset{(2.6)\&(2.7)}{\geq} \mu\theta^2 \exp\left(-\frac{\mu^2\theta^2}{3}\right) \exp\left(-\gamma\theta^2 \sum_{k=0}^{n} \left((3k+1)^2 + (3k+2)^2\right)\right)$$

$$\geq \mu\theta^2 \exp\left(-\frac{(6n+3)^2\theta^2}{3}\right) \exp\left(-\gamma\theta^2 \left(6n^3 + 18n^2 + 17n + 5\right)\right) \quad \text{(by } 0 \leq \mu \leq 6n+3\text{)}$$

$$= \mu\theta^2 \exp\left(-\theta^2 n^3 \left(\left(\frac{12}{n} + \frac{12}{n^2} + \frac{3}{n^3}\right) + \gamma\left(6 + \frac{18}{n} + \frac{17}{n^2} + \frac{5}{n^3}\right)\right)\right)$$

$$\geq \mu\theta^2 \exp\left(-cn^3\theta^2\right) \quad \text{(by } n \geq 168\text{)}, \tag{3.9}$$

where $c = 3.832$. Applying (3.9) to (3.8), we find that when $n \geq 168$ and $0 \leq \mu \leq 6n+3$,

$$\begin{aligned}
I_n^{(1)}(\mu) &= \int_0^{\frac{\pi}{6n+4}} \theta \sin(\mu\theta) \prod_{k=0}^{n} \cos((3k+1)\theta)\cos((3k+2)\theta)\mathrm{d}\theta \\
&\geq \int_0^{\frac{1}{3n+2}} \theta \sin(\mu\theta) \prod_{k=0}^{n} \cos((3k+1)\theta)\cos((3k+2)\theta)\mathrm{d}\theta \\
&\geq \int_0^{\frac{1}{3n+2}} \mu\theta^2 \exp\left(-cn^3\theta^2\right) \mathrm{d}\theta \\
&= \left\{\int_0^{\infty} - \int_{\frac{1}{3n+2}}^{\infty}\right\} \mu\theta^2 \exp\left(-cn^3\theta^2\right) \mathrm{d}\theta \\
&= \frac{\mu}{2c^{\frac{3}{2}} n^{\frac{9}{2}}} \left(\int_0^{\infty} v^{\frac{1}{2}} e^{-v} \mathrm{d}v - \int_{\frac{cn^3}{(3n+2)^2}}^{\infty} v^{\frac{1}{2}} e^{-v} \mathrm{d}v\right) \\
&= \frac{\mu}{2c^{\frac{3}{2}} n^{\frac{9}{2}}} \left(\frac{\sqrt{\pi}}{2} - \int_{\frac{cn^3}{(3n+2)^2}}^{\infty} v^{\frac{1}{2}} e^{-v} \mathrm{d}v\right).
\end{aligned}$$

Observe that when $n \geq 168$,

$$\frac{cn^3}{(3n+2)^2} \geq \frac{c \cdot 168^3}{(3 \times 168 + 2)^2},$$

so

$$\int_{\frac{cn^3}{(3n+2)^2}}^{\infty} v^{\frac{1}{2}} e^{-v} \mathrm{d}v \leq \int_{\frac{c \cdot 168^3}{(3 \times 168 + 2)^2}}^{\infty} v^{\frac{1}{2}} e^{-v} \mathrm{d}v \leq 1.29 \times 10^{-30}.$$

Consequently, when $n \geq 168$ and $0 \leq \mu \leq 6n+3$,

$$I_n^{(1)}(\mu) \geq \frac{\frac{\sqrt{\pi}}{2} - 1.29 \times 10^{-30}}{2 \times 3.832^{\frac{3}{2}}} \cdot \frac{\mu}{n^{\frac{9}{2}}} \geq \frac{0.8862\mu}{15.2 n^{\frac{9}{2}}} \geq \frac{0.0583\mu}{n^{\frac{9}{2}}}. \tag{3.10}$$

We now turn to estimate the value of $I_n^{(2)}(\mu)$, which is defined by

$$I_n^{(2)}(\mu) = \int_{\frac{\pi}{6n+4}}^{\frac{\pi}{2}} \theta \sin(\mu\theta) \prod_{k=0}^{n} \cos((3k+1)\theta)\cos((3k+2)\theta)\mathrm{d}\theta. \tag{3.11}$$

When $\frac{\pi}{6n+4} \le \theta \le \frac{\pi}{2}$, by (2.9), (2.11) and (2.12), we deduce that

$$\begin{aligned}
&\left|\prod_{k=0}^{n} \cos((3k+1)\theta)\cos((3k+2)\theta)\right| \\
&\overset{(2.9)}{\le} \exp\Bigg(-\frac{1}{2}\sum_{k=0}^{n}\left(\sin^2((3k+1)\theta)+\sin^2((3k+2)\theta)\right) \\
&\qquad\qquad -\frac{1}{4}\sum_{k=0}^{n}\left(\sin^4((3k+1)\theta)+\sin^4((3k+2)\theta)\right)\Bigg) \\
&= \exp\left(-\frac{1}{2}\left(\sum_{k=1}^{3n+2}\sin^2(k\theta)-\sum_{k=1}^{n}\sin^2(3k\theta)\right)-\frac{1}{4}\left(\sum_{k=1}^{3n+2}\sin^4(k\theta)-\sum_{k=1}^{n}\sin^4(3k\theta)\right)\right) \\
&\overset{(2.11)\&(2.12)}{=} \exp\Bigg(-\frac{11(n+1)}{16}+\frac{3\sin((6n+5)\theta)}{16\sin(\theta)}-\frac{\sin((6n+5)2\theta)}{64\sin(2\theta)} \\
&\qquad\qquad -\frac{3\sin((2n+1)3\theta)}{16\sin(3\theta)}+\frac{\sin((2n+1)6\theta)}{64\sin(6\theta)}\Bigg) := E(n).
\end{aligned}$$

We proceed to prove that

$$E(n) < \exp\left(-0.163n-0.031\right) \quad \text{for} \quad \frac{\pi}{6n+4} \le \theta \le \frac{\pi}{2} \quad \text{and} \quad n \ge 168. \tag{3.12}$$

The proof of (3.12) is divided into two steps. When $\frac{\pi}{6n+4} \le \theta \le \frac{\pi}{6}$, using (2.8) and (2.10), we obtain

$$\begin{aligned}
E(n) &\le \exp\left(-\frac{11(n+1)}{16}+\frac{3}{16\sin(\theta)}+\frac{1}{64\sin(2\theta)}+\frac{3}{16\sin(3\theta)}+\left|\frac{\sin((2n+1)6\theta)}{64\sin(6\theta)}\right|\right) \\
&\overset{(2.8)\&(2.10)}{\le} \exp\Bigg(-\frac{11(n+1)}{16}+\frac{3}{16\left(\frac{\pi}{6n+4}\left(1-\frac{\left(\frac{\pi}{6n+4}\right)^2}{6}\right)\right)}+\frac{1}{64\left(\frac{\pi}{3n+2}\left(1-\frac{\left(\frac{\pi}{3n+2}\right)^2}{6}\right)\right)} \\
&\qquad\qquad +\frac{3}{16\left(\frac{3\pi}{6n+4}\left(1-\frac{\left(\frac{3\pi}{6n+4}\right)^2}{6}\right)\right)}+\frac{2n+1}{64}\Bigg) \quad \left(\text{by } \frac{\pi}{6n+4} \le \theta \le \frac{\pi}{6}\right).
\end{aligned} \tag{3.13}$$

Applying

$$1-\frac{\left(\frac{\pi}{6n+4}\right)^2}{6} \ge 1-\frac{\left(\frac{\pi}{3n+2}\right)^2}{6} \ge 1-\frac{\left(\frac{3\pi}{6n+4}\right)^2}{6}$$

to (3.13), we derive that

$$E(n) \leq \exp\left(-\frac{42n+43}{64} + \frac{3}{16\left(\frac{\pi}{6n+4}\left(1-\frac{\left(\frac{3\pi}{6n+4}\right)^2}{6}\right)\right)} + \frac{1}{64\left(\frac{\pi}{3n+2}\left(1-\frac{\left(\frac{3\pi}{6n+4}\right)^2}{6}\right)\right)} + \frac{3}{16\left(\frac{3\pi}{6n+4}\left(1-\frac{\left(\frac{3\pi}{6n+4}\right)^2}{6}\right)\right)}\right)$$

$$= \exp\left(-\frac{42n+43}{64} + \frac{33}{128\left(\frac{\pi}{6n+4}\left(1-\frac{\left(\frac{3\pi}{6n+4}\right)^2}{6}\right)\right)}\right)$$

$$= \exp\left(-\frac{42n+43}{64} + \frac{33(6n+4)}{128\pi\left(1-\frac{6\pi^2}{(12n+8)^2}\right)}\right).$$

Note that when $n \geq 168$,

$$1-\frac{6\pi^2}{(12n+8)^2} \geq 1-\frac{6\pi^2}{(12\times 168+8)^2} = 1-\frac{3\pi^2}{2048288},$$

so when $\frac{\pi}{6n+4} \leq \theta \leq \frac{\pi}{6}$ and $n \geq 168$,

$$\begin{aligned} E(n) &\leq \exp\left(-\frac{42n+43}{64} + \frac{33(6n+4)}{128\pi\left(1-\frac{3\pi^2}{2048288}\right)}\right) \\ &= \exp\left(\left(-\frac{21}{32} + \frac{99}{64\pi\left(1-\frac{3\pi^2}{2048288}\right)}\right)n - \frac{43}{64} + \frac{33}{32\pi\left(1-\frac{3\pi^2}{2048288}\right)}\right) \\ &< \exp\left(-0.163n - 0.343\right). \end{aligned} \tag{3.14}$$

When $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$, by (2.10), we deduce that

$$\begin{aligned} E(n) &\leq \exp\left(-\frac{11(n+1)}{16} + \frac{3}{16\sin(\theta)} + \left|\frac{\sin((6n+5)2\theta)}{64\sin(2\theta)}\right| + \left|\frac{3\sin((2n+1)3\theta)}{16\sin(3\theta)}\right| + \left|\frac{\sin((2n+1)6\theta)}{64\sin(6\theta)}\right|\right) \\ &\overset{(2.10)}{\leq} \exp\left(-\frac{11(n+1)}{16} + \frac{3}{16\sin(\frac{\pi}{6})} + \frac{6n+5}{64} + \frac{3(2n+1)}{16} + \frac{2n+1}{64}\right) \\ &= \exp\left(-\frac{3}{16}n - \frac{1}{32}\right) < \exp\left(-0.187n - 0.031\right). \end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15) yields (3.12). Applying (3.12) to (3.11), and in view of (2.8) and (3.10), we derive that when $n \geq 168$,

$$
\begin{aligned}
|I_n^{(2)}(\mu)| &\overset{(2.8)}{<} \mu \exp\left(-0.163n - 0.031\right) \int_{\frac{\pi}{6n+4}}^{\frac{\pi}{2}} \theta^2 \mathrm{d}\theta \\
&\leq \frac{\mu\pi^3}{3}\left(\frac{1}{8} - \frac{1}{(6n+4)^3}\right) \exp\left(-0.163n - 0.031\right) \\
&= \frac{\mu\pi^3}{3}\left(\frac{1}{2} - \frac{1}{6n+4}\right)\left(\frac{1}{2^2} + \frac{1}{2(6n+4)} + \frac{1}{(6n+4)^2}\right) \exp\left(-0.163n - 0.031\right) \\
&< \frac{\mu\pi^3}{3} \cdot \frac{3}{4} \cdot \left(\frac{1}{2} - \frac{1}{6n+4}\right) \exp\left(-0.163n - 0.031\right) \\
&\overset{(3.10)}{\leq} \frac{\pi^3 n^{\frac{9}{2}}}{4 \times 0.0583}\left(\frac{1}{2} - \frac{1}{6n+4}\right) \exp\left(-0.163n - 0.031\right) I_n^{(1)}(\mu).
\end{aligned}
$$

Define

$$
f(n) := \frac{\pi^3 n^{\frac{9}{2}}}{4 \times 0.0583}\left(\frac{1}{2} - \frac{1}{6n+4}\right) \exp\left(-0.163n - 0.031\right).
$$

To show (3.7), it remains to show that $f(n) < 1$ for $n \geq 168$. We claim that $f'(n) < 0$ for $n \geq 168$. Since $f(n) > 0$ for $n \geq 168$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}n} f(n) = \frac{\mathrm{d}}{\mathrm{d}n} e^{\ln f(n)} = f(n) \frac{\mathrm{d}}{\mathrm{d}n} \ln f(n). \tag{3.16}
$$

Observe that when $n \geq 168$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}n} \ln f(n) &= \frac{9}{2n} + \frac{6}{(3n+1)(6n+4)} - 0.163 \\
&\leq \frac{9}{2 \times 168} + \frac{6}{(3 \times 168 + 1)(6 \times 168 + 4)} - 0.163 < -0.13 < 0.
\end{aligned}
$$

Hence, we derive from (3.16) that $f'(n) < 0$ for $n \geq 168$, and the claim is proved. Consequently, $f(n) \leq f(168) < 0.851$ when $n \geq 168$. Therefore, (3.7) is valid, and so (3.4) is valid. This leads to (3.2) holds for $n \geq 168$ and $\frac{3n^2}{2} \leq m \leq \frac{3(n+1)^2}{2}$. Using Maple, we can check that (3.2) also holds for $n < 168$ and $\frac{3n^2}{2} \leq m \leq \frac{3(n+1)^2}{2}$. Thus the lemma is proved. ∎

We conclude this paper with the proof of Theorem 1.2.

Proof of Theorem 1.2. When $n \geq 0$, we first show that $B_n(q)$ is a symmetric polynomial. Replacing q by q^{-1} in (3.1), we deduce that

$$
B_n(q^{-1}) = \prod_{k=0}^{n} (1 + q^{-(3k+1)})(1 + q^{-(3k+2)})
$$

$$= q^{-d_n} \prod_{k=0}^{n} (1 + q^{(3k+1)})(1 + q^{(3k+2)})$$
$$= q^{-d_n} B_n(q).$$

To wit,

$$B_n(q) = q^{d_n} B_n(q^{-1}),$$

from which, it follows that $B_n(q)$ is symmetric.

We proceed to show that the polynomial $B_n(q)$ is unimodal by induction on n. When $n = 0$, we have

$$B_0(q) = (1 + q)(1 + q^2) = 1 + q + q^2 + q^3.$$

Clearly, the coefficients of $B_0(q)$ are unimodal.

Suppose that $B_{n-1}(q)$ is unimodal for $n \geq 1$, namely, for $n \geq 1$ and $1 \leq m \leq \lfloor \frac{d_{n-1}}{2} \rfloor$,

$$a_{n-1}(m) \geq a_{n-1}(m-1). \tag{3.17}$$

We intend to show that $B_n(q)$ is unimodal. Since $B_n(q)$ is a symmetric polynomial, it suffices to show that for $n \geq 1$ and $1 \leq m \leq \lfloor \frac{d_n}{2} \rfloor$,

$$a_n(m) \geq a_n(m-1). \tag{3.18}$$

Observe that

$$B_n(q) = \left(1 + q^{3n+1}\right) \left(1 + q^{3n+2}\right) B_{n-1}(q),$$

which implies the following recurrence relation:

$$a_n(m) = a_{n-1}(m) + a_{n-1}(m - 3n - 1) + a_{n-1}(m - 3n - 2) + a_{n-1}(m - 6n - 3). \tag{3.19}$$

It's evident from (3.17) and (3.19) that (3.18) holds for $n \geq 1$ and $1 \leq m \leq \lfloor \frac{d_{n-1}}{2} \rfloor$. In view of Lemma 3.1, we see that (3.18) also holds for $n \geq 1$ and $\lceil \frac{d_{n-1}}{2} \rceil \leq m \leq \lfloor \frac{d_n}{2} \rfloor$. Hence, we conclude that (3.18) is valid for $n \geq 1$ and $1 \leq m \leq \lfloor \frac{d_n}{2} \rfloor$, and so $B_n(q)$ is unimodal. Thus, we complete the proof of Theorem 1.2. ∎

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