# An extension of Arakelyan's Theorem

Spyros Pasias

Orcid:0000-0001-6611-0155

sp00@aubmed.ac.cy

American University of Beirut Mediterraneo

**Abstract**

Arakelyan's Theorem provides conditions on a relatively closed subset $F$ of a domain $G \subset \mathbb{C}$, such that any continuous function $f : F \to \mathbb{C}$ that is analytic in $F^{\circ}$, can be approximated by analytic functions defined on $G$. In this paper we will extend Arakelyan's theorem by adding the extra requirement that the analytic functions that approximate $f$ may also be chosen to be bounded on a relatively closed subset $C \subset G$. In [1] the same problem has been considered but for the specific case that $G = \mathbb{C}$. In this paper we will extend the result in [1] and show that is true for an arbitrary $G$, provided that $F$ and $C$ satisfy certain topological conditions in $G$. Additionally, we will show that the result holds always true when $G$ is simply connected.

Keywords: Arakelyan's Theorem, $G-$holes, $G-$hole independent, $G-$bounded, $G-$proper intersection

# Introduction

In this paper we will generalize an important theorem in the field of Complex approximation, namely we will generalize *Arakelyan's Theorem*. It is well known by the Weirstrass Approximation Theorem that in a real closed interval, any continuous function can be approximated uniformly by polynomials on an open set. In the complex case things are different. Since polynomials are analytic, by Weierstrass Theorem it follows that if $f$ is uniformly approximated by polynomials, then $f$ better be an analytic function. However, $f$ being analytic is not enough to guarantee uniform approximation by polynomials. This is indeed the case if we let $f(z) = 1/z$ defined on the annulus $F = \{z \in \mathbb{C} : \frac{1}{2} \leq z \leq 1\}$. Even though $f$ is analytic in $F$, in this case $f$ cannot be approximated on $F$ by polynomials. Of course this happens because $f$ cannot be analytically continued inside the unit circle, which occurs because $f$ has a pole in a bounded connected component(a hole) in the complement of $F$.

A special case of *Runge's Theorem* states that any function $f$ that is analytic on a domain $G$ which contains a compact set $F$, can be uniformly approximated by polynomials provided $F$ has a connected complement. Runge's Theorem [see [2], p.94] is a classical theorem and it essentially serves as the starting point of Complex Approximation. The theorem in all of its generality does not require $F$ to have a connected complement, and thus the approximation in general is not by polynomials but rather by rational functions. The theorem also asserts that the poles of those rational functions may be chosen freely to lie anywhere in the respective connected components on $G \setminus F$ they lie in. As a corollary when $G \setminus F$ has a single complementary component the pole may be chosen to be at "$\infty$", and thus $f$ may be approximated by polynomials. Another well known theorem is *Mergelyan's Theorem* [[2],p.97] which improves Runge's Theorem when $F$ has a connected complement. The theorem states that any function that is continuous on a compact set $F \subset \mathbb{C}$ and analytic on $F^o$ can be uniformly approximated by polynomials on $F$, provided our set $F$ has a connected complement.

Arakalyan's Theorem on the other hand deals with analytic functions defined in more general sets $F$ that need not be compact nor have connected complement. Consequently we don't expect the approximation to be by polynomials.

To this end, let $G \subset \mathbb{C}$ be an arbitrary domain and $F \subset G$ a relatively closed subset of $G$.

Arakelyan's Theorem provides conditions that determine whether a function $f \in A(F)$ can be approximated on $F$ by functions $g \in Hol(G)$.

$$Hol(G) = \{g : g \text{ analytic in } G\}$$
$$A(F) = \{f : f \text{ analytic on } F^o, \text{ continuous on F}\}$$

The main result in [1] generalizes Arakelyan's Theorem when $G = \mathbb{C}$. In [1] the authors consider two closed sets $F$ and $C$ and find necessary and sufficient conditions so that every function defined on $F \subset \mathbb{C}$ can be approximated by entire functions that are bounded on $C$. In this paper we will extend this result even further by proving the result holds for a general $G$. In this paper we would like to approximate functions in $A(F)$ by functions in $Hol(G)$ and require the approximating functions to be bounded on any chosen relatively closed subset $C$. This can be achieved provided $C$ and $F$ satisfy certain topological conditions in $G$, or if $G$ is a simply connected domain

The results in this paper fit in the category of joint approximation. We provide some relevant articles where similar problems have been considered. In particular, articles ([5]-[8]) are devoted to various problems regarding bounded approximation by polynomials. Whereas articles ([9]-[11]) are devoted to problems regarding the joint approximation by analytic functions in Banach Spaces.

# Description of Arakelyan sets

In the proof of Arakelyan's Theorem in ([2], p. 142-144) the definition used for $F$ to be an Arakelyan set is different than definition we will use in this paper but it turns out that both definitions are equivalent. The definition we will use in this paper can be thought of as a variation of proposition 2.1 found in ([3], p.263). The definition proven in ([3], p.263) was also used in [1] to prove the extension of Arakelyan's Theorem for the case $G = \mathbb{C}$.

**Definition 1.** Let $G \subset \mathbb{C}$ be an arbitrary domain and $F \subset G$ be a relatively closed subset of $G$. A non empty connected component $g$ of $G \setminus F$ is called a $G-$hole of $F$ if it can be enclosed in a compact subset $L \subset G$. A subset $S \subset G$ that can be contained in the interior of compact subset of $G$ is called $G-$bounded. Otherwise the set $S$ is called $G-$unbounded.

**Remark 1.** It is important to note that a $G-$ hole of $F$ is simply a bounded connected component of the complement of $F$ (i.e a hole of $F$) whose boundary does not intersect with the boundary of $G$.

Now that we defined what a hole is in the setting of Arakelyan's theorem on an arbitrary domain, we are ready to define Arakelyan sets.

**Definition 2.** Let $G$ be an arbitrary domain and let $F$ be a relatively closed subset, then we call $F$ an Arakelyan set if:

1. The set $F$ has no $G-$holes.

2. For any connected compact set $K \subset G$ such that $\partial K$ is the union of finitely many disjoint jordan curves[1], the set $H \equiv \{\bigcup\{h\} : h \text{ is a } G\text{-hole of } F \cup K\}$ is $G-$bounded.

**Remark 2.** Clearly since the boundary of $G-$holes does not intersect with the boundary of $G$, it follows that if $F \cup K$ has only a finite number of $G-$holes and $F$ has no $G-$holes, then $F$ is an Arakelyan set. Additionally, it follows that if $F$ has no $G-$holes and it fails to be an Arakelyan set, then either $H$ must be unbounded or $\partial H \cap \partial G \neq \emptyset$.

Now, in order to compare our definition with the more topologically flavored definition presented in ([2], p. 142), we will need to use some tools from topology. To this end let us recall the *Alexandroff compactification* of an arbitrary domain $G \subset \mathbb{C}$. The set $G^*$ is defined by introducing a point $\{*\}$ so that $G^* = G \cup \{*\}$. The topology of $G^*$ is defined to consist from the open sets of $G$ and in addition all the sets that are complements of compact subsets $K \subset G$. Under this topology one may check that $G^*$ is compact.
We are ready now to provide the definition of an Arakleyan set as presented in ([2], p. 142).

**Definition 3.** The conditions given on the set $F$ to be an Arakelyan set are:

1. $G^* \setminus F$ is connected.

2. $G^* \setminus F$ is locally connected at $\{*\}$.

[1]Note that we put the requirement on $\partial K$, so that locally, the neighborhood around any point on $\partial K$ looks like an open arc.

The following proposition introduced in ([2], p. 133) provides us with an intuitive approach for determining whether $G^* \setminus F$ is connected.

**Proposition 1.** *The space $G^* \setminus F$ is connected if and only if each component of the open set $G \setminus F$ has an accumulation point on $\partial G$ or is unbounded.*

*Proof.* The proof can be found in ([2], p. 133-134). $\square$

Even though the conditions of Definition 3 given above seem different from the ones in Definition 2, actually they are the same.

**Proposition 2.** *Definition 2 and Definition 3 describing Arakelyan sets are equivalent.*

*Proof.* Indeed, let $G$ be an arbitrary domain and $F$ a relatively closed subset of $G$. By Proposition 1, the set $G^* \setminus F$ is connected if and only if each component of the open set $G \setminus F$ has an accumulation point on $\partial G$ or is unbounded. Therefore, $G^* \setminus F$ is connected if and only if every hole (component of $G \setminus F$) is not a $G-$hole. Therefore, conditions 1 of both definitions are equivalent.

Now we will show that conditions 2 of both definitions are equivalent as well. Indeed, suppose condition 2 in definition 2 holds. To show that condition 2 in definition 3 holds, it is sufficient to show that, for every compact set $K \subset G$, there is a compact set $Q \subset G$, such that every $G-$hole of $F \cup K$ is contained in $Q$. Let $K$ be a compact set in $G$. There is a compact set $B \subset G$, containing $K$, whose boundary consists of finitely many disjoint analytic Jordan curves. By our assumption there is a compact set $Q \subset G$ containing $K$, such that every $G-$hole of $F \cup B$ is contained in $Q$. Let $k$ be a $G-$hole $F \cup K$. If $k \subset B$, then clearly $k \subset Q$. Suppose $k \not\subseteq Q$ and let $\tilde{k}$ be a component of $k \setminus B$. Then $\tilde{k} \subset G \setminus (F \cup B)$. Let $h$ be the component of $G \setminus (F \cup B)$ containing $\tilde{k}$. Then $h \subset k$, so $h$ can certainly be contained in a compact subset of $G$. That is $h$ is a $G-$hole of $F \setminus B$. Thus, $h \subset Q$ and consequently $\tilde{k} \subset Q$. We have shown that every component of $k \setminus B$ is contained in $Q$. Since $B \subset Q$, it follows that $k \subset Q$. Therefore, every $G-$hole of $F \cup K$ is contained in $Q$ as required.

The other direction is obvious since for any compact set $K \subset G$ that fails condition 2 of definition 2, the neighborhood $G^* \setminus (F \cup K) \subset G^* \setminus F$ is not locally connected at $\infty$. The proof is complete. $\square$

Now we are ready to formally state Arakelyan's Theorem.

**Theorem 1** (Arakelyan's Theorem)**.** *Let $F$ and $G$ be as above, then any $f \in A(F)$ can be uniformly approximated by functions $g \in Hol(G)$ if and only if $F$ is an Arakelyan set.*

*Proof.* Arakelyan's Theorem was first stated and proved in [4] by the Armenian Mathematician Arakelyan. An additional proof may be found in ([2], p. 142-144). □

# Main result

In [1] the specific case where $G = \mathbb{C}$ Arakelyan's Theorem has been extended to the following:

**Theorem 2** (Extension of Arakelyan's Theorem when $G = \mathbb{C}$)**.** *Let $F$ and $C$ be closed sets in the complex plane, with $C \neq \mathbb{C}$. In order that every function $f \in A(F)$ can be approximated uniformly on $F$ by entire functions, each of which is bounded on $C$, it is necessary and sufficient that:*

1. *$F$ be an Arakeljan set.*

2. *There exists an Arakeljan set $C_1$, $C_1 \neq \mathbb{C}$, so that $C \subset C_1$ and $F \cap C_1$ is a bounded set.*

Our goal now is to move past the restriction $G = \mathbb{C}$ and extend Arakelyan's Theorem for a general domain $G$.

The proof of the extended version of Arakelyan's Theorem for $G = \mathbb{C}$ stated above is based on the following two lemmas.

**Lemma 1.** *Let $f$ be an entire function, and let $C$ a closed subset in $\mathbb{C}$. In order that $f$ be bounded on $C$ it is necessary and sufficient that there exists a closed Arakelyan set $C_1 \subset \mathbb{C}$ so that $C \subset C_1$ and $f$ is bounded on $C_1$.*

**Lemma 2.** *The union of two disjoint Arakelyan sets $E$ and $F$ is again an Arakelyan set.*

Our goal now is to move past the restriction $G = \mathbb{C}$ and extend Arakelyan's Theorem for a general domain $G$. In order to prove the Extension of Arakelyan's Theorem we have to generalize Lemma 1 and Lemma 2 for a general domain $G$.

**Lemma 3.** *Let $G \subset \mathbb{C}$ be an arbitrary domain and let $f \in Hol(G)$. Suppose $C$ is a relatively closed subset of $G$. In order that $f$ be bounded on $C$ it is necessary and sufficient that there exists an Arakelyan set $C_1 \subset G$ so that $C \subset C_1$ and $f$ is bounded on $C_1$.*

*Proof.* The proof of sufficiency is obvious so we will only prove necessity. Therefore, let us assume that there exists an $M > 0$ such that $|f(z)| < M$ for all $z \in C$, our goal is to show that there is an Arakelyan set $C_1$ containing $C$ such that $f$ is bounded on $C_1$. Now since $G$ is open and $C$ is a relatively closed subset of $G$, by continuity of $f$ on $C$, it follows that for each $z \in C$ there exists an open disk $U_z(\delta_z) \subset G$ centered at $z$ of radius $\delta_z < 1$, so that $|f(t)| < M + 1$ for all $t \in U_z(\delta_z) \cap G$. Furthermore, the open cover $\{U_z(\delta_z) \cap G\}$ , $z \in C$ of the relatively closed set $C$ has a locally finite subcover which we denote by $\{U_{z_n}(\delta_{z_n})\}$, $n = 1, 2, 3, .....$ Hence we have

1. $C \subset \bigcup_{n=1}^{\infty} U_{z_n}(\delta_{z_n})$

2. For any compact subset $K$ of $G$ only a finite number of disks $\{U_{z_n}(\delta_{z_n})\}$ intersect $K$. Write $E = (\bigcup_{n=1}^{\infty} \overline{U_{z_n}(\delta_{z_n})}) \cap G$. By 2, $E$ is a relatively closed subset of $G$.

3. $|f(z)| \leq M + 1, \quad z \in E$.

Let $C_1$ be the intersection of $G$ with the union of $E$ and its $G-$holes (if any exist). $C_1$ is the union of a locally finite collection of closed sets intersected with the open set $G$, thus it is a relatively closed subset of $G$. Furthermore by construction $C_1$ does not have any $G-$holes. From (3), by the maximum principle, we have

$$|f(z)| \leq M + 1, \quad z \in C_1.$$

By (1), we also have $C \subset C_1$. Hence the lemma will be proved once we show $C_1$ is an Arakelyan set. Recall that to show that $C_1$ is Arakelyan set we must show that:

a) The set $C_1$ has no $G-$holes

b) For any connected compact set $K \subset G$ such that $\partial K$ is the union of finitely disjoint jordan curves the set $H \equiv \{h : h \text{ is a } G- \text{ hole of } C_1 \cup K\}$ can be contained in a compact set $L \subset G$.

Condition $(a)$ is satisfied by construction of the set $C_1$, thus by Remark 2 it suffices to prove that for any such connected compact set $K \subset G$ such that $\partial K$ is the union of disjoint jordan curves, $C_1 \cup K$ has only a finite number of $G-$holes.

To this end, let $K$ be such a compact subset of $G$. By (2) in the sequence $\{\overline{U_n}\}$, there exists only a finite number of disks $\overline{U_{n_1}}, \overline{U_{n_2}}, ....\overline{U_{n_k}}$, each of which intersects $K$. The set $(\partial K) \setminus \bigcup_{m=1}^{m=k} \overline{U_{n_m}}$ is the union of a finite number of disjoint open intervals $I_1, I_2, ..., I_p$ on $\partial K$. For any $G-$hole $h$ of $K \cup C_1$, there exists at least one interval $I_k$ $(1 \leq k \leq p)$ so that $I_k \subset \partial h$. (Otherwise $h$ would be a $G-$hole of $C_1$, which is impossible, since $C_1$ is without $G-$holes).

If $h_1, h_2$ are two distinct $G-$holes of $K \cup C_1$ and $I_{k_1} \subset \partial h_1$ $(1 \leq k_1 \leq p), I_{k_2} \subset \partial h_2$ $(1 \leq k_2 \leq p)$, then $k_1 \neq k_2$, since otherwise we would have $h_1 \cap h_2 \neq \emptyset$ which is impossible since $h_1$ and $h_2$ are distinct $G-$holes.

Consequently, the number of $G-$holes of $K \cup C_1$ cannot be more than the number of intervals $I_k$. Thus the number of $G-$holes of $C_1 \cup K$ is finite, hence by Remark 2 it follows that $C_1$ is an Arakelyan set and the proof is complete. $\square$

In order to state the corresponding analogue of Lemma 2 for the case $G = \mathbb{C}$, we first need to introduce the following definition.

**Definition 4**. Let $E$ and $F$ be relatively closed subsets of $G$. We say that $E$ and $F$ are $G-$hole independent if, the intersection of any $G-$unbounded component $U$ of $G \setminus E$, with any $G-$unbounded component $V$ of $G \setminus F$, is a $G-$bounded set $U \cap V$.

The definition above is an essential element of the extension of Arakelyan's Theorem as it turns out to be the necessary condition that our domains $F$ and $C$ must satisfy in $G$ so that the Extension of Arakelyan's Theorem holds.

**Lemma 4.** *Let $G$ be an arbitrary open domain in $\mathbb{C}$ and suppose $E$ and $F$ are two disjoint Arakelyan sets in $G$ that are $G-$hole independent. Then the set $E \cup F$ is an Arakelyan set in $G$.*

*Proof.*

$$\begin{aligned} G \setminus (E \cup F) &= (G \setminus E) \cap (G \setminus F) \\ &= \bigcup_{i=1}^{\infty} E_i \cap \bigcup_{j=1}^{\infty} F_j \\ &= \bigcup_{i=1}^{\infty} G_i. \end{aligned}$$

Where $E_i$ and $F_j$ are the components of $G \setminus E$ and $G \setminus F$ respectively and $G_i = \bigcup_{j=1}^{\infty}(E_i \cap F_j)$. Now, suppose for a contradiction that $E \cup F$ has a $G-$hole $h$. Since, the $G_i = \bigcup_{j=1}^{\infty}(E_i \cap F_j)$ are disjoint, it follows $h$ belongs to some component $C$ of $E_i \cap F_j$. Since the $E_i \cap F_j$ are disjoint it follows that $C$ is in fact a component of $G \setminus (E \cup F)$. Thus $h = C$. Now since $E$ and $F$ are Arakelyan sets it follows that none of them has $G-$holes, thus $E_i$ and $F_j$ are $G-$unbounded components of $G \setminus E$ and $G \setminus F$ respectively. Now, of course since $E$ and $F$ are $G-$hole independent it follows that $C$, and thus $h$ are $G-$unbounded. This contradicts that $h$ is a $G-$hole of $E \cup F$, hence $E \cup F$ doesn't have $G-$holes as required.

Therefore, it remains to verify the latter condition of definition 2 describing Arakelyan sets. To this end, assume for a contradiction that there exists a connected compact subset $K \subset G$ where $\partial K$ is the union of finitely many disjoint analytic jordan curves, and such that $H$, the union of all $G-$ holes of $E \cup F \cup K$ is either unbounded or $\partial H \cap \partial G \neq \emptyset$. Then by Remark 2 it follows that the set $E \cup F \cup K$ has infinitely many $G-$holes. Consequently, the set $H$ must have infinitely many components i.e $H = \bigcup_{i=1}^{\infty}\{h_i\}$ where each $h_i$ represents a $G-$hole of $E \cup F \cup K$. Now for fixed $i$, let $a_k^i$, $k = 1, 2, ...$ be all the $G-$holes of $\overline{h_i} \cup K$. The connected compact set $\overline{h_i} \cup \bigcup_k \overline{a_k^i}$ has no $G-$holes. Denote by $d_i$ the interior of this connected compact set which contains $h_i$. Now $\partial d_i$ consists of an open arc on $\partial K$[2] and a connected compact set which we denote by $K_i$. That is, $K_i = \overline{(\partial d_i) \setminus \partial K}$. Clearly, $K_i \subset E \cup F$. Hence $K_i$ lies completely either on $E$ or in $F$. Let $i_n$, $n = 1, 2, ...$, be all the natural numbers for which $K_{i_n} \subset E$ and let $i_l$, $l = 1, 2, ...$, be the remaining natural numbers such that $K_{i_l} \subset F$. Now by our assumption the set $H = \bigcup_{i=1}^{\infty}\{h_i\}$ is either unbounded or $\partial H \cap \partial G \neq \emptyset$ or both. The case where $H$ is unbounded is already dealt with on [1]. For the case $\partial H \cap \partial G \neq \emptyset$ we simply note that since $H$ has an accumulation point on $\partial G$, it follows that one or both of

[2]Since $\partial K$ is the union of finitely many disjoint jordan curves and $K$ is connected

the families sets $\{h_{i_n}\}$ or $\{h_{i_l}\}$ has an accumulation point on $\partial G$. Without loss of generality let us assume that the family $\{h_{i_n}\}$ has an accumulation point on $\partial G$. Now let us consider the holes of $\{h_{i_n}\} \cup K$. Since $K_{i_n} \subset E$ and $h_{i_n}$ is a $G-$hole of $E \cup F \cup K$, we see that there exists a $G-$hole $V_{i_n}$ of $E \cup K$ so that $h_{i_n} \subset V_{i_n} \subset d_{i_n}$. Hence the set $\bigcup_{n=1}^{\infty}\{V_{i_n}\}$ has an accumulation point in $\partial G$, which implies that the set $E$ is not an Arakelyan set. This is a contradiction hence the lemma is proved. □

**Remark 3.** Note that in the lemma above we must necessarily require $E$ and $F$ to be $G-$hole independent. Indeed, if we let $G$ to be the punctured unit disc and $E$ and $F$ circles with radii $\frac{1}{2}$ and $\frac{1}{4}$ respectively. Then we can easily verify that $E$ and $F$ are disjoint Arakelyan sets whose union $E \cup F$ has the annuls $A = \{z \in \mathbb{C} : \frac{1}{4} < |z| < \frac{1}{2}\}$ as a $G-$hole. This happens of course because they are not $G-$hole independent.

**Definition 5.** Let $F$ and $C$ be Arakelyan subsets of a domain $G$. We say that $F \cap C$ is a $G-$proper intersection if there is no connected component $\gamma$ of $\partial(F \cap C)$ such that $\gamma \subset C^o$ and $\mathbb{C} \setminus \gamma$ has a bounded component $h$ satisfying $g \subset h$, where $g$ is a bounded component of $\mathbb{C} \setminus G$.

Before we proceed to the proof of our main result we will introduce a final lemma necessary for the proof.

**Lemma 5.** *Let $G \subset \mathbb{C}$ be a domain and let $F$ and $C$ be Arakelyan subsets of $G$ that are $G-$hole independent and $F \cap C$ is a $G-$proper intersection that is also $G-$bounded. Then there exists a compact subset $L \subset G$ such that $F \cap C \subset L^o$, and $F$ is $G-$hole independent with $C \setminus L^o$.*

*Proof.* Since $F \cap C$ is $G-$bounded there exists a compact set $L$ such that $L^o \supset F \cap C$. Now since $F \cap C$ is a $G-$proper intersection and $C$ an Arakelyan set we can choose $L$ so that $C \setminus L^o$ is also an Arakelyan set and satisfies the following:
$G \setminus (C \setminus L^o) = \bigcup_{j=1}^{\infty} h_{i_j} \cup \bigcup_{k=1}^{\infty} h_{i_k} \cup L^o$ where $G \setminus C = \bigcup_{i=1}^{\infty} h_i = \bigcup_{j=1}^{\infty} h_{i_j} \cup \bigcup_{k=1}^{\infty} h_{i_k}$ and for each $i_j$ and $i_k$, we have that $h_{i_j} \cap L \neq \emptyset$ and $h_{i_k} \cap L = \emptyset$. Hence, it follows that any $G-$unbounded component of $C \setminus L^o$ is either of the form $\bigcup_{j=1}^{\infty} h_{i_j} \cup L^o$ or $h_{i_k}$. Now, since $F \cap C$ is a $G-$proper intersection, it follows that $F$ and $C \setminus L^o$ must be $G-$hole independent because any $G-$unbounded component $s$ of $G \setminus F$ that intersects $L^o \setminus F$, must

also intersect $h_{i_j}$ for some $i_j$, and this intersection is $G-$unbounded since $F$ and $C$ are $G-$hole independent. Therefore any $G-$unbounded component of $C \setminus L^o$ intersected with any $G-$unbounded component of $F$ is $G-$unbounded as required. The proof is complete.

□

Now that we have the three lemmas in our arsenal we are ready to state and prove the Extension of Arakelyan's Theorem.

**Remark 4.** Note that in the lemma above we must necessarily require $F \cap C$ is a $G-$proper intersection. Indeed, if we let $G$ to be the punctured unit disc and $F$ be the circle of radius $\frac{1}{2}$ and $C = \{z \in \mathbb{C} : \frac{1}{4} \leq |z| \leq \frac{1}{2}\}$. Then we can easily verify that there is no compact set $L \subset G$ such that $L^o \supset F \cap C$ and $F$ and $C \setminus L^o$ are are $G-$hole independent. This happens of course because $F \cap C$ is not a $G-$proper intersection. It is also worth to note the following case. Let $F$ and $G$ as above and $C = F$. In this case, the set $C \cap F$ is a $G-$proper intersection since $C^o = \emptyset$. Clearly for any compact $L$ satisfying $L^o \supset F \cap C$, the sets $F$ and $C \setminus L^o = \emptyset$ are $G-$hole independent since $F$ has no $G-$holes.

**Theorem 3** (Extension of Arakelyan's Theorem)**.** *Let $G \subset \mathbb{C}$ be an arbitrary domain and suppose $F$ and $C$ are relatively closed subsets of $G$ that are $G-$hole independent and $F \cap C$ is a $G-$proper intersection. In order that every function $f \in A(F)$ can be approximated uniformly on $F$ by functions $g \in Hol(G)$, each of which is bounded on $C$, it is necessary and sufficient that:*

1. *$F$ is an Arakelyan set.*
2. *There exists an Arakelyan set $C_1 \neq G$, so that $C \subset C_1$ and $F \cap C$ is $G-$bounded.*

*Proof of Extension of Arakelyan's Theorem.* **(Necessity.)** Assume that any $f \in A(F)$ can be approximated uniformly by functions $g \in Hol(G)$, each of which is bounded on $C$. By Arakelyan's Theorem it follows that $F$ is an Arakelyan set. If $\overline{F} \cap \partial G \neq \emptyset$, choose $p \in \overline{F} \cap \partial G$ and set:

$$\phi(z) = \begin{cases} z & \text{if } \overline{F} \cap \partial G = \emptyset \\ (z-p)^{-1} & \text{if } \overline{F} \cap \partial G \neq \emptyset. \end{cases}$$

Clearly the function $\phi(z)$ belongs to the class $Hol(G)$; thus there exists a function $f \in Hol(G)$ such that

$$|\phi(z) - f(z)| < 1, \quad z \in F. \tag{1}$$

and $f$ is also bounded on C. By Lemma 3 there exists a closed Arakelyan set $C_1 \subset \mathbb{C}$ such that that $C \subset C_1$ and $f(z)$ is bounded on $C_1$. By (1) clearly $F \cap C_1$ is $G-$bounded. The proof of necessity is complete.

**(Sufficiency.)** We have Arakelyan sets $F$ and $C_1$, $C \subset C_1$, such that $F \cap C$ is $G-$bounded. Now we note that by construction of $C_1$, it may be chosen so that its $G-$unbounded components are arbitrarily close to those of $C$, thus we may assume without loss of generality that $F$ and $C_1$ are $G-$hole independent as well. Now by assumption of the theorem there exists a compact subset $L$ in $G$ and so large that:

$$F \cap C_1 \subset L^o \tag{2}$$

$$(\partial L) \setminus C_1 \neq \emptyset \tag{3}$$

Note that (3) follows since $C_1 \neq G$ is a closed set without $G-$holes.

The set $C_1 \setminus L^o$ is also without $G-$holes, because otherwise there would exist a $G-$hole $g$ of $C_1 \setminus L^o$. This leads to a contradiction since then, $g$ would be a $G-$hole of $C_1$ which is clearly impossible because it is an Arakelyan set.

The set $C_1 \setminus L^o$ is also an Arakelyan set. Otherwise, there would exist a compact set $K \subset G$ so that the set $H \equiv \{h : h \text{ is a } G-\text{hole of } K \cup (C_1 \setminus L^o)\}$ is either unbounded or $\partial H \cap \partial G \neq \emptyset$. But this is a contradiction because then the set of $G-$holes of the set $C_1 \setminus L^o$ union the compact set $K \cup L$ would also have to either be unbounded or accumulate to a point in $\partial G$. Indeed, the claim follows, since $(K \cup L) \cup (C_1 \setminus L^o) = C_1 \cup K$. This contradicts that $C_1$ is an Arakelyan set, as such, by the contradiction we conclude $C_1 \setminus L^o$ is an Arakelyan set as well.

Moreover, since by assumption $F \cap C$ is a $G-$proper intersection, it follows by lemma 5 that the Arakelyan sets $F$ and $C_1 \setminus L^o$ are $G-$hole independent as well. Now since $F$ and $C_1 \setminus L^o$ are disjoint by (2), it follows by lemma 4 that $F \cup (C_1 \setminus L^o)$ is also an Arakelyan set.

Now, let $\phi(z) \in A(F)$ be any function and define a function $h(z)$ by $h(z) = \phi(z)$ on $F$ and $h(z) = 0$ on $C_1 \setminus L^o$. Clearly by (2) it is evident that $h(z) \in A(F \cup (C_1 \setminus L^o))$. Therefore by

Arakelyan's theorem $h(z)$ is uniformly approximable by functions in $A(G)$. Therefore for any $\epsilon > 0$, there exists a function $f \in A(G)$ such that $|h(z) - f(z)| < \epsilon$ for any $z \in F \cup (C_1 \setminus L^o)$. Hence we have that $|\phi(z) - f(z)| < \epsilon$ on $F$ and $|f(z)| < \epsilon$ on $C_1 \setminus L^o$. The function $f(z)$ is bounded on $C_1 \setminus L^o$, moreover $f(z)$ is bounded on the compact set $C_1 \cap L^o$ thus, $f(z)$ uniformly approximates $\phi(z)$ on $F$ and is bounded on $C$. The proof is complete. $\square$

The following lemma is proven in [[3], p.265] and replaces lemma 4 for the case $G$ is simply connected.

**Lemma 6.** *Let $G \subset \mathbb{C}$ be a simply connected open set $\{F_n\}_{n=1}^{\infty}$ is a locally finite family of pairwise disjoint Arakelyan sets in $G$, then the union $\bigcup_{n=1}^{\infty} F_n$ is also an Arakelyan set in $G$.*

**Corollary 1** (Extension of Arakelyan's Theorem when $G$ is simply connected)**.** *Let $G \subset \mathbb{C}$ be an arbitrary simply connected domain and suppose $F$ and $C$ are relatively closed subsets of $G$. In order that every function $f \in A(F)$ can be approximated uniformly on $F$ by functions $g \in Hol(G)$, each of which is bounded on $C$, it is necessary and sufficient that:*

1. *$F$ is an Arakelyan set.*

2. *There exists an Arakelyan set $C_1 \neq G$, so that $C \subset C_1$ and $F \cap C_1$ is $G-$bounded.*

*Proof.* The proof is identical to the one above where of course we omit the parts about sets being $G-$hole independent, since they are unnecessary because we replace lemma 4 by lemma 6. $\square$

To conclude we ask the following open question that occurs naturally. Does the Extension of Arakelyan's Theorem still hold if we require that the approximating functions be uniformly bounded on $C$? This question remains unanswered even for the case $G = \mathbb{C}$

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