Isolated hypersurface singularities, spectral invariants, and quantum cohomology

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Abstract

We study the relation between isolated hypersurface singularities (e.g. ADE) and the quantum cohomology ring by using spectral invariants, which are symplectic measurements coming from Floer theory. We prove, under the assumption that the quantum cohomology ring is semi-simple, that (1) if the smooth Fano variety degenerates to a Fano variety with an isolated hypersurface singularity, then the singularity has to be an A_m-singularity, (2) if the symplectic manifold contains an A_m-configuration of Lagrangian spheres, then there are consequences for the Hofer geometry, and that (3) the Dehn twist reduces spectral invariants.

Contents

1 Introduction

1.1 Context

Degeneration is a theme that originates in classical algebraic geometry that is still actively studied in the context of various modern topics such as the minimal model program [KM98], Kähler–Einstein metric [DK01], mirror symmetry, and the SYZ conjecture [Gr13]. Its importance in symplectic topology was noticed by Arnold [Arn95] and Donaldson [Don00], especially that Lagrangians can appear as vanishing cycles. Seidel largely developed this idea and obtained various results on the symplectic aspect of the Dehn twist [Sei97, Sei99, Sei00, Sei08]. In this paper, we study degeneration by symplectic topology, and vice versa. Note that such an attempt was also made by Biran which was highlighted in his ICM address [Bir02, Section 5.2].

Understanding the type of singularities an algebraic variety can degenerate to is an important subject in algebraic geometry. *Isolated hypersurface singularities* have been fundamental in the study of singularities since [Mil68]. Any isolated hypersurface singularity can be associated to a positive integer called *modality*, which roughly speaking, expresses the complexity of the singularity. Arnold classified isolated hypersurface singularities up to modality two [Arn76, AGLV93] and according to his classification, the ones with modality zero are called the *simple (ADE) singularities*. Up to right equivalence, A_m, D_m ($m \geqslant 4$), and E_m ($m = 6, 7, 8$) singularities are given as

$$x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + p(y, z) = 0$$

where $p(y, z)$ is

$$y^2 + z^{m+1},\ yz^2 + z^{m-1},\ y^3 + z^4,\ y^3 + yz^3,\ y^3 + z^5,$$

respectively. The vanishing cycle of an A, D, E singularity forms a configuration of Lagrangian spheres that intersect as expressed in the A, D, E type Dynkin diagram (Figure 1), respectively. We call these configurations of Lagrangian spheres ADE configurations.

A_n

D_n

E_6

E_7

E_8

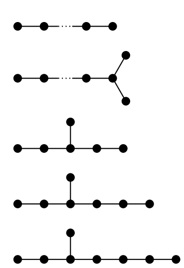

Figure 1: Dynkin diagrams of type A_n, D_n, E_6, E_7, E_8.

Isolated hypersurface singularities of modality one consists of three types, namely the *parabolic (or simple elliptic) singularities* $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, *hyperbolic singularities* $T_{p,q,r}$, and *14 exceptional singularities*. Similarly to the simple singularities, all the isolated hypersurface singularities of positive modality also give rise to configurations of Lagrangian spheres by taking the vanishing cycles. We refer to Section 2.5 for further information.

Note that for surfaces, i.e. complex dimension two, simple singularities have many different characterizations such as du Val singularities, rational double points, Kleinian singularities [Rei]. For this case, algebraic geometers have a fairly good understanding of degenerations. In fact, for Fano surfaces, i.e. the del Pezzo surfaces, du Val classified all the possible simple singularities that can occur on singular del Pezzo surfaces [DV34]. On the other hand, very little is known for higher dimensional spaces, and the importance of studying the higher dimensional case is emphasized by Arnold in [Arn95].

Another object that has been of interest to both algebraic and symplectic geometers is the *quantum cohomology*. After its introduction in string theory by Vafa and Witten [Vaf91, Wit91], the algebro-geometric formulation was found by Kontsevich–Manin in [KM94], shortly followed by the symplectic formulation in [RT95] due to Ruan–Tian. An important case is when the quantum cohomology ring is semi-simple; the (small) quantum cohomology ring[1] $QH(X, \omega)$ of a symplectic manifold (X, ω) is semi-simple if it splits into a direct sum of finitely many fields $\{Q_j\}_{1 \leqslant j \leqslant k}$:

$$QH(X, \omega) = \bigoplus_{1 \leqslant j \leqslant k} Q_j.$$

Examples of symplectic manifolds equipped with monotone symplectic forms[2] that have semi-simple quantum cohomology rings over $\mathbb{C}$-coefficients[3] include

- the complex projective space $\mathbb{C}P^n$ [MS04],
- the quadric hypersurfaces Q^n [Abr00],
- the del Pezzo surfaces $\mathbb{D}_k := \mathbb{C}P^2 \# k(\overline{\mathbb{C}P^2})$ with $0 \leqslant k \leqslant 4$ (where k is the number of the points blown up in such a way that the symplectic manifold become monotone) [BM04, CM95],
- the complex Grassmannians $Gr_{\mathbb{C}}(k, n)$ (i.e. the space of complex k dimensional linear subspace in $\mathbb{C}^n$) [Abr00],
- some homogeneous spaces [CMP10, Per] and some generalized Grassmannians [Gra],
- products of monotone symplectic manifolds with semi-simple quantum cohomology rings [EP08].

In addition to the above, examples of symplectic manifolds equipped with generic symplectic forms that have semi-simple quantum cohomology rings include

- any symplectic toric Fano manifold [FOOO10, OT09, Ush11],

[1] In this paper, quantum cohomology ring will always refer to the small one unless mentioned otherwise. See Section 2.3 for further comments on different notions of semi-simplicity for small and big quantum cohomology rings.

[2] Recall that a symplectic manifold (X, ω) is monotone if we have $[\omega]|_{\pi_2(X)} = \kappa_X \cdot c_1(TX)|_{\pi_2(X)}$ for some $\kappa_X > 0$.

[3] Note that semi-simplicity depends on the choice of coefficients. For example, over $\mathbb{F}_p$, a field of characteristic p, the quantum cohomology ring of $\mathbb{C}P^n$ is not semi-simple when p divides $n+1$. In this paper, we work over $\mathbb{C}$-coefficients.

- 36 out of the 59 Fano 3-folds with no odd rational cohomology [Cio05],
- one-point blow up of any of the above [Ush11].

The semi-simplicity of the (small/big) quantum cohomology ring has also shared interest in algebraic and symplectic geometry; for example, see [BM04, Dub96, KM94] for the algebraic geometry side, [EP03, EP08, EP09, BC, FOOO10] for the symplectic side. See Section 2.3 for more information on the above examples and the relation between different notions of semi-simplicity. The semi-simplicity also has important implication for physics; we refer the readers to [HKK$^+$].

In this paper, we study the interaction between isolated hypersurface singularities and the semi-simplicity of quantum cohomology rings, which are both objects lying in the intersection of algebraic and symplectic geometry. Our method is based on Floer theory, more precisely the theory of spectral invariants, which allows us to have less dimensional restrictions than in the current algebraic geometry.

1.2 Isolated hypersurface singularities and quantum cohomology

We have seen in the previous section that the theories of singularities and the quantum cohomology are both of interest to algebraic and symplectic geometers. However, the interaction between the two theories has not been studied.[4] Our first main result, which can be formulated in terms of algebraic and symplectic geometry, claims that semi-simplicity of the quantum cohomology ring excludes most of the isolated hypersurface singularities. We first state the algebro-geometric version.

Theorem A (Algebraic geometry version). *Let X be a complex n dimensional smooth Fano variety. Assume $QH(X,\omega)$ is semi-simple, where ω is the anti-canonical form of X. If X degenerates to a Fano variety with an isolated hypersurface singularity, then the singularity has to be*

- *an A_m-singularity with $m \geqslant 1$, if n is even.*
- *an A_m-singularity with $m = 1, 2$, if n is odd and $\frac{\dim_{\mathbb{C}} X+1}{2r_X} \notin \mathbb{Z}$ where r_X is the Fano index.*

Remark 1.2.1. Here are some remarks on Theorem A:

1. A smooth Fano variety X carries a natural symplectic form called the anti-canonical form which comes from the projective embedding $f : X \hookrightarrow \mathbb{C}P^N$ for some $N \in \mathbb{N}$. From the symplectic perspective, this is a monotone symplectic form, see (2.4.2).
2. It would be very interesting to study the case of other classes of singularities, for example cyclic quotient singularities. See Remark 3.3.7.
3. It would be interesting to study the remaining case, i.e. when n is odd and $\frac{\dim_{\mathbb{C}} X+1}{2r_X} \in \mathbb{Z}$ where r_X is the Fano index.

[4] The author thanks Kaoru Ono and Rahul Pandharipande for pointing this out.

4. Theorem A is expected to be slightly generalized, see Conjecture 3.3.1.

To prove Theorem A, we reduce it to the following Theorem B, which could be regarded as the symplectic version of Theorem A (We point out that this translation from algebraic geometry to symplectic topology is not immediate. See Section 3.3 for further information).

Theorem B (Symplectic topology version)**.** *Let (X,ω) be a real $2n$ dimensional closed monotone symplectic manifold. Assume $QH(X,\omega)$ is semi-simple. Then (X,ω) does not contain*

- *a D_4-configuration of Lagrangian spheres, if n is even.*
- *an A_3-configuration of Lagrangian spheres, if n is odd and $\frac{n+1}{2N_X} \notin \mathbb{Z}$ where N_X is the minimal Chern number.*

As we pointed out in Section 1.1, the classification of singularities for varieties is extremely important in algebraic geometry and mainly due to the lack of methods, very little is known about the possible singularity types for the higher dimensional varieties. Theorem A has no dimensional restriction, as our approach is based on Floer theory.

We now look at relation to other works.

Simple singularities on surfaces. As we mentioned earlier, isolated hypersurface singularities that can occur on surfaces are fairly well understood.

- For the Fano case (i.e. del Pezzo surfaces), the simple (ADE) singularities that can occur on singular Fano surfaces (i.e. singular del Pezzo surfaces) were completely classified by du Val [DV34] (see also [Sta21, Section 1]). Denote the smooth del Pezzo surface of degree $9-k$ by $\mathbb{D}_k$. According to it, $\mathbb{D}_k$ can degenerate to a singular Fano surface with D or E type singularities when $5 \leqslant k \leqslant 8$, while it can only degenerate to a singular Fano surface with A type singularities when $0 \leqslant k \leqslant 4$. The quantum cohomology ring $QH(\mathbb{D}_k)$ is semi-simple (when $\mathbb{D}_k$ is equipped with a monotone symplectic form) if and only if $0 \leqslant k \leqslant 4$, so this is consistent with Theorem A.
- For the Calabi–Yau case, it is also well-known that D,E and the 14 exceptional singularities can appear in degenerations of the K3 surface. However, Calabi–Yau manifolds, i.e. $c_1|_{\pi_2} = 0$, cannot have semi-simple quantum cohomology rings. Once again, this is consistent with Theorem A.

Compactification of Milnor fibers. First of all, notice that Theorem B immediately implies the following.

Corollary 1.2.2. *The Milnor fiber of an isolated hypersurface singularity that is not of type A cannot be compactified to a symplectic manifold with semi-simple quantum cohomology ring.*

Milnor fiber is, loosely speaking, a smoothing of the singularity (see Definition 2.5.2).

- In [Kea15], where Keating studied the symplectic topology of the Milnor fibers of isolated hypersurface singularities of positive modality, an important step was to find compactifications of the Milnor fibers to del Pezzo surfaces $\mathbb{D}_k$. In [Kea15, Proposition 5.19], Keating proves that the Milnor fibers of $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ (isolated hypersurface singularities of modality one) can be compactified to $\mathbb{D}_6, \mathbb{D}_7, \mathbb{D}_8$, respectively. This is compatible with Corollary 1.2.2, as $\mathbb{D}_6, \mathbb{D}_7, \mathbb{D}_8$ do not have semi-simple quantum cohomology rings.

- It is also well-known that the Milnor fiber of the 14 exceptional singularities (modality one) can be compactified to the K3 surface [Pin77, Nik79, Dol96], see also [KMU13]. This is also consistent with Corollary 1.2.2.

1.3 A_m singularities and Hofer geometry

Theorem B implies that there is a possibility that a symplectic manifold with semi-simple quantum cohomology ring can contain an A_m-configuration. In fact, this can happen; for example, del Pezzo surfaces $\mathbb{D}_3$ and $\mathbb{D}_4$ have an A_2-configuration and an A_4-configuration, respectively. If we are in such a situation, we get some implication for the Hofer geometry. Before stating the result, recall that the set of Hamiltonian diffeomorphisms, denoted by $\mathrm{Ham}(X, \omega)$, form a group and has a remarkable bi-invariant metric called the Hofer metric [Hof93]. The study of geometric properties of the group $\mathrm{Ham}(X, \omega)$ with respect to the Hofer metric has been an important subject of the field [Pol01]. For readers who are not familiar with the subject, we refer to [Kaw, Section 1.1] for a rapid overview of the aspects that are relevant to this paper. Our second main result is the following.

Theorem C. *Let (X, ω) be a real $2n$ dimensional closed monotone symplectic manifold with even n. Assume $QH(X, \omega)$ is semi-simple. If (X, ω) contains an A_m-configuration, then there are $m-1$ pairwise distinct Entov–Polterovich quasimorphisms on $\widetilde{\mathrm{Ham}}(X, \omega)$.*

Remark 1.3.1. Entov–Polterovich quasimorphisms are special maps on $\widetilde{\mathrm{Ham}}(X, \omega)$, i.e. the universal cover of $\mathrm{Ham}(X, \omega)$, constructed from Floer theory that have powerful applications to Hofer geometry. See Section 2.2 for further information.

Application. The del Pezzo surface $\mathbb{D}_4$ has semi-simple quantum cohomology ring, and by combining some toric degeneration and (complex) 2-dimensional techniques with Theorem C, we obtain the following result on the Hofer geometry for $\mathbb{D}_4$.

Theorem 1.3.2 (Kapovich–Polterovich question, Entov–Polterovich–Py question)**.** *There are four pairwise distinct Entov–Polterovich quasimorphisms on* $\mathrm{Ham}(\mathbb{D}_4)$*. Thus,* $\mathrm{Ham}(\mathbb{D}_4)$ *admits a quasi-isometric embedding of* $\mathbb{R}^4$*. Moreover, there are three linearly independent quasimorphisms on* $\mathrm{Ham}(\mathbb{D}_4)$ *that are both C^0 and Hofer-Lipshitz continuous. In particular, the group* $\mathrm{Ham}(\mathbb{D}_4)$ *is not quasi-isometric to the real line* $\mathbb{R}$ *with respect to the Hofer metric.*

Remark 1.3.3. The Kapovich–Polterovich question, which asks whether for a closed symplectic manifold (X, ω), the group $\mathrm{Ham}(X, \omega)$ is quasi-isometric to the real line with respect to the Hofer metric, has been an important open problem in Hofer geometry for a long time, and at the time of writing, it has been answered in the negative for symplectic manifolds that satisfy some dynamical condition [Ush13], the monotone $S^2 \times S^2$ [FOOO19] (see also [EliPol]), and the 2-sphere [CGHS, PS] and the del Pezzo surfaces $\mathbb{D}_3, \mathbb{D}_4$ [Kaw]. Theorem 1.3.2 improves what was known about the Kapovich–Polterovich, and the Entov–Polterovich–Py questions for $\mathbb{D}_4$ from [Kaw, Theorem C(2),D,E].

Remark 1.3.4. It seems likely that the number of pairwise distinct Entov–Polterovich quasimorphisms on $\mathrm{Ham}(\mathbb{D}_4)$ in Theorem 1.3.2 can be improved from four to six, see Remark 3.4.2.

1.4 Dehn twist and spectral invariants

The proofs of Theorems A, B, and C are based on the theory of spectral invariants (see Section 2.1 for the definition of spectral invariants). As mentioned earlier in Section 1.1, configurations of Lagrangian spheres were used by Seidel to study the Dehn twist. Recall that the Dehn twist is a (class of) symplectomorphism(s) that is defined for a Lagrangian sphere. By using some ingredients of the proof of Theorem B, we get the following result which describes the effect of the Dehn twist on spectral invariants.

Theorem D. *Let (X, ω) be a real $2n$ dimensional closed monotone symplectic manifold. Assume $QH(X, \omega)$ is semi-simple and also either one of the following:*

- *n is even,*
- *n is odd and $\frac{n+1}{2N_X} \notin \mathbb{Z}$ where N_X is the minimal Chern number.*

If (X, ω) contains an A_2-configuration of Lagrangian spheres $\{L, L'\}$, then we have

$$\overline{\ell}_{\tau_L(L')}(H) \leqslant \max\{\overline{\ell}_L(H), \overline{\ell}_{L'}(H)\}$$

for any Hamiltonian H, where τ_L is the Dehn twist about L.

Remark 1.4.1.

1. The function $\overline{\ell}_L$ is the asymptotic Lagrangian spectral invariant associated to a Lagrangian L. For the precise definition, see (2.2.3) in Section 2.1.
2. Strictly speaking, the Dehn twist τ_L about a Lagrangian sphere L is usually referred to a class of symplectomorphisms, i.e. $\tau_L \in \pi_0(\mathrm{Symp}(X, \omega))$. In Theorem D, we consider any representative of the class and denote it by τ_L by abuse of notation. Any two representatives of the Dehn twist are Hamiltonian isotopic and thus the spectral invariants corresponding to $\ell_{\tau_L(L')}$ might have a shift up to the Hofer norm of the Hamiltonian isotopy between them [LZ18, Proposition 2.6]. Nevertheless, the *asymptotic* spectral invariant $\overline{\ell}_{\tau_L(L')}$ does not depend on the choice of the representative of the Dehn twist and is well-defined.
3. It is not difficult to find examples (e.g. in $\mathbb{D}_4$) where we have a strict inequality in Theorem D.

To put Theorem D into perspective, following the success of the barcode theory in symplectic topology, there has been a lot of work on the filtration beyond the level of Floer homology, namely for Fukaya categories, e.g. [BCZ, Amb]. Cone-attaching is a fundamental algebraic operation in the A_∞-category theory, which in the context of Fukaya category has a geometric interpretation, namely the Lagrangian cobordism, e.g. the Dehn twist of a Lagrangian sphere. Biran–Cornea studied the filtration of Seidel's Floer-theoretic long exact sequence involving the Dehn twist [BC21], but the precise impact of the Dehn twist on spectral invariants was not clear. Thus, Theorem D could be regarded as the first step in the study of spectral invariants in the filtered A_∞-categorical setting.

1.5 Structure of the paper

To prove Theorem A, we reduce it to its symplectic counterpart, namely Theorem B, by some elementary algebro-geometric argument in Section 3.3. Theorem B is proven by the *spectral rigidity* (i.e. symplectic rigidity in terms of spectral invariants) of Lagrangian spheres. The key two lemmas are Lemma 3.1.4 and Lemma 3.2.1: the former describes some spectral rigidity of a Lagrangian sphere, and the latter describes a property of idempotents corresponding to Lagrangian spheres forming an A_2-configuration. Although Theorems C and D stem from different perspectives compared to Theorems A and B (where Theorem C has some Hamiltonian dynamical flavor and Theorem D comes from the filtered A_∞-categorical perspective), their proofs are based on the spectral rigidity of Lagrangian spheres that was studied to prove Theorem B.

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2 Preliminaries

2.1 Spectral invariant theory

It is well-known that on a closed symplectic manifold (X,ω)[5], for a non-degenerate Hamiltonian $H := \{H_t : X \to \mathbb{R}\}_{t\in[0,1]}$ and a choice of a nice coefficient field $\Lambda^{\downarrow}$, such as the downward Laurent coefficients $\Lambda^{\downarrow}_{\mathrm{Lau}}$ for the monotone case

$$\Lambda^{\downarrow}_{\mathrm{Lau}} := \{\sum_{k\leqslant k_0} b_k t^k : k_0 \in \mathbb{Z}, b_k \in \mathbb{C}\},$$

or the downward Novikov coefficients $\Lambda^{\downarrow}_{\mathrm{Nov}}$ for the general case

$$\Lambda^{\downarrow}_{\mathrm{Nov}} := \{\sum_{j=1}^{\infty} a_j T^{\lambda_j} : a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j\to-\infty} \lambda_j = +\infty\},$$

one can construct a filtered Floer homology group $\{HF^{\tau}(H) := HF^{\tau}(H;\Lambda^{\downarrow})\}_{\tau\in\mathbb{R}}$. Note that in this paper, we only use Novikov coefficients, i.e.

$$\Lambda^{\downarrow} = \Lambda^{\downarrow}_{\mathrm{Nov}}.$$

For two numbers $\tau < \tau'$, the groups $HF^{\tau}(H;\Lambda^{\downarrow})$ and $HF^{\tau'}(H;\Lambda^{\downarrow})$ are related by a map induced by the inclusion map on the chain level:

$$i_{\tau,\tau'} : HF^{\tau}(H;\Lambda^{\downarrow}) \longrightarrow HF^{\tau'}(H;\Lambda^{\downarrow}),$$

and especially we have

$$i_{\tau} : HF^{\tau}(H;\Lambda^{\downarrow}) \longrightarrow HF(H;\Lambda^{\downarrow}),$$

where $HF(H;\Lambda^{\downarrow})$ is the Floer homology group. There is a canonical ring isomorphism called the Piunikhin–Salamon–Schwarz (PSS)-map [PSS96], [MS04]

$$PSS_{H;\Lambda} : QH(X,\omega;\Lambda) \xrightarrow{\sim} HF(H;\Lambda^{\downarrow}),$$

where $QH(X,\omega;\Lambda)$ denotes the quantum cohomology ring of (X,ω) with Λ-coefficients, i.e.

$$QH(X,\omega;\Lambda) := H^*(X;\mathbb{C}) \otimes \Lambda.$$

Here, Λ is the Novikov coefficients (the universal Novikov field) Λ_{Nov}

$$\Lambda_{\mathrm{Nov}} := \{\sum_{j=1}^{\infty} a_j T^{\lambda_j} : a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j\to+\infty} \lambda_j = +\infty\}.$$

From now on, we will always take the the universal Novikov field to set-up the quantum cohomology ring, so we will often abbreviate it by $QH(X,\omega)$, i.e.

$$QH(X,\omega) := QH(X,\omega;\Lambda_{\mathrm{Nov}}).$$

[5]Although the results in this section hold for general closed symplectic manifolds, we will only be using the monotone case due to some Floer-theoretic constraints that will appear later, which is not from the spectral invariant theory.

The ring structure of $QH(X,\omega)$ is given by the quantum product, which is a quantum deformation of the intersection product

$$- * - : QH(X,\omega) \times QH(X,\omega) \to QH(X,\omega).$$

The spectral invariants, which were introduced by Schwarz [Sch00] and developed by Oh [Oh05] following the idea of Viterbo [Vit92], are real numbers $\{c(H,a) \in \mathbb{R}\}$ associated to a pair of a Hamiltonian H and a class $a \in QH(X,\omega)$ in the following way:

$$c(H,a) := \inf\{\tau \in \mathbb{R} : PSS_{H;\Lambda}(a) \in \mathrm{Im}(i_\tau)\}.$$

Remark 2.1.1. Although the Floer homology is only defined for a non-degenerate Hamiltonian H, the spectral invariants can be defined for any Hamiltonian by using the following *Hofer continuity property*:

$$\int_0^1 \min_{x\in X} (H_t(x) - G_t(x))\, dt \leqslant c(H,a) - c(G,a) \leqslant \int_0^1 \max_{x\in X} (H_t(x) - G_t(x))\, dt \quad (2.1.1)$$

for any $a \in QH(X,\omega)$, H and G.

Spectral invariants satisfy the *triangle inequality*: for Hamiltonians H, G and $a, b \in QH(X,\omega)$, we have

$$c(H,a) + c(G,b) \geqslant c(H\#G, a*b) \quad (2.1.2)$$

where $H\#G(t,x) := H_t(x) + G_t((\phi_H^t)^{-1}(x))$ and it generates the path $t \mapsto \phi_H^t \circ \phi_G^t$ in $\mathrm{Ham}(X,\omega)$.

When we take the zero function as the Hamiltonian, we have the *valuation property*: for any $a \in QH(X;\Lambda)\backslash\{0\}$,

$$c(0,a) = \nu(a) \quad (2.1.3)$$

where 0 is the zero-function and $\nu : QH(X;\Lambda) \to \mathbb{R}$ is the natural valuation function

$$\begin{aligned} &\nu : QH(X;\Lambda) \to \mathbb{R} \\ &\nu(a) := \nu(\sum_{j=1}^{\infty} a_j T^{\lambda_j}) := \min\{\lambda_j : a_j \neq 0\}. \end{aligned} \quad (2.1.4)$$

Note that from the triangle inequality (2.1.2) and the valuation property (2.1.3), for any $a \in QH(X;\Lambda)\backslash\{0\}$, $\lambda \in \Lambda$ and a Hamiltonian H, we have

$$c(H, \lambda \cdot a) = c(H,a) + \nu(\lambda). \quad (2.1.5)$$

Analogous invariants for Lagrangian Floer homology, namely the Lagrangian spectral invariants, were defined in [Lec08, LZ18, FOOO19, PS]. We summarize some basic properties of Lagrangian spectral invariants from these references. Once again, given a pair of a (non-degenerate) Hamiltonian H and a class $a \in HF(L)$[6], we define

$$\ell(H,\alpha) := \inf\{\tau \in \mathbb{R} : PSS_{L,H}(\alpha) \in \mathrm{Im}(i_\tau^L)\}$$

[6]The Lagrangian Floer homology for L without a Hamiltonian term $HF(L)$ stands for the Lagrangian quantum cohomology [BC08], which is also written as $QH(L)$ in the literature.

where

$$PSS_{L,H} : HF(L) \to HF(L,H),$$
$$i^L_\tau : HF^\tau(L,H) \to HF^\tau(L,H).$$

In this paper, we pay particular attention to the case where $\alpha = 1_L$. In this case, we simply denote

$$\ell_L(H) := \ell(H, 1_L).$$

Analogously to the Hamiltonian case (c.f. (2.1.1)), we have the *Lagrangian control property* for ℓ_L:

$$\int_0^1 \min_{x\in L} H_t(x)dt \leqslant \ell_L(H) \leqslant \int_0^1 \max_{x\in L} H_t(x)dt \tag{2.1.6}$$

Properties analogous to (2.1.2), (2.1.3), (2.1.5) also hold for Lagrangian spectral invariants.

Note that both Hamiltonian and Lagrangian spectral invariants satisfy the homotopy invariance, i.e. if two normalized Hamiltonians H and G generate homotopic Hamiltonian paths $t \mapsto \phi^t_H$ and $t \mapsto \phi^t_G$ in $\mathrm{Ham}(X,\omega)$, then

$$c(H,-) = c(G,-).$$

Thus, one can define spectral invariants on $\widetilde{\mathrm{Ham}}(X,\omega)$:

$$\begin{gathered} c : \widetilde{\mathrm{Ham}}(X,\omega) \times QH(X,\omega) \to \mathbb{R} \\ c(\widetilde{\phi}, a) := c(H,a) \end{gathered} \tag{2.1.7}$$

where the path $t \mapsto \phi^t_H$ represents the class of paths $\widetilde{\phi}$. Similarly, one can define

$$\ell : \widetilde{\mathrm{Ham}}(X,\omega) \times HF(L) \to \mathbb{R}.$$

Hamiltonian and Lagrangian Floer homologies are related by the closed-open and open-closed maps

$$\begin{gathered} \mathcal{CO}^0 : QH(X,\omega) \to HF(L), \\ \mathcal{OC}^0 : HF(L) \to QH(X,\omega), \end{gathered} \tag{2.1.8}$$

which are defined by counting certain holomorphic curves. The closed-open map $\mathcal{CO}^0$ is a ring homomorphism and the open-closed map $\mathcal{OC}^0$ defines a module action. As they are defined by counting certain holomorphic curves, which have positive ω-energy, they have the following effect on spectral invariants.

Proposition 2.1.2 ([BC08, LZ18, FOOO19])**.** *Let H be any Hamiltonian.*

1. *For any $a \in QH(X,\omega)$, we have*

$$c(H,a) \geqslant \ell(H, \mathcal{CO}^0(a)).$$

2. *For any $\alpha \in HF(L)$, we have*

$$\ell(H,\alpha) \geqslant c(H, \mathcal{OC}^0(\alpha)).$$

It is a basic property of the open-closed map (c.f. [BC12, Section 2.5.2]) that

$$\mathcal{OC}^0(1_L) = [L], \tag{2.1.9}$$

and thus, by Proposition 2.1.2, we have

$$\ell(H, 1_L) \geqslant c(H, [L]). \tag{2.1.10}$$

2.2 Entov–Polterovich quasimorphisms and (super)heaviness

Based on spectral invariants, Entov–Polterovich built two theories, namely the theory of (Calabi) quasimorphisms and the theory of (super)heaviness, which we briefly review in this section.

Quasimorphisms. Entov–Polterovich constructed a special map on $\widetilde{\mathrm{Ham}}(X,\omega)$ called the quasimorphism for under some assumptions. Recall that a quasimorphism μ on a group G is a map to the real line $\mathbb{R}$ that satisfies the following two properties:

1. There exists a constant $C > 0$ such that
$$|\mu(f \cdot g) - \mu(f) - \mu(g)| < C$$
for any $f, g \in G$.

2. For any $k \in \mathbb{Z}$ and $f \in G$, we have
$$\mu(f^k) = k \cdot \mu(f).$$

The following is Entov–Polterovich's construction of quasimorphisms on $\widetilde{\mathrm{Ham}}(X,\omega)$.

Theorem 2.2.1 ([EP03])**.** *Suppose $QH(X,\omega;\Lambda)$ has a field factor, i.e.*
$$QH(X,\omega) = Q \oplus A$$
where Q is a field and A is some algebra. Decompose the unit 1_X of $QH(X,\omega)$ with respect to this split, i.e.
$$1_X = e + a.$$
Then, the asymptotic spectral invariant of $\widetilde{\phi}$ with respect to e defines a quasimorphism, i.e.
$$\begin{aligned} \zeta_e : \widetilde{\mathrm{Ham}}(X,\omega) &\longrightarrow \mathbb{R} \\ \zeta_e(\widetilde{\phi}) := \lim_{k\to+\infty} \frac{c(\widetilde{\phi}^k, e)}{k} &= \lim_{k\to+\infty} \frac{c(H^{\#k}, e)}{k} \end{aligned} \tag{2.2.1}$$
where H is any mean-normalized Hamiltonian such that the path $t \mapsto \phi_H^t$ represents the class $\widetilde{\phi}$ in $\widetilde{\mathrm{Ham}}(X,\omega)$.

Remark 2.2.2. By slight abuse of notation, we will also see ζ_e as a function on the set of time-independent Hamiltonians:
$$\begin{aligned} \zeta_e : C^\infty(X) &\longrightarrow \mathbb{R} \\ \zeta_e(H) &:= \lim_{k\to+\infty} \frac{c(H^{\#k}, e)}{k}. \end{aligned} \tag{2.2.2}$$

Remark 2.2.3. The Lagrangian spectral invariants do not appear in the result of Entov–Polterovich, but we define the *asymptotic Lagrangian spectral invariants*, as we will use them later on in the proofs.

$$
\begin{aligned}
\overline{\ell}_L &: \widetilde{\mathrm{Ham}}(X,\omega) \longrightarrow \mathbb{R} \\
\overline{\ell}_L &:= \lim_{k\to+\infty} \frac{\ell(\widetilde{\phi}^k, 1_L)}{k}
\end{aligned}
\tag{2.2.3}
$$

As mentioned in the introduction, Entov–Polterovich quasimorphisms are useful to study the Hofer geometry. For example, in some cases Entov–Polterovich quasimorphisms on $\widetilde{\mathrm{Ham}}(X,\omega)$ descend to $\mathrm{Ham}(X,\omega)$, e.g. when $X = S^2, \mathbb{C}P^2, S^2 \times S^2$. Denote one by $\zeta_e : \mathrm{Ham}(X,\omega) \to \mathbb{R}$. Then, by using the homogeneity of ζ_e and the Hofer Lipschitz continuity, we can prove the Hofer diameter conjecture by

$$
\lim_{k\to+\infty} d_{\mathrm{Hof}}(\mathrm{id}, \phi^k) \geqslant \lim_{k\to+\infty} |\zeta_e(\phi^k)| = \lim_{k\to+\infty} k \cdot |\zeta_e(\phi)| = +\infty. \tag{2.2.4}
$$

Superheaviness. Entov–Polterovich introduced a notion of symplectic rigidity for subsets in (X,ω) called (super)heaviness.

Definition 2.2.4 ([EP09],[EP06])**.** *Take an idempotent $e \in QH(X,\omega)$ and denote the asymptotic spectral invariant with respect to e by ζ_e. A subset S of (X,ω) is called*

1. *e-heavy if for any time-independent Hamiltonian $H : X \to \mathbb{R}$, we have*

$$
\inf_{x\in S} H(x) \leqslant \zeta_e(H),
$$

2. *e-superheavy if for any time-independent Hamiltonian $H : X \to \mathbb{R}$, we have*

$$
\zeta_e(H) \leqslant \sup_{x\in S} H(x).
$$

REMARK 2.2.5. Note that if a set S is e-superheavy, then it is also e-heavy.

The following is an easy corollary of the definition of superheaviness which is useful.

Proposition 2.2.6 ([EP09])**.** *Assume the same condition on $QH(X,\omega)$ as in Theorem 2.2.1. Let S be a subset of X that is e-superheavy. For a time-independent Hamiltonian $H : X \to \mathbb{R}$ whose restriction to S is constant, i.e. $H|_S \equiv r,\ r \in \mathbb{R}$, we have*

$$
\zeta_e(H) = r.
$$

In particular, two disjoint subsets of (X,ω) cannot be both e-superheavy.

Proof. The first part is an immediate consequence of the definition of (super)heaviness. As for the second part, suppose we have two disjoint sets A, B in (X,ω) that are both e-superheavy. Consider a Hamiltonian H that is

$$
H|_A = 0,\ H|_B = 1.
$$

Then, by superheaviness, we have

$$
1 = \inf_{x\in B} H(x) \leqslant \zeta_e(H) \leqslant \sup_{x\in A} H(x) = 0,
$$

which is a contradiction. ∎

We end this section by giving a criterion for heaviness, proved by Fukaya–Oh–Ohta–Ono (there are earlier results with less generality, c.f. [Alb05]) using the closed-open map

$$\mathcal{CO}^0 : QH(X,\omega) \to HF(L).$$

Theorem 2.2.7 ([FOOO19, Theorem 1.6])**.** *Assume $HF(L) \neq 0$. If*

$$\mathcal{CO}^0(e) \neq 0$$

for an idempotent $e \in QH(X,\omega)$, then L is e-heavy.

Remark 2.2.8. When ζ_e is homogeneous, e.g. when e is a unit of a field factor of $QH(X,\omega)$ and ζ_e is an Entov–Polterovich quasimorphism, then heaviness and superheaviness are equivalent so Theorem 2.2.7 will be good enough to obtain the superheaviness of L.

2.3 Semi-simplicity of the quantum cohomology ring

In this section, we review the notion of semi-simplicity of the quantum cohomology ring, both in the context of algebraic and symplectic geometry. Indeed, the semi-simplicity of the quantum cohomology ring is an algebraic structure that has been studied and used widely across algebraic geometry [BM04, Dub96, Man99] and symplectic topology [BC, EP03, FOOO10], even though the definitions are slightly different. See the final paragraph for the conclusion. Recall that the (small) quantum cohomology ring[7] of a symplectic manifold (X,ω) is semi-simple when it splits into a direct sum of finite fields:

$$QH(X,\omega) = \bigoplus_{1\leqslant j\leqslant k} Q_j$$

where each Q_j is a field. For examples of symplectic manifolds whose quantum cohomology rings are semi-simple, we refer to Section 1.1.

Remark 2.3.1.

1. By the definition of semi-simplicity, it follows immediately that if $QH(X,\omega)$ is semi-simple, then there is no nilpotent in $QH(X,\omega)$.

2. Also, note that symplectic manifolds with semi-simple quantum cohomology ring provide examples to which one can apply Entov–Polterovich's construction of quasimorphisms (Theorem 2.2.1).

3. In this paper, we take the universal Novikov field Λ to define the quantum cohomology ring, but the quantum cohomology ring can defined by other coefficient fields, e.g. the field of Laurent series. The choice of a coefficient field does not impact the semi-simplicity [EP08, Proposition 2.1].

[7]We emphasize that in this paper, quantum cohomology ring always refers to the small one unless mentioned otherwise.

In fact, Usher proves that the existence of one symplectic form for which the (small) quantum cohomology ring is semi-simple implies semi-simplicity for a generic choice of a symplectic form [Ush11, Proposition 7.11] (However, having semi-simple quantum cohomology ring for a generic symplectic form does not imply that for all symplectic forms we have semi-simple quantum cohomology ring [OT09]). Usher also proves that this generic semi-simplicity for the (small) quantum cohomology ring implies the generic semi-simplicity for the big quantum cohomology ring [Ush11, Proposition 7.5].[8] In practice, the equivalent properties in [Ush11, Theorem 7.8, Proposition 7.11] are easier to understand and are more useful.

Moreover, he proves that the notion of semi-simplicity of the big quantum cohomology ring used in the algebraic geometry community, e.g. [BM04, Dub96, Man99], which is stated in terms of the Frobenius manifold, is equivalent to the symplectic definition of the generic semi-simplicity of the big quantum cohomology ring [Ush11, Section 7.3.3].

In summary, the monotone semi-simplicity, namely the assumption of Theorem A, implies the semi-simplicity of the big quantum cohomology ring used in the algebraic geometry community. Thus, strictly speaking, the assumption of Theorem A is stronger than the notion of semi-simplicity commonly used in algebraic geometry.

2.4 Degeneration

In this section, we review some basics of degeneration. A recommended reference for the topic of this section is [Eva].

Definition 2.4.1. *Let X be a smooth variety. A degeneration of X is a flat family $\pi : \mathcal{X} \to \mathbb{C}$ such that*

- *The only singular fiber is $X_0 := \pi^{-1}(0)$.*
- *Some regular fiber is isomorphic to X.*
- *The variety $\mathcal{X}$ is smooth away from the singular locus of X_0.*

If there is a degeneration of a variety X such that the central fiber is $X_0 := \pi^{-1}(0)$, we say that X degenerates to X_0. Note that up to here, the notion of degeneration is completely in the realm of algebraic geometry, but if we are in a situation where the following is valid, then we can start seeing the variety X as a symplectic manifold.

- There is a relatively ample line bundle $\mathcal{L} \to \mathcal{X}$ with respect to $\pi : \mathcal{X} \to \mathbb{C}$ (also said π-relatively ample line bundle).

Remark 2.4.2. In the proof of Theorem A, we will consider the anti-canonical bundle $-\Omega_{\mathcal{X}}$ (sometimes also written $-K_{\mathcal{X}}$) on $\mathcal{X}$. We will make this precise, as the variety $\mathcal{X}$ is singular. Recall that if a complex n-dimensional variety Y is smooth, then the canonical line bundle is $\Omega_Y := \bigwedge^n T^*Y$. When Y is singular, the same object cannot be defined. In order to circumvent this problem, Grothendieck introduced the dualizing complex $\omega_Y^\bullet$, which turns out to be a dualizing sheaf $\omega_Y = \omega_Y^\bullet$ if Y is Cohen–Macaulay.

[8] For the definition of the generic semi-simplicity for the big (resp. small) quantum cohomology ring, see [Ush11, Section 7.2] (resp. [Ush11, Definition 7.4]).

Furthermore, if Y is Gorenstein, then ω_Y is an invertible sheaf, i.e. a line bundle. Note that if Y is smooth, we have $\omega_Y = K_Y$, thus the dualizing sheaves can be considered as the canonical bundles for singular varieties.

In this paper, we will only be concerned with singular varieties $\mathcal{X}, X_0$ that have at most isolated hypersurface singularities, which are Gorenstein, so we can define the line bundles $\omega_{\mathcal{X}}, \omega_{X_0}$ as well as their inverse line bundles $-\omega_{\mathcal{X}}, -\omega_{X_0}$. For a Gorenstein variety Y, we say that it is Fano if the line bundle $-\omega_Y$ is ample.

The π-relative ample line bundle $\mathcal{L} \to \mathcal{X}$ defines the following map:

$$\begin{tikzcd} X_t \arrow[r, "f_t"] \arrow[d, hook, "i_t"] & \{t\} \times \mathbb{C}P^n \arrow[d, hook] \\ \mathcal{X} \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{P}(\pi_*\mathcal{L}) = \mathbb{C} \times \mathbb{C}P^n \arrow[dl, "\mathrm{pr}_1"] \\ \mathbb{C} & \end{tikzcd} \quad . \tag{2.4.1}$$

The restriction of the form $f^*\omega_{\mathrm{FS}}$ on each fiber $X_t,\ t \neq 0$ gives a symplectic form on it. Moreover, by using the form $f^*\omega_{\mathrm{FS}}$ on $\mathcal{X}$, one can define a symplectic parallel transport for $\pi : \mathcal{X} \to \mathbb{C}$, and from a standard argument, it follows that all the smooth fibers are symplectomorphic with respect to the symplectic form $\omega_t := (f|_{X_t})^*\omega_{\mathrm{FS}}$ on X_t (c.f. [Eva, Lemma 1.1]).

One point we need to be careful about is that if we are interested in the monotone symplectic form on $X = X_1$, then we need to find a π-relatively ample line bundle $\mathcal{L} \to \mathcal{X}$ that restricts to the anti-canonical bundle on $X_t := \pi^{-1}(t),\ t \neq 0$. When the anti-canonical bundle $-K_X$ of a variety X is ample, i.e. X is Fano, the sections of some power $(-K_X)^{\otimes m},\ m \in \mathbb{N}$ give rise to an embedding

$$f : X \hookrightarrow \mathbb{P}(V),$$

where V is the dual of the space of sections, and the pull-back $f^*\omega_{\mathrm{FS}}$ is monotone. To see this,

$$\begin{aligned} f^*[\omega_{\mathrm{FS}}] &= f^*c_1(\mathcal{O}(1)) \\ &= c_1(((\bigwedge^n T^*X)^{-1})^{\otimes m}) \\ &= m \cdot c_1((\bigwedge^n T^*X)^{-1}) \\ &= -m \cdot c_1(\bigwedge^n T^*X) \\ &= m \cdot c_1(TX). \end{aligned} \tag{2.4.2}$$

This is precisely the point where one needs to be careful about when reducing Theorem A to Theorem B. See the proof of Theorem A on this.

Remark 2.4.3. In this paper, we only study Fano varieties/monotone symplectic manifolds, but for general type varieties/negative monotone symplectic manifolds, there are examples of degenerations in [ES20, EU21] for which the relative canonical bundle is not relatively ample. Thus, to study these degenerations one needs to use a symplectic form that is not negative monotone.

2.5 Isolated hypersurface singularities

In this section, we gather some facts about the isolated hypersurface singularities. A recommended reference with more details is [Kea15, Section 1,2].

As we mentioned in the introduction (Section 1.1), isolated hypersurface singularities have been used to produce Lagrangian spheres. The modality is a non-negative integer associated to an isolated hypersurface singularity, which could be thought of as the number of complex parameters of the miniversal deformation space (we refer to [Kea15] for more details). Arnold classified isolated hypersurface singularities up to modality two. The modality zero case is precisely the simple singularities, which have three types A_m, D_m $(m \geqslant 4)$, and E_m $(m = 6, 7, 8)$ and they are locally expressed as

$$x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + p(y, z) = 0 \tag{2.5.1}$$

where $p(y, z)$ is

$$y^2 + z^{m+1},\ yz^2 + z^{m-1},\ y^3 + z^4,\ y^3 + yz^3,\ y^3 + z^5,$$

for A_m, D_m, E_6, E_7, and E_8, respectively. Now, suppose the variety X_0 has an A_m singularity. The vanishing cycle of the A_m singularity in its smoothing (X, ω) forms a collection of Lagrangian spheres

$$\mathscr{S}_{A_m} := \{S_j\}_{1 \leqslant j \leqslant m}$$

satisfying the following intersection property:

$$\#\left(S_i \cap S_j\right) = \begin{cases} 1 \text{ if } |i-j| = 1 \\ 0 \text{ otherwise.} \end{cases} \tag{2.5.2}$$

One can see the intersection pattern of (2.5.2) in the Dynkin diagram of the type A_m (Figure 1), where each dot in the diagram corresponds to a Lagrangian sphere and a segment between two dots implies that the Lagrangian spheres corresponding to the dots intersect transversally at one point. We call such a collection of Lagrangian spheres an A_m configuration. Similarly, we define D_m $(m \geqslant 4)$, and E_m $(m = 6, 7, 8)$ configurations of Lagrangian spheres to be the collections of Lagrangian spheres that satisfy the intersection patterns of the D_m and E_m type Dynkin diagrams, respectively (Figure 1). These configurations appear as the vanishing cycles of singularities of type D_m $(m \geqslant 4)$, and E_m $(m = 6, 7, 8)$, respectively.

As for the isolated hypersurface singularities of modality one, which were also classified by Arnold, there are the following three types:

1. (parabolic or simple elliptic singularities) $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$.

2. (hyperbolic singularities) $T_{p,q,r}$, where

$$\begin{aligned} T_{p,q,r} &= x_1^2 + x_2^2 + \cdots + x_{n-2}^2 + h(x, y, z), \\ h(x, y, z) &= x^p + y^q + z^r + axyz,\ a \neq 0 \end{aligned} \tag{2.5.3}$$

   with integers p, q, r such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

Note that the three pairs $(p, q, r) = (3, 3, 3), (2, 4, 4), (2, 3, 6)$, which are the solutions to $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, are precisely the three parabolic singularities;

$$\widetilde{E}_6 = T_{3,3,3},\ \widetilde{E}_7 = T_{2,4,4},\ \widetilde{E}_8 = T_{2,3,6}.$$

3. 14 exceptional singularities.

Other than the simple singularities and the three types of modality one singularities, all the other isolated hypersurface singularities have modality greater than or equal to two.

EXAMPLE 2.5.1. There are other famous classes of singularities such as the *Brieskorn–Pham singularities*. The Brieskorn–Pham singularities are isolated hypersurface singularities which include a part of simple singularities, parabolic (or simple elliptic) singularities, the 14 exceptional unimodal singularities. Thus, Brieskorn–Pham singularities are covered in Theorems A, B. See [FU11, Kea21] for some symplectic results related to Brieskorn–Pham singularities.

In [Kea15], Keating executed a detailed study of the vanishing cycles and the Milnor fibers for these singularities. We recall the definition of the *Milnor fiber*.

Definition 2.5.2. *The Milnor fiber of a hypersurface singularity $h = 0$, where h is the polynomial expressing the singularity (e.g.* (2.5.1), (2.5.3)*), is the intersection of the affine hypersurface $h^{-1}(t) \subset \mathbb{C}^{n+1}$ for a small $|t|$ and a small ball $B(0; \varepsilon)$ (the ball of radius $\varepsilon > 0$ around the origin), i.e. $h^{-1}(t) \cap B(0; \varepsilon)$.*[9]

Note that Milnor fibers are Liouville domains. We collect some of Keating's results that will be used in the proof of Theorem A.

Proposition 2.5.3. *The vanishing cycles of the parabolic singularities form a configuration of Lagrangian spheres with the intersection property as in the Dynkin diagram of Gabrielov [Kea15, Figure 4].*

Proposition 2.5.4 ([Kea15, Lemma 2.12 (resp. 2.13), Corollary 2.17])**.** *Take any isolated hypersurface singularity with positive modality. Then, the Milnor fiber (resp. vanishing cycles) of one of the three parabolic singularities $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ can be symplectically embedded to the Milnor fiber (resp. the vanishing cycles) of the taken isolated hypersurface singularity with positive modality.*

Proposition 2.5.4 will be used in the second part of the proof of Theorem A.

REMARK 2.5.5. Note that Proposition 2.5.4 was used by Keating in the first line of the proof of [Kea15, Theorem 5.7], which might be instructive for the reader.

[9] The definition depends on the choices of t and ε, but it is unique up to diffeomorphism. From a symplectic viewpoint, different choices of t and ε will give non-symplectomorphic Milnor fibers, but they both have completions that are symplectomorphic.

3 Proofs

3.1 Spectral rigidity of Lagrangian spheres

In this section, we study properties of spectral invariants for Lagrangian spheres, which will be relevant in the later sections. Note that in this section, we do not assume semi-simplicity but we assume that (X, ω) is a real $2n$-dimensional closed monotone symplectic manifold with even n (which means that the results in this section are not directly relevant for the odd n case). The monotonicity is assumed for technical reasons.

In [BM16, Theorem 3.3], Biran–Membrez proved that for an even dimensional monotone Lagrangian sphere L in (X, ω) (which forces the real dimension of (X, ω) to be $2n$ with even n) satisfies the following property: the cohomology class $[L] \in QH(X, \omega)$, which is the Poincaré dual of the homology class represented by L, satisfies the cubic equation

$$[L]^3 = 4\beta_L [L] \tag{3.1.1}$$

for some $\beta_L \in \Lambda$. When $\beta_L \neq 0$, then the cubic equation (3.1.1) implies that

$$e_{\pm}^L = \pm \frac{1}{4\sqrt{\beta_L}}[L] + \frac{1}{8\beta_L}[L]^2 \tag{3.1.2}$$

gives two orthogonal idempotents of $QH(X, \omega)$. In fact, the idempotents $e_{\pm}^L$ are not only idempotents of $QH(X, \omega)$, but are units of field factors of $QH(X, \omega)$.

Claim 3.1.1 ([San21, Proposition 5.8])**.** *The idempotents $e_{\pm}^L$ are units of field factors of $QH(X, \omega)$, i.e.*

$$e_{\pm}^L \cdot QH(X, \omega) = \Lambda \cdot e_{\pm}^L.$$

Although this is already proven in [San21, Proposition 5.8] (for the monotone case for technical reasons), we explain Claim 3.1.1 with an additional assumption that $QH(X, \omega)$ is semi-simple, which is the situation we consider in the rest of the paper, as in this case the argument is elementary.

Proof. We know that $e_{\pm}^L$ are idempotents of $QH(X, \omega)$ which we assume to be semi-simple, so what we want to check is that $e_{\pm}^L$ are not sums of finer idempotents, i.e. $e_{\pm}^L = e_1 + e_2 + \cdots$ where e_j are idempotents. In Sanda's proof of Biran–Membrez's Lagrangian cubic equation ([San21, Proof of Proposition 5.7], which works for the monotone case), it is shown that for any Lagrangian sphere L, we have

$$\begin{aligned} \mathcal{CO}^0 \circ \mathcal{OC}^0(1_L) &= 2p_L, \\ \mathcal{CO}^0 \circ \mathcal{OC}^0(p_L) &= \beta_L 1_L, \end{aligned} \tag{3.1.3}$$

where p_L is the Poincaré dual of the point class of $HF(L)$, and β_L is defined by the equation $p_L^2 = \beta_L 1_L$. The equations (3.1.3) imply that the map $\mathcal{CO}^0 \circ \mathcal{OC}^0$ is an isomorphism (between Λ-vector spaces);

$$\begin{tikzcd} & QH(X,\omega) \arrow[dr, "\mathcal{CO}^0"] & \\ HF(L) \arrow[ur, "\mathcal{OC}^0"] \arrow[rr, "\sim"] & & HF(L). \end{tikzcd} \tag{3.1.4}$$

Thus,

$$\mathcal{OC}^0 : HF(L) \to \mathrm{Im}(\mathcal{OC}^0)$$

is also an isomorphism (between Λ-vector spaces). We have $\mathrm{Im}(\mathcal{OC}^0) = (e_+^L + e_-^L) \cdot QH(X, \omega)$, which follows from the following two facts:

1. $\mathcal{OC}^0(1_L) = 2\sqrt{\beta_L}(e_+^L - e_-^L),\ \mathcal{OC}^0(p_L) = 2\beta_L(e_+^L + e_-^L)$.
2. $\mathcal{OC}^0$ is a $QH(X,\omega)$-module map, i.e. $\alpha \cdot \mathcal{OC}^0(a) = \mathcal{OC}^0(\mathcal{CO}^0(\alpha) \cdot a)$ for any $\alpha \in QH(X,\omega)$, $a \in HF(L)$ (combined with $e_\pm^L \in \mathrm{Im}(\mathcal{OC}^0)$ from the first item).

Remark 3.1.2. Note that in the first item, we used that $[L] = 2\sqrt{\beta_L}(e_+^L - e_-^L)$ and

$$\mathcal{OC}^0(p_L) = \frac{1}{2}\mathcal{OC}^0(\mathcal{CO}^0 \circ \mathcal{OC}^0(1_L)) = \frac{1}{2}\mathcal{OC}^0(1_L) \cdot \mathcal{OC}^0(1_L) = \frac{1}{2}[L]^2 = 2\beta_L(e_+^L + e_-^L).$$

We know that $HF(L) \simeq \Lambda \oplus \Lambda$ as a Λ-vector space, i.e. $\dim_\Lambda HF(L) = 2$, and thus, by the isomorphism $\mathcal{OC}^0 : HF(L) \xrightarrow{\sim} \mathrm{Im}(\mathcal{OC}^0)$ as Λ-vector spaces, we have $\dim_\Lambda (e_+^L + e_-^L) \cdot QH(X,\omega) = 2$. This implies that $e_\pm^L$ cannot further split to finer idempotents, i.e. they satisfy $e_\pm^L \cdot QH(X,\omega) = \Lambda \cdot e_\pm^L$. □

Remark 3.1.3. It is useful to keep in mind that, when $\beta_L \neq 0$, the class $[L]$ can be expressed by the two idempotents in (3.1.2) as follows:

$$[L] = 2\sqrt{\beta_L}(e_+^L - e_-^L). \tag{3.1.5}$$

Lemma 3.1.4. *Let L an even-dimensional monotone Lagrangian sphere in a closed monotone symplectic manifold (X, ω). Assume $\beta_L \neq 0$. The spectral invariants for $1_L \in HF(L)$ and $e_\pm^L \in QH(X,\omega)$ are related as follows:*

$$\overline{\ell}_L(H) = \max\{\zeta_{e_+^L}(H),\ \zeta_{e_-^L}(H)\} \tag{3.1.6}$$

for any Hamiltonian H. In particular, L is $e_\pm^L$-superheavy, i.e. superheavy with respect to both $e_\pm^L$.

Proof of Lemma 3.1.4. From (2.1.10), we have

$$\ell(H, 1_L) \geqslant c(H, [L])$$

for any Hamiltonian H. By using (3.1.5), we further get

$$\begin{aligned} \ell(H, 1_L) &\geqslant c(H, [L]) \\ &= c(H, 2\sqrt{\beta_L}(e_+^L - e_-^L)) \\ &= c(H, e_+^L - e_-^L) + \nu(2\sqrt{\beta_L}). \end{aligned} \tag{3.1.7}$$

Note that the last equality uses (2.1.5). By using (3.1.3), we can see that the closed-open map

$$\mathcal{CO}^0 : QH(X,\omega) \longrightarrow HF(L)$$

satisfies

$$\begin{aligned}\mathcal{CO}^0(e_{\pm}^L) &= \pm\frac{1}{4\sqrt{\beta_L}}\mathcal{CO}^0\circ\mathcal{OC}^0(1_L)+\frac{1}{8\beta_L}(\mathcal{CO}^0\circ\mathcal{OC}^0(1_L))^2\\ &= \pm\frac{1}{4\sqrt{\beta_L}}2p_L+\frac{1}{8\beta_L}(2p_L)^2\\ &= \pm\frac{1}{2\sqrt{\beta_L}}p_L+\frac{1}{8\beta_L}4\beta_L\cdot 1_L\\ &= \pm\frac{1}{2\sqrt{\beta_L}}p_L+\frac{1}{2}1_L.\end{aligned}\tag{3.1.8}$$

Thus, we have

$$\mathcal{CO}^0(e_+^L+e_-^L)=1_L.$$

This implies

$$c(H,e_+^L+e_-^L)\geqslant \ell(H,1_L).\tag{3.1.9}$$

Inequalities (3.1.7) and (3.1.9) imply

$$c(H,e_+^L+e_-^L)\geqslant \ell(H,1_L)\geqslant c(H,e_+^L-e_-^L)+\nu(2\sqrt{\beta_L}).$$

By homogenizing this, we get

$$\zeta_{e_+^L+e_-^L}(H)\leqslant \overline{\ell}_L(H)\leqslant \zeta_{e_+^L-e_-^L}(H).\tag{3.1.10}$$

We claim the following.

Claim 3.1.5. *We have*

$$\zeta_{e_+^L+e_-^L}(H)=\zeta_{e_+^L-e_-^L}(H)=\max\{\zeta_{e_+^L}(H),\ \zeta_{e_-^L}(H)\}$$

for every Hamiltonian H.

Claim 3.1.5 will be proved shortly after the proof. The inequality (3.1.10) and Claim 3.1.5 imply

$$\overline{\ell}_L(H)=\max\{\zeta_{e_+^L}(H),\ \zeta_{e_-^L}(H)\}$$

for any H, which completes the proof of (3.1.6). Now, by combining (2.1.6) and (3.1.6), we obtain

$$\max\{\zeta_{e_+^L}(H),\ \zeta_{e_-^L}(H)\}=\overline{\ell}_L(H)\leqslant \ell_L(H)\leqslant\int_0^1\max_{x\in L}H_t(x)dt$$

for every Hamiltonian H, which implies that L is superheavy with respect to both e_+^L and e_-^L. This completes the proof of Lemma 3.1.4. □

Proof of Claim 3.1.5. We first prove $\zeta_{e_+^L+e_-^L}(H)\geqslant\max\{\zeta_{e_+^L}(H),\ \zeta_{e_-^L}(H)\}$ for any H. By the triangle inequality (2.1.2), we get

$$\begin{aligned}c(H,e_+^L+e_-^L)+\nu(e_{\pm}^L) &= c(H,e_+^L+e_-^L)+c(0,e_{\pm}^L)\\ &\geqslant c(H,e_{\pm}^L)\end{aligned}$$

for any Hamiltonian H, and thus

$$\zeta_{e_+^L+e_-^L}(H)\geqslant \zeta_{e_{\pm}^L}(H)$$

for any Hamiltonian H. Therefore,

$$\zeta_{e_+^L+e_-^L}(H) \geqslant \max\{\zeta_{e_+^L}(H),\ \zeta_{e_-^L}(H)\} \tag{3.1.11}$$

for any Hamiltonian H.

Next, we prove $\zeta_{e_+^L+e_-^L}(H) \leqslant \max\{\zeta_{e_+^L}(H),\ \zeta_{e_-^L}(H)\}$ for any H. The characteristic exponent property of spectral invariants (c.f. [EP03, Section 2.6.4]) implies

$$c(H, e_+^L + e_-^L) \leqslant \max\{c(H, e_+^L),\ c(H, e_-^L)\},$$

and thus by homogenizing, we obtain

$$\zeta_{e_+^L+e_-^L}(H) \leqslant \max\{\zeta_{e_+^L}(H),\ \zeta_{e_-^L}(H)\} \tag{3.1.12}$$

for any Hamiltonian H. From (3.1.11) and (3.1.12), we have

$$\zeta_{e_+^L+e_-^L}(H) = \max\{\zeta_{e_+^L}(H),\ \zeta_{e_-^L}(H)\} \tag{3.1.13}$$

for any Hamiltonian H. By an analogous argument, one can also prove

$$\zeta_{e_+^L-e_-^L}(H) = \max\{\zeta_{e_+^L}(H),\ \zeta_{e_-^L}(H)\} \tag{3.1.14}$$

for any Hamiltonian H. This completes the proof of Claim 3.1.5. □

Now, we consider the situation where there are several Lagrangian spheres, especially when they are (homologically) intersecting. Biran–Membrez (as well as Sanda) proved that if two Lagrangian spheres L and L' are (co)homologically intersecting, i.e.

$$[L] \cdot [L'] \neq 0,$$

then we have

$$\beta_L = \beta_{L'}. \tag{3.1.15}$$

This implies that if there is an ADE configuration

$$\mathscr{S} := \{S_1, \cdots, S_m\},$$

then all the β_{S_j} coincide. In such a case, we simply denote

$$\beta = \beta_{\mathscr{S}} := \beta_{S_j}. \tag{3.1.16}$$

3.2 Proof of Theorem B

In this section, we prove Theorem B. We separate the proof depending on the parity of n.

Proof of Theorem B: case of even n. We start with an important lemma, which will also be used in the proof of Theorem C.

Lemma 3.2.1. *Assume $\beta_L, \beta_{L'} \neq 0$. If $[L] \cdot [L'] \neq 0$, then*

$$\{e_+^L,\ e_-^L\} \cap \{e_+^{L'},\ e_-^{L'}\} \neq \emptyset.$$

Remark 3.2.2. In Lemma 3.2.1, the Lagrangian spheres L, L' do not necessarily have to form an A_2 configuration. However, if they do form an A_2 configuration, then we can say furthermore that only one of the two idempotents is shared by L and L', i.e.

$$\{e_+^L,\ e_-^L\} \neq \{e_+^{L'},\ e_-^{L'}\}.$$

For this, see the proof of Proposition 3.5.1. Also note that if $[L] \cdot [L'] = 0$, then

$$\{e_+^L,\ e_-^L\} \cap \{e_+^{L'},\ e_-^{L'}\} = \emptyset,$$

as they are all orthogonal.

We postpone the proof of Lemma 3.2.1 until the end of the proof of Theorem B. We argue by contradiction; assume that (X, ω) contains a D_4 configuration of Lagrangian spheres. As in Figure 2, there is a Lagrangian sphere S that intersects three other Lagrangian spheres S_1, S_2, S_3,

$$|S \cap S_j| = 1,\ 1 \leqslant j \leqslant 3.$$

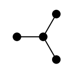
D_4 

Figure 2: D_4-configuration: The sphere S corresponds to the sphere in the middle.

Since n is even and L is a Lagrangian sphere, we have $[L] \cdot [L] = -2$, and thus $[L] \neq 0$. Therefore, by the semi-simplicity of $QH(X, \omega)$, $[L]$ is not a nilpotent, and since $[L]^3 = 4\beta_L[L]$ by the cubic equation (3.1.1), we deduce that $\beta_L \neq 0$. By the intersection property (3.1.15), all the Lagrangian spheres involved in the D_m, E_m configuration have the same β_L, which is non-zero, i.e.

$$\beta := \beta_S = \beta_{S_j} \neq 0,\ \forall j.$$

Thus, by Claim 3.1.1, each Lagrangian sphere produces two idempotents that are units of field factors of $QH(X, \omega)$ as (3.1.2):

$$\begin{aligned} e_\pm^S &= \pm \frac{1}{4\sqrt{\beta}}[S] + \frac{1}{8\beta}[S]^2, \\ e_\pm^{S_j} &= \pm \frac{1}{4\sqrt{\beta}}[S_j] + \frac{1}{8\beta}[S_j]^2. \end{aligned} \tag{3.2.1}$$

From Lemma 3.2.1, S and S_1 share at least one idempotent, i.e.

$$\{e_+^S,\ e_-^S\} \cap \{e_+^{S_1},\ e_-^{S_1}\} \neq \emptyset. \tag{3.2.2}$$

We assume

$$e_-^S = e_-^{S_1} \tag{3.2.3}$$

without loss of generality. Applying Lemma 3.2.1 to S and S_2, we get

$$\begin{aligned} e_-^S &\notin \{e_+^{S_2},\ e_-^{S_2}\}, \\ e_+^S &\in \{e_+^{S_2},\ e_-^{S_2}\}, \end{aligned} \tag{3.2.4}$$

as

- S and S_2 share at least one idempotent,
- S_1 and S_2 are disjoint, so they cannot share any idempotent (if they did, then this will contradict the superheaviness property Proposition 2.2.6),
- and we have assumed (3.2.3).

Once again, without loss of generality we can assume

$$e_+^S = e_+^{S_2}. \tag{3.2.5}$$

Finally, we apply Lemma 3.2.1 to S and S_3:

$$\{e_+^S,\ e_-^S\} \cap \{e_+^{S_3},\ e_-^{S_3}\} \neq \emptyset. \tag{3.2.6}$$

Property (3.2.6), combined with (3.2.3) and (3.2.5), implies

$$\{e_-^{S_1},\ e_+^{S_2}\} \cap \{e_+^{S_3},\ e_-^{S_3}\} \neq \emptyset. \tag{3.2.7}$$

In view of Proposition 2.2.6, property (3.2.7) contradicts the superheaviness properties (by Lemma 3.1.4, S_1 is $e_-^{S_1}$-superheavy, S_2 is $e_+^{S_2}$-superheavy, and S_3 is $e_\pm^{S_3}$-superheavy), as S_1 and S_2 are both disjoint from S_3, i.e.

$$(S_1 \cup S_2) \cap S_3 = \emptyset.$$

This completes the proof of Theorem B for even n. □

We prove Lemma 3.2.1.

Proof of Lemma 3.2.1. First of all, the assumption $[L] \cdot [L'] \neq 0$ implies $\beta_L = \beta_{L'} \neq 0$. We call this β, i.e.

$$\beta := \beta_L = \beta_{L'} \neq 0.$$

We have

$$\begin{aligned} \frac{1}{2\beta^{1/2}}[L] &= e_+^L - e_-^L, \\ \frac{1}{2\beta^{1/2}}[L'] &= e_+^{L'} - e_-^{L'}, \end{aligned} \tag{3.2.8}$$

and $[L] * [L'] \neq 0$ implies

$$(e_+^L - e_-^L) * (e_+^{L'} - e_-^{L'}) \neq 0. \tag{3.2.9}$$

By developing the left hand side, we get

$$e_+^L * e_+^{L'} - e_-^L * e_+^{L'} - e_-^L * e_-^{L'} + e_-^L * e_-^{L'} \neq 0. \tag{3.2.10}$$

This implies that at least one of the four terms is non-zero. We can assume $e_+^L * e_+^{L'} \neq 0$ without loss of generality. We prove $e_+^L = e_+^{L'} = e_+^L * e_+^{L'}$. As e_+^L is a unit of a field factor, i.e. $e_+^L \cdot QH(X,\omega) = \Lambda \cdot e_+^L$ (Claim 3.1.1), we have

$$e_+^L * e_+^{L'} = \alpha \cdot e_+^L$$

for some $\alpha \in \Lambda\backslash\{0\}$. The left hand side $e_+^L * e_+^{L'}$ is an idempotent, so we have

$$(\alpha \cdot e_+^L)^2 = (e_+^L * e_+^{L'})^2 = e_+^L * e_+^{L'} = \alpha \cdot e_+^L,$$

which is

$$\alpha^2 = \alpha,$$

and thus, $\alpha = 1$. Therefore,

$$e_+^L * e_+^{L'} = e_+^L.$$

By applying the same argument to $e_+^{L'}$, we get

$$e_+^L * e_+^{L'} = e_+^{L'},$$

which is

$$e_+^L = e_+^{L'} (= e_+^L * e_+^{L'}).$$

Thus,

$$\{e_+^L,\ e_-^L\} \cap \{e_+^{L'},\ e_-^{L'}\} \neq \emptyset.$$

This finishes the proof of the lemma. $\square$

Proof of Theorem B: case of odd n. Assume n is odd and (X, ω) satisfies $\frac{n+1}{2N_X} \notin \mathbb{Z}$. Then, any Lagrangian sphere L in (X, ω) satisfies a $\dim_\Lambda HF(L) = 2$ as a vector space ([BM16]) and $pt_L^2 = 0$, both for degree reasons. Thus, 1_L is the only idempotent of $HF(L)$. By using the semi-simplicity, we decompose the unit 1_X into a sum of units of field factors:

$$1_X = \sum_{1 \leqslant j \leqslant l} e_j. \tag{3.2.11}$$

As $\mathcal{CO}^0(1_X) = 1_L$, $\mathcal{CO}^0 \colon QH(X, \omega) \to HF(L)$ is a ring homomorphism, and 1_L is the only idempotent of $HF(L)$, there exists a unique unit e_{j_0} of a field factor such that

$$\mathcal{CO}^0(e_j) = \delta_{j,j_0} \cdot 1_L,$$

where $\delta_{j,j_0} = 1$ if $j = j_0$ and $\delta_{j,j_0} = 0$ otherwise. We denote this distinguished idempotent corresponding to L by $e^L := e_{j_0}$.

Now, we consider an A_2-configuration L, L'. As they intersect at one point, we have $HF(L, L') \neq 0 (= \Lambda \cdot \langle L \cap L' \rangle)$. Together with $\mathcal{CO}^0(e^L) = 1_L$ and $\mathcal{CO}^0(e^{L'}) = 1_{L'}$, we obtain $e^L * e^{L'} \neq 0$ (see, for example, [San21, Lemma 4.7]). As $e^L, e^{L'}$ are units of fields factors of $QH(X, \omega)$, we conclude that $e^L = e^{L'}$ (by the same argument that starts right after the equation (3.2.10)). This implies that L and L' are both $e^L = e^{L'}$-superheavy.

Now, we suppose there is an A_3-configuration L_1, L_2, L_3 and deduce a contradiction. The pairs L_1, L_2 and L_2, L_3 both form A_2-configurations. Thus, from the previous argument, we have

$$e^{L_1} = e^{L_2} = e^{L_3}.$$

Thus, the three Lagrangian spheres L_1, L_2, L_3 are all superheavy with respect to the same idempotent $e^{L_1} = e^{L_2} = e^{L_3}$. This contradicts that $L_1 \cap L_3 = \emptyset$ (Proposition 2.2.6). Thus, there cannot be any A_3-configuration when n is odd and (X, ω) satisfies $\frac{n+1}{2N_X} \notin \mathbb{Z}$. $\square$

Remark 3.2.3. We can also prove the even n case of Theorem B by an argument closer to the above argument for the odd n case.

3.3 Proof of Theorem A: From SG to AG

In this section, we prove Theorem A by reducing it to its symplectic counterpart, namely Theorem B. Before we start the proof, we mention the following expected statement that extends Theorem A, which will hold as soon as [AFOOO] is established.

Conjecture 3.3.1 (Algebraic geometry version: expected)**.** *Let X be a complex n dimensional smooth Fano variety. Assume either one of the following two:*

- *$QH(X, \omega)$ is semi-simple, where ω is the anti-canonical form.*
- *$n > 2$ and $QH(X, \omega)$ is semi-simple for a generic choice of a symplectic form ω.*

If X degenerates to a Fano variety with an isolated hypersurface singularity, then the singularity has to be

- *an A_m-singularity with $m \geqslant 1$, if n is even.*
- *an A_m-singularity with $m = 1, 2$, if n is odd and $\frac{\dim_{\mathbb{C}} X+1}{2r_X} \notin \mathbb{Z}$ where r_X is the Fano index.*

Remark 3.3.2.

1. As we have pointed out in Section 2.3, the two versions of the semi-simplicity that we pose in Conjecture 3.3.1, namely the monotone one and the generic one, will imply the semi-simplicity that is commonly used in the community of algebraic geometry.
2. Conjecture 3.3.1 does not hold with the the generic semi-simplicity (i.e. the second) assumption when $n = 2$. In fact, the del Pezzo surface $\mathbb{D}_5 = \mathbb{C}P^2 \# 5 \cdot (\overline{\mathbb{C}P^2})$ is known to have semi-simple quantum cohomology ring with respect to a generic symplectic form, but it can degenerate to a singular del Pezzo surface with a D_5 singularity.

Proof of Theorem A. As we have pointed out in Section 2.4, in order to reduce Theorem A to Theorem B, we need to prove that there is a π-relative ample line bundle that gives us a monotone symplectic form on the general fibers (in the neighborhood of the origin).

Suppose X degenerates to a Fano variety X_0 with hypersurface singularities, that is, we have a degeneration $\pi : \mathcal{X} \to \mathbb{C}$ of X whose central fiber X_0 is Fano (in the sense of Remark 2.4.2) and has hypersurface singularities. We claim the following in this situation.

Claim 3.3.3. *The variety $\mathcal{X}/\mathbb{C}(= \mathcal{X})$ is Gorenstein. Thus, the dualizing sheaf $\Omega_{\mathcal{X}/\mathbb{C}} = \Omega_{\mathcal{X}}$ is a line bundle over $\mathcal{X}$ (Remark 2.4.2). There exists a Zariski-open neighborhood $U \subset \mathbb{C}$ of the origin such that the restriction of $\Omega_{\mathcal{X}}$ to $\mathcal{X}|_U = \pi^{-1}(U)$ is π-relative ample line bundle.*

Remark 3.3.4.

1. In this case, $\mathbb{C}$ satisfies $K_{\mathbb{C}} = 0$, so we have

$$\Omega_{\mathcal{X}/\mathbb{C}} = \Omega_{\mathcal{X}}.$$

2. A Zariski-open subset of $\mathbb{C}$ is the complement of finitely many points.

Before we prove Claim 3.3.3, we continue with the proof of Theorem A. Take a Zariski-open subset U of $\mathbb{C}$ as in Claim 3.3.3. Then, we have a π-relative ample line bundle $\Omega_{\mathcal{X}}|_{\mathcal{X}|_U}$ on $\mathcal{X}|_U$. As explained in Section 2.4, this defines a projective embedding

$$f_t : X_t \hookrightarrow \mathbb{C}P^N$$

for every $t \in U\backslash\{0\}$ which gives us a symplectic form $\omega_t := (f_t)^*\omega_{\mathrm{FS}}$, where ω_{FS} is the Fubini–Study form on $\mathbb{C}P^N$. Now, the symplectic form ω_t is a monotone form on $X_t,\ t \neq 0$, as $\Omega_{\mathcal{X}}|_{\mathcal{X}|_U} \to \mathcal{X}|_U$ is π-relatively ample, and $\Omega_{\mathcal{X}}|_{X_t} = \Omega_{X_t} = K_{X_t}$, where K_{X_t} is the canonical line bundle (see also (2.4.2)). Now, from the degeneration, we obtain a configuration of Lagrangian spheres in (X_t, ω_t) as the vanishing cycles of the hypersurface singularities. We deal with simple singularities and isolated hypersurface singularities of positive modality separately.

Case of simple singularities, i.e. modality zero. First, assume n is even. We argue by contradiction; assume that X degenerates to a Fano variety X_0 with a D or E singularity. Then, as we have discussed above, (X, ω) contains a D or E configuration of Lagrangian spheres. In either case, as in Figure 3, there is a Lagrangian sphere S that intersects three other Lagrangian spheres S_1, S_2, S_3,

$$|S \cap S_j| = 1,\ 1 \leqslant j \leqslant 3.$$

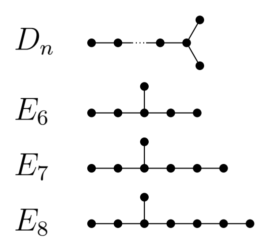


Figure 3: The sphere S corresponds to, the sphere at the end of the straight line in the D_n diagram, and the third sphere in the E_6, E_7, E_8 diagrams, respectively.

By Theorem B, these configurations do not occur. Thus, DE singularities cannot occur on X. Next, assume n is even and $\frac{\dim_{\mathbb{C}} X+1}{2r_X} \notin \mathbb{Z}$ where r_X is the Fano index. Similarly to the even n case, by Theorem B, an A_3-configuration is prohibited, so A_1 and A_2 singularities are the only two simple singularities that can occur on X.

Case of positive modality. The argument goes similarly to the case of modality zero, i.e. simple singularities. We argue by contradiction; assume that X degenerates to a Fano variety X_0 with a positive modality singularity. Then (X, ω) contains a configuration of Lagrangian spheres coming from a positive modality singularity. From Proposition 2.5.4, we know that the vanishing cycle of an isolated hypersurface singularity with positive modality includes the vanishing cycle of one of the three parabolic singularities $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$. Keating studied the geometry of the vanishing cycles of the three parabolic singularities $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ and proved that the Lagrangian spheres that appear in the vanishing cycles of the three parabolic singularities intersect as in the Dynkin diagram of Gabrielov [Kea15, Figure 4]. In the Dynkin diagram of Gabrielov [Kea15, Figure 4], one can find a collection of four Lagrangian spheres $\{S, S_1, S_2, S_3\}$ just as in the case of simple singularities: there is a Lagrangian sphere S that intersects three other Lagrangian spheres S_1, S_2, S_3,

$$|S \cap S_j| = 1,\ 1 \leqslant j \leqslant 3.$$

Remark 3.3.5. The Lagrangian spheres S, S_1, S_2, S_3 correspond to the Lagrangian spheres 4,1,2,3, respectively, in Keating's numbering in [Kea15, Figure 4].

The rest of the proof is exactly the same as the modality zero case. We complete the proof of Theorem A by proving Claim 3.3.3. As we assume that X_0 is Fano, which is equivalent to the ampleness of the line bundle $\Omega_{X_0} = \Omega_{\mathcal{X}/\mathbb{C}}|_{X_0}$, Claim 3.3.3 is a direct consequence of [Gro61, Théorème 4.7.1] (see also [Laz04, Theorem 1.2.17]). Note that [Gro61, Théorème 4.7.1] requires the morphism $\pi : \mathcal{X} \to \mathbb{C}$ to be proper and $\Omega_{\mathcal{X}}$ to be a line bundle. The properness of π is satisfied, as all the fibers of $\pi : \mathcal{X} \to \mathbb{C}$ are compact. As for verifying that the dualizing sheaf $\Omega_{\mathcal{X}}$ is indeed a line bundle, it suffices to show that $\mathcal{X}$ is Gorenstein (see Remark 2.4.2), and this follows from the following proposition.

Proposition 3.3.6. *Let X, Y be varieties and $\pi : X \to Y$ a flat morphism. If Y is Gorenstein and all the fibers $\pi^{-1}(y),\ y \in Y$ are Gorenstein, then X is also Gorenstein.*

In our case

- $X_t,\ t \neq 0$ is a smooth variety, and smooth varieties are Gorenstein,
- X_0 has at most isolated hypersurface singularities, so it is Gorenstein,

and thus all the fibers of the flat morphism of the degeneration $\pi : \mathcal{X} \to \mathbb{C}$ are Gorenstein. Thus, $\Omega_{\mathcal{X}}$ is a line bundle from Proposition 3.3.6. We have completed the proof of Theorem A.

□

Remark 3.3.7. It would be very interesting to study the spectral rigidity of singularities that are not isolated hypersurface singularities, for example cyclic quotient singularities. They are 'similar' to the A_m-singularities in the sense that their vanishing cycles are A_m-configurations attached to a certain singular Lagrangian called the Lagrangian pinwheel, c.f. [Eva, Section 1.2.3]. In this case, it is interesting to see if a property similar to Lemma 3.2.1 would hold between a Lagrangian pinwheel and a Lagrangian sphere. Studying the spectral rigidity of Lagrangian pinwheels is also an interesting topic. If one could prove results similar to Lemma 3.1.4 for Lagrangian pinwheels, it will bring new applications to Hofer geometry.

3.4 Proof of Theorem C

In this section, we prove Theorem C.

Proof of Theorem C. Denote the Lagrangian spheres forming the A_m-configuration by

$$\mathscr{S}_{A_m} := \{S_1, \cdots, S_m\}.$$

As $QH(X, \omega)$ is semi-simple, there is no nilpotent, thus any Lagrangian sphere L with $[L] \neq 0 \in QH(X)$ is not a nilpotent. This implies $\beta_L \neq 0$, where β_L is the scalar in the cubic equation. Thus, for the A_m-configuration $\mathscr{S}_{A_m}$ we have

$$\beta = \beta_{\mathscr{S}_{A_m}} = \beta_{S_j} \neq 0$$

by (3.1.16). Thus, by (3.1.2), each Lagrangian sphere S_j gives rise to two idempotents $e_{\pm}^{S_j}$, where

$$e_{\pm}^{S_j} = \pm \frac{1}{4\sqrt{\beta}}[S_j] + \frac{1}{8\beta}[S_j]^2. \tag{3.4.1}$$

From Lemma 3.1.4, we have for every j,

$$\overline{\ell}_{S_j}(H) = \max\{\zeta_{e_+^{S_j}}(H),\ \zeta_{e_-^{S_j}}(H)\} \tag{3.4.2}$$

for any Hamiltonian H, and S_j is $e_{\pm}^{S_j}$-superheavy for any j. By Lemma 3.2.1, the two idempotents corresponding to the Lagrangian spheres that are next to each other in the A_m-configuration, say S_j and S_{j+1}, share one of the two; without loss of generality we can assume

$$e_+^{S_j} = e_-^{S_{j+1}}. \tag{3.4.3}$$

To prove Theorem C, it is enough to prove the following.

Claim 3.4.1. *The Entov–Polterovich quasimorphisms*

$$\{\zeta_{e_+^{S_j}}\}_{1 \leqslant j \leqslant m-1}$$

are pairwise distinct.

We prove this claim. We need to prove that for any i, j such that $1 \leqslant i < j \leqslant m-1$, we have $\zeta_{e_+^{S_i}} \neq \zeta_{e_+^{S_j}}$. For such i, j, we have that

- S_i is $e_+^{S_i}$-superheavy,
- S_{j+1} is $e_+^{S_j}$-superheavy, as $e_+^{S_j} = e_-^{S_{j+1}}$.

As $j - i \geqslant 1$, we have $S_i \cap S_{j+1} = \emptyset$. Thus,

$$\zeta_{e_+^{S_i}} \neq \zeta_{e_+^{S_j}}.$$

This proves Claim 3.4.1 and thus completes the proof of Theorem C.

□

We now prove Theorem 1.3.2. In [Kaw, Theorem C(2)], the author proved the existence of three Entov–Polterovich quasimorphisms on $\widetilde{\mathrm{Ham}}(\mathbb{D}_4)$. The proof uses a set of Lagrangian configurations arising from a toric degeneration studied by Y. Sun in [Sun20] that consists of a Lagrangian torus, an A_1-configuration, and an A_2-configuration. Here, we manage to improve [Kaw, Theorem C(2)] by completing the union of the A_1 configuration and the A_2 configuration to an A_4 configuration by finding an additional Lagrangian sphere using some four (real) dimensional technique. This allows us to apply Theorem C and complete the proof.

Proof of Theorem 1.3.2. First, we find four linearly independent Entov–Polterovich quasimorphisms on $\widetilde{\mathrm{Ham}}(\mathbb{D}_4)$, the universal cover of $\mathrm{Ham}(\mathbb{D}_4)$. By considering the degeneration of the del Pezzo surface $\mathbb{D}_4$ studied in [Sun20, Appendix B, case of X_6], one obtains a monotone Lagrangian torus L, an A_2 configuration $\mathscr{S}_{A_2} := \{S_1, S_2\}$, an A_1 configuration $\mathscr{S}_{A_1} := \{S_3\}$, that are mutually disjoint (i.e. $L \cap S_1 = \emptyset$, $L \cap (S_2 \cup S_3) = \emptyset$, $S_1 \cap (S_2 \cup S_3) = \emptyset$). By looking at the moment polytope of the degeneration, one can see that the homology classes represented by the A_2 configuration $\mathscr{S}_{A_2}$ and the A_1 configuration $\mathscr{S}_{A_1}$ are as follows [10] :

$$\begin{aligned} [S_1] &= E_2 - E_3, \\ [S_2] &= E_3 - E_4, \\ [S_3] &= H - E_2 - E_3 - E_4. \end{aligned} \tag{3.4.4}$$

Now, consider the class $E_4 - E_1$. By [BLW14, Lemma 5.3], one can take a Lagrangian sphere S that represents the class $E_4 - E_1$, i.e.

$$[S] = E_4 - E_1,$$

that is disjoint from the Lagrangian sphere S_1, i.e. $S_1 \cap S = \emptyset$. By computing the intersection number, one can see that the Lagrangian spheres S_1, S_2, S, S_3 form a *partial* A_4 *configuration* in the sense that they satisfy the intersection property (2.5.2) expect for S, which instead satisfies the following weaker, homological version of the intersection property

$$[S] \cdot [S_j] = \begin{cases} 1 \text{ if } j = 2, 3 \\ 0 \text{ if } j = 1. \end{cases} \tag{3.4.5}$$

However, this is enough to obtain three Entov–Polterovich quasimorphisms

$$\zeta_{e_+^{S_1}}, \zeta_{e_+^{S_2}}, \zeta_{e_+^{S}} = \zeta_{e_-^{S_3}}$$

which come from the Lagrangian spheres involved in the partial A_4 configuration, and they are linearly independent, as the Lagrangian spheres S_1, S_2, S, S_3 above satisfy $S_1 \cap S = \emptyset$, $S_2 \cap S_3 = \emptyset$. This is enough to conclude $\zeta_{e_+^{S_1}} \neq \zeta_{e_+^{S_2}}$, $\zeta_{e_+^{S_2}} \neq \zeta_{e_+^{S}}$. As it was shown in [Kaw, Proof of Theorem C], there is an Entov–Polterovich quasimorphism ζ_{e^L} for which the monotone Lagrangian torus L is e^L-superheavy. As the Lagrangian torus L, the A_2 configuration $\mathscr{S}_{A_2}$, and the A_1 configuration $\mathscr{S}_{A_1}$ are all mutually disjoint, we conclude that

$$\{\zeta_j\}_{1 \leqslant j \leqslant 4} := \{\zeta_{e^L}, \zeta_{e_+^{S_1}}, \zeta_{e_+^{S_2}}, \zeta_{e_+^{S}} = \zeta_{e_-^{S_3}}\} \tag{3.4.6}$$

[10] This was communicated to the author by Yuhan Sun.

are four linearly independent Entov–Polterovich quasimorphisms on $\widetilde{\mathrm{Ham}}(\mathbb{D}_4)$. Evans proved that $\mathrm{Symp}_0(\mathbb{D}_4) = \mathrm{Ham}(\mathbb{D}_4)$ is weakly contractible, i.e. $\pi_k(\mathrm{Ham}(\mathbb{D}_4)) = \{\mathrm{id}\}$ for all $k \in \mathbb{N}$ [Eva11, Theorem 1.3]. Thus, any quasimorphism on $\widetilde{\mathrm{Ham}}(\mathbb{D}_4)$ descends to a quasimorphism on $\mathrm{Ham}(\mathbb{D}_4)$ and thus, $\{\zeta_j\}_{1\leqslant j\leqslant 4}$ defines four pairwise distinct Entov–Polterovich quasimorphisms on $\mathrm{Ham}(\mathbb{D}_4)$. As each of the four Entov–Polterovich quasimorphisms is Hofer Lipschitz continuous, we conclude that $\mathbb{R}^4$ embeds quasi-isometrically to $\mathrm{Ham}(\mathbb{D}_4)$, which in particular answers the Kapovich–Polterovich question for $X = \mathbb{D}_4$ in the negative. The three quasimorphisms that are C^0 and Hofer-Lipschitz continuous can be obtained as in [Kaw22, Theorem 22]. This completes the proof of Theorem 1.3.2. $\square$

Remark 3.4.2. It seems very likely that one can furthermore prove that quasimorphisms $\zeta_{e_-^{S_1}}$, $\zeta_{e_+^{S_3}}$ are pairwise distinct to the quasimorphisms in (3.4.6) and obtain six pairwise distinct quasimorphisms on $\mathrm{Ham}(\mathbb{D}_4)$. We will just briefly outline the argument: given an A_m-configuration $\{L_j\}_{1\leqslant j\leqslant m}$ in a del Pezzo surface, one can construct a new Lagrangian sphere L_{m+1} that completes the A_m configuration into a $m+1$-gon by taking a Lagrangian sphere representing the class $[L_{m+1}] = -(\sum_{j=1}^{m}[L_j])$ (Seidel considers this configuration of Lagrangian spheres in [Sei08, Example 1.10]). It should be able to show that L_{m+1} can be taken so that it is disjoint to $\{L_j\}_{2\leqslant j\leqslant m-1}$ by a similar argument to above, which shows that $\zeta_{e_-^{S_1}}$, $\zeta_{e_+^{S_3}}$ are pairwise distinct to the quasimorphisms in (3.4.6) by superheaviness.

3.5 Proof of Theorem D

In this section, we prove Theorem D. For the definition and the basic properties of the Dehn twist, we refer to [Sei97].

Proof of Theorem D. Case of even n: First, notice that as β is a symplectic invariant and $\tau_L \in \mathrm{Symp}(X)$, we have

$$\beta_{\tau_L(L')} = \beta_L.$$

Note that we have $\beta_L \neq 0$, as $QH(X,\omega)$ is semi-simple. We denote them all by β, i.e.

$$\beta := \beta_{\tau_L(L')} = \beta_L \neq 0.$$

From (3.1.2), the idempotents induced by $\tau_L(L')$ are as follows.

$$\begin{aligned} e_{\pm}^{\tau_L(L')} &= \pm\frac{1}{4\sqrt{\beta}}[\tau_L(L')] + \frac{1}{8\beta}[\tau_L(L')]^2 \\ &= \pm\frac{1}{4\sqrt{\beta}}(\tau_L)_*[L'] + \frac{1}{8\beta}\left((\tau_L)_*[L']\right)^2. \end{aligned} \tag{3.5.1}$$

The Picard–Lefschetz formula implies

$$(\tau_L)_*[L'] = [L'] - (-1)^{n(n-1)/2}([L]\cdot[L'])[L], \tag{3.5.2}$$

and we also have

$$\begin{aligned} [L] &= 2\sqrt{\beta}(e_+^L - e_-^L), \\ [L'] &= 2\sqrt{\beta}(e_+^{L'} - e_-^{L'}). \end{aligned} \tag{3.5.3}$$

As Lagrangian spheres $\{L, L'\}$ form an A_2 configuration, i.e. $|L \cap L'| = 1$, there are two possibilities for the intersection number, i.e. $[L] \cdot [L'] = \pm 1$. We have the following relation between the homological intersection number $[L] \cdot [L']$ and the idempotents corresponding to L and L'.

Proposition 3.5.1. *Let (X, ω) is a $2n$-dimensional monotone symplectic manifold with even n. Suppose there is an A_2 configuration $\{L, L'\}$. The idempotents corresponding to L and L', namely $\{e_+^L,\ e_-^L\}$ and $\{e_+^{L'},\ e_-^{L'}\}$, and the intersection number $[L] \cdot [L']$ have the following relationship:*

1. $(-1)^{n(n-1)/2}[L] \cdot [L'] = -1$ *if and only if* $e_+^L = e_-^{L'}$ *or* $e_-^L = e_+^{L'}$.
2. $(-1)^{n(n-1)/2}[L] \cdot [L'] = 1$ *if and only if* $e_+^L = e_+^{L'}$ *or* $e_-^L = e_-^{L'}$.

Remark 3.5.2. As pointed out in Remark 3.2.2, only one of $\{e_+^L,\ e_-^L\}$ and $\{e_+^{L'},\ e_-^{L'}\}$ is shared. See the proof of Proposition 3.5.1 for this.

We prove Proposition 3.5.1 at the end of this section, after the proof of Theorem D. First, we assume $(-1)^{n(n-1)/2}[L] \cdot [L'] = -1$. From Proposition 3.5.1, this implies that we have either $e_+^L = e_-^{L'}$ or $e_-^L = e_+^{L'}$. We assume the former, as the proof for the latter goes identically.

Putting equations (3.5.2) and (3.5.3) together, we get

$$\begin{aligned} (\tau_L)_*[L'] &= [L'] + [L] \\ &= 2\sqrt{\beta}\left((e_+^{L'} - e_-^{L'}) + (e_+^L - e_-^L)\right) \\ &= 2\sqrt{\beta}(e_+^{L'} - e_-^L). \end{aligned} \tag{3.5.4}$$

From (3.5.4), we get

$$\begin{aligned} e_\pm^{\tau_L(L')} &= \pm\frac{1}{4\sqrt{\beta}}[\tau_L(L')] + \frac{1}{8\beta}[\tau_L(L')]^2 \\ &= \pm\frac{1}{4\sqrt{\beta}}2\sqrt{\beta}(e_+^{L'} - e_-^L) + \frac{1}{8\beta}\left(2\sqrt{\beta}(e_+^{L'} - e_-^L)\right)^2 \\ &= \pm\frac{1}{2}(e_+^{L'} - e_-^L) + \frac{1}{2}(e_+^{L'} + e_-^L) \\ &= \begin{cases} e_+^{L'} & \text{if } \pm = +, \\ e_-^L & \text{if } \pm = -. \end{cases} \end{aligned} \tag{3.5.5}$$

The equation (3.5.5) and Lemma 3.1.4 imply

$$\overline{\ell}_{\tau_L(L')} = \max\{\zeta_{e_-^L}, \zeta_{e_+^{L'}}\}. \tag{3.5.6}$$

Comparing it with

$$\begin{aligned} \overline{\ell}_L &= \max\{\zeta_{e_+^L}, \zeta_{e_-^L}\}, \\ \overline{\ell}_{L'} &= \max\{\zeta_{e_+^{L'}}, \zeta_{e_-^{L'}}\}, \end{aligned} \tag{3.5.7}$$

we obtain

$$\overline{\ell}_{\tau_L(L')} \leqslant \max\{\overline{\ell}_L, \overline{\ell}_{L'}\}. \tag{3.5.8}$$

The other case where we have $(-1)^{n(n-1)/2}[L]\cdot[L'] = +1$ can be dealt exactly in the same way.

Case of odd n: As we have seen in the proof of Theorem B for the case of odd n, for any Lagrangian sphere L, we can find a distinguished unique unit e^L, which satisfies

$$\overline{\ell}_L = \zeta_{e^L}.$$

We have also seen that for any Lagrangian spheres satisfying $HF(L, L') \neq 0$, we have $e^L = e^{L'}$. Now, for an A_2-configuration L, L', we have $HF(L, L') \neq 0$ and $HF(L, \tau_L(L')) \neq 0$, so we have

$$e^L = e^{L'} = e^{\tau_L(L')}.$$

Thus,

$$\overline{\ell}_{\tau_L(L')} = \overline{\ell}_L = \overline{\ell}_{L'} (= \zeta_{e^L} = \zeta_{e^{L'}} = \zeta_{e^{\tau_L(L')}}),$$

so we have

$$\overline{\ell}_{\tau_L(L')} = \max\{\overline{\ell}_L, \overline{\ell}_{L'}\}. \tag{3.5.9}$$

We have completed the proof of Theorem D. □

We end this section with a proof of Proposition 3.5.1 which was used in the proof of Theorem D.

Proof of Proposition 3.5.1. First of all, the assumption $[L]\cdot[L'] \neq 0$ implies $\beta_L = \beta_{L'} \neq 0$. We define

$$\beta := \beta_L = \beta_{L'} \neq 0.$$

By using

$$\begin{aligned} [L] &= 2\beta^{1/2}(e_+^L - e_-^L), \\ [L'] &= 2\beta^{1/2}(e_+^{L'} - e_-^{L'}), \end{aligned} \tag{3.5.10}$$

we have

$$\begin{aligned} [L]\cdot[L'] &= \int_X 4\beta(e_+^L - e_-^L)\cdot(e_+^{L'} - e_-^{L'}) \\ &= 4\beta\int_X (e_+^L\cdot e_+^{L'} - e_-^L\cdot e_+^{L'} - e_+^L\cdot e_+^{L'} + e_-^L\cdot e_-^{L'}). \end{aligned} \tag{3.5.11}$$

We will show that only one of the four terms survive. This is because only one of the two idempotents between $\{e_+^L,\ e_-^L\}$ and $\{e_+^{L'},\ e_-^{L'}\}$ is shared, i.e. $\{e_+^L,\ e_-^L\}\cap\{e_+^{L'},\ e_-^{L'}\} \neq \emptyset$ and $\{e_+^L,\ e_-^L\} \neq \{e_+^{L'},\ e_-^{L'}\}$. To see this, first from Lemma 3.2.1, we know that $\{e_+^L,\ e_-^L\}\cap\{e_+^{L'},\ e_-^{L'}\} \neq \emptyset$. However, we also have $\{e_+^L,\ e_-^L\} \neq \{e_+^{L'},\ e_-^{L'}\}$, as if we had $\{e_+^L,\ e_-^L\} = \{e_+^{L'},\ e_-^{L'}\}$, then (3.5.10) implies

$$[L] = \pm[L'].$$

On one hand, L, L' forming an A_2 configuration implies

$$[L]\cdot[L'] = \pm 1,$$

but on the other hand, L being an even-dimensional Lagrangian sphere implies

$$[L]\cdot[L'] = [L]\cdot(\pm[L]) = \pm 2,$$

which is a contradiction. Thus, we have proven that only one of the two idempotents between $\{e_+^L,\ e_-^L\}$ and $\{e_+^{L'},\ e_-^{L'}\}$ is shared. From the proof of Lemma 3.2.1, when two idempotents from $\{e_+^L,\ e_-^L\}$ and $\{e_+^{L'},\ e_-^{L'}\}$ do not coincide, then they are orthogonal to each other, so only one of the four terms in (3.5.11) remains.

We have

$$\begin{aligned}\int_X e_\pm^L &= \int_X \left(\pm\frac{1}{4\sqrt{\beta}}[L] + \frac{1}{8\beta}[L]^2\right)\\ &= \int_X \frac{1}{8\beta}[L]^2\\ &= \frac{1}{8\beta}\int_X [L]\cup[L]\\ &= \frac{1}{8\beta}[L]\cdot[L]\\ &= \frac{1}{8\beta}\cdot(-1)^{n(n-1)/2}\chi(L)\\ &= (-1)^{n(n-1)/2}\cdot\frac{1}{4\beta}.\end{aligned} \tag{3.5.12}$$

Note that the second and the third equality follow from degree reasons, and the fifth equality uses the fact that L is a Lagrangian sphere, $\chi(L)$ is the Euler characteristic of L, i.e. two. The same property applies for L':

$$\int_X e_\pm^{L'} = (-1)^{n(n-1)/2}\cdot\frac{1}{4\beta}. \tag{3.5.13}$$

Now, it follows from (3.5.11), (3.5.12), and (3.5.13) that the intersection number $[L]\cdot[L']$ satisfies the following:

- $e_+^L = e_-^{L'}$ or $e_-^L = e_+^{L'}$ if and only if $[L]\cdot[L'] = -(-1)^{n(n-1)/2}$.
- $e_+^L = e_+^{L'}$ or $e_-^L = e_-^{L'}$ if and only if $[L]\cdot[L'] = (-1)^{n(n-1)/2}$.

This completes the proof of Proposition 3.5.1.

□

REMARK 3.5.3. A recent work of Biran–Cornea [BC21, Lemma 5.3.1] contains a result that is based on a spirit similar to Theorem D. Although they work in a different setting, namely with exact Lagrangians in a Liouville manifold, it would be interesting to study if one can obtain a similar inequality to Theorem D by their method.

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