

THE UNIPOTENT MIXING CONJECTURE

VALENTIN BLOMER AND PHILIPPE MICHEL

ABSTRACT. We show that shifted pairs of discrete or continuous low-lying horocycles equidistribute in the product space of two modular curves.

Dedicated to Peter Sarnak with admiration

1. INTRODUCTION

1.1. Main results. A classical result of Sarnak [Sar81] says in a special case that the image of a low-lying horocycle

$$\{x + i/T \mid x \in [0, 1]\}$$

of height $1/T$ equidistributes on the modular curve $X = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ with respect to the usual hyperbolic probability measure

$$d\mu_X(z) = \frac{3}{\pi} \frac{dx dy}{y^2}$$

as $T \rightarrow \infty$.

Similarly, one can consider for a prime q , say, “discrete” low-lying horocycles

$$H_q := \left\{ \frac{a+i}{q} \mid a \pmod{q} \right\}.$$

Except for the point $iq \in \mathbb{H}$, this is the image of $\mathrm{SL}_2(\mathbb{Z})i$ by the Hecke correspondence T_q (which has degree $q+1$), and non-trivial bounds for eigenvalues of Hecke operators imply that these discrete low-lying horocycles also equidistribute on X as $q \rightarrow \infty$.

It is natural to investigate whether equidistribution persists when some additional constraints are imposed on the entries a . Identifying the set of a ’s with integers contained in the interval $[0, q-1]$, one can restrict a to belong to a subinterval $I \subset [0, q-1]$ (see [Str04]) or vary a along a subsequence of integers of a special shape like $a = [n^c]$, $c = 1.1$ [Ven10] or ask for a to vary along the primes.

This last case was studied by Sarnak and Ubis [SU15] who proved, assuming the Ramanujan-Petersson-Selberg conjecture that, as $q \rightarrow \infty$, the “prime” q -Hecke points

$$H_q^p := \left\{ \frac{p+i}{q}, \ 1 \leq p \leq q-1, \ p \text{ prime} \right\}.$$

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become dense in X ; more precisely any weak limit, μ^P say, of the uniform probability measures supported by the H_q^P satisfies

$$\frac{1}{5}\mu_X \leq \mu^P \leq \frac{9}{5}\mu_X.$$

As for the proof, the combinatorial decomposition of the characteristic function of the primes lead naturally to handling sums of “Type I” and “Type II”. The treatment of type I sums is closely related to restricting a to a sub-interval while the treatment of type II leads a new equidistribution problem: the joint equidistribution of sets of the shape

$$H_{q,(b_1,b_2)}(N) := \left\{ \left(\frac{b_1 a + i}{q}, \frac{b_2 a + i}{q} \right) \mid 1 \leq a \leq N \right\} \subset X \times X$$

on the product $X \times X$; here $0 < b_1 < b_2$ are distinct integers bounded by a fixed power of q and with N satisfying $b_1 N, b_2 N \leq q$.

In the present paper we investigate the distribution properties of the above sets when we let a vary over the *complete set* of congruence classes modulo q and look at how

$$H_{q,b} := \left\{ \left(\frac{a+i}{q}, \frac{ab+i}{q} \right) \mid a \pmod{q} \right\}$$

distributes in the product $X \times X$ as $q \rightarrow \infty$ as we allow b to possibly vary with q . Clearly if $b = 1$ we cannot expect full equidistribution to hold: $H_{q,b}$ is simply trapped in the diagonally embedded copy of X in $X \times X$ (and equidistributes along it). A similar phenomenon will occur for any fixed (i.e. independent of q) residue class b . One might optimistically conjecture that something like $q\|b/q\| \rightarrow \infty$ (where $\|\cdot\|$ denotes the distance in \mathbb{R}/\mathbb{Z}) may suffice to ensure equidistribution, but the situation is more subtle as we shall see in a moment.

It turns out to be useful to introduce the lattice

$$(1.1) \quad \Lambda_{q,b} = \{(n_1, n_2) \in \mathbb{Z}^2, n_1 + n_2 b \equiv 0 \pmod{q}\} \subseteq \mathbb{Z}^2.$$

This lattice has covolume q and we denote by

$$s(q; b) = \min_{\mathbf{0} \neq \mathbf{n} \in \Lambda_{q,b}} \|\mathbf{n}\|$$

its minimum (i.e. the minimal euclidean norm of a non-zero element). It is well-known from the geometry of numbers that

$$(1.2) \quad s(q; b) \ll q^{1/2}.$$

In particular $s(q; b) \rightarrow \infty$ implies $q \rightarrow \infty$. We are now ready to state our first main result.

Theorem 1.1. *Let q be a large prime and $b \in (\mathbb{Z}/q\mathbb{Z})^\times$. Then for pairs (q, b) such that $s(q; b) \rightarrow \infty$ the set*

$$(1.3) \quad H_{q,b} = \left\{ \left(\frac{a+i}{q}, \frac{ab+i}{q} \right) \mid a \pmod{q} \right\} \subseteq X \times X$$

becomes equidistributed with respect to the product of hyperbolic measures $\mu_{X \times X} = \mu_X \otimes \mu_X$.

Once the case of two factors is available, it is possible to obtain equidistribution for more ([EL19, Corollary 1.5]):

Corollary 1.2. For $d \geq 2$ and a d -tuple of congruence classes $\mathbf{b} = (b_1, \dots, b_d) \in ((\mathbb{Z}/q\mathbb{Z})^\times)^d$ let

$$s(q; \mathbf{b}) := \min_{i \neq j} s(q; b_i b_j^{-1}).$$

Then for pairs (q, \mathbf{b}) such that $s(q, \mathbf{b}) \rightarrow \infty$ the set

$$H_{q, \mathbf{b}} := \left\{ \left(\frac{ab_1 + i}{q}, \dots, \frac{ab_d + i}{q} \right) \mid a \pmod{q} \right\} \subseteq X^d$$

becomes equidistributed with respect to the product of hyperbolic measures $\mu_{X^d} = \mu_X^{\otimes d}$.

Remark 1.1. Theorem 1.1 and Corollary 1.2 hold true with X replaced by any fixed modular curve $X_0(D) = \Gamma_0(D) \backslash \mathbb{H}$ for $D \geq 1$ and

$$\Gamma_0(D) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \ c \equiv 0 \pmod{D} \right\}.$$

In fact, for the proof we will need such an extension to a modular curve of higher level.

Remark 1.2. If we denote for a continuous, compactly supported function $\varphi : X \times X \rightarrow \mathbb{C}$ by

$$(1.4) \quad \mathcal{W}_\varphi(b; q) := \frac{1}{q} \sum_{a \pmod{q}} \varphi\left(\frac{a+i}{q}, \frac{ab+i}{q}\right)$$

the corresponding Weyl sum, then one approach to Theorem 1.1 is to show that

$$(1.5) \quad \mathcal{W}_\varphi(b; q) = \int_{X \times X} \varphi(z_1, z_2) d\mu_{X \times X}(z_1, z_2) + o_\varphi(1).$$

If we assume the Ramanujan-Petersson conjecture for the Hecke eigenvalues of Maaß cusp forms for the group $\mathrm{SL}_2(\mathbb{Z})$ and φ is cuspidal (in both components), we prove (1.5) with a polylogarithmic rate in the form

$$o_\varphi(1) \ll_{\varphi, \varepsilon} s(q; b)^{-1+\varepsilon} + \log^{-\delta} p$$

for some fixed $\delta > 0$ and any $\varepsilon > 0$. See Section 1.2 for more details.

Remark 1.3. The condition $s(q; b) \rightarrow \infty$ means that b/q must not be too close to a rational number with fixed denominator (like $2/5$), so b/q must be, in a rather weak sense, “badly approximable”. We indicate in Remark 4.2 below why this condition is probably sharp.

Remark 1.4. Theorem 1.1 can probably be generalized to composite numbers q at the cost of slightly more work. For instance, some care must be taken if q has many small prime factors which might collide with the auxiliary primes to be chosen later.

We prove a similar theorem for “continuous” low-lying pairs of horocycles.

Theorem 1.3. Let $T > 1$, $y \in [1, 2]$ and write $y = a/q + O(1/qQ)$ for positive coprime integers a, q with $q \leq Q := T^{0.99}$. Let $I \subseteq \mathbb{R}$ be a fixed non-empty interval. Then

$$\left\{ \left(\frac{x+i}{T}, \frac{xy+i}{T} \right) \mid x \in I \right\} \subseteq X \times X$$

equidistributes as $T \rightarrow \infty$ for pairs (y, T) with $q \rightarrow \infty$.

The condition on y is similar: y must be in some weak sense badly approximable by rational numbers with small denominator. There is some flexibility in the definition of Q as the proof shows. Also Theorem 1.3 can be generalized to d factors in a similar way.

These results are new examples of a growing family of equidistribution results inspired by Sarnak's classical theorem [Sar81] mentioned above, as well as by Duke's celebrated equidistribution theorems for Heegner points and closed geodesics with increasing discriminants on modular curves [Duk88]; the latter needs crucially subconvexity for twisted L -functions [DFI93] as an input.

A new generation of equidistribution problems was posed by the second author and Venkatesh in their ICM address [MV06] in the context of Duke's theorem. Let $H(D)$ be the set of Heegner points in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$. Let $[\mathfrak{b}]$ be an element in the class group of K (which acts on $H(D)$), and consider the set of pairs

$$\{(z, [\mathfrak{b}] \star z) \mid z \in H(D)\} \in X \times X.$$

As $D \rightarrow \infty$, does this equidistribute in the product space $X \times X$? Similarly to the assumption $s(q; b) \rightarrow \infty$ in Theorem 1.1, a necessary condition to escape from the diagonal is that the norm of the smallest ideal in $[\mathfrak{b}]$ tends to infinity, and it is reasonable to conjecture that this is sufficient. This is (a special case of) the *mixing conjecture*. This conjecture is still open but two different conditional proofs were given by Khayutin [Kha19] and Blomer-Brumley-Khayutin [BBK]. One of the key ingredients in the former is a deep result in homogeneous dynamics due to Einsiedler and Lindenstrauss [EL19] classifying under suitable condition joinings in products of locally homogeneous spaces. As we will see below this result also play a crucial role in the present paper. Further variations on this theme can be found in [AES16, AEW22, ALMW22, BB], and we recommend in particular the Bourbaki seminar by M. Aka [Aka21].

Remark 1.5. The astute reader might have noticed that H_q is essentially the set $H(-4q^2)$ of Heegner points of the highly non-maximal order of conductor $-4q^2$.

More precisely, if $q \equiv 3 \pmod{4}$, we have $\#H(-4q^2) = \frac{1}{2}(q+1)$, and H_q corresponds to the q primitive quadratic forms $F_{q,a} := (q^2, 2qa, a^2 + 1)$ for $a \pmod{q}$. The forms F_{q,a_1} and F_{q,a_2} are equivalent if and only if $a_1 a_2 \equiv -1 \pmod{q}$ or $a_1 \equiv a_2 \pmod{q}$, so that H_q covers $H(-4q^2)$ twice except for the form $F_{q,0}$ which corresponds uniquely to the identity in $H(-4q^2)$.

If $q \equiv 1 \pmod{4}$, then $\#H(-4q^2) = \frac{1}{2}(q-1)$, and the same analysis holds, except that the two forms $(q^2, 2qa, a^2 + 1)$ with $a^2 \equiv -1 \pmod{q}$ are not primitive any more, and the corresponding two representatives of the same point in $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ are not counted by $H(-4q^2)$.

However the group action in $H(-4q^2)$ is quite different from the multiplication in H_q .

We recall that Khayutin's work [Kha19] is tailored for fundamental discriminants and uses quite crucially the fact that the underlying order is maximal (or at least not far away from being maximal, cf. [Kha19, Section 1.8.2]). While our proof also builds crucially on the work of Einsiedler-Lindenstrauss, our approach is different from Khayutin's: instead of bounding a correlation function between two measures, we prove, by direct arguments, that the Weyl sums attached to a well chosen cuspidal test function φ converge to the correct limit (i.e. 0); the classification theorem of Einsiedler-Lindenstrauss then allows us to bootstrap this partial information to full equidistribution.

Remark 1.6. Unlike other examples, our result does not require any splitting conditions: in our case such conditions are in fact automatically satisfied. As such, this seems to be one of the first *unconditional* instances of a mixing type conjecture (see also [Saw21] as well as [ST17] for proofs of function field versions of this conjecture).

Remark 1.7. Lindenstrauss, Mohammadi and Wang [LMW] have obtained very general and effective forms of equidistribution with polynomial decay rates for certain types of one parameter unipotent flows on products of modular curves. It is not clear to us whether this would cover all cases of Theorem 1.3 (as the analysis in [LMW] depends on the injectivity radius of the base point x_0) and then whether this could be transferred to the discrete case treated in Theorem 1.1.

1.2. Principle of the proof. By the spectral decomposition and Weyl's equidistribution criterion, it is sufficient to bound non-trivially the Weyl sum $\mathcal{W}_\varphi(b; q)$ for $\varphi = f_1 \otimes f_2$ when f_1 and f_2 are either constant functions or (non-constant) Hecke eigenforms. The case where f_1 or f_2 is constant follows from the equidistribution of (1.3) when projected to each factor X . To handle the remaining sums and prove Theorem 1.1 we use different techniques depending on how fast $s(q; b)$ approaches ∞ within the range (1.2), a principle that is familiar from [EMV13], [Kha19] or [BBK].

More precisely, let $\theta = 7/64$ be the best known approximation towards the Ramanujan-Petersson conjecture [Kim03] and suppose we are in the range

$$s(q; b) \leq q^{1/2-2\theta-\eta}$$

for some fixed $\eta > 0$. Bounds for the Weyl sums can be achieved by resolving some versions of the shifted convolution problem (see Proposition 2.1). In this regime, the Ramanujan-Petersson conjecture is not necessary and one can show that equidistribution holds with a polynomial decay rate $o_\varphi(1) \ll q^{-\delta(\eta)}$ in the notation of (1.5).

In the remaining range

$$q^{1/2-2\theta-\eta} \leq s(q; b) \ll q^{1/2},$$

methods from harmonic analysis do not work so well at the moment. Using sieve methods instead together with the analytic theory of multiplicative functions, we are able to bound non-trivially the Weyl sums (1.5) when f_1 and f_2 are cuspidal and both satisfy the Ramanujan-Petersson conjecture. The Ramanujan-Petersson conjecture is needed to implement the sieving argument as it insures that the (multiplicative) Hecke eigenvalue functions $n \mapsto \lambda_{f_i}(n)$ are bounded in absolute value by the divisor function. This reduces the problem to that of obtaining a sufficiently good bound for sums of the Hecke eigenvalues along the primes, namely

$$(1.6) \quad \sum_{\substack{p \leq z \\ p \text{ prime}}} \frac{|\lambda_{f_1}(p)| + |\lambda_{f_2}(p)|}{p} \leq (2 - \delta) \log \log z$$

for some fixed $\delta > 0$ and sufficiently large z . Such a bound, which follows from suitable approximations to the Sato-Tate conjecture for cusp forms, yields a power saving in $\log p$ on the size of the Weyl sum (in a way similar to [Hol10]). Unfortunately, if f_1 or f_2 is an Eisenstein series, the distribution properties for their Hecke eigenvalues do not allow to obtain (1.6). Nevertheless it will be useful to remember that if f_1 and f_2 are both *CM forms* (cusp forms attached to Hecke characters of quadratic fields) the Ramanujan-Petersson conjecture holds and (1.6) is unconditional.

To deal with the rest of the spectrum we take a very different route and make use of a powerful measure classification theorem of Einsiedler-Lindenstrauss [EL19, Thm 1.4]. Let $\mu_{q,b}$ denote the uniform probability measure on $X \times X$ supported by $H_{q,b}$. As we explain below in Section 5, $\mu_{q,b}$ is the projection to $X \times X$ of a measure $\mu_{q,b}^S$ on a suitable S -arithmetic quotient

$$\Gamma_S \times \Gamma_S \backslash G(\mathbb{Q}_S) \times G(\mathbb{Q}_S), \quad G = \mathrm{PGL}_2$$

and (by non-trivial bounds for Hecke eigenvalues in the discrete case or Sarnak's theorem in the continuous case) any weak- \star limit of the $\mu_{q,b}^S$ projects to the Haar measure on each factor, i.e. it defines a joining of the Haar measures. Moreover, these limits are invariant under the action of a rank 2 diagonalisable subgroup, namely the diagonal subgroup of $G(\mathbb{Q}_S) \times G(\mathbb{Q}_S)$ generated by

$$(t_1, t_1) \text{ and } (t_2, t_2) \text{ where } t_i = \begin{pmatrix} q_i^{-1} & \\ & q_i \end{pmatrix}, \quad i = 1, 2$$

for q_1, q_2 two primes distinct from q . This is an immediate consequence of the invariance of $H_{q,b}$ under multiplicative shifts that we have already observed:

$$H_{q,b} = \left\{ \left(\frac{aq_i^2 + i}{q}, \frac{aq_i^2 b + i}{q} \right) \mid a \pmod{q} \right\}, \quad i = 1, 2.$$

By the measure classification theorem, any such joining is a convex combination of *algebraic* measures which in the present situation are either the (image of the) Haar measure on the full product space or the Haar measure along diagonal G -orbits in the product. We will have to exclude the latter possibility. This will follow – in a rather subtle way – from what we have already proved. Testing the measure against a carefully selected test function $\varphi = f_1 \otimes f_2$, where both f_1 and f_2 are suitable vectors in a certain CM automorphic representation (which satisfies the Ramanujan conjecture), we show that the corresponding Weyl sum would have a *positive* limit if the measure were not the full product measure.

Remark 1.8. This principle of using measure rigidity to bootstrap the evaluation of Weyl sums for a small portion of the spectrum to full an equidistribution statement has occurred in the past notably in [ELMV11] (for the Siegel Eisenstein series for SL_3). This principle is likely to be used again in higher rank situations in connection to functoriality: for instance we hope that the results of this paper will be useful for equidistribution problems associated with GSp_4 (on using Saito-Kurokawa lifts).

Remark 1.9. Notice that the two entries $(a \pmod{q}, ab \pmod{q})$ in Theorem 1.1 are related by a linear equation. Another interesting question occurring in classical problems from analytic number theory is whether joint equidistribution holds for tuples of discrete horocycles whose entries are related by more general algebraic equations. For instance, the second named author together with Einsiedler and Lindenstrauss ([ELM18]) established joint equidistribution in the case of monomial relations: given two fixed integers $1 \leq k < l$, then as $q \rightarrow \infty$ amongst the primes and for any $b \in (\mathbb{Z}/q\mathbb{Z})^\times$, the set

$$H_{k,l,q,b} := \left\{ \left(\frac{a^k + i}{q}, \frac{ba^l + i}{q} \right) \mid a \pmod{q} \right\}$$

becomes equidistributed on $X \times X$. The proof uses again crucially the measure classification results of [EL19], but the application is *simpler*: the sets $H_{k,l,q,b}$ are then invariant under the rank 2 subgroup generated by (t_1^k, t_1^l) and (t_2^k, t_2^l) , and the fact that $1 \leq k < l$ excludes the possibility of having measures supported along diagonal G -orbits. The combination of

the present results with those of [ELM18] will be the topic of a future work, joint with Einsiedler and Lindenstrauss.

2. SHIFTED CONVOLUTION PROBLEMS

In this section we prepare the stage for a proof of Theorem 1.1 in the “low regime” case, i.e. when

$$s(q; b) \leq q^{1/2-2\theta-\delta}$$

for some $\delta > 0$.

We start with a variation of [Blo04, Theorem 1.3] with an additional summation over h .

Proposition 2.1. *Let $N_1 N_2, d, l_1, l_2 \in \mathbb{N}$ with $(N_1 N_2 l_1 l_2, d) = 1$, $H, M_1, M_2, P_1, P_2 \geq 1$. Let f, g be two (holomorphic or Maaß) cuspidal newforms of levels N_1, N_2 respectively and central characters χ_f, χ_g with Hecke eigenvalues $\lambda_f(m), \lambda_g(m)$. Let G be a smooth function supported on $[M_1, 2M_1] \times [M_2, 2M_2]$ satisfying $\|G^{(i,j)}\|_\infty \ll_{i,j} (P_1/M_1)^i (P_2/M_2)^j$ for $i, j \in \mathbb{N}_0$. For $H \leq h \leq 2H$ let $|\alpha(h)| \leq 1$. Then*

$$\begin{aligned} \mathcal{D} &:= \sum_{\substack{H \leq h \leq 2H \\ d|h, (d, h/d)=1}} \alpha(h) \left| \sum_{l_1 m_1 \pm l_2 m_2 = h} \lambda_f(m_1) \lambda_g(m_2) G(m_1, m_2) \right| \\ &\ll (l_1 M_1 + l_2 M_2)^{1/2+\theta+\varepsilon} \left(\frac{H}{d} \right)^{1/2} \left(1 + \frac{H/d}{l_1 l_2 (1 + H/l_2 M_2)} \right)^{1/2} \end{aligned}$$

with an implied constant depending on ε and polynomially on P_1, P_2 and the conductors of f and g .

The proof depends on the following bound for averages of twisted Kloosterman sums. Here we work unconditionally and denote by $\theta \leq 7/64$ an admissible exponent towards the Ramanujan-Petersson conjecture.

Lemma 2.1. *Let $P_0, P_1, P_2, S, H, Q \geq 1$, $d, N \in \mathbb{N}$ with $(d, N) = 1$. Let χ be a (possibly trivial) Dirichlet character modulo N . Let u be a smooth function with support in $[H, 2H] \times [S, 2S] \times [Q, 2Q]$ satisfying $\|u^{(ijk)}\|_\infty \ll (P_0/H)^i (P_1/S)^j (P_2/Q)^k$ for $0 \leq i, j, k \leq 2$. Let $a(h)$ and $b(s)$ for $H \leq h \leq 2H$, $S \leq s \leq 2S$ be sequences of complex numbers. Then*

$$\begin{aligned} \sum_s \sum_{N|q} \sum_{\substack{d|h \\ (d, h/d)=1}} a(h) b(s) S_\chi(\pm h, s, q) u(h, s, q) &\ll Q \left(\sum_s |b(s)|^2 \right)^{1/2} \left(1 + \frac{HS}{Q^2} + \frac{S}{N} \right)^{1/2} \\ &\times d^\theta \left(\sum_{d|h} |a(h)|^2 \right)^{1/2} \left(1 + \frac{H/d}{N(1 + HS/Q^2)} \right)^{1/2} \left(1 + \left(\frac{HS}{Q} \right)^{-\theta} \right) (NHSQ)^\varepsilon \end{aligned}$$

with an implied constant that depends on ε and polynomially on P_0, P_1, P_2 .

Proof. We follow the proof of [Blo04, Proposition 3.5]. Instead of $U_h(t, q)$ we work with

$$U(t_0, t_1; q) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u\left(x, y, \frac{4\pi\sqrt{xy}}{q}\right) \frac{4\pi\sqrt{xy}}{q} e(-t_0 x - t_1 y) dx dy$$

which satisfies

$$\frac{\partial^n}{\partial q^n} U(t_0, t_1; q) \ll_n \left(1 + \frac{H|t_0|}{P_0} \right)^{-2} \left(1 + \frac{S|t_1|}{P_1} \right)^{-2} HSQ \left(\frac{QP_2}{\sqrt{HS}} \right)^n$$

and

$$qu(h, s, q) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U\left(t_0, t_1; \frac{4\pi\sqrt{hs}}{q}\right) e(t_0 h + t_1 s) dt_0 dt_1.$$

As in [Blo04, Proposition 3.5] we now apply the Kuznetsov formula, Cauchy-Schwarz and the spectral large sieve for cusp forms of level N and character χ . Note that, as $(d, N) = (d, h/d) = 1$, we can extract the divisibility condition $d \mid h$ at the cost of a factor d^θ in the h -sum and reduce the length of the h -sum to H/d . We then arrive in the same way at [Blo04, (3.19)] except that the factor Th^θ is replaced with

$$d^\theta \left(T^2 + \frac{H/d}{N}\right)^{1/2} \left(\sum_{d|h} |a(h)|^2\right)^{1/2},$$

and the result follows with the same choice of T as in [Blo04, (3.20)]. \square

For the **proof of Proposition 2.1** we follow literally the proof of [Blo04, Theorem 1.3] in [Blo04, Section 4] with the only minor modification that f and g may be two different cusp forms of potentially levels, they may have non-trivial central characters, and that the summation condition can have either sign. We keep the extra sum over h outside until the very end when we apply our Lemma 2.1 with $S = Q^2/l_2 M_2$ and $N = N_1 N_2 l_1 l_2$ as a replacement for [Blo04, Proposition 3.5]. This replaces the factor h^θ in [Blo04, (4.19)] with

$$d^\theta \left(\frac{H}{d}\right)^{1/2} \left(1 + \frac{H/d}{l_1 l_2 (1 + H/l_2 M_2)}\right)^{1/2},$$

so that

$$\mathcal{D} \ll (l_1 M_1 + l_2 M_2)^{1/2+\theta+\varepsilon} \left(\frac{H}{d}\right)^{1/2} \left(1 + \frac{H/d}{l_1 l_2 (1 + H/l_2 M_2)}\right)^{1/2}$$

as desired. \square

3. APPLICATION OF A SIEVE

For $X, Y \gg 1$ and $q \in \mathbb{N}$ we denote by $\mathcal{C}_q(X, Y)$ the set of subsets $\mathcal{S} \subseteq \mathbb{N}^2$ contained in a ball of radius X^{100} about the origin and satisfying

$$\sum_{\substack{\mathbf{n}=(n_1, n_2) \in \mathcal{S} \\ d_1 | n_1, d_2 | n_2}} 1 = \frac{X}{d_1 d_2} + O(Y)$$

for all $(d_1 d_2, q) = 1$. A typical example is $\mathcal{S} = \Lambda \cap \mathcal{R}$ where $\Lambda \subseteq \mathbb{Z}^2$ is a sublattice of index q and shortest nonzero vector of length s , $\mathcal{R} \subseteq \mathbb{R}_{>0}^2$ a non-empty, bounded, connected and simply-connected set with piecewise smooth boundary $\partial\mathcal{R}$, with

$$X = \frac{\text{vol}(\mathcal{R})}{q}, \quad Y = \frac{\text{length}(\partial\mathcal{R})}{s}.$$

Note that for $(d_1 d_2, q) = 1$, the pairs $(n_1, n_2) \in \Lambda$ with $d_j \mid n_j$ for $j = 1, 2$ form a sublattice of index $d_1 d_2$.

Proposition 3.1. *Let $\mathcal{S} \in \mathcal{C}_q(X, Y)$. Let λ_1, λ_2 be two non-negative multiplicative functions satisfying $\lambda_j(n) \leq \tau_k(n)$ for some $k \in \mathbb{N}$. Fix $0 < \gamma < 1/2$ and let $z = X^\gamma$. Then*

$$\sum_{\substack{(n_1, n_2) \in \mathcal{S} \\ (n_1 n_2, q) = 1}} \lambda_1(n_1) \lambda_2(n_2) \ll_{k, \gamma, \varepsilon} \frac{X}{(\log z)^2} \exp\left(\sum_{p \leq z} \frac{\lambda_1(p) + \lambda_2(p)}{p}\right) + \frac{X^{1+\varepsilon}}{z^{1/4}} + X^\varepsilon Y z^3$$

for any $\varepsilon > 0$.

For $z \geq 1$ and $q \in \mathbb{N}$ let $\mathcal{P}_{q, z}$ be the set of primes $p \leq z$, $p \nmid q$. For $\mathcal{S} \in \mathcal{C}_q(X, Y)$, $a_1, a_2 \in \mathbb{N}$ and $y_1, y_2 \geq 1$ define

$$S(\mathbf{a}, \mathbf{y}) := \sum_{\substack{\mathbf{n} = (n_1, n_2) \in \mathcal{S} \\ a_1 | n_1, a_2 | n_2 \\ (n_1/a_1, \mathcal{P}_{q, y_1}) = (n_2/a_2, \mathcal{P}_{q, y_2}) = 1}} 1.$$

As usual we write $P^+(n)$ for the largest prime factor of n (with the convention $P^+(1) = 1$). From a standard sieve, e.g. Selberg's sieve, we obtain

Lemma 3.1. *For $\mathcal{S} \in \mathcal{C}_q(X, Y)$, $y_1, y_2 \geq 1$, $a_1, a_2 \in \mathbb{N}$ with $(q, a_1 a_2) = 1$ we have*

$$S(\mathbf{a}, \mathbf{y}) \ll \frac{X}{a_1 a_2 \log(1 + y_1) \log(1 + y_2)} + Y y_1^2 y_2^2.$$

For the **proof of Proposition 3.1** we decompose a general n_1 appearing in the sum as

$$n_1 = p_1^{e_1} \cdots p_{k_1}^{e_{k_1}} \cdot p_{k_1+1}^{e_{k_1+1}} \cdots p_r^{e_r} = a_1 b_1$$

with

$$p_1 < p_2 < \cdots, \quad p_j \nmid q$$

and similarly for $n_2 = a_2 b_2$ where k_j is maximal with $a_j \leq z = X^\gamma$. Let

$$\xi := (\log z)(\log \log z).$$

We distinguish the following four cases for n_1 :

$$\begin{aligned} \text{(I)} \quad & p_{k_1+1} \geq z^{1/2}, \quad \text{(II)} \quad p_{k_1+1} < z^{1/2}, a_1 \leq z^{1/2}, \\ \text{(III)} \quad & p_{k_1+1} < \xi, a_1 > z^{1/2}, \quad \text{(IV)} \quad \xi \leq p_{k_1+1} \leq z^{1/2}, a_1 > z^{1/2} \end{aligned}$$

and similarly for n_2 . This set-up goes back originally to Erdős [Erd52] and was refined by Wolke [Wol71], Nair-Tenenbaum [NT98], Khayutin [Kha19] and others.

1) Consider first pairs (n_1, n_2) such that n_1 is in case (II). Then $p_{k_1+1}^{e_{k_1+1}} \geq z^{1/2}$, so n_1 is divisible by a prime power $p^e \geq z^{1/2}$ with $p < z^{1/2}$. Let $e_0 = e_0(p, z) \geq 2$ be the smallest positive integer with $p^{e_0} \geq z^{1/2}$. Estimating $\lambda_1(n_1) \lambda_2(n_2)$ trivially, by Lemma 3.1 the contribution of such pairs is at most

$$\begin{aligned} (3.1) \quad & \ll X^\varepsilon \sum_{\substack{p \leq z^{1/2} \\ p \nmid q}} S(p^{e_0}, 1, 1, 1) \ll X^{1+\varepsilon} \left(\sum_{p \leq z^{1/4}} \frac{1}{z^{1/2}} + \sum_{z^{1/4} \leq p \leq z^{1/2}} \frac{1}{p^2} \right) + z^{1/2} X^\varepsilon Y \\ & \ll \frac{X^{1+\varepsilon}}{z^{1/4}} + z^{1/2} X^\varepsilon Y. \end{aligned}$$

The same argument works with exchanged indices if n_2 is in case (II).

2) Next consider pairs (n_1, n_2) such that n_1 is in case (III). Then n_1 is divisible by a number $z^{1/2} < a_1 \leq z$ such that $P^+(a_1) < \xi$. Since

$$\sum_{\substack{a \leq x \\ P^+(a) \leq \log x \log \log x}} 1 \ll x^\varepsilon,$$

estimating $\lambda_1(n_1)\lambda_2(n_2)$ trivially, the contribution of such pairs is by Lemma 3.1 and partial summation at most

$$(3.2) \quad \ll X^\varepsilon \sum_{\substack{z^{1/2} \leq a \leq z \\ P^+(a) \leq \xi \\ (a, q) = 1}} S(a, 1, 1, 1) \ll \frac{X^{1+\varepsilon}}{z^{1/2}} + X^\varepsilon Y.$$

The same argument works with exchanged indices if n_2 is in case (III).

3) Finally we consider the situation when both n_1 and n_2 are in cases (I) or (IV).

Suppose first that n_1 is in case (I). Then $z^{\Omega(b_1)/2} \leq b_1 \ll X^{O(1)}$, so that $\Omega(b_1) \ll 1$ and hence $\lambda_1(b_1) \ll 1$. We conclude that n_1 is divisible by some

$$a_1 \leq z, \quad (n_1/a_1, \mathcal{P}_{q, z^{1/2}}) = 1$$

and $\lambda_1(n_1) \ll \lambda_1(a_1)$. The same argument works if n_2 is in case (I) with exchanged indices.

Next suppose that n_1 is in case (IV). Then we localize p_{k_1+1} into intervals of the shape $z^{1/(r+1)} \leq p_{k_1+1} \leq z^{1/r}$ for $2 \leq r \leq \log z / \log \xi$. In each such interval we have $p_{k_1} \leq p_{k_1+1} \leq z^{1/r}$. We conclude that for each fixed r , the number n_1 is divisible by some

$$z^{1/2} < a_1 \leq z, \quad P^+(a_1) \leq z^{1/r}, \quad (n_1/a_1, P_{q, z^{1/(r+1)}}) = 1,$$

and since $\Omega(n_1/a_1) \leq r$, we have $\lambda_1(n_1) \ll \lambda_1(a_1)k^r$ since $\lambda_j(n) \leq \tau_k(n)$. The same argument works if n_2 is in case (IV) with exchanged indices.

We conclude that in either case the contribution of such pairs is

$$\begin{aligned} & \sum_{r_1, r_2 \leq \log z / \log \xi} k^{r_1+r_2} \sum_{\substack{\delta_{r_1} \neq 1, z^{1/2} \leq a_1 \leq z \\ \delta_{r_2} \neq 1, z^{1/2} \leq a_2 \leq z \\ P^+(a_1), P^+(a_2) \leq z^{1/r} \\ (a_1 a_2, q) = 1}} \lambda_1(a_1) \lambda_2(a_2) S(a_1, a_2, z^{1/(r_1+1)}, z^{1/(r_2+1)}) \\ & \ll \sum_{r_1, r_2 \leq \log z / \log \xi} k^{r_1+r_2} r_1 r_2 \sum_{\substack{\delta_{r_1} \neq 1, z^{1/2} \leq a_1 \leq z \\ \delta_{r_2} \neq 1, z^{1/2} \leq a_2 \leq z \\ P^+(a_1), P^+(a_2) \leq z^{1/r}}} \frac{X \lambda_1(a_1) \lambda_2(a_2)}{a_1 a_2 (\log z)^2} + X^\varepsilon Y z^3 \end{aligned}$$

where we used once again Lemma 3.1. By [Wol71, Lemma 3] we have

$$\sum_{\substack{\delta_{r_1} \neq 1, z^{1/2} \leq a \leq z \\ P^+(a) \leq z^{1/r}}} \frac{\lambda_j(a)}{a} \ll \exp \left(\sum_{p \leq z} \frac{\lambda_j(p)}{p} - cr \log r \right)$$

for some constant $c > 0$, $r \ll \log z / \log \log z$ (this condition is needed in the proof and requires us to treat case (III) separately) and $j = 1, 2$, so that the total contribution of

pairs in the present case is

$$(3.3) \quad \ll \frac{X}{(\log z)^2} \exp \left(\sum_{p \leq z} \frac{\lambda_1(p) + \lambda_2(p)}{p} \right) + X^\varepsilon Y z^3.$$

Combining (3.1), (3.2) and (3.3), we complete the proof. \square

We will apply Proposition 3.1 to the multiplicative functions $n \mapsto |\lambda_{f_i}(n)|$, $i = 1, 2$, where f_1, f_2 are cuspidal (holomorphic or Maaß) Hecke eigenforms with Hecke eigenvalues $\lambda_{f_i}(n)$ satisfying the Ramanujan-Petersson conjecture, so that

$$(3.4) \quad |\lambda_{f_1}(n)|, |\lambda_{f_2}(n)| \leq \tau(n)$$

for all $n \geq 1$.

Lemma 3.2. *Let f be a cuspidal newform with Hecke eigenvalues λ_f satisfying the Ramanujan-Petersson conjecture.*

– If f is a dihedral form (in which case the Ramanujan-Petersson conjecture is automatic), then for any $z \geq 5$ we have

$$(3.5) \quad \sum_{p \leq z} \frac{|\lambda_f(p)|}{p} \leq \frac{3}{4} \log \log z + O(1 + \log(\text{cond } f)).$$

– If f is not a dihedral form, then for any $z \geq 5$ we have

$$(3.6) \quad \sum_{p \leq z} \frac{|\lambda_f(p)|}{p} \leq \frac{17}{18} \log \log z + O(1 + \log(\text{cond } f)).$$

Proof. Let N denote the level of f and χ be its nebentypus. If f is dihedral, we recall that it comes from some Hecke character (not of order 1 or 2 since f is cuspidal) of a quadratic number field K/\mathbb{Q} and hence $\lambda_f(p)$ vanishes if p is inert in K . For prime $p \nmid N$ split in K we have

$$|\lambda_f(p)| \leq \frac{1 + |\lambda_f(p)|^2}{2} = \frac{3}{2} + \frac{1}{2} \lambda_{f_2}(p)$$

where f_2 is the theta series induced from the square of the Hecke character mentioned above. We conclude that

$$\sum_{p \leq z} \frac{|\lambda_f(p)|}{p} \leq \frac{3}{2} \sum_{\substack{p \leq z, (p, N)=1 \\ p \text{ split}}} \frac{1}{p} w\left(\frac{p}{z}\right) + \frac{1}{2} \sum_{\substack{p \leq z, (p, N)=1 \\ p \text{ split}}} \frac{\lambda_{f_2}(p)}{p} w\left(\frac{p}{z}\right) + O(\log N)$$

for a smooth non-negative function w supported on $[0, 2)$ and equal to 1 on $[0, 1]$.

The desired bound (3.5) follows after integration by parts from Perron's formula applied to $-\text{d} \log \zeta_K(s)$ and $-\text{d} \log L(s, f_2)$ which are holomorphic and logarithmically bounded in a standard (Hadamard/de la Vallée-Poussin type) neighbourhood of the line $\Re s = 1$ except for a pole at $s = 1$ (for the former) with residue $+1$ and (possibly) a pole at some Siegel zero with residue -1 (which if it exists, then contributes a negative amount).

If f is not dihedral, the bound (3.6) is similar, but requires a few higher symmetric power L -functions of f (cf. [EMS84]). The starting point is the following computation, valid for $p \nmid N$ and whenever $|\lambda_f(p)| \leq 2$, namely

$$|\lambda_f(p)| \leq 1 + \frac{1}{2} \lambda_f(p^2) \overline{\chi}(p) - \frac{1}{18} \lambda_f(p^2)^2 \overline{\chi}^2(p) = \frac{17}{18} + \frac{4}{9} \lambda_{\text{sym}^2 f}(p) \overline{\chi}(p) - \frac{1}{18} \lambda_{\text{sym}^4 f}(p) \overline{\chi}^2(p).$$

We have therefore

$$\begin{aligned} \sum_{\substack{p \leq z \\ p \nmid N}} \frac{|\lambda_f(p)|}{p} &\leq \sum_{p \nmid N} \frac{|\lambda_f(p)|}{p} w\left(\frac{p}{z}\right) \\ &\leq \frac{17}{18} \log \log z + O(1) + \frac{4}{9} \sum_{p \nmid N} \frac{\lambda_{\text{sym}^2 f}(p) \bar{\chi}(p)}{p} w\left(\frac{p}{z}\right) - \frac{1}{18} \sum_{p \nmid N} \frac{\lambda_{\text{sym}^4 f}(p) \bar{\chi}^2(p)}{p} w\left(\frac{p}{z}\right). \end{aligned}$$

The bounds for the last two sums are then consequences of the holomorphy and the logarithmic bounds satisfied by $-\text{d} \log L(s, \text{sym}^{2\nu} f \times \bar{\chi}^\nu)$ for $\nu = 1, 2$ when s is within the standard neighbourhood of the line $\Re s = 1$.

For $\nu = 1$, the required properties follow from the cuspidality of $L(s, \text{sym}^2 f \times \bar{\chi})$ (since f is non-dihedral), and the absence of a Siegel zero established in [HL94, Ban97]. For $\nu = 2$, we know from [Kim03, KS02] that $L(s, \text{sym}^4 f \times \bar{\chi}^2)$ is automorphic but not necessarily cuspidal. It is cuspidal unless f is tetrahedral or octahedral. In the tetrahedral case, we have by [KS02, §3.2]

$$L(s, \text{sym}^4 f \times \bar{\chi}^2) = L(s, \text{sym}^2 f \times \bar{\chi}) L(s, \chi_3) L(s, \bar{\chi}_3)$$

for $\chi_3 = \chi_{3,f}$ a non-trivial character of order 3. In particular $L(s, \text{sym}^4 f \times \bar{\chi}^2)$ has no zeros or poles in a standard region along the line $\Re s = 1$. In the octahedral case we have [KS02, Thm. 3.3.7]

$$L(s, \text{sym}^4 f \times \bar{\chi}^2) = L(s, \pi(\chi)) L(s, \text{sym}^2 f \times \eta \chi_f^{-1})$$

where $\eta = \eta_f$ is a quadratic character and $\pi(\chi)$ is the dihedral representation induced from a non-trivial cubic Hecke character $\chi = \chi_f$ of the quadratic field determined by η . It follows that $L(s, \text{sym}^4 f \times \bar{\chi}^2)$ has no zeros or poles in a standard region along the line $\Re s = 1$.

Finally if f is not of the above type then $L(s, \text{sym}^4 f \times \bar{\chi}^2)$ is cuspidal and has no zeros or poles in a standard region along the line $\Re s = 1$ except for a possible Siegel zero. If χ_f is trivial, the existence of a Siegel zero was ruled out by Ramakrishnan and Wang in [RW03, Theorem B']. As was pointed out to us by D. Ramakrishnan, the proof extends to the case of a general nebentypus with adequate modifications: let Π be the degree 9 isobaric sum

$$\Pi = 1 \boxplus \text{sym}^2 f \times \bar{\chi} \boxplus \text{sym}^4 f \times \bar{\chi}^2;$$

its degree 81 Rankin-Selberg L -function (which has non-negative coefficients) factors as

$$\begin{aligned} (3.7) \quad L(s, \Pi \times \Pi) &= \zeta(s) L(s, \text{sym}^2 f \times \bar{\chi})^4 L(s, \text{sym}^4 f \times \bar{\chi}^2)^4 L(s, \text{sym}^6 f \times \bar{\chi}^3)^2 \\ &\quad \times L(s, \text{sym}^2 f \times \text{sym}^2 f \times \bar{\chi}^2) L(s, \text{sym}^4 f \times \text{sym}^4 f \times \bar{\chi}^4) \\ &= L(s, \text{sym}^4 f \times \bar{\chi}^2)^4 L_2(s) \end{aligned}$$

say. Following the proof of [RW03, Theorem B'] it is sufficient to prove that $L_2(s)$ is holomorphic along the interval $(1/2, 1)$ (cf. [RW03, Prop. 5.21]). Using the factorisation

$$L(s, \text{sym}^3 f; \text{sym}^2 \times \bar{\chi}^3) = L(s, \text{sym}^2 f \otimes \bar{\chi}) L(s, \text{sym}^6 f \otimes \bar{\chi}^3),$$

one can rewrite $L_2(s)$ into the form

$$\begin{aligned} L_2(s) &= \zeta(s) L(s, \text{sym}^2 f \times \bar{\chi})^2 L(s, \text{sym}^3 f; \text{sym}^2 \times \bar{\chi}^3)^2 \\ &\quad \times L(s, \text{sym}^2 f \times \text{sym}^2 f \times \bar{\chi}^2) L(s, \text{sym}^4 f \times \text{sym}^4 f \times \bar{\chi}^4), \end{aligned}$$

and it remains to prove the holomorphy of $L(s, \text{sym}^3 f; \text{sym}^2 \times \overline{\chi}^3)$ along $(1/2, 1)$. One then proceeds as in [RW03, Lem. 5.25 & §7]. The only difference is that, when χ^3 is non-trivial, one has to use the work of Takeda [Tak14] in place of the work of Bump-Ginzburg [BG92] regarding the holomorphy of the incomplete L -function $L^S(s, \text{sym}^3 f; \text{sym}^2 \times \overline{\chi}^3)$ (where S is the union of the archimedean places and the finite places where f is ramified). \square

Remark 3.1. In this paper we will need only the case where f has trivial nebentypus.

Remark 3.2. In the non-CM case the Sato-Tate conjecture predicts the constant $8/(3\pi)$ in place of $17/18$ in (3.6), and it is known when f is holomorphic [CHT08, HSBT10, Tay08, NT21]. In the CM case, the best possible constant is $2/3$ which is attained if f comes from a character of order 3.

Corollary 3.1. *Let f_1, f_2 be cuspidal newforms with Hecke eigenvalues $(\lambda_{f_i}(n))_{n \geq 1}$, $i = 1, 2$, satisfying the Ramanujan-Petersson conjecture (3.4). Then, with the notation of Proposition 3.1 we have*

$$\sum_{\substack{(n_1, n_2) \in \mathcal{S} \\ (n_1 n_2, q) = 1}} |\lambda_{f_1}(n_1) \lambda_{f_2}(n_2)| \ll_{f_1, f_2, \gamma, \varepsilon} \frac{X}{(\log z)^{1/9}} + \frac{X^{1+\varepsilon}}{z^{1/4}} + X^\varepsilon Y z^3$$

for any $\varepsilon > 0$.

4. EQUIDISTRIBUTION FOR THE CUSPIDAL SPECTRUM

In this section, we establish the equidistribution statements of Theorems 1.1 and 1.3 for pairs of cuspidal newforms (assuming the Ramanujan-Petersson conjecture).

4.1. The discrete case. For $q \in \mathbb{N}$ and $b \in (\mathbb{Z}/q\mathbb{Z})^\times$ we recall the definition (1.1) of the lattice

$$\Lambda_{q;b} = \{(n_1, n_2) \in \mathbb{Z}^2, n_1 + b n_2 \equiv 0 \pmod{q}\}$$

and its minimum $s = s(b; q)$. We start with the following simple result.

Lemma 4.1. *Let $(db, q) = 1$. Then*

$$d^{-1}s(b; q) \leq s(bd; q) \leq ds(b; q)$$

and

$$d^{-1}s(b; q) \leq s(b\bar{d}; q) \leq ds(b; q)$$

and $s(b; q) = s(\bar{b}; q)$.

Proof. Clearly $(n_1, n_2) \in \Lambda_{q;b}$ implies $(n_1 d, n_2) \in \Lambda_{q;bd}$, so that $s(bd; q) \leq ds(b; q)$. Similarly, $s(b\bar{d}; q) \leq ds(b; q)$. Replacing b with $b\bar{d}$ and bd we obtain the inequalities in the other direction. The last statement follows from exchanging n_1 and n_2 . \square

With future applications in mind, we consider slightly more general Weyl sums than in (1.4).

Theorem 4.1. *Let q be a large prime, $b \in (\mathbb{Z}/q\mathbb{Z})^\times$ and $s = s(b; q)$.*

Let f_j , $j = 1, 2$, be two L^2 -normalized cuspidal Hecke-Maaß newforms of levels N_j with Hecke eigenvalues $\lambda_j(n)$ for which the Ramanujan-Petersson conjecture holds.

Let $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^\times$, $r_0 \in \mathbb{Q}$ where the denominator of r_0 is coprime to $N_1 N_2 q$, and let

$$(4.1) \quad \mathcal{W}_{f_1, f_2}(b; q; x_0, y_0, r_0) := \frac{1}{q} \sum_{a \pmod{q}} f_1\left(\frac{a+i}{q}\right) f_2\left(\frac{ba + x_0 + y_0 i}{q} + r_0\right).$$

One has

$$\mathcal{W}_{f_1, f_2}(b; q; x_0, y_0, r_0) \ll_{f_1, f_2, x_0, y_0, r_0, \varepsilon} s^{\varepsilon-1} + (\log q)^{-1/9}$$

for any $\varepsilon > 0$ with an implied constant depending polynomially on $|x_0|$, $y_0 + y_0^{-1}$, the denominator of r_0 and the conductors of f_1 and f_2 . In other words, for fixed (x_0, y_0, r_0) and fixed cusp forms f_1, f_2 we obtain decay as soon as $s \rightarrow \infty$.

Remark 4.1. The reason to include general triples (x_0, y_0, r_0) (and not just the triple $(0, 1, 0)$ as in (1.4)) is that we will later apply this to pairs of cusp forms (f_1, f_2) where f_2 , lifted to a function on an S -adic quotient of

$$\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{Q}_{q_1}) \times \mathrm{PGL}_2(\mathbb{Q}_{q_2})$$

for two fixed primes q_1, q_2 , is acted on by a fixed group element $g \in \mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{Q}_{q_1}) \times \mathrm{PGL}_2(\mathbb{Q}_{q_2})$. When translated back to classical language, this action has the effect of introducing the extra parameters (x_0, y_0, r_0) . See Section 5 for details.

We remark that a similar bound can be obtained for more general test functions, for instance Maaß forms of fixed weights k_j , or even more generally automorphic forms whose archimedean component is a fixed test function (for instance compactly supported) in the Kirillov model.

Remark 4.2. To see that the condition $s \rightarrow \infty$ cannot be dropped completely or replaced with a simpler condition of the kind $q \|b/q\| \rightarrow \infty$, consider the case where $b = (q+1)/2$ (for odd q). Then the Fourier expansion (4.4) below yields essentially

$$\frac{1}{q} \sum_{\substack{n_1, n_2 \asymp q \\ q | n_1 - b n_2}} \lambda_1(n_1) \lambda_2(n_2).$$

The congruence is equivalent to $q \mid 2n_1 - n_2$, and so we obtain a diagonal term

$$\frac{1}{q} \sum_{n \asymp q} \lambda_1(a) \lambda_2(2a)$$

which in the case $f_1 = f_2$ and $\lambda_2(2) \neq 0$ does not decay in q . In other words, the underlying condition for equidistribution is really of diophantine nature.

Proof. We start with the Fourier expansion

$$(4.2) \quad f_j(z) = \sqrt{y} \sum_{n \geq 1} \lambda_j(n) \left(\frac{2 \cosh(\pi t_j)}{L(1, \mathrm{sym}^2 f_j)} \right)^{1/2} K_{it_j}(2\pi n y) (e(nx) + \epsilon_j e(-nx))$$

where ϵ_j is the parity, $\lambda_j(n)$ is the n -th Hecke eigenvalue (we have assumed that the f_j are newforms) and ${}^1 t_j \geq 0$ is the spectral parameter of f_j for $j = 1, 2$. For notational simplicity let us write

$$L_j = L(1, \mathrm{sym}^2 f_j)^{1/2}, \quad K_{it}^*(x) = \cosh(\pi t)^{1/2} K_{it}(2\pi x).$$

¹by the Selberg eigenvalue conjecture, although this is not essential for the argument, unlike the Ramanujan-Petersson conjecture at finite places

We use the simple bound

$$x^j \frac{d^j}{dx^j} K_{it}^*(x) \ll_{j,A,\varepsilon,t} x^{-\varepsilon} (1+x)^{-A}$$

for $j \in \mathbb{N}_0$, $A, \varepsilon > 0$ with polynomial dependence in t . In the following we use the convention that all implied constants may depend polynomially on $\text{cond}(f_j)$ and $|x_0|$, $y_0 + y_0^{-1}$, $\text{den}(r_0)$ without further mention.

Summing over $a \pmod{q}$, we conclude

$$\begin{aligned} \mathcal{W}_{f_1, f_2}(b; q; x_0, y_0, r_0) &= \frac{2|y_0|^{1/2}}{L_1 L_2 q} \left((1 + \epsilon_1 \epsilon_2) \sum_{q|n_1 + bn_2} + (\epsilon_1 + \epsilon_2) \sum_{q|n_1 - bn_2} \right) \lambda_1(n_1) \lambda_2(n_2) \\ &\quad \times e\left(\frac{n_2 x_0}{q} + n_2 r_0\right) K_{it_1}^*\left(\frac{n_1}{q}\right) K_{it_2}^*\left(\frac{|y_0| n_2}{q}\right). \end{aligned}$$

This vanishes unless $\epsilon_1 = \epsilon_2 = \epsilon$, say, in which case we get

$$\mathcal{W}_{f_1, f_2}(b; q; x_0, y_0, r_0) = \frac{1}{q} \sum_{\pm} \sum_{q|n_1 \pm bn_2} \lambda_1(n_1) \lambda_2(n_2) e(n_2 r_0) G\left(\frac{n_1}{q}, \frac{n_2}{q}\right)$$

with

$$(4.3) \quad G(x_1, x_2) = \frac{4e(x_0 x_2)}{L_1 L_2} K_{it_1}^*(x_1) K_{it_2}^*(|y_0| x_2)$$

satisfying

$$|x_1|^{j_1} |x_2|^{j_2} \frac{d^{j_1}}{dx_1^{j_1}} \frac{d^{j_2}}{dx_2^{j_2}} G(x_1, x_2) \ll_{\varepsilon, A, j_1, j_2, x_0, y_0} |x_1 x_2|^{-\varepsilon} (1 + |x_1| + |x_2|)^{-A}.$$

By trivial estimates we can truncate the n_1, n_2 -sum at $q^{1+\varepsilon}$ at the cost of a negligible error. Moreover, if $r_0 = c_0/d_0$, we can replace the additive character $n_2 \mapsto e(n_2 r_0)$ with (a linear combination of) Dirichlet characters χ modulo d_0 , so that it suffices to bound

$$(4.4) \quad \frac{1}{q} \left| \sum_{\substack{q|n_1 \pm bn_2 \\ n_1, n_2 \ll q^{1+\varepsilon}}} \lambda_1(n_1) \lambda_2(n_2) \chi(n_2) G\left(\frac{n_1}{q}, \frac{n_2}{q}\right) \right|$$

for some choice of \pm .

The character χ may not be primitive; if it is induced by the primitive character χ^* , we obtain by Möbius inversion and the Hecke multiplicativity relations

$$\begin{aligned} &\frac{1}{q} \left| \sum_{\substack{q|n_1 \pm bn_2 \\ n_1, n_2 \ll q^{1+\varepsilon}}} \lambda_1(n_1) \lambda_2(n_2) \chi^*(n_2) \sum_{f|(n_2, d_0)} \mu(f) G\left(\frac{n_1}{q}, \frac{n_2}{q}\right) \right| \\ &\leq \frac{1}{q} \sum_{f|d_0} \left| \sum_{\substack{q|n_1 \pm bfn_2 \\ n_1, fn_2 \ll q^{1+\varepsilon}}} \lambda_1(n_1) \sum_{g|(f, n_2)} \mu(g) \lambda_2\left(\frac{f}{g}\right) \lambda_2\left(\frac{n_2}{g}\right) \chi^*(n_2) G\left(\frac{n_1}{q}, \frac{fn_2}{q}\right) \right| \\ &\leq \frac{1}{q} \sum_{g|f|d_0} \left| \lambda_2\left(\frac{f}{g}\right) \right| \left| \sum_{\substack{q|n_1 \pm bfgn_2 \\ n_1, n_2 fg \ll q^{1+\varepsilon}}} \lambda_1(n_1) \lambda_2(n_2) \chi^*(n_2) G\left(\frac{n_1}{q}, \frac{gfn_2}{q}\right) \right|. \end{aligned}$$

The point of this manœuvre is to show that in (4.4) we may assume without loss of generality that χ is primitive. Since $(d_0, N_2) = 1$, the function $n \mapsto \lambda_2 \chi(n)$ describes the Hecke eigenvalues of the *newform* $f_2 \times \chi$, which allows us to apply Proposition 2.1. To this end we have to replace b by some fixed multiple, but by Lemma 4.1 we have $s(b; q) \asymp s(fgb; q)$, so that the subsequent analysis remains unchanged. We may therefore return to (4.4) under the assumption that χ is primitive.

The (n_1, n_2) -sum runs through a lattice $\Lambda_{q, \pm b}$ with basis

$$\begin{pmatrix} \mp b \\ 1 \end{pmatrix}, \quad \begin{pmatrix} q \\ 0 \end{pmatrix}$$

of volume q and minima $0 < s(b; q) = s(-b; q) \ll q^{1/2}$. Let us fix one sign and drop it from the notation. Let

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

be a reduced basis of $\Lambda_{q, b}$ with the first vector of minimal length so that $|x_1| + |x_2| \ll s$. Both x_1, x_2 are nonzero (since $q \nmid b$) and coprime (since s is minimal). In terms of this basis, the congruence condition reads

$$(4.5) \quad n_1 x_2 \equiv n_2 x_1 \pmod{q}.$$

We now use two different methods to estimate the n_1, n_2 -sum depending on the size of s .

If s is not too big, then we interpret the n_1, n_2 -sum as a shifted convolution problem and apply first Proposition 2.1 to the inner sum with summation condition (4.5) which we write as an equality

$$n_1 x_2 - n_2 x_1 = h$$

with $q \mid h$ (see [BFK⁺17] for a similar argument).

Since $s \ll q^{1/2}$ and $n_1, n_2 \ll q^{1+\varepsilon}$, we have automatically that $(h/q, q) = 1$. We need to treat the diagonal term Δ with $h = 0$ separately. In this case $x_1 \mid n_1, x_2 \mid n_2$, and by Rankin-Selberg theory (and the Ramanujan conjecture, although this could be dispensed with) it is easy to see the contribution is

$$(4.6) \quad \Delta \ll \frac{1}{s^{1-\varepsilon}}.$$

Let us call $\mathcal{W}_{f_1, f_2}^*(b; q; x_0, y_0, r_0)$ the remaining portion of (4.4). Applying a smooth partition of unity to the n_1, n_2, h -sums, we obtain by Proposition 2.1 with

$$d = q, \quad M_1, M_2 \ll q^{1+\varepsilon}, \quad H \ll l_1 M_1 + l_2 M_2, \quad l_1, l_2 \ll s$$

the upper bound

$$(4.7) \quad \mathcal{W}_{f_1, f_2}^*(b; q; x_0, y_0, r_0) \ll s^{1+\theta} q^{\theta-1/2+\varepsilon}.$$

Note that the θ -dependence in Proposition 2.1 comes from the entire spectrum of forms of level $x_1 x_2$, not from the two forms f_1, f_2 .

Alternatively, we apply absolute values to the inner sum in (4.4), and we can then apply Proposition 3.1 with $z = q^\gamma$ for some very small $\gamma > 0$ and

$$\mathcal{S} = \Lambda \cap \mathcal{R}$$

for suitable sets $\mathcal{R} \subseteq \mathbb{R}_{>0}^2$. More precisely, we estimate the contribution of the terms with $q \mid n_1 n_2$ trivially by $O(q^{\varepsilon-1})$. For the remaining terms we apply a decomposition into annuli

$(k-1)q \leq \sqrt{n_1^2 + n_2^2} \leq kq$ for $k = 1, 2, \dots, q^\varepsilon$, and by Proposition 3.1 we obtain the bound

$$\mathcal{W}_{f_1, f_2}(b; q; x_0, y_0, r_0) \ll q^\varepsilon \left(\frac{1}{q} + \frac{1}{z^{1/4}} + \frac{z^3}{s} \right) + \frac{1}{(\log q)^2} \exp \left(\sum_{p \leq z} \frac{|\lambda_1(p)| + |\lambda_2(p)|}{p} \right).$$

Employing Lemma 3.2, we obtain

$$(4.8) \quad \mathcal{W}_{f_1, f_2}(b; q; x_0, y_0, r_0) \ll \frac{1}{(\log q)^{1/9}} + \frac{q^{3\gamma+\varepsilon}}{s}.$$

Combining (4.6), (4.7) and (4.8), we complete the proof of Theorem 4.1. \square

Remark 4.3. The bound (4.8) is almost sufficient alone. The shifted convolution argument is only needed when s grows below the q^ε -scale, say logarithmically with q . On the other hand, however, the shifted convolution argument is more robust and provides a power saving.

4.2. The continuous case. For the continuous joint equidistribution problem in Theorem 1.3 we need the following analogue of Theorem 4.1.

Theorem 4.2. *Let f_1, f_2 be two L^2 -normalized cuspidal Hecke-Maaß newforms of levels N_j with Hecke eigenvalues $\lambda_j(n)$ for which the Ramanujan-Petersson conjecture holds. Let $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^\times$, $r_0 \in \mathbb{Q}$ such that the denominator of r_0 is coprime to $N_1 N_2$. Let $T > 1$, $I \subseteq (0, \infty)$ a fixed compact interval, $y \in I$ and W a fixed smooth weight function with compact support in \mathbb{R} and write $y = a/q + O(1/qQ)$ for positive coprime integers a, q with $q \leq Q := T^{0.99}$. Then*

$$\int W(x) f_1\left(x + \frac{i}{T}\right) f_2\left(xy + r_0 + \frac{x_0 + iy_0}{T}\right) dx \ll_{\varepsilon, f_1, f_2, x_0, y_0, r_0} \frac{1}{(\log T)^{1/9}} + \frac{1}{q^{\varepsilon-1}}$$

for any $\varepsilon > 0$ with an implied constant depending polynomially on the conductors of f_1 and f_2 and on $|x_0|$, $y_0 + 1/y_0$ and the denominator of r_0 .

Proof. We start again with the Fourier expansion (4.2) and perform the x -integration. As in (4.4) this leaves us with bounding

$$\mathcal{R}_{f_1, f_2}(y, T) := \frac{1}{T} \sum_{\pm} \left| \sum_{n_1, n_2 \ll T^{1+\varepsilon}} \hat{W}(n_1 \pm n_2 y) \lambda_1(n_1) \lambda_2(n_2) \chi(n_2) G\left(\frac{n_1}{T}, \frac{n_2}{T}\right) \right|$$

where \hat{W} denotes the Fourier transform of W , G is as in (4.3) and χ is some primitive character whose conductor divides the denominator of r_0 . Inserting the rational approximation, the n_1, n_2 -sum is, up to a negligible error, restricted by

$$qn_1 \pm an_2 \ll H := T^\varepsilon \left(q + \frac{T}{Q} \right).$$

For now, let us only assume $\log Q \asymp \log T$. Applying Proposition 2.1 with $d = 1$ and treating the diagonal contribution $qn_1 \pm an_2 = 0$ separately as in (4.6), we obtain the first bound

$$(4.9) \quad \begin{aligned} \mathcal{R}_{f_1, f_2}(y, T) &\ll \frac{1}{q^{1-\varepsilon}} + \frac{T^\varepsilon}{T} (qT)^{1/2+\theta} \left(q + \frac{T}{Q} \right)^{1/2} \left(1 + \frac{T}{qQ} \right)^{1/2} \\ &\ll \frac{1}{q^{1-\varepsilon}} + T^\varepsilon (qT)^\theta \left(\frac{q}{T^{1/2}} + \frac{T^{1/2}}{Q} \right). \end{aligned}$$

On the other hand, we can apply absolute values to $\mathcal{R}_{f_1, f_2}(y, T)$ and use Proposition 3.1. To this end, let $\mathcal{S}_\pm(a, q, H', T')$ for $H' \in \mathbb{N}$ and $T' \geq 1$ be the number of (n_1, n_2) such that $T' < n_2 \leq 2T'$ and $H' < qn_1 \pm an_2 \leq 2H'$. Then

$$\sum_{\substack{\lambda \in \mathcal{S}_\pm(a, q, H', T') \\ d_1 | \lambda_1, d_2 | \lambda_2}} 1 = H' \left(\frac{T'}{qd_1 d_2} + O(1) \right),$$

so that

$$\mathcal{S}_\pm(a, q, H', T') \in \mathcal{C}_1(H'T'/q, H').$$

Recall that $\widehat{W}(x)G(x_1, x_2) \ll_A (1+x)^{-A} |x_1 x_2|^{-\varepsilon} (1+|x_1|)^{-A} (1+|x_2|)^{-A}$. Now from Theorem 3.1 and Lemma 3.2 we conclude for some sufficiently small $\gamma > 0$ similarly as in the preceding proof

$$(4.10) \quad \begin{aligned} \mathcal{R}_{f_1, f_2}(y, T) &\ll T^{-10} + \frac{1}{T} \left(\frac{(q + T/Q)T}{q(\log T)^{1/9}} \right) + T^\varepsilon \left(\frac{(q + T/Q)T}{qT^{\gamma/4}} \right) + \left(q + \frac{T}{Q} \right) T^{3\gamma} \\ &\ll \frac{1}{(\log T)^{1/9}} + \frac{T}{qQ} + \frac{q}{T^{1-3\gamma-\varepsilon}}. \end{aligned}$$

We may now choose, for instance, $Q = T^{0.99}$ (with γ as above sufficiently small) and combine (4.9) and (4.10) to obtain the desired bound

$$\mathcal{R}_{f_1, f_2}(y, T) \ll \frac{1}{(\log T)^{1/9}} + \frac{1}{q^{\varepsilon-1}} + T^\varepsilon \min_{q \leq Q} \left((qT)^\theta \frac{q}{T}, \frac{T^{4\gamma}}{q} \right),$$

and the last term is absorbed in the previous two terms.

5. APPLICATION OF THE JOININGS THEOREM OF EINSIEDLER-LINDENSTRAUSS

In this section we use the powerful measure classification theorem of Einsiedler-Lindenstrauss [EL19].

5.1. Passage to an S -arithmetic quotient. For any place v of \mathbb{Q} we denote by \mathbb{Q}_v the corresponding completion and for a finite place $v = p$ by $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ the closure of \mathbb{Z} . We denote by

$$\mathbb{A} = \mathbb{R} \times \mathbb{A}_f = \mathbb{R} \times \prod'_p \mathbb{Q}_p$$

the ring of adèles (the first component is always the infinite place).

We fix an integer $D \geq 1$ and a non-empty finite set of places $S = \{\infty\} \cup S_f$ containing ∞ and such that the primes in S are coprime to D ; we set

$$\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v, \quad \mathbb{Z}_S = \prod_{p \in S} \mathbb{Z}_p, \quad \mathbb{Z}[1/S] = \mathbb{Z} \left[\prod_{p \in S} \frac{1}{p} \right],$$

and

$$\mathbb{A}^{(S)} = \prod'_{v \notin S} \mathbb{Q}_v, \quad \widehat{\mathbb{Z}}^{(S)} = \prod_{p \notin S} \mathbb{Z}_p.$$

Let $G = \mathrm{PGL}_2$ and let $K(D)^{(S)}$ be the open-compact subgroup

$$K(D)^{(S)} = \prod_{p \notin S} K(D)_p \subseteq G(\mathbb{A}^{(S)})$$

where for $p \notin S$ we put

$$K(D)_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{G}(\mathbb{Z}_p), c \in D\mathbb{Z}_p \right\} \subseteq \mathrm{G}(\mathbb{Z}_p);$$

in particular one has $K(D)_p = \mathrm{G}(\mathbb{Z}_p)$ for $p \nmid D$. We also set

$$K_\infty = \mathrm{PSO}_2(\mathbb{R}) \subset \mathrm{G}(\mathbb{R}).$$

By the strong approximation property for SL_2 and the fact that $\det(K(D)^{(S)}) = \widehat{\mathbb{Z}}^{(S)\times} / (\widehat{\mathbb{Z}}^{(S)\times})^2$ we have

$$\mathrm{G}(\mathbb{Q})\mathrm{G}(\mathbb{Q}_S)K(D)^{(S)} = \mathrm{G}(\mathbb{A})$$

and

$$\mathrm{G}(\mathbb{Q}) \backslash \mathrm{G}(\mathbb{A}) / K(D)^{(S)} \simeq \Gamma(D)_S \backslash \mathrm{G}(\mathbb{Q}_S),$$

where

$$\Gamma(D)_S := \mathrm{G}(\mathbb{Q}) \cap K(D)^{(S)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{G}(\mathbb{Z}[1/S]), c \in D\mathbb{Z}[1/S] \right\} \subseteq \mathrm{G}(\mathbb{Z}[1/S])$$

is a discrete subgroup of $\mathrm{G}(\mathbb{Q}_S)$. In the sequel we set

$$X(D)^S = \Gamma(D)_S \backslash \mathrm{G}(\mathbb{Q}_S).$$

For $S = \{\infty\}$ and $D = 1$, the space $X(1)^\infty$ is a covering of the level 1 modular curve

$$X = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \simeq \mathrm{G}(\mathbb{Z}) \backslash \mathrm{G}(\mathbb{R}) / K_\infty \simeq X(1)^\infty / K_\infty$$

with compact fibers. More generally the S -arithmetic quotient $X(D)^S$ is a covering of the usual modular curve $X_0(D)$ of level D . The compact fiber of a point $z \in \mathbb{H}$ is

$$\Gamma(D)_S g_z K_\infty \mathrm{G}(\mathbb{Z}_S)$$

where $g_z \in \mathrm{G}(\mathbb{R})$ is such that $g_z.i = z$. We denote by

$$\pi_S : X(D)^S \rightarrow X_0(D) \rightarrow X_0(1) = X$$

the composite of this projection and the usual finite covering. Since the fibers of this map are compact, $\Gamma(D)_S \subseteq \mathrm{G}(\mathbb{Q}_S)$ has finite covolume (i.e. is a lattice).

5.2. Lifting the measures. To simplify notations, we omit the dependency in D and write X^S for $X(D)^S$, Γ_S for $\Gamma(D)_S$ etc.

Given a prime $q \notin S$ and $a \in \mathbb{Z}$ we set

$$u_{a/q} := \begin{pmatrix} 1 & a/q \\ 0 & 1 \end{pmatrix}.$$

We define

$$x_{q,a}^S := \Gamma_S \left(u_{a/q} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, u_{a/q}, \dots, u_{a/q} \right) \in X^S.$$

We observe that since $q \notin S$ the Γ_S -coset $x_{q,a}^S$ depends only on the congruence class $a \pmod{q}$: for $p \in S$ and $k \in \mathbb{Z}$ we have

$$u_{(a+qk)/q} = u_k \cdot u_{a/q} \in \Gamma_S u_{a/q}.$$

Let

$$H_q^S := \{x_{q,a}^S \mid a \in \mathbb{Z}/q\mathbb{Z}\} \subseteq X^S.$$

By construction we have

$$\pi_S(H_q^S) = H_q.$$

Let us pass to the product: given $b \in (\mathbb{Z}/q\mathbb{Z})^\times$ we set

$$H_{q,b}^S := \{(x_{q,a}^S, x_{q,ab}^S) \mid a \in \mathbb{Z}/q\mathbb{Z}\} \subseteq X^S \times X^S$$

so that

$$(\pi_S \times \pi_S)(H_{q,b}^S) = H_{q,b} \in X \times X.$$

Let us denote by $\mu_{q,b}^S$ the uniform probability measure on $X^S \times X^S$ supported on $H_{q,b}^S$.

There are multiple advantages of lifting the whole situation to the S -arithmetic group quotient $X^S = \Gamma_S \backslash G(\mathbb{Q}_S)$.

- The space X^S is endowed with an action of $G(\mathbb{Q}_S)$ by right multiplication (noted $x.g = \Gamma_S u g$ for $x = \Gamma_S u \in X^S$ and $g \in G(\mathbb{Q}_S)$) and a left action on its space of functions: for $f : X^S \rightarrow \mathbb{C}$

$$g.f : x \in X^S \mapsto f(x.g) \in \mathbb{C}.$$

- We can test the lifted measures μ_q^S and $\mu_{q,b}^S$ against more general functions, e.g. corresponding to automorphic forms not necessarily of weight 0.
- The lifted measures have additional invariance from the places in S .

5.3. Properties of the measures $\mu_{q,b}^S$. We now specify the set of places S we will use: given $D \geq 1$, let q_1, q_2 be two fixed primes coprime with D and put

$$S = \{\infty, q_1, q_2\}.$$

In this section we verify that any weak- \star limit of the $\mu_{q,b}^S$ is a joining. Let $\mu_{q,b,i}^S, i = 1, 2$, denote the image of $\mu_{q,b}^S$ on X^S under the first and second coordinates projections $\pi_i : X^S \times X^S \rightarrow X^S$.

Lemma 5.1. *As $q \rightarrow \infty$, the measures $\mu_{q,b,i}^S, i = 1, 2$, converge to the Haar probability measure μ_G^S on X^S .*

Proof. For q large enough, we have $q \notin S$ and $(q, D) = 1$, and the two projections of $H_{q,b}^S$ are given by the set H_q^S . Since q is prime, H_q^S is the q -Hecke orbit of the identity class $\Gamma_S \text{Id} \in X^S$ minus one point, namely $x_{q,\infty}^S$ defined like $x_{q,a}^S$ but with $u_{a/q}$ replaced by $\text{diag}(q, q^{-1})$; equidistribution H_q^S follows from the equidistribution of q -Hecke orbit. \square

Remark 5.1. In particular any weak- \star limit of the $(\mu_{q,b}^S)_q$ (the limit of a converging subsequence) is a probability measure.

We now make use of the auxiliary places q_1 and q_2 . We introduce the two diagonal matrices

$$t_1 = \begin{pmatrix} q_1^{-1} & \\ & q_1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} q_2^{-1} & \\ & q_2 \end{pmatrix}$$

which we view as embedded diagonally into $G(\mathbb{Q}_S)$ and also diagonally embedded into $G(\mathbb{Q}_S) \times G(\mathbb{Q}_S)$, i.e. via $t_1^\Delta = (t_1, t_1), t_2^\Delta = (t_2, t_2) \in G(\mathbb{Q}_S) \times G(\mathbb{Q}_S)$.

Lemma 5.2. *The set $H_{q,b}^S$ is invariant under right multiplication by the elements t_1^Δ, t_2^Δ .*

Proof. For any $a \in \mathbb{Z}/q\mathbb{Z}$ we have

$$\begin{aligned} x_{q,a}^S \cdot t_1 &= \Gamma_S \left(u_{a/q} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} q_1^{-1} & \\ & q_1 \end{pmatrix}, u_{a/q} \begin{pmatrix} q_1^{-1} & \\ & q_1 \end{pmatrix}, \dots, u_{a/q} \begin{pmatrix} q_1^{-1} & \\ & q_1 \end{pmatrix} \right) \\ &= \Gamma_S t_1 \left(u_{aq_1^2/q} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, u_{aq_1^2/q}, \dots, u_{aq_1^2/q} \right) \\ &= \Gamma_S \left(u_{aq_1^2/q} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, u_{aq_1^2/q}, \dots, u_{aq_1^2/q} \right) = x_{q,aq_1^2}^S \end{aligned}$$

where we used that $\text{diag}(q_1^{-1}, q_1)$ commutes with $\text{diag}(1, q)$ and $t_1 \in \Gamma_S$. Also recall that q_1 is coprime to q (since q is a sufficiently large prime). Likewise

$$x_{q,ab}^S \cdot t_1 = x_{q,aq_1^2b}^S$$

and so

$$H_{q,b}^S \cdot t_1^\Delta = \{ (x_{q,aq_1^2}^S, x_{q,aq_1^2b}^S) \mid a \in \mathbb{Z}/q\mathbb{Z} \} = H_{q,b}^S.$$

The same computation applies to t_2 . □

In $G(\mathbb{R} \times \mathbb{Q}_{q_1} \times \mathbb{Q}_{q_2})$ we use the factorisation

$$\begin{aligned} t_1 &= \left(\begin{pmatrix} q_1^{-1} & \\ & q_1 \end{pmatrix}, \begin{pmatrix} q_1^{-1} & \\ & q_1 \end{pmatrix}, \text{Id}_2 \right) \left(\text{Id}_2, \text{Id}_2, \begin{pmatrix} q_1^{-1} & \\ & q_1 \end{pmatrix} \right) = t'_1 k_1, \\ t_2 &= \left(\begin{pmatrix} q_2^{-1} & \\ & q_2 \end{pmatrix}, \text{Id}_2, \begin{pmatrix} q_2^{-1} & \\ & q_2 \end{pmatrix} \right) \left(\text{Id}_2, \begin{pmatrix} q_2^{-1} & \\ & q_2 \end{pmatrix}, \text{Id}_2 \right) = t'_2 k_2, \end{aligned}$$

say, so that t'_1, t'_2 satisfy assumption \mathcal{A}' of [EL19, Definition 1.3], while k_1, k_2 are contained in a compact subgroup of the group of diagonal matrices of determinant 1. Applying now [EL19, Cor. 3.4] (see also [Kha19, Thm. 4.4] for the formulation we use here), we obtain

Proposition 5.1. *Let μ_∞ be a weak- \star limit of the measures $\mu_{q,b}^S$ parametrized by pairs (b, q) with $q \rightarrow \infty$. Then μ_∞ is a (t_1^Δ, t_2^Δ) -invariant probability measure, and any of its ergodic components is of the shape*

- either $\mu_{G \times G}$, the (image of the) product Haar measure $\mu_{G(\mathbb{Q}_S) \times G(\mathbb{Q}_S)}$ on $X^S \times X^S$;
- or $\mu_{G,h}$, the (image of the) Haar measure $\mu_{G(\mathbb{Q}_S)}$ on a $G(\mathbb{Q}_S)$ -orbit of the shape

$$(\Gamma_S \times \Gamma_S) G^\Delta(\mathbb{Q}_S)(1, h) \subset X^S \times X^S, \quad h \in G(\mathbb{Q}_S),$$

where G^Δ denotes the image of the diagonal embedding $\Delta : g \in G \mapsto (g, g) \in G \times G$.

More precisely, there exists a probability measure λ on $G(\mathbb{Q}_S)$ such that μ_∞ is a convex combination of $\mu_{G \times G}$ and of

$$\mu_{G^\Delta} := \int_{G(\mathbb{Q}_S)} \mu_{G,h} d\lambda(h),$$

i.e. there exists $c \in [0, 1]$ such that

$$\mu_\infty = (1 - c) \mu_{G \times G} + c \mu_{G^\Delta}.$$

5.4. Excluding the diagonal measures. In this section we use the bounds for certain well chosen Weyl sums established in Theorem 4.1 to show that the coefficient c of μ_{G^Δ} in the decomposition of is zero, effectively concluding the proof of Theorem 1.1. The proof of Corollary 1.2 is a consequence of this result and [EL19, Cor 1.5].

Consider a subsequence of the $\mu_{q,b}^S$ converging to some measure of the shape

$$\mu_\infty = (1 - c)\mu_{G \times G} + c\mu_{G^\Delta}$$

and suppose that $c > 0$.

Let Π be a cuspidal automorphic representation such that its space $\Pi^{K^{(S)}}$ of $K^{(S)}$ -invariant vectors occurs in X^S , and let φ_1, φ_2 be two smooth L^2 -normalized vectors in Π . Consider the function on $X^S \times X^S$ given by

$$\varphi = \varphi_1 \otimes \overline{\varphi}_2 : (g, g') \in X^S \times X^S \mapsto \varphi_1(g)\overline{\varphi}_2(g').$$

Then for $h \in G(\mathbb{Q}_S)$ we have

$$\mu_{G,h}(\varphi) = \int_{X^S} \varphi_1(g)\overline{\varphi}_2(g.h)dg = \langle \varphi_1, h.\varphi_2 \rangle$$

where we have set

$$\langle \varphi_1, \varphi_2 \rangle = \mu_G^S(\varphi_1\overline{\varphi}_2) = \int_{X^S} \varphi_1(g)\overline{\varphi}_2(g)dg.$$

We have therefore

$$\mu_{G^\Delta}(\varphi) = \int_{G(\mathbb{Q}_S)} \langle \varphi_1, h.\varphi_2 \rangle d\lambda(h) = \langle \varphi_1, \lambda * \varphi_2 \rangle$$

where

$$\lambda * \varphi_2 : g \mapsto \int_{G(\mathbb{Q}_S)} \varphi_2(g.h)d\lambda(h)$$

The integral $\lambda * \varphi_2$ converges since the measure λ is finite and cusp forms have rapid decay. Roughly speaking, we will choose $\varphi_1 = \lambda * \varphi_2$. On the one hand, Theorem 4.1 will imply that under certain conditions $\mu_\infty(\varphi) = 0$. On the other hand, we will show under certain conditions $\lambda * \varphi_2 \neq 0$, which is only possible if $c = 0$. We now make this precise.

We make the following choices: let $K = \mathbb{Q}(\sqrt{229})$ (notice that 229 is a prime fundamental discriminant); this quadratic field has class number 3. Let χ be a class group character of order 3; it lifts to an automorphic form f_χ of level 229 with central character the Legendre symbol $\chi_{229} = (\frac{229}{\cdot})$. Since $229 \equiv 1 \pmod{4}$, there exists primitive Dirichlet character ψ modulo 229 of order 4 and its square is χ_{229} . The automorphic form

$$(5.1) \quad f = f_\chi \times \psi$$

is an automorphic form of level 229^2 with trivial central character and Laplace eigenvalue $1/4$ (i.e. spectral parameter $t = 0$). Let

$$q_1 = 37, \quad q_2 = 53.$$

These are split primes in K lying below principal ideals. Moreover, q_1 and q_2 are 4-th power residues modulo 229, so ψ is trivial on q_1, q_2 . We conclude that the automorphic representation Π generated by f satisfies the following properties:

- it is cuspidal;
- it has trivial central character;
- it satisfies the Ramanujan conjecture at all places;

- it is ramified only at 229 which is disjoint from S ;
- it has trivial Langlands parameters at all three places in $S = \{\infty, q_1, q_2\}$.

Of course these properties could have been obtained in a rather general way, the explicit construction above is only for illustrative purposes. The key point is the last property, for the following reason: We consider the representation $\Pi_S = \Pi_\infty \otimes \Pi_{q_1} \otimes \Pi_{q_2}$. Since Π is locally unramified principal series at all places in S , we can consider its induced model \mathcal{I}_S which is the space of functions $F : G(\mathbb{Q}_S) \rightarrow \mathbb{C}$ that transform like

$$(5.2) \quad F(zn[x]a[y]g) = |y|_S^{1/2} F(g), \quad z \in Z(G), \quad n[x] = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \quad a[y] = \text{diag}(y, 1)$$

where we use that the Langlands parameters at places in S are trivial. (Note that y is only well-defined up to units in the non-archimedean case, but this is irrelevant for the transformation rule.) We have a $G(\mathbb{Q}_S)$ -equivariant vector space homomorphism $\iota : \Pi_S \rightarrow \mathcal{I}_S$.

The image of the weight 0 Maaß form f defined in (5.1), lifted to a vector in the adelic representation Π , projected onto Π_S in the model \mathcal{I}_S , corresponds to the function F on $G(\mathbb{Q}_S)$ that is 1 on $\text{PSO}(2) \times G(\mathbb{Z}_{q_1}) \times G(\mathbb{Z}_{q_2})$ which is obviously non-negative on all of $G(\mathbb{Q}_S)$ by (5.2). In particular, $\lambda * F$ is not the zero function, so there exists some $\delta > 0$ with

$$0 < \delta < \|\lambda * \iota^{-1} F\|^2$$

where $\iota^{-1} F$ is an L^2 -normalized spherical vector in Π_S (or in Π).

For $R > 0$ let $B_R \subseteq G(\mathbb{Q}_S)$ denote a ball of radius R about the origin. Choose $R > 0$ so that

$$\int_{B_R} d\lambda(g) \geq 1 - \delta,$$

and let us denote by λ_R the measure restricted to B_R . We now choose

$$\varphi_2 = \iota^{-1} F, \quad \varphi_1 = \lambda_R * \varphi_2.$$

Note that φ_1 is a smooth vector since λ_R is compactly supported. Setting as above $\varphi = \varphi_1 \otimes \bar{\varphi}_2$, we have

$$\mu_\infty(\varphi) = (1 - c)\mu_{G \times G}(\varphi) + c\mu_{G^\Delta}(\varphi) = c\mu_{G^\Delta}(\varphi) = c\langle \lambda_R * \varphi_2, \lambda * \varphi_2 \rangle$$

Since $\|\varphi_2\|^2 = 1$ and λ is a probability measure, we obtain by Cauchy-Schwarz and the definition of λ_R that

$$|\langle \lambda_R * \varphi_2, \lambda * \varphi_2 \rangle - \langle \lambda * \varphi_2, \lambda * \varphi_2 \rangle| \leq \delta,$$

so that

$$(5.3) \quad \mu_\infty(\varphi) \geq c(\|\lambda * \varphi_2\|^2 - \delta).$$

On the other hand, setting $g_{q,a} = u_{a/q} \text{diag}(1, q)$ we have

$$(5.4) \quad \mu_{q,b}^S(\varphi_1 \otimes \bar{\varphi}_2) = \frac{1}{q} \sum_{a \pmod{q}} \varphi_1(g_{q,a}) \bar{\varphi}_2(g_{q,ba}) \rightarrow \mu_\infty(\varphi)$$

as $q \rightarrow \infty$ (over primes). The left hand side equals

$$\int_{\|g\| \leq R} \frac{1}{q} \sum_{a \pmod{q}} \varphi_2(g_{q,a}g) \overline{\varphi_2(g_{q,ba})} d\lambda(g).$$

By strong approximation, we can translate this back into classical language. Clearly,

$$\varphi_2(g_{q,ba}) = f\left(\frac{ba+i}{q}\right).$$

Moreover, any $g \in G(\mathbb{Q}_S)$ with $\|g\| \leq R$ can be written $g = g_\infty \times g_{q_1} \times g_{q_2}$ with

$$g_\infty \in \begin{pmatrix} y_0 & x_0 \\ 0 & 1 \end{pmatrix} K_\infty, \quad g_{q_j} \in \begin{pmatrix} q_j^{r_j} & \xi_j \\ 0 & 1 \end{pmatrix} G(\mathbb{Z}_{q_j})$$

for some $y_0 \in \mathbb{R}^\times$, $x_0 \in \mathbb{R}$, $r_j \in \mathbb{Z}$, $\xi_j \in \mathbb{Z}[q_j^{-1}]$, then

$$\varphi_2(g_{q,ag}) = f\left(\begin{pmatrix} Y & r_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/q & a/q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 & x_0 \\ 0 & 1 \end{pmatrix} . i\right)$$

for some $r_0 \in \mathbb{Z}[q_1^{-1}, q_2^{-1}]$ and $Y = q_1^{-r_1} q_2^{-r_2}$. Matrix multiplication shows that this equals

$$f\left(\frac{aY + Yx_0 + iy_0Y}{q} + r_0\right).$$

Thus we see that with the notation (4.1)

$$\frac{1}{q} \sum_{a \pmod{q}} \varphi_2(g_{q,ag}) \overline{\varphi_2(g_{q,ba})} = \mathcal{W}_{\bar{f},f}(\bar{b}Y, q; x_0Y, y_0Y, r_0).$$

The entries x_0Y, y_0Y, r_0, Y are bounded in terms of R and as $q \rightarrow \infty$ (through primes), we have

$$s(q; \bar{b}Y) \asymp_Y s(q; b) \rightarrow \infty,$$

by our assumption. By Theorem 4.1 and Lemma 4.1 we obtain that

$$\mathcal{W}_{\bar{f},f}(\bar{b}Y, q; x_0Y, y_0Y, r_0) \rightarrow 0,$$

uniformly for $\|g\| \leq R$ and therefore (5.4) tends to zero.

This contradicts (5.3) unless $c = 0$ and this completes the proof of Theorem 1.1.

The proof of Theorem 1.3 is similar, based on Theorem 4.2 instead which we need to apply with yr_0 instead of y . If $y = a/q + O(1/qQ)$, then $yr_0 = ar_0/q + O(r_0/qQ)$, so as long as r_0 is fixed, this changes (as in the previous proof) only the implicit constants.

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MATHEMATISCHES INSTITUT, ENDENICHER ALLEE 60, 53115 BONN, GERMANY

Email address: `blomer@math.uni-bonn.de`

EPFL-SB-MATH-TAN, STATION 8, 1015 LAUSANNE, SWITZERLAND

Email address: `philippe.michel@epfl.ch`