

# A $(\phi_{\frac{n}{s}}, \phi)$ -POINCARÉ INEQUALITY IN JOHN DOMAIN

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**Abstract** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with  $n \geq 2$  and  $s \in (0, 1)$ . Assume that  $\phi : [0, \infty) \rightarrow [0, \infty)$  be a Young function obeying the doubling condition with the constant  $K_\phi < 2^{\frac{n}{s}}$ . We demonstrate that  $\Omega$  supports a  $(\phi_{\frac{n}{s}}, \phi)$ -Poincaré inequality if it is a John domain. Alternately, assume further that  $\Omega$  is a bounded domain that is quasiconformally equivalent to some uniform domain when  $n \geq 3$  or a simply connected domain when  $n = 2$ . We demonstrate  $\Omega$  is a John domain if a  $(\phi_{\frac{n}{s}}, \phi)$ -Poincaré inequality holds.

## 1. INTRODUCTION

Let  $n \geq 2$  and  $\Omega \subset \mathbb{R}^n$  be a bounded domain. Suppose that  $\phi$  is a Young function in  $[0, \infty)$ , that is,  $\phi \in C[0, \infty)$  is convex and satisfies  $\phi(0) = 0, \phi(t) > 0$  for  $t > 0$  and  $\lim_{t \rightarrow \infty} \phi(t) = \infty$ . For any  $s \in (0, 1)$ , define the intrinsic fractional Orlicz-Sobolev space  $\dot{V}_*^{s,\phi}(\Omega)$  as the collection of all measurable functions  $u$  in  $\Omega$  with the semi-norm

$$\|u\|_{\dot{V}_*^{s,\phi}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{|x-y|<\frac{1}{2}d(x,\partial\Omega)} \phi\left(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\right) \frac{dx dy}{|x-y|^n} \leq 1 \right\} < \infty.$$

Modulo constant functions,  $\dot{V}_*^{s,\phi}(\Omega)$  is a Banach space. When  $s = 1$ , we usually consider the classical Orlicz-Sobolev space  $W^{1,\phi}(\Omega)$ , whose sharp embedding has been solved in [11](see also [3] for an alternate formulation of the solution).

Alberico et al. [4] established an imbedding of  $\dot{V}_*^{s,\phi}(\mathbb{R}^n)$  into certain Orlicz target space. Recall that for any Young function  $\psi$ , the Orlicz space  $L^\psi(\Omega)$  is the collection of all  $u \in L^1_{\text{loc}}(\Omega)$  whose norm

$$\|u\|_{L^\psi(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \psi\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\} < \infty.$$

The following is a more thorough description.

**Theorem 1.1.** *Let  $\phi$  be a Young function satisfying*

$$(1.1) \quad \int_0^t \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau < \infty$$

and

$$(1.2) \quad \int_0^\infty \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau = \infty.$$

Define  $\phi_{\frac{n}{s}} := \phi \circ H^{-1}$ , where

$$(1.3) \quad H(t) = \left( \int_0^t \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau \right)^{\frac{n-s}{n}} \quad \forall t \geq 0.$$

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Then  $V_*^{s,\phi}(\mathbb{R}^n) \subset L^{\phi_{n/s}}(\mathbb{R}^n)$ , that is, for any  $u \in V_*^{s,\phi}(\mathbb{R}^n)$  with  $|\{x \in \mathbb{R}^n | u(x) > t\}| < \infty$  for every  $t > 0$ , one has  $u \in L^{\phi_{n/s}}(\mathbb{R}^n)$  with  $\|u\|_{L^{\phi_{n/s}}(\mathbb{R}^n)} \leq C\|u\|_{V_*^{s,\phi}(\mathbb{R}^n)}$ , where  $C$  is a constant independent of  $u$ .

They also showed that  $L^{\phi_{n/s}}(\mathbb{R}^n)$  is optimal target spaces for the imbedding of  $\dot{V}_*^{s,\phi}(\mathbb{R}^n)$  in the sense that if  $\dot{V}_*^{s,\phi}(\mathbb{R}^n) \subset L^A(\mathbb{R}^n)$  holds for another Orlicz space  $L^A(\mathbb{R}^n)$ , then  $L^{\phi_{n/s}}(\mathbb{R}^n) \subset L^A(\mathbb{R}^n)$ .

We are interested in bounded domains which supports the imbedding  $V_*^{s,\phi}(\Omega) \subset L^{\phi_{n/s}}(\Omega)$  or  $(\phi_{\frac{n}{s}}, \phi)$ -Poincaré inequality, that is, there exists a constant  $C \geq 1$  such that

$$(1.4) \quad \|u - u_\Omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \leq C\|u\|_{\dot{V}_*^{s,\phi}(\Omega)},$$

for every  $u \in L^1(\Omega)$ , where  $u_E = \int_E u = \frac{1}{|E|} \int_E u dx$  denotes the average of  $u$  in the set of  $E$  with  $|E| > 0$ .

The major finding of this article is to characterize the imbedding  $V_*^{s,\phi}(\Omega) \subset L^{\phi_{n/s}}(\Omega)$  via John domains under specific doubling assumption in  $\phi$ ; see Theorem 1.2 below. Remember that a bounded domain  $\Omega \subset \mathbb{R}^n$  is called as a  $c$ -John domain with respect to some  $x_0 \in \Omega$  for some  $c > 0$  if for each  $x \in \Omega$ , there is a rectifiable curve  $\gamma : [0, T] \rightarrow \Omega$  parameterized by arc-length such that  $\gamma(0) = x$ ,  $\gamma(T) = x_0$  and  $d(\gamma(t), \Omega^\complement) > ct$  for all  $t > 0$ . For further research on  $c$ -John domains, see [36, 38, 8, 37, 6, 7, 9] and references therein. We say that a Young function  $\phi$  has the doubling property ( $\phi \in \Delta_2$ ) if

$$(1.5) \quad K_\phi := \sup_{t>0} \frac{\phi(2t)}{\phi(t)} < \infty.$$

Note that if a Young function  $\phi \in \Delta_2$  with  $K_\phi < 2^{\frac{n}{s}}$ , then  $\phi$  satisfies (1.1) and (1.2); see Lemma 2.3.

**Theorem 1.2.** *Let  $0 < s < 1$ . Suppose  $\phi$  is a Young function and  $\phi \in \Delta_2$  with  $K_\phi < 2^{\frac{n}{s}}$  in (1.5).*

- (i) *If  $\Omega \subset \mathbb{R}^n$  is a  $c$ -John domain, then  $\Omega$  supports the  $(\phi_{\frac{n}{s}}, \phi)$ -Poincaré inequality (1.4) with the constant  $C$  depending on  $n, s, c$  and  $K_\phi$ .*
- (ii) *Assume further that  $\Omega \subset \mathbb{R}^n$  is a bounded simply connected planar domain, or a bounded domain which is a quasiconformally equivalent to some uniform domain when  $n \geq 3$ . If  $\Omega$  supports the  $(\phi_{\frac{n}{s}}, \phi)$ -Poincaré inequality, then  $\Omega$  is a  $c$ -John domain, where the constant  $c$  depend on  $n, s, C, K_\phi$  and  $\Omega$ .*

Theorem 1.2 extends several known results in the literature; for details see the following remark.

**Remark 1.1.** (i) For  $1 \leq p < n$ ,  $c$ -John domain  $\Omega$  supports Sobolev  $\dot{W}^{1,p}$ -imbedding or  $(\frac{np}{n-p}, p)$ -Poincaré inequality:

$$(1.6) \quad \|u - u_\Omega\|_{L^{np/(n-p)}(\Omega)} \leq C\|u\|_{\dot{W}^{1,p}(\Omega)} \quad \forall u \in \dot{W}^{1,p}(\Omega),$$

where the constant  $C$  depends on  $n, p$  and  $c$ ; see Reshetnyak [38] and Martio [37] for  $1 < p < n$  and Borjarski [5] (and also Hajłasz [24]) for  $p = 1$ . Conversely, further assume that  $\Omega$  is a bounded simply connected planar domain or a domain that is quasiconformally equivalently to some uniform domain when  $n \geq 3$ . Buckley and Koskela [7] proved that if (1.6) holds, then  $\Omega$  is a  $c$ -John domain.

(ii) For  $0 < s < 1$  and  $1 \leq p < \infty$ , the intrinsic fractional Sobolev space  $\dot{W}_*^{s,p}(\Omega)$  consists of all functions  $u \in L^1_{\text{loc}}(\Omega)$  with

$$\|u\|_{\dot{W}_*^{s,p}(\Omega)} := \left( \int_{\Omega} \int_{|x-y|<\frac{1}{2}d(x,\partial\Omega)} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dx dy \right)^{1/p} < \infty.$$

In the special case  $\phi(t) = t^p$  with  $p \geq 1$ ,  $\dot{V}_*^{s,\phi}(\Omega)$  is exactly  $\dot{W}_*^{s,p}(\Omega)$ .

For  $s \in (0, 1)$  and  $1 \leq p < n/s$ , Dyda-Ihnatsyeva-Vähäkangas [17] for  $p = 1$  and Hurri-Syrjänen-Vähäkangas [26] for  $1 < p < n/s$  proved that  $c$ -John domain  $\Omega$  supports the following fractional  $(np/(n-sp), p)_s$ -Poincaré inequality (or fractional Sobolev embedding  $\dot{W}_*^{s,p}(\Omega) \hookrightarrow L^{\frac{np}{n-sp}}(\Omega)$ ), which means that for any  $u \in \dot{W}_*^{s,p}(\Omega)$ ,

$$(1.7) \quad \|u - u_\Omega\|_{L^{np/(n-sp)}(\Omega)} \leq C\|u\|_{\dot{W}_*^{s,p}(\Omega)},$$

holds, where  $C$  depends on  $n, s, p$  and  $c$ . On the other hand, additionally assume that  $\Omega$  is a bounded simply connected planar domain or a domain that is quasiconformally equivalently to some uniform domain when  $n \geq 3$ . They [17, 26] also proved that if (1.7) holds, then  $\Omega$  is a  $c$ -John domain.

If  $1 \leq p < \frac{n}{s}$ , it's easy to see that  $\phi_{\frac{n}{s}}(t) = Ct^{\frac{np}{n-sp}}$  for any  $t \geq 0$ , where  $C$  is a positive constant. If  $\phi(t) = t^p$  with  $p \geq 1$  and  $0 < s < 1$ , then the  $(\phi_{\frac{n}{s}}, \phi)$ -Poincaré inequality is the classical fractional  $(\frac{np}{n-sp}, p)$ -Poincaré inequality.

(iii) Analogue results to (ii) were established for the intrinsic fractional Hajlasz-Sobolev space  $\dot{M}_*^{s,p}(\Omega)$ ; see [42] for details.

We also note that the imbeddings of the fractional Sobolev space  $\dot{W}^{s,p}(\Omega)$  and fractional Orlicz-Sobolev space  $\dot{V}^{s,\phi}(\Omega)$  were taken into account in the citations [4, 29, 30, 41]. Define the fractional Orlicz-Sobolev space  $\dot{V}^{s,\phi}(\Omega)$  consisting of all functions  $u \in L^1_{\text{loc}}(\Omega)$  with

$$\|u\|_{\dot{V}^{s,\phi}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(x) - u(y)|}{\lambda|x-y|^s} \right) \frac{dxdy}{|x-y|^n} \leq 1 \right\} < \infty.$$

The  $\dot{V}^{s,\phi}(\Omega)$ -(semi)norm is evidently derived by substituting the whole domain  $\Omega$  for the range  $B(x, \frac{1}{2} \text{dist}(x, \partial\Omega))$  for the variable  $y$  in the  $\dot{V}_*^{s,\phi}(\Omega)$ -(semi)norm. It goes without saying that  $\dot{V}^{s,\phi}(\mathbb{R}^n) = \dot{V}_*^{s,\phi}(\mathbb{R}^n)$ . For general domain  $\Omega$ , one always has  $\dot{V}^{s,\phi}(\Omega) \subset \dot{V}_*^{s,\phi}(\Omega)$  with a normal bound, but the reverse side is not true necessarily. When  $\phi(t) = t^p$  with  $p \geq 1$ ,  $\dot{V}^{s,\phi}(\Omega)$  is the fractional Sobolev space  $\dot{W}^{s,p}(\Omega)$ , which consists of all functions  $u \in L^1_{\text{loc}}(\Omega)$  with

$$\|u\|_{\dot{W}^{s,p}(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p} < \infty.$$

**Remark 1.2.** (i) Let  $s \in (0, 1)$  and  $1 \leq p < n/s$ . It was shown in [29, 30, 41] that a domain  $\Omega$  supports the  $\dot{W}^{s,p}$ -imbedding

$$\|u - u_\Omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \leq C\|u\|_{\dot{V}^{s,\phi}(\Omega)} \quad \forall u \in \dot{V}^{s,\phi}(\Omega).$$

if and only if  $\Omega$  is Ahlfors  $n$ -regular, that is, there exists a constant  $c > 0$  such that

$$B(x, r) \cap \Omega \geq Cr^n \quad \forall x \in \Omega, 0 < r < 2 \text{diam } \Omega.$$

Note that in the case  $|\Omega| = \infty$  we set  $u_\Omega = 0$ .

(ii) Assume that  $s \in (0, 1)$  and Young function  $\phi$  satisfies (1.1) and (1.2). It was shown in [4] that Lipschitz domain  $\Omega$  supports  $\dot{V}^{s,\phi}(\Omega)$ -imbedding

$$\|u - u_\Omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \leq C\|u\|_{\dot{V}^{s,\phi}(\Omega)} \quad \forall u \in \dot{V}^{s,\phi}(\Omega).$$

But it is not clear whether Ahlfors  $n$ -regular domains characterize  $\dot{V}^{s,\phi}(\Omega)$ -imbedding domains.

The paper is organized as follows. The proof of Theorem 1.2(i) is given in section 2, which uses Boman's chain property, the embedding  $\dot{V}_*^{s,\phi}(Q) \hookrightarrow L^{\phi_{n/s}}(Q)$  for cubes  $Q \subset \mathbb{R}^n$  and the vector-valued inequality in Orlicz norms for the Hardy-Littlewood maximum operators. We also give some property of  $\phi \in \Delta_2$  with  $K_\phi < 2^{\frac{n}{s}}$  in section 2. Conversely, under the condition (2.1), together with

the aid of some ideas from [7, 25, 34, 40, 41], we obtain the *LLC(2)* property of  $\Omega$ , and then prove Theorem 1.2(ii) by a capacity argument; see Section 3 for details.

## 2. PROOF OF THEOREM 1.2(i)

First we give the embedding  $C_c^\infty(\Omega) \subset \dot{V}_*^{s,\phi}(\Omega)$ . It's easy to know

$$(2.1) \quad C_\phi := \sup_{t>0} \int_0^t \frac{\phi(\rho)}{\phi(t)} \frac{d\rho}{\rho} < \infty.$$

In fact, since for practically all  $t \geq 0$   $\phi'(t) \geq 0$  and  $\phi'$  is increasing, we know

$$\frac{\phi(\rho)}{\rho} = \frac{\phi(\rho) - \phi(0)}{\rho} \leq \phi'(\rho).$$

Hence

$$\int_0^t \frac{\phi(\rho)}{\phi(t)} \frac{d\rho}{\rho} \leq \frac{1}{\phi(t)} \int_0^t \phi'(\rho) d\rho \leq 1,$$

that is,  $C_\phi \leq 1$ .

**Lemma 2.1.** *Let  $0 < s < 1$ , and  $\phi$  be a Young function satisfying (2.1). For any bounded domain  $\Omega \subset \mathbb{R}^n$ , we have  $C_c^\infty(\Omega) \subset \dot{V}^{s,\phi}(\Omega) \subset \dot{V}_*^{s,\phi}(\Omega)$ .*

*Proof.*  $\forall u \in C_c^1(\Omega)$ ,  $L := \|u\|_{L^\infty(\Omega)} + \|Du\|_{L^\infty(\Omega)}$ , and  $W \subset \Omega$  such that  $V = \text{supp } u \Subset W \Subset \Omega$ , then

$$\begin{aligned} H &:= \int_{\Omega} \int_{\Omega} \phi\left(\frac{|u(x) - u(y)|}{\lambda|x-y|^s}\right) \frac{dxdy}{|x-y|^n} \\ &\leq \int_W \int_W \phi\left(\frac{L|x-y|}{\lambda|x-y|^s}\right) \frac{dxdy}{|x-y|^n} + 2 \int_V \int_{\Omega \setminus W} \phi\left(\frac{L}{\lambda|x-y|^s}\right) \frac{dxdy}{|x-y|^n}. \end{aligned}$$

By (2.1), we have

$$\begin{aligned} \int_W \int_W \phi\left(\frac{L|x-y|}{\lambda|x-y|^s}\right) \frac{dxdy}{|x-y|^n} &\leq \int_W \int_{B(x, 2 \operatorname{diam} W)} \phi\left(\frac{L|x-y|^{1-s}}{\lambda}\right) \frac{dy}{|x-y|^n} dx \\ &= n\omega_n \int_W \int_0^{2 \operatorname{diam} W} \phi\left(\frac{L\rho^{1-s}}{\lambda}\right) \frac{d\rho}{\rho} dx \\ &= n\omega_n \frac{1}{1-s} \int_W \int_0^{\frac{L(2 \operatorname{diam} W)^{1-s}}{\lambda}} \phi(\mu) \frac{d\mu}{\mu} dx \\ &\leq C_\phi n\omega_n \frac{1}{1-s} \phi\left(\frac{L(2 \operatorname{diam} W)^{1-s}}{\lambda}\right) |W|. \end{aligned}$$

And

$$\begin{aligned} \int_V \int_{\Omega \setminus W} \phi\left(\frac{L}{\lambda|x-y|^s}\right) \frac{dxdy}{|x-y|^n} &\leq \int_V \int_{\Omega \setminus B(y, \operatorname{dist}(V, W))} \phi\left(\frac{L}{\lambda|x-y|^s}\right) \frac{dx}{|x-y|^n} dy \\ &\leq n\omega_n \int_V \int_{\operatorname{dist}(V, W)}^{\infty} \phi\left(\frac{L}{\lambda\rho^s}\right) \frac{d\rho}{\rho} dy \\ &= n\omega_n \frac{1}{s} \int_V \int_0^{\frac{L}{\lambda \operatorname{dist}(V, W)}} \phi(\mu) \frac{d\mu}{\mu} dy \end{aligned}$$

$$\leq C_\phi n \omega_n \frac{1}{s} \phi \left( \frac{L}{\lambda \operatorname{dist}(V, W^C)^s} \right) |V|.$$

If  $\lambda$  is so large, we have  $H \leq 1$ , with  $u \in \dot{V}^{s,\phi}(\Omega)$ , so  $C_c^1(\Omega) \subset \dot{V}^{s,\phi}(\Omega)$ . Combining  $C_c^\infty(\Omega) \subset C_c^1(\Omega)$  and  $\dot{V}^{s,\phi}(\Omega) \subset \dot{V}_*^{s,\phi}(\Omega)$ , we get the result desired.  $\square$

To prove Theorem 1.2(i), we need the embedding  $\dot{V}^{s,\phi}(Q) \hookrightarrow L^{\phi_{n/s}}(Q)$  in all cubes  $Q \subset \mathbb{R}^n$ . So firstly, we give some lemmas we needed.

**Lemma 2.2.** *Let  $\phi \in \Delta_2$  be a Young function, then  $\forall c > 1, \phi(cx) \leq c^{K_\phi-1} \phi(x)$ .*

*Proof.* By the increasing property of  $\phi'$ ,

$$\phi(2x) - \phi(x) = \int_x^{2x} \phi'(t) dt \geq \phi'(x), \quad \forall x > 0.$$

$\phi \in \Delta_2, \phi(2x) - \phi(x) \leq (K_\phi - 1)\phi(x)$ , so

$$(\ln \phi)'(x) = \frac{\phi'(x)}{\phi(x)} \leq \frac{K_\phi - 1}{x}.$$

For any  $c > 1$ , we have

$$\ln \left( \frac{\phi(cx)}{\phi(x)} \right) = \int_x^{cx} (\ln \phi)'(t) dt \leq \int_x^{cx} \frac{K_\phi - 1}{t} dt = \ln(c^{K_\phi-1}).$$

So  $\phi(cx) \leq c^{K_\phi-1} \phi(x)$ .  $\square$

**Lemma 2.3.** *Let  $\phi \in \Delta_2$  be a Young function satisfying  $K_\phi < 2^{\frac{n}{s}}$ , then  $\phi$  satisfies (1.1) and (1.2).*

*Proof.* By the definition of the  $K_\phi$  in (1.5), we get  $\phi(2t) \leq K_\phi \phi(t)$ . Hence

$$\begin{aligned} \int_{\frac{t}{2}}^t \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau &= \int_{\frac{t}{4}}^{\frac{t}{2}} \left( \frac{2\tau}{\phi(2\tau)} \right)^{\frac{s}{n-s}} 2d\tau \\ &\geq \int_{\frac{t}{4}}^{\frac{t}{2}} \left( \frac{2\tau}{K_\phi \phi(\tau)} \right)^{\frac{s}{n-s}} 2d\tau. \end{aligned}$$

Then

$$\int_{\frac{t}{4}}^{\frac{t}{2}} \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau \leq \frac{K_\phi^{\frac{s}{n-s}}}{2^{\frac{n}{n-s}}} \int_{\frac{t}{2}}^t \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau.$$

By induction, we have

$$\begin{aligned} \int_{\frac{t}{2^m}}^{\frac{t}{2^{m-1}}} \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau &\leq \frac{K_\phi^{\frac{s}{n-s}}}{2^{\frac{n}{n-s}}} \int_{\frac{t}{2^{m-1}}}^{\frac{t}{2^{m-2}}} \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau \\ &\leq \left( \frac{K_\phi^{\frac{s}{n-s}}}{2^{\frac{n}{n-s}}} \right)^{m-1} \int_{\frac{t}{2}}^t \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau. \end{aligned}$$

Change  $m$  from 1 to  $\infty$  and sum up, we can get

$$\int_0^t \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau \leq \sum_{m=1}^{\infty} \left( \frac{K_\phi^{\frac{s}{n-s}}}{2^{\frac{n}{n-s}}} \right)^{m-1} \int_{\frac{t}{2}}^t \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau.$$

The series convergences because if the range of the  $K_\phi$ , and

$$\begin{aligned} \left( \frac{t}{\phi(t)} \right)'(t) &= \frac{\phi(t) - t\phi'(t)}{\phi^2(t)} \\ &= \frac{\frac{\phi(t) - \phi(0)}{t} - \phi'(t)}{\phi^2(t)} \\ &= \frac{\phi'(\xi) - \phi'(t)}{\phi^2(t)}, \end{aligned}$$

where  $0 < \xi < t$ , by the convexity of  $\phi$ , we know  $\left( \frac{t}{\phi(t)} \right)'(t) \leq 0$ , then (1.1) follows from decreasing property of  $\frac{\tau}{\phi(\tau)}$ , actually,

$$\int_{\frac{t}{2}}^t \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau \leq \left( \frac{\frac{t}{2}}{\phi(\frac{t}{2})} \right)^{\frac{s}{n-s}} \frac{t}{2} < \infty.$$

Similarly,

$$\begin{aligned} \int_0^{2^m} \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau &\geq \int_0^{2^{m-1}} \left( \frac{2\tau}{K_\phi \phi(\tau)} \right)^{\frac{s}{n-s}} 2d\tau \\ &= \frac{2^{\frac{n}{n-s}}}{K_\phi^{\frac{s}{n-s}}} \int_0^{2^{m-1}} \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau \\ &\geq \dots \\ &\geq \left( \frac{2^{\frac{n}{n-s}}}{K_\phi^{\frac{s}{n-s}}} \right)^m \int_0^1 \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau, \quad \forall m \in \mathbb{N}. \end{aligned}$$

Let  $m \rightarrow \infty$  we get (1.2).  $\square$

**Lemma 2.4.** *Let  $\phi \in \Delta_2$  be a Young function satisfying  $K_\phi < 2^{\frac{n}{s}}$ , then*

$$(2.2) \quad \frac{H(A)}{A} \leq C \frac{1}{\phi(A)^{\frac{s}{n}}}.$$

*Proof.* By Lemma 2.3, we have

$$\int_0^t \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau \leq C \int_{\frac{t}{2}}^t \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau \leq C \left( \frac{\frac{t}{2}}{\phi(\frac{t}{2})} \right)^{\frac{s}{n-s}} \frac{t}{2}.$$

Then

$$\frac{H(A)}{A} = \frac{\left( \int_0^A \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau \right)^{\frac{n}{n-s}}}{A} \leq \frac{\left( C \left( \frac{\frac{A}{2}}{\phi(\frac{A}{2})} \right)^{\frac{s}{n-s}} \frac{A}{2} \right)^{\frac{n}{n-s}}}{A} \leq \frac{\left( C \left( \frac{\frac{A}{2}}{K_\phi \phi(A)} \right)^{\frac{s}{n-s}} \frac{A}{2} \right)^{\frac{n}{n-s}}}{A} \leq C \frac{1}{\phi(A)^{\frac{s}{n}}}.$$

$\square$

With above lemmas, we proved  $\dot{V}^{s,\phi}(Q) \hookrightarrow L^{\phi_{n/s}}(Q)$ .

**Lemma 2.5.** *Let  $0 < s < 1$  and  $\phi \in \Delta_2$  be a Young function satisfying  $K_\phi < 2^{\frac{n}{s}}$ , then there exists a constant  $C_1 = C_1(n, s)$  such that*

$$(2.3) \quad \int_Q \phi_{\frac{n}{s}} \left( \frac{u(x) - u_Q}{\lambda} \right) dx \leq \int_Q \int_Q \phi \left( \frac{C|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{dxdy}{|x - y|^n}.$$

for all cubes  $Q \subset \mathbb{R}^n$ ,  $u \in \dot{V}^{s,\phi}(Q)$  and  $\lambda \geq C_1 \|u\|_{\dot{V}^{s,\phi}(Q)}$ .

*Proof.* Denote a cube centered at the origin with sides of length 2 paralleled to the axes by  $Q(0, 1)$ . At first we prove that

$$(2.4) \quad \int_{Q(0,1)} \phi_{\frac{n}{s}} \left( \frac{|u(x) - u_{Q(0,1)}|}{\lambda} \right) dx \leq \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{dxdy}{|x - y|^n},$$

where  $u \in \dot{V}^{s,\phi}(Q(0, 1))$ ,  $\lambda \geq C_1 \|u\|_{\dot{V}^{s,\phi}(Q(0,1))}$ .

$K_\phi < 2^{\frac{n}{s}}$ , by Lemma 2.3 and [4], we have

$$\|u\|_{L^{\phi_{\frac{n}{s}}}(Q(0,1))} \leq C_1 \|u\|_{\dot{V}^{s,\phi}(Q(0,1))}.$$

where

$$u \in \dot{V}_\perp^{s,\phi}(Q(0, 1)), C_1 = C_1(n, s),$$

$$\dot{V}_\perp^{s,\phi}(Q(0, 1)) := \{u \in \dot{V}^{s,\phi}(Q(0, 1)) : u_{Q(0,1)} = 0\}.$$

Replacing  $u$  by  $u - u_{Q(0,1)}$ , we have

$$\|u - u_{Q(0,1)}\|_{L^{\phi_{\frac{n}{s}}}(Q(0,1))} \leq C_1 \|u - u_{Q(0,1)}\|_{\dot{V}^{s,\phi}(Q(0,1))},$$

where  $u \in \dot{V}^{s,\phi}(Q(0, 1))$ . When  $\|u\|_{\dot{V}^{s,\phi}(Q(0,1))} = 0$ ,  $u$  is constant in  $Q(0, 1)$ , the equality holds. Suppose that  $\|u\|_{\dot{V}^{s,\phi}(Q(0,1))} \neq 0$ , then

$$\int_{Q(0,1)} \phi_{\frac{n}{s}} \left( \frac{|u - u_{Q(0,1)}|}{C_1 \|u\|_{\dot{V}^{s,\phi}(Q(0,1))}} \right) dx \leq 1.$$

Fixed  $u_0 \in \dot{V}^{s,\phi}(Q(0, 1))$ , let

$$M := \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1|u_0(x) - u_0(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^n} \neq 0.$$

Let  $\bar{\phi} = \frac{\phi}{M}$ , then  $\bar{\phi}_{\frac{n}{s}}(t) = \frac{1}{M} \phi_{\frac{n}{s}}(\frac{t}{M^{\frac{s}{n}}})$  and  $C_1 = C_1(n, s)$ , so

$$\|u - u_{Q(0,1)}\|_{L^{\bar{\phi}_{\frac{n}{s}}}(Q(0,1))} \leq C_1 \|u\|_{\dot{V}^{s,\bar{\phi}}(Q(0,1))}.$$

Then we get

$$\int_{Q(0,1)} \phi_{\frac{n}{s}} \left( \frac{|u - u_{Q(0,1)}|}{C_1 M^{\frac{s}{n}} \|u\|_{\dot{V}^{s,\bar{\phi}}(Q(0,1))}} \right) dx \leq M.$$

And  $C_1 \|u_0\|_{\dot{V}^{s,\bar{\phi}}(Q(0,1))} \leq 1$ , otherwise,

$$\begin{aligned} 1 &< \int_{Q(0,1)} \int_{Q(0,1)} \bar{\phi} \left( \frac{C_1|u_0(x) - u_0(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^n} \\ &= \frac{1}{M} \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1|u_0(x) - u_0(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^n} = 1, \end{aligned}$$

we get a contradiction.

Specially, when  $u = u_0$ , we have

$$\begin{aligned} & \int_{Q(0,1)} \phi_{\frac{n}{s}} \left( \frac{|u_0 - u_{Q(0,1)}|}{\left( \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |u_0(x) - u_0(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^n} \right)^{\frac{s}{n}}} \right) dx \\ & \leq \int_{Q(0,1)} \phi_{\frac{n}{s}} \left( \frac{|u_0 - u_{Q(0,1)}|}{C_1 M^{\frac{s}{n}} \|u_0\|_{\dot{V}^{s,\phi}(Q(0,1))}} \right) dx \\ & \leq \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |u_0(x) - u_0(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^n}. \end{aligned}$$

By the arbitrariness of  $u_0$ , we have

$$\begin{aligned} & \int_{Q(0,1)} \phi_{\frac{n}{s}} \left( \frac{|u - u_{Q(0,1)}|}{\left( \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |u(x) - u(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^n} \right)^{\frac{s}{n}}} \right) dx \\ & \leq \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |u(x) - u(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^n}. \end{aligned}$$

Replacing  $u$  by  $\frac{u}{\lambda}$ ,

$$(2.5) \quad \int_{Q(0,1)} \phi_{\frac{n}{s}} \left( \frac{|u - u_{Q(0,1)}|}{\lambda \left( \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |u(x) - u(y)|}{\lambda |x-y|^s} \right) \frac{dxdy}{|x-y|^n} \right)^{\frac{s}{n}}} \right) dx$$

$$(2.6) \quad \leq \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |u(x) - u(y)|}{\lambda |x-y|^s} \right) \frac{dxdy}{|x-y|^n}.$$

Let  $\lambda \geq C_1 \|u\|_{\dot{V}^{s,\phi}(Q(0,1))}$ , then

$$\int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |u(x) - u(y)|}{\lambda |x-y|^s} \right) \frac{dxdy}{|x-y|^n} \leq 1,$$

so

$$\begin{aligned} \int_{Q(0,1)} \phi_{\frac{n}{s}} \left( \frac{|u - u_{Q(0,1)}|}{\lambda} \right) dx & \leq \int_{Q(0,1)} \phi_{\frac{n}{s}} \left( \frac{|u - u_{Q(0,1)}|}{\lambda \left( \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |u(x) - u(y)|}{\lambda |x-y|^s} \right) \frac{dxdy}{|x-y|^n} \right)^{\frac{s}{n}}} \right) dx \\ & \leq \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |u(x) - u(y)|}{\lambda |x-y|^s} \right) \frac{dxdy}{|x-y|^n}. \end{aligned}$$

Now we prove the case of general cube  $Q$ . Let  $Q$  be a cube with  $a$  as the center and  $2l$  as the side length, then there is an orthogonal transformation  $T$ , and  $T(Q-a) = Q(0, l)$ .  $\forall u \in \dot{V}^{s,\phi}(Q)$  and  $u$  is not a constant. Let  $v(x) = \frac{u(T^{-1}(lx)+a)}{l^s}$ , where  $x \in Q(0, 1)$ , then  $v \in \dot{V}^{s,\phi}(Q(0, 1))$  and  $v_{Q(0,1)} = \frac{u_Q}{l^s}$ . And

$$\begin{aligned} & \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |v(x) - v(y)|}{\lambda |x-y|^s} \right) \frac{dxdy}{|x-y|^n} \\ & = \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 \left| \frac{u(T^{-1}(lx)+a)}{l^s} - \frac{u(T^{-1}(ly)+a)}{l^s} \right|}{\lambda |x-y|^s} \right) \frac{dxdy}{|x-y|^n}, \end{aligned}$$

by transformation  $z_1 = T^{-1}(lx) + a, z_2 = T^{-1}(ly) + a$ , we have  $|x - y| = \left| \frac{T(z_1 - a)}{l} - \frac{T(z_2 - a)}{l} \right| = \frac{|z_1 - z_2|}{l}$ , so

$$\int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |v(x) - v(y)|}{\lambda |x - y|^s} \right) \frac{dxdy}{|x - y|^n} = \int_Q \int_Q \phi \left( \frac{C_1 |u(z_1) - u(z_2)|}{\lambda |z_1 - z_2|^s} \right) \frac{dz_1 dz_2}{l^n |z_1 - z_2|^n},$$

and

$$\begin{aligned} & \int_{Q(0,1)} \phi_{\frac{n}{s}} \left( \frac{|v - v_{Q(0,1)}|}{\lambda \left( \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |u(x) - u(y)|}{\lambda |x - y|^s} \right) \frac{dxdy}{|x - y|^n} \right)^{\frac{s}{n}}} \right) dx \\ &= \int_{Q(0,1)} \phi_{\frac{n}{s}} \left( \frac{|v - v_{Q(0,1)}|}{\lambda \left( \int_Q \int_Q \phi \left( \frac{C_1 |u(z_1) - u(z_2)|}{\lambda |z_1 - z_2|^s} \right) \frac{dz_1 dz_2}{l^n |z_1 - z_2|^n} \right)^{\frac{s}{n}}} \right) dx. \end{aligned}$$

By transformation  $y = T^{-1}(lx) + a$ , we get

$$\begin{aligned} & \int_{Q(0,1)} \phi_{\frac{n}{s}} \left( \frac{|v - v_{Q(0,1)}|}{\lambda \left( \int_{Q(0,1)} \int_{Q(0,1)} \phi \left( \frac{C_1 |u(x) - u(y)|}{\lambda |x - y|^s} \right) \frac{dxdy}{|x - y|^n} \right)^{\frac{s}{n}}} \right) dx \\ &= \int_Q \phi_{\frac{n}{s}} \left( \frac{|u(y) - u_Q|}{\lambda \left( \int_Q \int_Q \phi \left( \frac{C_1 |u(z_1) - u(z_2)|}{\lambda |z_1 - z_2|^s} \right) \frac{dz_1 dz_2}{|z_1 - z_2|^n} \right)^{\frac{s}{n}}} \right) \frac{dy}{l^n}. \end{aligned}$$

By (2.5), we have

$$\begin{aligned} & \int_Q \phi_{\frac{n}{s}} \left( \frac{|u(y) - u_Q|}{\lambda \left( \int_Q \int_Q \phi \left( \frac{C_1 |u(z_1) - u(z_2)|}{\lambda |z_1 - z_2|^s} \right) \frac{dz_1 dz_2}{|z_1 - z_2|^n} \right)^{\frac{s}{n}}} \right) dy \\ &\leq \int_Q \int_Q \phi \left( \frac{C_1 |u(z_1) - u(z_2)|}{\lambda |z_1 - z_2|^s} \right) \frac{dz_1 dz_2}{|z_1 - z_2|^n}. \end{aligned}$$

Let  $\lambda \geq C_1 \|u\|_{\dot{V}^{s,\phi}(Q)}$ , then

$$\int_Q \int_Q \phi \left( \frac{C_1 |u(x) - u(y)|}{\lambda |x - y|^s} \right) \frac{dxdy}{|x - y|^n} \leq 1.$$

Hence,

$$\begin{aligned} \int_Q \phi_{\frac{n}{s}} \left( \frac{|u - u_Q|}{\lambda} \right) dx &\leq \int_Q \phi_{\frac{n}{s}} \left( \frac{|u - u_Q|}{\lambda \left( \int_Q \int_Q \phi \left( \frac{C_1 |u(x) - u(y)|}{\lambda |x - y|^s} \right) \frac{dxdy}{|x - y|^n} \right)^{\frac{s}{n}}} \right) dx \\ &\leq \int_Q \int_Q \phi \left( \frac{C_1 |u(x) - u(y)|}{\lambda |x - y|^s} \right) \frac{dxdy}{|x - y|^n}. \end{aligned}$$

□

We also need the Fefferman-Stein type vector-valued inequality for Hardy-Littlewood maximum operator in Orlicz space. Denote by  $\mathcal{M}$  the Hardy-Littlewood maximum operator,

$$\mathcal{M}(g)(x) = \sup_{x \in Q} \fint_Q |g| dx$$

with the supremum taken over all cubes  $Q \subset \mathbb{R}^n$  containing  $x$ . The Young function  $\phi$  is in  $\nabla_2$  if there exist a  $a > 1$ , such that

$$\phi(x) \leq \frac{1}{2a} \phi(ax), \quad \forall x \geq 0.$$

**Lemma 2.6.** If  $\phi \in \Delta_2$  be a Young function satisfying  $K_\phi < 2^{\frac{n}{s}}$ , then  $\phi_{\frac{n}{s}} \in \Delta_2 \cap \nabla_2$ .

*Proof.* We know

$$\begin{aligned} H(2t) &= \left( \int_0^{2t} \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau \right)^{\frac{n-s}{n}} \\ &\geq \left( \int_0^t \left( \frac{2\tau}{K_\phi \phi(\tau)} \right)^{\frac{s}{n-s}} 2d\tau \right)^{\frac{n-s}{n}} = \frac{2}{K_\phi^{\frac{s}{n}}} H(t). \end{aligned}$$

Letting  $2y = H(2t)$ , we have  $K_\phi^{\frac{s}{n}} y \geq H\left(\frac{H^{-1}(2y)}{2}\right)$ . Therefore,

$$H^{-1}(2y) \leq 2H^{-1}(K_\phi^{\frac{s}{n}} y) \leq 2^2 H^{-1}(K_\phi^{\frac{s}{n}} \frac{K_\phi^{\frac{s}{n}}}{2} y) \leq \dots \leq 2^{m+1} H^{-1}(K_\phi^{\frac{s}{n}} \left(\frac{K_\phi^{\frac{s}{n}}}{2}\right)^m y).$$

Because of the range of  $K$ , we get  $\frac{K_\phi^{\frac{s}{n}}}{2} < 1$ . Let  $m$  so big that  $K_\phi^{\frac{s}{n}} \left(\frac{K_\phi^{\frac{s}{n}}}{2}\right)^m < 1$ . Then we have  $H^{-1}(2y) < CH^{-1}(y)$ . So  $H^{-1} \in \Delta_2$  and  $\phi_{\frac{n}{s}} = \phi \circ H^{-1} \in \Delta_2$ .

By the decreasing property of  $\frac{\tau}{\phi(\tau)}$ ,

$$\begin{aligned} H(2^{\frac{n}{s}}x) &= \left( \int_0^{2^{\frac{n}{s}}x} \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} d\tau \right)^{\frac{n-s}{n}} \\ &= \left( \int_0^x \left( \frac{2^{\frac{n}{s}}\tau}{\phi(2^{\frac{n}{s}}\tau)} \right)^{\frac{s}{n-s}} 2^{\frac{n}{s}} d\tau \right)^{\frac{n-s}{n}} \\ &\leq \left( \int_0^x \left( \frac{\tau}{\phi(\tau)} \right)^{\frac{s}{n-s}} 2^{\frac{n}{s}} d\tau \right)^{\frac{n-s}{n}} \\ &= 2^{\frac{n-s}{s}} H(x). \end{aligned}$$

So  $2^{\frac{n}{s}}x \leq H^{-1}(2^{\frac{n-s}{s}}H(x))$ , then  $2^{\frac{n}{s}}H^{-1}(x) \leq H^{-1}(2^{\frac{n-s}{s}}x)$ .

And we have

$$2^{\frac{n}{s}}\phi \circ H^{-1}(x) \leq \phi(2^{\frac{n}{s}}H^{-1}(x)) \leq \phi \circ H^{-1}(2^{\frac{n-s}{s}}x).$$

Letting  $a = 2^{\frac{n-s}{s}} > 1$ , we have  $\phi_{\frac{n}{s}}(x) \leq \frac{1}{2a}\phi_{\frac{n}{s}}(ax)$  and  $\phi_{\frac{n}{s}} \in \nabla_2$ .  $\square$

**Remark 2.1.** If  $K_\phi \geq 2^{\frac{n}{s}}$ , there exists  $\phi \in \Delta_2$  such that  $\phi_{\frac{n}{s}} \notin \Delta_2$ . [4] Example 6.4: Let  $\phi$  with

$$\phi(t) = \begin{cases} t^{\frac{n}{s}} (\log \frac{1}{t})^{\alpha_0} & \text{near zero,} \\ t^{\frac{n}{s}} (\log t)^\alpha & \text{near infinity,} \end{cases}$$

where  $\alpha_0 > \frac{n}{s} - 1$ ,  $\alpha \leq \frac{n}{s} - 1$ . And connected by a convex function, then

$$\phi_{\frac{n}{s}}(t) \text{ is equivalent to } \begin{cases} e^{-t^{\frac{n}{s(\alpha_0+1)-n}}} & \text{near zero,} \\ e^{t^{\frac{n}{n-s(\alpha+1)}}} & \text{near infinity, } \alpha < \frac{n}{s} - 1, \\ e^{e^t^{\frac{n}{n-s}}} & \text{near infinity, } \alpha = \frac{n}{s} - 1, \end{cases}$$

so  $\phi_{\frac{n}{s}} \notin \Delta_2$ .

We then propose a few lemmas that might be utilized to support the assertion of Theorem 1.2(i).

**Lemma 2.7** ([16]). *Let  $\psi \in \Delta_2 \cap \nabla_2$  be a Young function. For any  $0 < q < \infty$ , there exists a constant  $C > 1$  depending on  $n, q, K_\psi$  and  $a$  such that for all sequences  $\{f_j\}_{j \in \mathbb{N}}$ , we have*

$$\int_{\mathbb{R}^n} \psi \left( \left[ \sum_{j \in \mathbb{N}} (\mathcal{M}(f_j))^2 \right]^{\frac{1}{q}} \right) dx \leq C(n, K_\psi, a) \int_{\mathbb{R}^n} \psi \left( \left[ \sum_{j \in \mathbb{N}} (f_j)^2 \right]^{\frac{1}{q}} \right) dx.$$

**Lemma 2.8.** *For any constant  $k \geq 1$ , sequence  $\{a_j\}_{j \in \mathbb{N}}$ , and cubes  $\{Q_j\}_{j \in \mathbb{N}}$  with  $\sum_j \chi_{Q_j} \leq k$ , we have*

$$\sum_j |a_j| \chi_{kQ_j} \leq C(k, n) \sum_j [\mathcal{M}(|a_j|^{\frac{1}{2}} \chi_{Q_j})]^2.$$

*Proof.* By the definition of  $\mathcal{M}$ , we know

$$\chi_{kQ_j} \leq k^n \mathcal{M}(\chi_{Q_j}).$$

So

$$\sum_j |a_j| \chi_{kQ_j} = \sum_j (|a_j|^{\frac{1}{2}} \chi_{kQ_j})^2 \leq k^{2n} \sum_j [\mathcal{M}(|a_j|^{\frac{1}{2}} \chi_{Q_j})]^2.$$

□

Now let us begin to give the proof of Theorem 1.2(i).

*Proof of Theorem 1.2(i).* Let  $\Omega$  be a c-John domain. By Boman [6] and Buckley [9],  $\Omega$  enjoys the following chain property: for every integer  $\kappa > 1$ , there exist a positive constant  $C(\kappa, \Omega)$  and a collection  $\mathcal{F}$  of the cubes such that

(i)  $Q \subset \kappa Q \subset \Omega$  for all  $Q \in \mathcal{F}$ ,  $\Omega = \cup_{Q \in \mathcal{F}} Q$  and

$$\sum_{Q \in \mathcal{F}} \chi_{\kappa Q} \leq C_{\kappa, c} \chi_{\Omega}.$$

(ii)  $Q_0 \in \mathcal{F}$  is a fixed cube. For any other  $Q \in \mathcal{F}$ , there exist a subsequence  $\{Q_j\}_{j=1}^N \subset \mathcal{F}$ , satisfying that  $Q = Q_N \subset C_{\kappa, c} Q_j$ ,  $C_{\kappa, c}^{-1} |Q_{j+1}| \leq |Q_j| \leq C_{\kappa, c} |Q_{j+1}|$  and  $|Q_j \cap Q_{j+1}| \geq C_{\kappa, c}^{-1} \min\{|Q_j|, |Q_{j+1}|\}$  for all  $j = 0, \dots, N-1$ .

Let  $\kappa = 5n$ , by (i)  $Q \subset 5nQ \subset \Omega$  for each  $Q \in \mathcal{F}$ ,

$$d(Q, \partial\Omega) \geq d(Q, \partial(5nQ)) \geq \frac{5n-1}{2} l(Q) \geq 2nl(Q),$$

and hence

$$|x - y| \leq \sqrt{nl(Q)} \leq nl(Q) \leq \frac{1}{2} d(Q, \partial\Omega) \leq \frac{1}{2} d(x, \partial\Omega), \quad \forall x, y \in Q \in \mathcal{F}.$$

Let  $u \in \dot{V}_*^{s, \phi}(\Omega)$ . Up to approximating by  $\min\{\max\{u, -N\}, N\}$ , we can assume that  $u \in L^\infty(\Omega)$ , and by the boundedness of  $\Omega$ ,  $u \in L^1(\Omega)$ .

By

$$\forall x, y \in Q, |x - y| \leq \frac{1}{2} d(x, \partial\Omega),$$

we know

$$\int_Q \int_Q \phi \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{dx dy}{|x - y|^n} \leq \int_Q \int_{B(x, \frac{1}{2}d(x, \partial\Omega))} \phi \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{dy dx}{|x - y|^n},$$

then  $\|u\|_{\dot{V}^{s,\phi}(Q)} \leq \|u\|_{\dot{V}_*^{s,\phi}(\Omega)}$ , so

$$(2.7) \quad \lambda \geq \|u\|_{\dot{V}^{s,\phi}(Q)} \text{ when } \lambda \geq \|u\|_{\dot{V}_*^{s,\phi}(\Omega)}.$$

Because of the convexity of  $\phi_{\frac{n}{s}}$ , we have

$$\begin{aligned} I &:= \int_{\Omega} \phi_{\frac{n}{s}} \left( \frac{|u(z) - u_{\Omega}|}{\lambda} \right) dz \\ &\leq \int_{\Omega} \phi_{\frac{n}{s}} \left( \frac{1}{2} \left( \frac{2|u(z) - u_{Q_0}| + 2|u_{\Omega} - u_{Q_0}|}{\lambda} \right) \right) dz \\ &\leq \frac{1}{2} \left[ \int_{\Omega} \phi_{\frac{n}{s}} \left( \frac{2|u(z) - u_{Q_0}|}{\lambda} \right) dz + |\Omega| \phi_{\frac{n}{s}} \left( \frac{2|u_{\Omega} - u_{Q_0}|}{\lambda} \right) \right]. \end{aligned}$$

By Jensen inequality,

$$|\Omega| \phi_{\frac{n}{s}} \left( \frac{2|u_{\Omega} - u_{Q_0}|}{\lambda} \right) \leq \int_{\Omega} \phi_{\frac{n}{s}} \left( \frac{2|u(z) - u_{Q_0}|}{\lambda} \right) dz.$$

In (i) we have  $\chi_{\Omega} \leq \sum_{Q \in \mathcal{F}} \chi_Q$ , so

$$\begin{aligned} I &\leq \int_{\Omega} \phi_{\frac{n}{s}} \left( \frac{2|u(z) - u_{Q_0}|}{\lambda} \right) dz \\ &\leq \sum_{Q \in \mathcal{F}} \int_Q \phi_{\frac{n}{s}} \left( \frac{2|u(z) - u_{Q_0}|}{\lambda} \right) dz \\ &\leq \frac{1}{2} \sum_{Q \in \mathcal{F}} \int_Q \phi_{\frac{n}{s}} \left( \frac{4|u(z) - u_Q|}{\lambda} \right) dz + \frac{1}{2} \sum_{Q \in \mathcal{F} \setminus \{Q_0\}} |Q| \phi_{\frac{n}{s}} \left( \frac{4|u_Q - u_{Q_0}|}{\lambda} \right) \\ &:= \frac{1}{2} I_1 + \frac{1}{2} I_2. \end{aligned}$$

By the inequality (2.3), (2.7) and

$$\forall x, y \in Q, |x - y| \leq \frac{1}{2} d(x, \partial\Omega),$$

we know

$$\begin{aligned} I_1 &\leq \sum_{Q \in \mathcal{F}} \int_Q \int_Q \phi \left( \frac{|u(x) - u(y)|}{\frac{\lambda}{4C_1} |x - y|^s} \right) \frac{dxdy}{|x - y|^n} \\ &\leq \sum_{Q \in \mathcal{F}} \int_Q \int_{B(x, \frac{1}{2}d(x, \partial\Omega))} \phi \left( \frac{|u(x) - u(y)|}{\frac{\lambda}{4C_1} |x - y|^s} \right) \frac{dydx}{|x - y|^n}. \end{aligned}$$

Using the  $\sum_{Q \in \mathcal{F}} \chi_Q \leq C_{\kappa, c} \chi_{\Omega}$  in (i) above,

$$\begin{aligned} I_1 &\leq C_{\kappa, c} \int_{\Omega} \int_{B(x, \frac{1}{2}d(x, \partial\Omega))} \phi \left( \frac{|u(x) - u(y)|}{\frac{\lambda}{4C_1} |x - y|^s} \right) \frac{dydx}{|x - y|^n} \\ &\leq \int_{\Omega} \int_{B(x, \frac{1}{2}d(x, \partial\Omega))} \phi \left( \frac{\tilde{C} |u(x) - u(y)|}{\lambda |x - y|^s} \right) \frac{dydx}{|x - y|^n}. \end{aligned}$$

For  $I_2$ , for each  $Q \in \mathcal{F}$ , by (ii)  $\forall Q \neq Q_0$ , we have  $Q = Q_N$ , and

$$\begin{aligned} |u_Q - u_{Q_0}| &\leq \sum_{j=0}^{N-1} |u_{Q_j} - u_{Q_{j+1}}| \\ &\leq \sum_{j=0}^{N-1} (|u_{Q_j} - u_{Q_{j+1} \cap Q_j}| + |u_{Q_{j+1}} - u_{Q_{j+1} \cap Q_j}|). \end{aligned}$$

For adjacent cubes  $Q_j, Q_{j+1}$ , one has

$$\begin{aligned} |Q_j - Q_{j+1}| &\geq C_{\kappa, c}^{-1} \min\{|Q_j|, |Q_{j+1}|\}, \\ C_{\kappa, c}^{-1} |Q_{j+1}| &\leq |Q_j| \leq C_{\kappa, c} |Q_{j+1}|. \end{aligned}$$

So

$$\begin{aligned} |u_{Q_j} - u_{Q_{j+1} \cap Q_j}| &\leq \frac{1}{|Q_{j+1} \cap Q_j|} \int_{Q_{j+1} \cap Q_j} |u(v) - u_{Q_j}| dv \\ &\leq \frac{C_{\kappa, c}^2}{|Q_j|} \int_{Q_j} |u(v) - u_{Q_j}| dv. \end{aligned}$$

Similarly,

$$|u_{Q_{j+1}} - u_{Q_{j+1} \cap Q_j}| \leq \frac{C_{\kappa, c}^2}{|Q_{j+1}|} \int_{Q_{j+1}} |u(v) - u_{Q_{j+1}}| dv.$$

As a result, we get

$$|u_Q - u_{Q_0}| \leq 2C_{\kappa, c}^2 \sum_{j=0}^N \int_{Q_j} |u(v) - u_{Q_j}| dv.$$

For each  $Q_j$ , by the convexity of  $\phi_{\frac{n}{s}}$  and Jenson inequality,

$$\begin{aligned} \int_{Q_j} \frac{|u(v) - u_{Q_j}|}{\lambda} dv &= \phi_{\frac{n}{s}}^{-1} \circ \phi_{\frac{n}{s}} \left( \int_{Q_j} \frac{|u(v) - u_{Q_j}|}{\lambda} dv \right) \\ &\leq \phi_{\frac{n}{s}}^{-1} \left( \int_{Q_j} \phi_{\frac{n}{s}} \left( \frac{|u(v) - u_{Q_j}|}{\lambda} \right) dv \right). \end{aligned}$$

By the inequality (2.3), (2.7) and  $\forall v, w \in Q_j, |v - w| \leq \frac{1}{2}d(v, \partial\Omega)$ ,

$$\begin{aligned} \int_{Q_j} \phi_{\frac{n}{s}} \left( \frac{|u(v) - u_{Q_j}|}{\lambda} \right) dv &\leq \int_{Q_j} \int_{Q_j} \phi \left( \frac{|u(v) - u(w)|}{\frac{\lambda}{C_1} |v - w|^s} \right) \frac{dv dw}{|v - w|^n} \\ &\leq \int_{Q_j} \int_{B(v, \frac{1}{2}d(v, \partial\Omega))} \phi \left( \frac{|u(v) - u(w)|}{\frac{\lambda}{C_1} |v - w|^s} \right) \frac{dw dv}{|v - w|^n} \\ &:= \int_{Q_j} f(v) dv. \end{aligned}$$

Hence

$$\int_{Q_j} \frac{|u(v) - u_{Q_j}|}{\lambda} dv \leq \phi_{\frac{n}{s}}^{-1} \left( \int_{Q_j} f(v) dv \right),$$

and

$$\frac{4|u_Q - u_{Q_0}|}{\lambda} \leq 8C_{\kappa,c}^2 \sum_{j=0}^N \phi_{\frac{n}{s}}^{-1} \left( \int_{Q_j} f(v) dv \right).$$

By Lemma 2.2,

$$\phi_{\frac{n}{s}} \left( 8C_{\kappa,c}^2 \sum_{j=0}^N \phi_{\frac{n}{s}}^{-1} \left( \int_{Q_j} f(v) dv \right) \right) \leq C \phi_{\frac{n}{s}} \left( \sum_{j=0}^N \phi_{\frac{n}{s}}^{-1} \left( \int_{Q_j} f(v) dv \right) \right).$$

Applying  $Q = Q_N \subset C_{\kappa,c} Q_j$  given in (ii),

$$|Q| \phi_{\frac{n}{s}} \left( \sum_{j=0}^N \phi_{\frac{n}{s}}^{-1} \left( \int_{Q_j} f(v) dv \right) \right) \leq \int_Q \phi_{\frac{n}{s}} \left( \sum_{P \in \mathcal{F}} \phi_{\frac{n}{s}}^{-1} \left( \int_P f(v) dv \right) \chi_{C_{\kappa,c} P} \right) (x) dx.$$

Using the  $\sum_{Q \in \mathcal{F}} \chi_Q \leq \sum_{Q \in \mathcal{F}} \chi_Q \leq C_{\kappa,c} \chi_{\Omega}$  in (i) above,

$$\begin{aligned} I_2 &\leq C \sum_{Q \in \mathcal{F}} \int_Q \phi_{\frac{n}{s}} \left( \sum_{P \in \mathcal{F}} \phi_{\frac{n}{s}}^{-1} \left( \int_P f(v) dv \right) \chi_{C_{\kappa,c} P} \right) (x) dx \\ &\leq \tilde{C} \int_{\Omega} \phi_{\frac{n}{s}} \left( \sum_{P \in \mathcal{F}} \phi_{\frac{n}{s}}^{-1} \left( \int_P f(v) dv \right) \chi_{C_{\kappa,c} P} \right) (x) dx. \end{aligned}$$

By Lemma 2.8,

$$I_2 \leq C \int_{\Omega} \phi_{\frac{n}{s}} \left( \sum_{P \in \mathcal{F}} \left\{ \mathcal{M} \left[ \left( \phi_{\frac{n}{s}}^{-1} \left( \int_P f(v) dv \right) \right)^{\frac{1}{2}} \chi_P \right] \right\}^2 \right) (x) dx.$$

By Lemma 2.6,  $\phi_{\frac{n}{s}} \in \Delta_2 \cap \nabla_2$ . Let  $\psi(t) := \phi_{\frac{n}{s}}(t^2)$ , then  $\psi \in \Delta_2 \cap \nabla_2$ . Applying Lemma 2.7 to  $q = 2$  and  $\psi$ , we obtain

$$I_2 \leq C \int_{\Omega} \phi_{\frac{n}{s}} \left( \sum_{P \in \mathcal{F}} \left( \phi_{\frac{n}{s}}^{-1} \left( \int_P f(v) dv \right) \chi_P \right) (x) dx.$$

Let  $a_P = \int_P f(v) dv$ . For each  $x \in \Omega$ , we have

$$\begin{aligned} \phi_{\frac{n}{s}} \left( \sum_{P \in \mathcal{F}} \left( \phi_{\frac{n}{s}}^{-1} (a_P) \right) \chi_P (x) \right) &= \phi_{\frac{n}{s}} \left( \frac{\sum_{P \in \mathcal{F}} \chi_P (x)}{\sum_{P \in \mathcal{F}} \chi_P (x)} \sum_{P \in \mathcal{F}} \left( \phi_{\frac{n}{s}}^{-1} (a_P) \right) \chi_P (x) \right) \\ &\leq \phi_{\frac{n}{s}} \left( \frac{C_{\kappa,c}}{\sum_{P \in \mathcal{F}} \chi_P (x)} \sum_{P \in \mathcal{F}} \left( \phi_{\frac{n}{s}}^{-1} (a_P) \right) \chi_P (x) \right) \\ &\leq \sum_{P \in \mathcal{F}} \frac{\chi_P (x)}{\sum_{P \in \mathcal{F}} \chi_P (x)} \phi_{\frac{n}{s}} (C_{\kappa,c} \phi_{\frac{n}{s}}^{-1} (a_P)) \\ &\leq \sum_{P \in \mathcal{F}} \chi_P (x) \tilde{C} a_P. \end{aligned}$$

So

$$I_2 \leq C \int_{\Omega} \sum_{P \in \mathcal{F}} a_P \chi_P (x) dx$$

$$\begin{aligned} &\leq C \sum_{P \in \mathcal{F}} a_P |P| = C \sum_{P \in \mathcal{F}} \int_P f(v) dv \\ &\leq C(n, C_{\kappa, c}, K_\phi) \int_{\Omega} \int_{B(v, \frac{1}{2}d(v, \partial\Omega))} \phi \left( \frac{C|u(v) - u(y)|}{\lambda|u - w|^s} \right) \frac{dwdv}{|u - w|^n}. \end{aligned}$$

In the end, we obtain

$$I \leq \int_{\Omega} \int_{B(v, \frac{1}{2}d(v, \partial\Omega))} \phi \left( \frac{C|u(v) - u(y)|}{\lambda|u - w|^s} \right) \frac{dwdv}{|u - w|^n},$$

where  $C = C(n, s, K_\phi)$ ,  $\lambda > 4C_1 \|u\|_{\dot{V}_*^{s,\phi}(\Omega)}$  and  $C \geq 4C_1$ . Let  $\lambda > C \|u\|_{\dot{V}_*^{s,\phi}(\Omega)}$ , we have  $I \leq 1$ .  $\square$

### 3. PROOF OF THEOREM 1.2(ii)

To prove Theorem 1.2 (ii), the most important method is getting the fact which Lemma 3.5 expressed. We first need to choose a special test function to estimate the relationship between its norms and its radius.

Let  $z \in \Omega$ ,  $d(z, \partial\Omega) \leq m < \text{diam } \Omega$ . Denote  $\Omega_{z,m}$  by a component of  $\Omega \setminus \overline{B_\Omega(z, m)}$ . For  $t > r \geq m$  with  $\Omega_{z,m} \neq \emptyset$ , define  $u_{z,r,t}$  in  $\Omega$  as

$$(3.1) \quad u_{z,r,t}(y) = \begin{cases} 0 & y \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z, r)] \\ \frac{|y-z|-r}{t-r} & y \in \Omega_{z,m} \cap [B(z, t) \setminus B(z, r)], \\ 1 & y \in \Omega_{z,m} \setminus B_\Omega(z, t), \end{cases}$$

where  $B_\Omega(z, t) = B(z, t) \cap \Omega$ .

It's not difficult to know the following property.

**Lemma 3.1.**  $u_{z,r,t}$  is Lipschitz with the Lipschitz constant  $\frac{1}{t-r}$ .

*Proof.* We spilt into three cases to prove it.

Case 1. For  $x \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z, r)]$ , it means that  $u_{z,r,t}(x) = 0$ . Since  $u_{z,r,t}(y) = u_{z,r,t}(x) = 0$  when  $y \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z, r)]$ , we only need to consider  $y \in \Omega_{z,m} \cap [B(z, t) \setminus B(z, r)]$  or  $y \in \Omega_{z,m} \setminus B_\Omega(z, t)$ . If  $y \in \Omega_{z,m} \cap [B(z, t) \setminus B(z, r)]$ , we know  $|x - z| \leq r$ . Hence

$$|u_{z,r,t}(x) - u_{z,r,t}(y)| = \frac{|y - z| - r}{t - r} \leq \frac{|y - z| - |x - z|}{t - r} \leq \frac{|x - y|}{t - r}.$$

If  $y \in \Omega_{z,m} \setminus B_\Omega(z, t)$ , we get  $|x - y| \geq t - r$ . Therefore,

$$|u_{z,r,t}(x) - u_{z,r,t}(y)| = 1 \leq \frac{|x - y|}{t - r}.$$

Case 2. For  $x \in \Omega_{z,m} \cap [B(z, t) \setminus B(z, r)]$ , then  $u_{z,r,t}(x) = \frac{|x-z|-r}{t-r}$ . If  $y \in \Omega_{z,m} \cap [B(z, t) \setminus B(z, r)]$  with  $u_{z,r,t}(y) = \frac{|y-z|-r}{t-r}$ ,

$$|u_{z,r,t}(x) - u_{z,r,t}(y)| = \left| \frac{|x-z|-r}{t-r} - \frac{|y-z|-r}{t-r} \right| \leq \frac{|x-z| - |y-z|}{t-r} \leq \frac{|x-y|}{t-r}.$$

If  $y \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z, r)]$  with  $u_{z,r,t}(y) = 0$ , we have  $|y - z| \leq r$ . Then

$$|u_{z,r,t}(x) - u_{z,r,t}(y)| = \frac{|x-z|-r}{t-r} \leq \frac{|y-z|-|x-z|}{t-r} \leq \frac{|x-y|}{t-r}.$$

If  $y \in \Omega_{z,m} \setminus B_\Omega(z, t)$  with  $u_{z,r,t}(y) = 1$ , then  $|y - z| \geq t$ . Together with  $|x - z| \leq t$ , we have

$$|u_{z,r,t}(x) - u_{z,r,t}(y)| = \left| \frac{|x-z|-r}{t-r} - 1 \right| = \left| \frac{|x-z|-t}{t-r} \right| = \frac{t - |x-z|}{t-r}$$

$$\leq \frac{|y - z| - |x - z|}{t - r} \leq \frac{|x - y|}{t - r}.$$

Case 3. For  $x \in \Omega_{z,m} \setminus B_\Omega(z, t)$ , then  $u_{z,r,t}(x) = 1$ . Since  $u_{z,r,t}(y) = u_{z,r,t}(x) = 1$  when  $y \in \Omega_{z,m} \setminus B_\Omega(z, t)$ , we only need to consider  $y \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z, r)]$  or  $y \in \Omega_{z,m} \cap [B(z, t) \setminus B(z, r)]$ . If  $y \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z, r)]$  with  $u_{z,r,t}(y) = 0$ , together with  $|x - y| \geq t - r$ , we know

$$|u_{z,r,t}(x) - u_{z,r,t}(y)| = 1 \leq \frac{|x - y|}{t - r}.$$

If  $y \in \Omega_{z,m} \cap [B(z, t) \setminus B(z, r)]$  with  $u_{z,r,t}(y) = \frac{|y-z|-r}{t-r}$ , then  $|y - z| \leq t$ . Moreover,  $|x - z| \geq t$ . Hence

$$\begin{aligned} |u_{z,r,t}(x) - u_{z,r,t}(y)| &= \left| 1 - \frac{|y - z| - r}{t - r} \right| \\ &\leq \frac{|x - z| - |y - z|}{t - r} \leq \frac{|x - y|}{t - r}. \end{aligned}$$

Combining above cases, we know  $u_{z,r,t}$  is Lipschitz with the Lipschitz constant  $\frac{1}{t-r}$ .  $\square$

Next we provide an estimation of the test function.

**Lemma 3.2.** *Let  $s \in (0, 1)$  and  $\phi$  be a Young function. For any bounded domain  $\Omega \subset \mathbb{R}^n$  and  $z \in \Omega$  with  $d(z, \partial\Omega) \leq m < \text{diam } \Omega$ . For  $t > r \geq m$ , we have  $u_{z,r,t} \in \dot{V}_*^{s,\phi}(\Omega)$  with*

$$\|u_{z,r,t}\|_{\dot{V}_*^{s,\phi}(\Omega)} \leq C \left( \phi^{-1} \left( \frac{1}{|\Omega_{z,m} \setminus B(z, r)|} \right) \right)^{-1} \frac{1}{(t-r)^s},$$

where  $C = C(n, s, C_\phi) \geq 1$ .

*Proof.* For any  $x \in \Omega$  and  $y \in B(x, \frac{1}{2}d(x, \partial\Omega)) \subset \Omega$ ,  $|u_{z,r,t}(x) - u_{z,r,t}(y)| \neq 0$  means that either  $x$  or  $y$  in  $\Omega_{z,m} \setminus B(z, r)$ .

$$\begin{aligned} H &:= \int_{\Omega} \int_{|x-y|<\frac{1}{2}d(x,\partial\Omega)} \phi \left( \frac{|u_{z,r,t}(x) - u_{z,r,t}(y)|}{\lambda|x-y|^s} \right) \frac{dydx}{|x-y|^n} \\ &\leq 2 \int_{\Omega_{z,m} \setminus B(z, r)} \int_{\Omega} \phi \left( \frac{|u_{z,r,t}(x) - u_{z,r,t}(y)|}{\lambda|x-y|^s} \right) \frac{dydx}{|x-y|^n} \\ &\leq 2 \int_{\Omega_{z,m} \setminus B(z, r)} \int_{B(x, t-r)} \phi \left( \frac{|x-y|^{1-s}}{\lambda(t-r)} \right) \frac{dydx}{|x-y|^n} \\ &\quad + 2 \int_{\Omega_{z,m} \setminus B(z, r)} \int_{\mathbb{R}^n \setminus B(x, t-r)} \phi \left( \frac{1}{\lambda|x-y|^s} \right) \frac{dydx}{|x-y|^n} \\ &:= 2H_1 + 2H_2. \end{aligned}$$

Using change of variable and (2.1), we have

$$\begin{aligned} H_1 &= \int_{\Omega_{z,m} \setminus B(z, r)} \int_0^{t-r} n\omega_n \phi \left( \frac{\rho^{1-s}}{\lambda(t-r)} \right) \frac{d\rho}{\rho} dx \\ &= \int_{\Omega_{z,m} \setminus B(z, r)} \int_0^{\frac{1}{\lambda(t-r)^s}} n\omega_n \frac{1}{1-s} \phi(\mu) \frac{d\mu}{\mu} dx \\ &\leq \int_{\Omega_{z,m} \setminus B(z, r)} \frac{C_\phi n\omega_n}{1-s} \phi \left( \frac{1}{\lambda(t-r)^s} \right) dx \end{aligned}$$

$$= \frac{C_\phi n \omega_n}{1-s} \phi\left(\frac{1}{\lambda(t-r)^s}\right) |\Omega_{z,m} \setminus B(z, r)|,$$

and

$$\begin{aligned} H_2 &= \int_{\Omega_{z,m} \setminus B(z, r)} \int_{t-r}^{\infty} n \omega_n \phi\left(\frac{1}{\lambda \rho^s}\right) \frac{d\rho}{\rho} dx \\ &= \int_{\Omega_{z,m} \setminus B(z, r)} \int_0^{\frac{1}{\lambda(t-r)^s}} n \omega_n \frac{1}{s} \phi(\mu) \frac{d\mu}{\mu} dx \\ &\leq \int_{\Omega_{z,m} \setminus B(z, r)} \frac{C_\phi n \omega_n}{s} \phi\left(\frac{1}{\lambda(t-r)^s}\right) dx \\ &= \frac{C_\phi n \omega_n}{s} \phi\left(\frac{1}{\lambda(t-r)^s}\right) |\Omega_{z,m} \setminus B(z, r)|. \end{aligned}$$

Let  $\lambda = M \left( \phi^{-1} \left( \frac{1}{|\Omega_{z,m} \setminus B(z, r)|} \right) \right)^{-1} \frac{1}{(t-r)^s}$ , where  $M \geq \max \left\{ \frac{4C_\phi n \omega_n}{1-s}, \frac{4C_\phi n \omega_n}{s}, 1 \right\}$ . We have  $H_1 \leq \frac{1}{4}, H_2 \leq \frac{1}{4}$ , hence  $H \leq 1$ . As a result

$$\|u_{z,r,t}\|_{\dot{V}_*^{s,\phi}(\Omega)} \leq C \left( \phi^{-1} \left( \frac{1}{|\Omega_{z,m} \setminus B(z, r)|} \right) \right)^{-1} \frac{1}{(t-r)^s}.$$

□

For  $x_0, z \in \Omega$ , let  $r > 0$  such that  $d(z, \partial\Omega) < r < |x_0 - z|$ . Define

$$\omega_{x_0,z,r}(y) := \frac{1}{r} \inf_{\gamma(x_0,y)} l(\gamma \cap B(z, r)), \quad \forall y \in \Omega,$$

where the infimum is taken over all reactable curves  $\gamma$  joining  $x_0$  and  $y$ .

**Lemma 3.3.** *s ∈ (0, 1) and φ be a Young function. For any bounded domain Ω ⊂ ℝ^n and x₀, z ∈ Ω and r > 0 with d(z, ∂Ω) ≤ r < |x₀ - z|, we have ω\_{x₀,z,r} ∈ ḽ\_\*^{s,φ}(Ω) with*

$$\|\omega_{x_0,z,r}\|_{\dot{V}_*^{s,\phi}(\Omega)} \leq C \left( \phi^{-1} \left( \frac{1}{r^n} \right) \right)^{-1} \frac{1}{r^s},$$

where  $C = C(n, s, C_\phi) \geq 1$ .

*Proof.* For  $x \in \Omega \setminus B(z, 6r), y \in B(x, \frac{1}{2}d(x, \partial\Omega))$ , we have

$$d(x, \partial\Omega) \leq |x - z| + d(z, \partial\Omega) \leq |x - z| + r,$$

and

$$\begin{aligned} |y - z| &\geq |x - z| - |y - x| \\ &\geq |x - z| - \frac{1}{2}(|x - z| + r) \\ &= \frac{1}{2}|x - z| - \frac{r}{2} \\ &\geq 3r - \frac{r}{2} \geq 2r. \end{aligned}$$

So  $B(x, \partial\Omega) \cap B(z, 2r) = \emptyset$ . Let  $\gamma_{x,y}$  be the segment joining  $x, y$  containing in  $B(x, \frac{1}{2}d(x, \partial\Omega))$ , then  $\gamma_{x,y} \subset \Omega \setminus B(z, r)$ . For any  $\gamma(x_0, x), \gamma(x_0, x) \cup \gamma_{x,y}$  is a curve joining  $x_0$  and  $y$ , with

$$l((\gamma(x_0, x) \cup \gamma_{x,y}) \cap B(z, r)) = l(\gamma(x_0, x) \cap B(z, r)).$$

Hence  $\omega_{x_0,z,r}(y) \leq \omega_{x_0,z,r}(x)$ .

Similarity  $\omega_{x_0,z,r}(x) \leq \omega_{x_0,z,r}(y)$ . So for any  $x \in \Omega \setminus B(z, 6r)$ ,  $y \in B(x, \frac{1}{2}d(x, \partial\Omega))$ , we have  $\omega_{x_0,z,r}(x) = \omega_{x_0,z,r}(y)$ .

For any  $x \in \Omega$ ,  $|x - y| < \frac{1}{2}d(x, \partial\Omega)$ , we know  $l(\gamma_{x,y} \cap B(z, r)) \leq |x - y|$ . Since  $\gamma(x_0, x) \cup \gamma_{x,y}$  is a curve joining  $x_0$  and  $y$ , we get

$$\omega_{x_0,z,r}(y) \leq \omega_{x_0,z,r}(x) + \frac{1}{r}|x - y|.$$

Similarity  $\omega_{x_0,z,r}(x) \leq \omega_{x_0,z,r}(y) + \frac{1}{r}|x - y|$ . So  $|\omega_{x_0,z,r}(y) - \omega_{x_0,z,r}(x)| \leq \frac{1}{r}|x - y|$ .

For  $x \in \Omega \cap B(z, 6r)$ , we have  $d(x, \partial\Omega) \leq 6r + d(z, \partial\Omega) < 8r$ .

$$\begin{aligned} H &:= \int_{\Omega} \int_{|x-y|<\frac{1}{2}d(x,\partial\Omega)} \phi\left(\frac{|\omega_{x_0,z,r}(x) - \omega_{x_0,z,r}(y)|}{\lambda|x-y|^s}\right) \frac{dydx}{|x-y|^n} \\ &= \int_{\Omega \cap B(z, 6r)} \int_{|x-y|<\frac{1}{2}d(x,\partial\Omega)} \phi\left(\frac{|\omega_{x_0,z,r}(x) - \omega_{x_0,z,r}(y)|}{\lambda|x-y|^s}\right) \frac{dydx}{|x-y|^n} \\ &\leq \int_{\Omega \cap B(z, 6r)} \int_0^{4r} n\omega_n \phi\left(\frac{\rho^{1-s}}{\lambda}\right) \frac{d\rho}{\rho} dx \\ &\leq \int_{\Omega \cap B(z, 6r)} \frac{C_\phi n\omega_n}{1-s} \phi\left(\frac{4^{1-s}}{\lambda r^s}\right) dx \\ &\leq \frac{C_\phi n\omega_n^2}{1-s} \phi\left(\frac{4^{1-s}}{\lambda r^s}\right) (6r)^n \end{aligned}$$

Let  $\lambda = M \left( \phi^{-1} \left( \frac{1}{r^n} \right) \right)^{-1} \frac{1}{r^s}$ , where  $M > \max \left\{ \frac{C_\phi n\omega_n^2 4^{1-s}}{1-s} 6^n, 4^{1-s} \right\}$ , then  $H \leq 1$ . So

$$\|\omega_{x_0,z,r}\|_{V_*^{s,\phi}(\Omega)} \leq C \left( \phi^{-1} \left( \frac{1}{r^n} \right) \right)^{-1} \frac{1}{r^s}.$$

□

**Lemma 3.4.** Let  $s \in (0, 1)$  and  $\phi \in \Delta_2$  be a Young function satisfying  $K_\phi < 2^{\frac{n}{s}}$  in (1.5), a bounded domain  $\Omega \subset \mathbb{R}^n$  supports the  $(\phi_{\frac{n}{s}}, \phi)$ -Poincaré inequality (1.4). Fix a point  $x_0$  so that  $r_0 := \max\{d(x, \partial\Omega) : x \in \Omega\} = d(x_0, \partial\Omega)$ . Assume that  $x, x_0 \in \Omega \setminus \overline{B(z, r)}$  for some  $z \in \Omega$  and  $r \in (0, 2 \operatorname{diam} \Omega)$ , there exists a positive constant  $b_0$  that  $x, x_0$  are contained in the same component of  $\Omega \setminus \overline{B(z, b_0 r)}$ .

*Proof.* Let  $b_{x,z,r} := \sup \{c \in (0, 1] : x, x_0 \text{ in the same component of } \Omega \setminus \overline{B(z, cr)}\}$ . We need prove that  $b_{x,z,r}$  has the positive low bound independent of  $x, z, r$ , that is

$$b = \inf \{b_{x,z,r} : \exists z \in \Omega, r \in (0, 2 \operatorname{diam} \Omega) \text{ such that } x, x_0 \in \Omega \setminus \overline{B(z, r)}\} > 0.$$

then let  $b_0 = \frac{b}{2}$ , we get the conclusion. Because it is a infimum problem, we may assume  $b_{x,z,r} \leq \frac{1}{10}$ .

We want to prove

$$\frac{r}{C} \left( \frac{1}{2} - 2b_{x,z,r} \right) \leq |\Omega_x|^{\frac{1}{n}} \leq 2Cb_{x,z,r}r, \quad C \geq 1.$$

then

$$b_{x,z,r} \geq \frac{1}{4(C^2 + 1)},$$

so  $b > 0$ .

First for fixed  $x, z, r$ , we have  $b_{x,z,r} > 0$ . By  $z \in \Omega$ , then there existing  $0 < \delta < 1$  such that  $B(z, \delta r) \subset \Omega$ , and  $x_0 \notin B(z, \delta r)$ . For  $h = \frac{\delta}{2}$ , and a curve  $\gamma(x, x_0)$  if

$$\gamma(x, x_0) \cap \overline{B(z, hr)} = \emptyset,$$

then  $x, x_0$  are contained in the same component of  $\Omega \setminus \overline{B(z, hr)}$ .

If  $\gamma(x, x_0) \cap \overline{B(z, hr)} \neq \emptyset$ , denote  $t_0 := \inf \{t \in [0, 1] : \gamma(x, x_0)(t) \in \partial B(z, \delta r)\}$ ,  $t_1 := \sup \{t \in [0, 1] : \gamma(x, x_0)(t) \in \partial B(z, \delta r)\}$  and  $A := \gamma(x, x_0)(t_0)$ ,  $B := \gamma(x, x_0)(t_1)$ . Then we have

$$\tilde{\gamma} = \gamma(x, x_0)|_{t \in (0, t_0)} \cup \widehat{AB} \cup \gamma(x, x_0)|_{t \in (t_1, 1)} \subset \Omega \setminus \overline{B(z, hr)}.$$

and  $x, x_0$  are contained in the same component of  $\Omega \setminus \overline{B(z, hr)}$ . So  $b_{x,z,r} \geq h > 0$ .

Set  $c_0 = 2b_{x,z,r} \leq \frac{1}{5}$ , then  $x_0 \notin \overline{B(z, c_0 r)}$ . Denote by  $\Omega_{x_0}$  the component of  $\Omega \setminus \overline{B(z, c_0 r)}$  containing  $x_0$ . By  $b_{x,z,r} < \frac{2}{3}c_0 < 1$ , we have  $x, x_0$  are not contained in the same component of  $\Omega \setminus \overline{B(z, \frac{2}{3}c_0 r)}$ . Now we prove that  $B(z, c_0 r) \cap \partial\Omega \neq \emptyset$ . If not, by  $z \in \Omega$ , we have

$$B(z, \frac{2}{3}c_0) \subset B(z, c_0 r) \subset \Omega.$$

From the above discussion, we get  $x, x_0$  are contained in the same component of  $\Omega \setminus \overline{B(z, \frac{2}{3}c_0 r)}$ , and we get contradiction. So  $B(z, c_0 r) \cap \partial\Omega \neq \emptyset$ . Then

$$\begin{aligned} r_0 = d(x_0, \partial\Omega) &\leq \max_{y \in B(z, c_0 r)} |x_0 - y| \leq r + c_0 r + d(x_0, B(z, r)) \leq \frac{6}{5}r + d(x_0, B(z, r)). \\ d(x_0, B(z, c_0 r)) &\geq |x_0 - z| - \frac{r}{5} = d(x_0, B(z, r)) + \frac{4}{5}r. \end{aligned}$$

So  $d(x_0, B(z, c_0 r)) \geq \frac{r_0}{2}$ , and

$$(3.2) \quad B(x_0, \frac{r_0}{2}) \subset \Omega_{x_0} \subset \Omega \setminus \Omega_x.$$

Define

$$\omega(y) := \frac{1}{c_0 r} \inf_{\gamma(x_0, y)} l(\gamma \cap B(z, c_0 r)), \quad \forall y \in \Omega.$$

Since  $B(z, c_0 r) \cap \partial\Omega \neq \emptyset$  and  $x_0 \notin \overline{B(z, c_0 r)}$ , we have  $d(z, \partial\Omega) < c_0 r < |x_0 - z|$ . By Lemma 3.3, we know

$$\|\omega\|_{\dot{V}_*^{s,\phi}(\Omega)} \leq C \left( \phi^{-1} \left( \frac{1}{(c_0 r)^n} \right) \right)^{-1} \frac{1}{(c_0 r)^s},$$

By the  $(\phi_{\frac{n}{s}}, \phi)$ -Poincaré inequality (1.4),

$$\|\omega - \omega_\Omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \leq C \|\omega\|_{\dot{V}_*^{s,\phi}(\Omega)} \leq C \left( \phi^{-1} \left( \frac{1}{(c_0 r)^n} \right) \right)^{-1} \frac{1}{(c_0 r)^s}.$$

On the other hand, by (3.2),  $y \in B(x_0, \frac{1}{2}r_0)$ ,  $\omega(y) = 0$ . Since  $\Omega$  is bounded,  $r_0 > 0$ , we have  $\frac{|\text{diam } \Omega|}{r_0^n} \leq C$ . Using the convexity of  $\phi_{\frac{n}{s}}$ ,

$$\int_{\Omega} \phi_{\frac{n}{s}} \left( \frac{|\omega(x)|}{\lambda} \right) dx \leq \frac{1}{2} \int_{\Omega} \phi_{\frac{n}{s}} \left( \frac{|\omega(x) - \omega_\Omega|}{\lambda} \right) dx + \frac{|\Omega|}{2} \phi_{\frac{n}{s}} \left( \frac{|\omega_{B(x_0, \frac{1}{2}r_0)} - \omega_\Omega|}{\lambda} \right).$$

By the Jensen inequality,

$$\begin{aligned} |\Omega| \phi_{\frac{n}{s}} \left( \frac{|\omega_{B(x_0, \frac{1}{2}r_0)} - \omega_\Omega|}{\lambda} \right) &\leq |\Omega| \int_{B(x_0, \frac{1}{2}r_0)} \phi_{\frac{n}{s}} \left( \frac{|\omega(x) - \omega_\Omega|}{\lambda} \right) dx \\ &\leq \frac{|\Omega|}{|B(x_0, \frac{1}{2}r_0)|} \int_{\Omega} \phi_{\frac{n}{s}} \left( \frac{|\omega(x) - \omega_\Omega|}{\lambda} \right) dx \\ &\leq 2^n C^n \int_{\Omega} \phi_{\frac{n}{s}} \left( \frac{|\omega(x) - \omega_\Omega|}{\lambda} \right) dx. \end{aligned}$$

As a result,

$$\int_{\Omega} \phi_{\frac{n}{s}} \left( \frac{|\omega(x)|}{\lambda} \right) dx \leq C \int_{\Omega} \phi_{\frac{n}{s}} \left( \frac{|\omega(x) - \omega_\Omega|}{\lambda} \right) dx,$$

and

$$(3.3) \quad \|\omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \leq C \|\omega - \omega_\Omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)}.$$

Since  $\forall y \in \Omega_x$ ,  $\omega(y) \geq 1$ , we have

$$\int_{\Omega} \phi_{\frac{n}{s}} \left( \frac{|\omega(x)|}{\lambda} \right) dx \geq \phi_{\frac{n}{s}} \left( \frac{1}{\lambda} \right) |\Omega_x|,$$

and

$$\|\omega\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \geq \left( \phi_{\frac{n}{s}}^{-1} \left( \frac{1}{|\Omega_x|} \right) \right)^{-1}.$$

So

$$C \phi^{-1} \left( \frac{1}{(c_0 r)^n} \right) (c_0 r)^s \leq \phi_{\frac{n}{s}}^{-1} \left( \frac{1}{|\Omega_x|} \right).$$

By (2.2),

$$\frac{H(A)}{A} \leq C \frac{1}{\phi(A)^{\frac{s}{n}}}.$$

Let

$$A = \phi^{-1} \left( \frac{1}{(c_0 r)^n} \right),$$

we have

$$\frac{\phi_{\frac{n}{s}}^{-1} \left( \frac{1}{(c_0 r)^n} \right)}{\phi^{-1} \left( \frac{1}{(c_0 r)^n} \right)} \leq C (c_0 r)^s.$$

So

$$\phi_{\frac{n}{s}}^{-1} \left( \frac{1}{(c_0 r)^n} \right) \leq C \phi_{\frac{n}{s}}^{-1} \left( \frac{1}{|\Omega_x|} \right).$$

By Lemma 2.6,  $\phi_{\frac{n}{s}} \in \Delta_2$ , and Lemma 2.2, we have

$$\frac{1}{(c_0 r)^n} \leq C \frac{1}{|\Omega_x|},$$

and

$$(3.4) \quad |\Omega_x|^{\frac{1}{n}} \leq C(c_0 r).$$

Let  $c_j > c_{j-i}$  for  $j \geq 1$  such that

$$|\Omega_x \setminus B(z, c_j r)| = \frac{1}{2} |\Omega_x \setminus B(z, c_{j-1} r)| = 2^{-j} |\Omega_x|.$$

For  $j \geq 0$  with  $\Omega_x \setminus \overline{B(z, c_j r)} \neq \emptyset$ , define  $v_j$  in  $\Omega$  as

$$v_j(y) = \begin{cases} 0 & y \in \Omega \setminus [\Omega_x \setminus B_\Omega(z, c_{j+1} r)] \\ \frac{|y-z|-c_j r}{c_{j+1} r - c_j r} & y \in \Omega_x \cap [B(z, c_j r) \setminus B(z, c_{j+1} r)], \\ 1 & y \in \Omega_x \setminus B_\Omega(z, c_j r), \end{cases}$$

Let  $\Omega_{z,x} = \Omega_x$ ,  $r = c_j r$  and  $t = c_{j+1} r$ , then  $v_j(y) = u_{z,c_j r, c_{j+1} r}(y)$  where  $u_{z,c_j r, c_{j+1} r}(y)$  is defined in (3.1). Applying Lemma 3.2, we have

$$\|v_j\|_{V_*^{s,\phi}(\Omega)} \leq C \left( \phi^{-1} \left( \frac{1}{|\Omega_x \setminus B(z, c_j r)|} \right) \right)^{-1} \frac{1}{(c_{j+1} r - c_j r)^s}.$$

Applying (3.2), we have  $v_j(y) = 0$  for  $y \in B(x_0, \frac{1}{2}r_0)$ . Similarly to (3.3), we have

$$(3.5) \quad \|v_j\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \leq C \|v_j - v_{j\Omega}\|_{L^{\phi_{\frac{n}{s}}}(\Omega)}.$$

And  $v_j(y) = 1$  for  $y \in \Omega_x \setminus B_\Omega(z, c_j r)$ , then we have

$$\|v_j\|_{L^{\phi_{\frac{n}{s}}}(\Omega)} \geq \left( \phi_{\frac{n}{s}}^{-1} \left( \frac{1}{|\Omega_x \setminus B_\Omega(z, c_j r)|} \right) \right)^{-1}.$$

By the  $(\phi_{\frac{n}{s}}, \phi)$ -Poincaré inequality (1.4), we have

$$\phi_{\frac{n}{s}}^{-1} \left( \frac{1}{|\Omega_x \setminus B_\Omega(z, c_j r)|} \right) \geq C \phi^{-1} \left( \frac{1}{|\Omega_x \setminus B(z, c_j r)|} \right) (c_{j+1} r - c_j r)^s.$$

By (2.2),

$$\frac{H(A)}{A} \leq C \frac{1}{\phi(A)^{\frac{s}{n}}}.$$

and let

$$A = \phi^{-1} \left( \frac{1}{|\Omega_x \setminus B(z, c_j r)|} \right),$$

then

$$(c_{j+1} r - c_j r)^s \leq C |\Omega_x \setminus B(z, c_j r)|^{\frac{s}{n}}.$$

So  $c_{j+1} - c_j \leq C |\Omega_x \setminus B(z, c_j r)|^{\frac{1}{n}} \leq C 2^{-\frac{j}{n}} |\Omega_x|^{\frac{1}{n}}$ .

Now we prove that  $\sup \{c_j\} > 1$ . If not, we have  $\forall c_j \leq 1$ . By  $x \in \Omega \setminus \overline{B(x, r)}$ , then  $\exists \delta > 0$  such that

$$B(x, \delta) \subset \Omega \setminus \overline{B(x, r)} \subset \Omega \setminus \overline{B(x, c_0 r)}.$$

By the connectivity of the  $B(x, \delta)$ , we have  $B(x, \delta) \subset \Omega_x$ . Then

$$B(x, \delta) \subset \Omega_x \setminus \overline{B(x, r)} \subset \Omega_x \setminus B(x, c_j r),$$

and

$$0 < |B(x, \delta)| \leq |\Omega_x \setminus \overline{B(x, r)}| \leq |\Omega_x \setminus B(x, c_j r)| = 2^{-j} |\Omega_x|.$$

Letting  $j \rightarrow \infty$ , we get a contradiction, and hence  $\sup \{c_j\} > 1$ . So there exists  $c_j$  such that  $c_j \geq \frac{1}{2}$ . Let  $j_0 = \inf \{j \geq 1 : c_j \leq \frac{1}{2}\}$ , then

$$\left(\frac{1}{2} - c_0\right)r \leq (c_{j_0} - c_0)r = \sum_{j=0}^{j_0-1} (c_{j+1} - c_j)r \leq C \sum_{j=0}^{j_0-1} 2^{-\frac{j}{n}} |\Omega_x|^{\frac{1}{n}} \leq 2C |\Omega_x|^{\frac{1}{n}}.$$

So  $\frac{r}{C} (\frac{1}{2} - 2b_{x,z,r}) \leq |\Omega_x|^{\frac{1}{n}}$ . By the (3.4), we have

$$\frac{r}{C} \left(\frac{1}{2} - 2b_{x,z,r}\right) \leq |\Omega_x|^{\frac{1}{n}} \leq C 2b_{x,z,r} r, \quad C \geq 1.$$

Then

$$b_{x,z,r} \geq \frac{1}{4(C^2 + 1)},$$

which implies  $b > 0$ .  $\square$

**Lemma 3.5.** *Let  $s \in (0, 1)$  and  $\phi \in \Delta_2$  be a Young function satisfying  $K_\phi < 2^{\frac{n}{s}}$  in (1.5). If a bounded domain  $\Omega \subset \mathbb{R}^n$  supports the  $(\phi_{\frac{n}{s}}, \phi)$ -Poincaré inequality (1.4), then the  $\Omega$  has the LLC(2) property, that is, there exists a constant  $\bar{b} \in (0, 1)$  such that for all  $z \in \mathbb{R}^n$  and  $r > 0$ , any pair of point in  $\Omega \setminus \overline{B(z, r)}$  can be joined in  $\Omega \setminus \overline{B(z, br)}$ .*

*Proof.* Fix  $x_0$  so that  $r_0 := \max(d(x, \partial\Omega) : x \in \Omega) = d(x_0, \partial\Omega)$  and  $b_0$  is the constant in Lemma 3.4. Then we spilt into three cases to prove it.

Case 1. For  $z \notin B\left(x_0, \frac{r_0}{8 \operatorname{diam} \Omega}\right)$ , we consider the radius  $r$ .

If  $r > \frac{16(\operatorname{diam} \Omega)^2}{r_0}$ , then  $\forall y \in B\left(z, \frac{r_0}{16 \operatorname{diam} \Omega}\right)$ , we have

$$|y - x_0| \geq |z - x_0| - |z - y| \geq \frac{r_0}{16 \operatorname{diam} \Omega} r > \operatorname{diam} \Omega.$$

By  $\Omega \subset B(x_0, \operatorname{diam} \Omega)$ , we get  $\Omega \cap \overline{B\left(z, \frac{r_0}{16 \operatorname{diam} \Omega}\right)} = \emptyset$ . Here, any pair of point in  $\Omega \setminus \overline{B(z, r)}$  can be joined in  $\Omega \setminus \overline{B(z, \frac{r_0}{16 \operatorname{diam} \Omega})} = \Omega$ .

If  $r \leq \frac{16(\operatorname{diam} \Omega)^2}{r_0}$  and  $d(z, \partial\Omega) > \frac{b_0 r_0}{32 \operatorname{diam} \Omega} r$ . When  $z \notin \Omega$ , then any pair of point in  $\Omega \setminus \overline{B(z, r)}$  can be joined in  $\Omega \setminus \overline{B\left(z, \frac{b_0 r_0}{32 \operatorname{diam} \Omega} r\right)} = \Omega$ . When  $z \in \Omega$ , then  $B\left(z, \frac{b_0 r_0}{64 \operatorname{diam} \Omega} r\right) \subset B\left(z, \frac{b_0 r_0}{32 \operatorname{diam} \Omega} r\right) \subset \Omega$ . Similar to the process of proving  $b_{x,z,r} > 0$  in Lemma 3.4, we know  $\Omega \setminus \overline{B\left(z, \frac{b_0 r_0}{64 \operatorname{diam} \Omega} r\right)}$  is a connected set. Here, any pair of point in  $\Omega \setminus \overline{B(z, r)}$  can be joined in  $\Omega \setminus \overline{B\left(z, \frac{b_0 r_0}{64 \operatorname{diam} \Omega} r\right)}$ .

If  $r \leq \frac{16(\operatorname{diam} \Omega)^2}{r_0}$  and  $d(z, \partial\Omega) \leq \frac{b_0 r_0}{32 \operatorname{diam} \Omega} r$ . Let  $y \in B\left(z, \frac{b_0 r_0}{16 \operatorname{diam} \Omega} r\right) \cap \Omega$ . By  $B\left(y, (1 - \frac{b_0}{2}) \frac{r_0}{8 \operatorname{diam} \Omega} r\right) \subset B\left(z, \frac{r_0}{8 \operatorname{diam} \Omega} r\right) \subset B(z, r)$ , we know

$$\forall x \in \Omega \setminus \overline{B(z, r)}, x, x_0 \in \Omega \setminus \overline{B\left(y, (1 - \frac{b_0}{2}) \frac{r_0}{8 \operatorname{diam} \Omega} r\right)}.$$

By Lemma 3.4,  $x, x_0$  are in the same component of  $\Omega \setminus \overline{B\left(y, b_0(1 - \frac{b_0}{2}) \frac{r_0}{8 \operatorname{diam} \Omega} r\right)}$ . By

$$\forall w \in B\left(z, \frac{b_0(1 - b_0)r_0}{16 \operatorname{diam} \Omega} r\right),$$

we have

$$|w - y| \leq |w - z| + |z - y| < \frac{b_0(1 - b_0)r_0}{16 \operatorname{diam} \Omega} r + \frac{b_0r_0}{16 \operatorname{diam} \Omega} r = b_0 \left(1 - \frac{b_0}{2}\right) \frac{r_0}{8 \operatorname{diam} \Omega} r.$$

Then

$$B\left(z, \frac{b_0(1 - b_0)r_0}{16 \operatorname{diam} \Omega} r\right) \subset B\left(y, b_0 \left(1 - \frac{b_0}{2}\right) \frac{r_0}{8 \operatorname{diam} \Omega} r\right),$$

and  $\Omega \setminus \overline{B\left(y, b_0 \left(1 - \frac{b_0}{2}\right) \frac{r_0}{8 \operatorname{diam} \Omega} r\right)} \subset \Omega \setminus \overline{B\left(z, \frac{b_0(1 - b_0)r_0}{16 \operatorname{diam} \Omega} r\right)}$ . Here, any pair of point in  $\Omega \setminus \overline{B(z, r)}$  can be joined in  $\Omega \setminus \overline{B\left(z, \frac{b_0(1 - b_0)r_0}{16 \operatorname{diam} \Omega} r\right)}$ .

Case 2. If  $z \in B\left(x_0, \frac{r_0}{8 \operatorname{diam} \Omega} r\right)$ , for any  $x \in \Omega \setminus \overline{B(z, r)}$ ,

$$r - \frac{r_0}{8 \operatorname{diam} \Omega} r \leq |x - z| - |x_0 - z| \leq |x - x_0| \leq \operatorname{diam} \Omega,$$

so

$$r \leq \frac{\operatorname{diam} \Omega}{1 - \frac{r_0}{8 \operatorname{diam} \Omega}} \leq 2 \operatorname{diam} \Omega.$$

Then

$$B\left(z, \frac{r_0}{8 \operatorname{diam} \Omega} r\right) \subset B\left(x_0, \frac{r_0}{4 \operatorname{diam} \Omega} r\right) \subset B\left(x_0, \frac{r_0}{2}\right) \subset B(x_0, r_0) \subset \Omega$$

Similar to the process of proving  $b_{x,z,r} > 0$  in Lemma 3.4, we have  $\Omega \setminus \overline{B\left(z, \frac{r_0}{8 \operatorname{diam} \Omega} r\right)}$  is a connected set. And by

$$\Omega \setminus \overline{B(z, r)} \subset \Omega \setminus \overline{B\left(z, \frac{r_0}{8 \operatorname{diam} \Omega} r\right)},$$

we know any pair of point in  $\Omega \setminus \overline{B(z, r)}$  can be joined in  $\Omega \setminus \overline{B\left(z, \frac{r_0}{8 \operatorname{diam} \Omega} r\right)}$ .

Combining above cases, we get the desired result with  $b = \min \left\{ \frac{r_0}{16 \operatorname{diam} \Omega}, \frac{b_0r_0}{64 \operatorname{diam} \Omega}, \frac{b_0(1 - b_0)r_0}{16 \operatorname{diam} \Omega} \right\}$ .  $\square$

*Proof of Theorem 1.2(ii).* Let  $\Omega \subset \mathbb{R}^n$  be a simply connected planar domain, or a bounded domain that is quasiconformally equivalent to some uniform domain when  $n \leq 3$ . Assume  $\Omega$  supports the  $(\phi_{\frac{n}{s}}, \phi)$ -Poincaré inequality.

By [7, 8],  $\Omega$  has a separation property with  $x_0 \in \Omega$  and some constant  $C_0 \geq 1$ , that is  $\forall x \in \Omega$ ,  $\exists$  a curve  $\gamma : [0, 1] \rightarrow \Omega$ , with  $\gamma(0) = x$ ,  $\gamma(1) = x_0$ , and  $\forall t \in [0, 1]$ , either  $\gamma([0, 1]) \subset \overline{B} := B(\gamma(t), C_0 d(\gamma(t), \Omega^C))$ , or  $\forall y \in \gamma([0, 1]) \setminus \overline{B}$  belongs to the different component of  $\Omega \setminus \overline{B}$ . For any  $x \in \Omega$ , let  $\gamma$  be a curve as above. By the arguments in [36], It suffices to prove there exists a constant  $C > 0$  so that

$$(3.6) \quad d(\gamma(t), \Omega^C) \geq C \operatorname{diam} \gamma([0, t]), \quad \forall t \in [0, 1].$$

Indeed, (3.6) could modify  $\gamma$  to get a John curve for  $x$ .

By Lemma 3.5,  $\Omega$  has the LLC(2) property. Let  $a = 2 + \frac{C_0}{b}$ , where  $b$  is the constant in Lemma 3.5.

For  $t \in [0, 1]$ . (1) If  $d(\gamma(t), \Omega^C) \geq \frac{d(x_0, \Omega^C)}{a}$ , then

$$\gamma([0, t]) \subset \Omega \subset B\left(\gamma(t), \frac{ad(\gamma(t), \Omega^C)}{d(x_0, \Omega^C)} \operatorname{diam} \Omega\right).$$

So

$$\operatorname{diam} \gamma([0, t]) \leq \frac{2ad(\gamma(t), \Omega^C)}{d(x_0, \Omega^C)} \operatorname{diam} \Omega.$$

and

$$d(\gamma(t), \Omega^C) \geq \frac{d(x_0, \Omega^C)}{2a \operatorname{diam} \Omega} \operatorname{diam} \gamma([0, t]).$$

(2) If  $d(\gamma(t), \Omega^C) < \frac{d(x_0, \Omega^C)}{a}$ , we prove that

$$\gamma([0, t]) \subset \overline{B(\gamma(t), (a-1)d(\gamma(t), \Omega^C))}.$$

Otherwise, there exists  $y \in \gamma([0, t]) \setminus \overline{B(\gamma(t), (a-1)d(\gamma(t), \Omega^C))}$ . By

$$|x_0 - y| \geq d(x_0, \Omega^C) - d(y, \Omega^C) > (a-1)d(\gamma(t), \Omega^C),$$

we know  $x_0, y \in \Omega \setminus \overline{B(\gamma(t), (a-1)d(\gamma(t), \Omega^C))}$ , by Lemma 3.5,  $x_0$  and  $y$  are contained in the same complement of  $\Omega \setminus \overline{B(\gamma(t), b(a-1)d(\gamma(t), \Omega^C))}$ . Since  $b(a-1) \geq C_0$ , then  $x_0$  and  $y$  are contained in the same complement of  $\Omega \setminus \overline{B(\gamma(t), C_0d(\gamma(t), \Omega^C))}$ , which is in contradiction with the separation property. Hence

$$\gamma([0, t]) \subset \overline{B(\gamma(t), (a-1)d(\gamma(t), \Omega^C))},$$

then

$$\operatorname{diam} \gamma([0, t]) \leq 2(a-1)d(\gamma(t), \Omega^C).$$

So

$$d(\gamma(t), \Omega^C) \geq \frac{1}{2(a-1)} \operatorname{diam} \gamma([0, t]).$$

Let  $C = \min \left\{ \frac{d(x_0, \Omega^C)}{2a \operatorname{diam} \Omega}, \frac{1}{2(a-1)} \right\}$ , then (3.6) holds. The proof is completed.  $\square$

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