Effective lower bounds for spectra of random covers and random unitary bundles

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Abstract

Let X be a finite-area non-compact hyperbolic surface. We study the spectrum of the Laplacian on random covering surfaces of X and on random unitary bundles over X. We show that there is a constant $c > 0$ such that, with probability tending to 1 as $n \to \infty$, a uniformly random degree-n Riemannian covering surface X_n of X has no Laplacian eigenvalues below $\frac{1}{4} - c\frac{(\log\log\log n)^2}{\log\log n}$ other than those of X and with the same multiplicities. We also show that with probability tending to 1 as $n \to \infty$, a random unitary bundle E_ϕ over X of rank n has no Laplacian eigenvalues below $\frac{1}{4} - c\frac{(\log\log n)^2}{\log n}$.

Contents

1 Introduction

Let X be a finite-area non-compact hyperbolic surface, i.e. a Riemannian surface with constant curvature -1. We study the Laplacian Δ_X on $L^2(X)$, whose spectrum $\operatorname{spec}(\Delta_X)$ is contained in $[0, \infty)$. The low-energy spectrum $\operatorname{spec}(\Delta_X) \cap [0, \frac{1}{4})$ consists of the trivial eigenvalue 0 (which is simple if and only if X is connected) and possibly finitely many non-zero eigenvalues of finite multiplicity. There is absolutely continuous spectrum in $[\frac{1}{4}, \infty)$ with possibly infinitely many eigenvalues embedded in the continuous spectrum. Of particular interest to us is the *spectral gap* $\inf(\operatorname{spec}(\Delta_X) \setminus \{0\})$.

We study the size of the spectral gap for random surfaces. Our model, studied first in [MN20, MNP22], is to fix a base surface X and consider uniformly random covers X_n of degree n. In this context, since the $\operatorname{spec}(\Delta_X) \subset \operatorname{spec}(\Delta_{X_n})$, we restrict our attention to *new* eigenvalues, those appearing in $\operatorname{spec}(\Delta_{X_n})$ which do not appear in $\operatorname{spec}(\Delta_X)$. Our covers will not need to be connected, but will be connected with high probability by a result of Dixon [Di69] (see Section 2). We say that a family of events (depending on n) happens asymptotically almost surely (a.a.s.) if they happen with probability tending to 1 as $n \to \infty$.

In this paper, we build upon the following theorem of Magee and the author, from [HM23].

Theorem 1.1 ([HM23, Theorem 1.1])**.** *Let X be a finite-area non-compact hyperbolic surface X, for any $\epsilon > 0$, a uniformly random degree n Riemannian cover X_n a.a.s. satisfies*

$$\operatorname{spec}(\Delta_{X_n}) \cap \left[0, \frac{1}{4} - \epsilon\right) = \operatorname{spec}(\Delta_X) \cap \left[0, \frac{1}{4} - \epsilon\right),$$

where the multiplicities are the same on either side.

The purpose of the current paper is to study which functions $\epsilon = \epsilon(n)$ one can take in Theorem 1.1. To this end, we show one can take $\epsilon \to 0$ at the rate $\frac{(\log\log\log n)^2}{\log\log n}$. We prove the following.

Theorem 1.2. *For any finite-area non-compact hyperbolic surface X, there exists a constant $c > 0$ such that a uniformly random degree n Riemannian cover X_n a.a.s. satisfies*

$$\operatorname{spec}(\Delta_{X_n}) \cap \left[0, \frac{1}{4} - c\frac{(\log\log\log n)^2}{\log\log n}\right) = \operatorname{spec}(\Delta_X) \cap \left[0, \frac{1}{4} - c\frac{(\log\log\log n)^2}{\log\log n}\right),$$

where the multiplicities are the same on either side.

As a consequence of Theorem 1.1, it was shown in [HM23, Section 8] that there exists a sequence of closed surfaces $\{X_i\}$ with genera $g_i \to \infty$ with first non-zero eigenvalue $\lambda_1(X_i) \to \frac{1}{4}$, resolving a conjecture of Buser [Bu84] (see also [LM22] for an alternative proof). Theorem 1.2 allows the convergence to $\frac{1}{4}$ to be made quantitative. Taking X to be, for example, the thrice punctured sphere which has $\lambda_1(X) > \frac{1}{4}$, Theorem 1.2 produces a family of covers X_n with Euler characteristic $-n$ and

$$\inf \operatorname{spec}(X_n) \geqslant \frac{1}{4} - c\frac{(\log\log\log n)^2}{\log\log n}.$$

Taking covers of even degree, as explained in [HM23, Section 8], one can then apply the compactification procedure of Buser, Burger and Dodzuik [BBD88] to get sequence of closed hyperbolic surfaces X_g with genus g and

$$\lambda_1(X_g) \geqslant \frac{1}{4} - c'\frac{(\log\log\log g)^2}{\log\log g},$$

giving a quantitative rate of converge to $\frac{1}{4}$ in the proof of Buser's conjecture [Bu84]. We refer the reader to [HM23, Section 1.1] for the history of this conjecture.

We are also interested in studying the analogous question for random rank n unitary bundles over X. The analogue of Theorem 1.1 in this context was proven by Zargar in [Za22], which we introduce now. Since X is a finite-area non-compact hyperbolic surface, X can be realized as $X = \Gamma/\mathbb{H}$ where Γ is a discrete torsion free subgroup of $\mathrm{PSL}_2(\mathbb{R})$, freely generated by $\gamma_1, \ldots, \gamma_d$

and any $\phi \in \mathrm{Hom}(\Gamma, \mathrm{U}(n))$ is determined by $\phi(\gamma_1), \ldots, \phi(\gamma_d)$. We equip $\mathrm{Hom}(\Gamma, \mathrm{U}(n))$ with a natural probability measure $\mathbb{P}_n$ by sampling each $\phi(\gamma_i) \in \mathrm{U}(n)$ independently with Haar probability. Let $\rho_\phi : \Gamma \to \mathrm{U}(n)$ be the random $\mathbb{C}^n$ representation obtained via $\mathrm{std}_n \circ \phi$ where std_n is the standard representation. We consider the associated (random) unitary bundle E_ϕ and the Laplacian Δ_ϕ on sections of E_ϕ. Then $\mathrm{spec}(\Delta_\phi) \cap [0, \frac{1}{4})$ consists of finitely many eigenvalues with finite multiplicity. The following was shown in [Za22].

Theorem 1.3 ([Za22, Theorem 1.2])**.** *For any finite-area non-compact hyperbolic surface X, for any $\kappa > 0$, a random unitary bundle E_ϕ over X of rank n has*

$$\inf \mathrm{spec} \Delta_\phi \geqslant \frac{1}{4} - \kappa,$$

a.a.s.

We note that [Za22, Theorem 1.2] also deals with flat bundles arising from other irreducible representations of $\mathrm{U}(n)$, subject to a condition on the signature. We prove the following quantitative version of Theorem 1.3.

Theorem 1.4. *For any finite-area non-compact hyperbolic surface X, there exists a constant $c > 0$ such that a random unitary bundle E_ϕ over X of rank n has*

$$\inf \mathrm{spec} \Delta_\phi \geqslant \frac{1}{4} - c\frac{(\log \log n)^2}{\log n},$$

a.a.s.

1.1 Other related works

Random regular graphs

Motivation for the results in this paper can be found in the setting of random regular graphs. A celebrated theorem of Friedman [Fr08], formerly Alon's conjecture, says that for any $\epsilon > 0$, a random d-regular graph on n vertices satisfies

$$\lambda_2, |\lambda_n| \leqslant 2\sqrt{d-1} + \epsilon, \tag{1.1}$$

with probability tending to 1 as $n \to \infty$. It was shown by Bordenave [Bo20] that one can take ϵ in the above to be $c\left(\frac{\log \log n}{\log n}\right)^2$. In an impressive work of Huang and Yau [HY21], it was shown that one can take $\epsilon = O(n^{-c})$ for some $c > 0$.

It was conjectured by Friedman [Fr03] that a version of Alon's conjecture should hold for random covers of a fixed graph. This was proved in a breakthrough work of Bordenave and Collins [BC19].

Random covers

The analogue of Theorem 1.1 for Schottky surfaces was proved by Magee and Naud in [MN21] following an intermediate result [MN20]. Random covers of compact surfaces were studied in [MNP22] by Magee, Naud and Puder. They show that for any $\epsilon > 0$, (a.a.s.) a uniformly random degree n cover has no new eigenvalues below $\frac{3}{16} - \epsilon$. Eigenvalue statistics for random covers have also been studied by Naud in [Na22].

Other models of random surfaces

There has been much interest in studying the geometry and spectral theory of random hyperbolic surfaces in various models. In [BM04] Brooks and Makover considered a combinatorial model of random surfaces, showing the existence of a non-explicit uniform spectral gap with high probability. Other works on the Brooks-Makover model include [Ga06, BCP21, SW22A].

Another model of random surfaces is the Weil-Petersson model, arising from sampling from moduli space with normalised Weil-Petersson volume. Lengths of pants decompositions for compact surfaces in this model were studied by Guth, Parlier and Young in [GPY11]. Mirzakhani [Mi13] was the first to study the spectrum of the Laplacian in this model proving that a random genus g compact surface has spectral gap of size at least $\frac{1}{4}\left(\frac{\log(2)}{2\pi+\log(2)}\right)^2 \approx 0.0024$ with probability tending to 1 as $g \to \infty$. This was improved to $\frac{3}{16}-\epsilon$ independently by Wu and Xue [WX21] and Lipnowksi and Wright [LW21], and subsequently to $\frac{2}{9}-\epsilon$ recently by Anantharaman and Monk [AM23].

For large volume non-compact surfaces, the number of cusps compared to the genus has an impact on the low-energy spectrum. It was shown by Zograf [Zo87] that there exists a constant $C>0$ such that any hyperbolic surface with genus g and n cusps has $\lambda_1 < C\frac{g-1}{n}$. If the number of cusps n grows slower than $\sqrt{g}$ a Weil-Petersson random surface has a uniform spectral gap with high probability [Hi22, SW22]. This fails to be true when the number of cusps grows faster than $\sqrt{g}$ [SW22], in this regime Weil-Petersson random surfaces have an arbitrarily small eigenvalue. At the other extreme, if the genus is fixed and the number of cusps n tends to infinity, the number of small eigenvalues is polynomial in n with high probability [HT22]. Other works on spectral theory of Weil-Petersson random surfaces include [GMST21, Mo21, Ru22].

Selberg's eigenvalue conjecture

Spectral gaps for certain arithmetic hyperbolic surfaces are of great interest in Number Theory, see e.g. [Sa03]. Let $N \geqslant 1$, the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N is

$$\Gamma(N) = \{T \in \mathrm{SL}_2(\mathbb{Z}) \mid T \equiv I \mod N\}.$$

Consider the quotient $X(N) \stackrel{\text{def}}{=} \Gamma(N)\backslash\mathbb{H}$. Selberg proved in [Se65] that

$$\lambda_1(X(N)) \geqslant \frac{3}{16},$$

for every $N \geqslant 1$ and made the conjecture that $\frac{3}{16}$ can be improved to $\frac{1}{4}$. After many intermediate results [GJ78, Iw89, LRS95, Sa95, Iw96, KS02], the best known lower bound is $\frac{975}{4096}$ due to Kim and Sarnak [Ki03]. In the context of the current paper, taking X to be a non-compact arithmetic surface with $\lambda_1(X) \geqslant \frac{1}{4}$, Theorem 1.2 shows that one can find a sequence of arithmetic (not necessarily congruence) surfaces $\{X_n\}_{n\in\mathbb{N}}$ with $\mathrm{Vol}(X_n) = n\mathrm{Vol}(X)$ and

$$\lambda_1(X_n) \geqslant \frac{1}{4} - c\frac{(\log\log\log n)^2}{\log\log n}.$$

1.2 A word on the proof

In [HM23], proof of Theorem 1.1 relies on the work of Bordenave and Collins in [BC19]. The results of [BC19] were recently extended in a quantitative manor in [BC23] which is a crucial

ingredient in the current paper. The method is similar for unitary bundles so we restrict the discussion here to covering surfaces.

Let $X = \Gamma\backslash\mathbb{H}$ be given. It is explained in Section 2 that one can parameterize degree n covering surfaces X_φ by $\varphi \in \mathrm{Hom}\left(\Gamma, S_n\right)$. In [HM23], problem of forbidding new eigenvalues of a cover X_φ is reduced to bounding, with high probability, the operator norm of a (random) operator of the form

$$\sum_{\gamma\in S} a_\gamma \otimes \rho_\varphi\left(\gamma\right), \tag{1.2}$$

where $a_\gamma \in M_m\left(\mathbb{C}\right)$, acting on $\mathbb{C}^m \otimes V_n^0$ where is the standard $n-1$ dimensional irreducible representation of S_n. Here m and $S \subset \Gamma$ are finite and fixed, depending on the ϵ one chooses in Theorem 1.1. The results of [BC19] can be used to this end.

In order to take $\epsilon(n) \to 0$ as $n \to \infty$ as in Theorem 1.2 one needs, amongst other things, to allow the size of the set S and the dimension m of the matrices a_γ to grow as a function of n, depending on ϵ. The work of Bordenave and Collins in [BC23], with an effective linearization trick (Lemma 3.5), is able to deal with precisely this situation. The proofs of Theorem 1.2 and Theorem 1.4 rely on effectivising the arguments in [HM23] in order to apply results from [BC23].

We briefly discuss the rates obtained in Theorem 1.2 and Theorem 1.4. Consider an operator P the form (1.2). Let $l = \sup_{\gamma\in S} \mathrm{wl}\left(\gamma\right)$ where $\mathrm{wl}\left(\gamma\right)$ is the word-length in Γ and $S \subset \Gamma$ be the index set. After applying a linearization trick, in order to apply the results of Bordenave and Collins [BC23] to bound the operator norm of P, we eventually require

$$l^2 \left|S\right|^{\lceil \log_2 l\rceil} l^{(\lceil \log_2 l\rceil - 1)} \ll \left(\log\left(n\right)\right)^{\frac{1}{4}}, \tag{1.3}$$

for permutations (c.f. Corollary 3.7) and

$$l^2 \left|S\right|^{\lceil \log_2 l\rceil} l^{(\lceil \log_2 l\rceil - 1)} \ll n^{\frac{1}{32d+160}}, \tag{1.4}$$

for unitaries (c.f. Corollary 3.6). It is shown in Section 5 that in order to forbid spectra below $\frac{1}{4} - \epsilon(n)$, one needs to consider an operator P of the form (1.2) where $\left|S\right|, l \ll \exp\left(\frac{2}{\epsilon(n)}\right)$. This, together with (1.3) and (1.4) governs the rates in Theorem 1.2 and Theorem 1.4.

1.3 Notation

In the proofs, there will be quite a few constants, some of which are important to track and some which are not. We choose to use the notation $C > 0$ throughout to denote some positive constant that only depends (possibly) on the fixed surface X, whose precise value is irrelevant. We warn that the value of C sometimes changes from line to line. Constants we do need to keep track of will be indexed by a subscript in order of their first appearance, e.g. c_1, or given as a numerical value.

For functions $f = f(n)$, $g = g(n)$ we use the Vinogradov notation $f \ll g$ to mean that there exists a constant $c > 0$ and an $N \in \mathbb{N}$ such that $f(n) \leqslant cg(n)$ for all $n \geqslant N$. We also write $f = o(1)$ to mean that $f(n) \to 0$ as $n \to \infty$.

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2 Set up

Let X be a fixed non-compact finite-area hyperbolic surface. To simplify notations, we assume X has only one cusp. This will not affect our arguments. We view X as

$$X = \Gamma \backslash \mathbb{H},$$

where Γ is a discrete, torsion free subgroup of $\mathrm{PSL}_2(\mathbb{R})$, freely generated by

$$\gamma_1, \ldots, \gamma_d \in \Gamma.$$

We let F be a Dirichlet domain for X. After possibly conjugating Γ in $\mathrm{PSL}_2(\mathbb{R})$, we can assume that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, generating the stability group of the cusp and that $F \subset \{x + iy \in \mathbb{H} \mid 0 \leqslant x \leqslant 1\}$. We then define

$$H(L) \stackrel{\text{def}}{=} \{x + iy \in F \mid y > L\}. \tag{2.1}$$

Random covers

For any $n \in \mathbb{N}$, let $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$ and S_n denote the group of permutations of $[n]$. Given any $\varphi \in \mathrm{Hom}(\Gamma, S_n)$ we define an action of Γ on $\mathbb{H} \times [n]$ by

$$\gamma(z, x) \stackrel{\text{def}}{=} (\gamma z, \varphi(\gamma)[x]).$$

We obtain a degree n covering space X_φ of X by

$$X_\varphi \stackrel{\text{def}}{=} \Gamma \backslash_\varphi (\mathbb{H} \times [n]). \tag{2.2}$$

Sampling φ uniformly randomly we obtain a uniformly random degree n cover. Note that X_φ need not be connected, indeed X_φ is connected if and only if Γ acts transitively on $[n]$ via φ. By a Theorem of Dixon [Di69], two uniformly random permutations in S_n generate A_n or S_n a.a.s. hence a uniformly random cover X_φ is connected a.a.s.

Let $V_n \stackrel{\text{def}}{=} \ell^2([n])$ and $V_n^0 \subset V_n$ the subspace of functions with zero mean. Then S_n acts on V_n via the standard representation std_n by 0-1 matrices, and V_n^0 is an $n-1$ dimensional irreducible component. Given a uniformly random $\varphi \in \mathrm{Hom}(\Gamma; S_n)$, we consider the random V_n^0 representation of Γ

$$\rho_\varphi : \Gamma \to \mathrm{End}\left(V_n^0\right),$$

given by

$$\rho_\varphi \stackrel{\text{def}}{=} \mathrm{std}_n \circ \varphi.$$

Random unitary bundles

Let $\mathrm{U}(n)$ denote the unitary group. Then a homomorphism $\phi : \Gamma \to \mathrm{U}(n)$ is determined uniquely by

$$\phi(\gamma_1), \ldots, \phi(\gamma_d) \in \mathrm{U}(n).$$

We therefore can equip $\mathrm{Hom}(\Gamma; \mathrm{U}(n))$ with a probability measure $\mathbb{P}_n$ by sampling the image of each generator independently with Haar probability. Given such a random $\phi \in \mathrm{Hom}(\Gamma; \mathrm{U}(n))$, we consider the random $\mathbb{C}^n$ representation of Γ

$$\rho_\phi : \Gamma \to \mathrm{U}(n),$$

given by

$$\rho_\phi \stackrel{\text{def}}{=} \text{std}_n \circ \phi,$$

where std_n is the standard representation of $\mathrm{U}(n)$. Consider the action of Γ on $\mathbb{H} \times \mathbb{C}^n$ by

$$\gamma(z, \mathbf{x}) \stackrel{\text{def}}{=} (\gamma z, \phi(\gamma)\mathbf{x}),$$

and let

$$E_\phi \stackrel{\text{def}}{=} \Gamma\backslash_\phi (\mathbb{H} \times \mathbb{C}^n)$$

denote the quotient by this action. Then sampling $\phi \in \text{Hom}(\Gamma; \mathrm{U}(n))$ with probability $\mathbb{P}_n$, we obtain a random rank-n unitary bundle over X.

2.1 Function spaces

Covers

For the convenience of the reader we recall the following function spaces from [HM23, Section 2.2]. We define $L^2_{\text{new}}(X_\varphi)$ to be the space of L^2 functions on X_φ orthogonal to all lifts of L^2 functions from X. Then

$$L^2(X_\varphi) \cong L^2_{\text{new}}(X) \oplus L^2(X).$$

Recall that we fixed F to be a Dirichlet fundamental domain for X. Let $C^\infty\left(\mathbb{H}; V_n^0\right)$ denote the smooth V_n^0-valued functions on $\mathbb{H}$. There is an isometric linear isomorphism between

$$C^\infty(X_\varphi) \cap L^2_{\text{new}}(X_\varphi),$$

and the space of smooth V_n^0-valued functions on $\mathbb{H}$ satisfying

$$f(\gamma z) = \rho_\varphi(\gamma) f(z),$$

for all $\gamma \in \Gamma$, with finite norm

$$\|f\|^2_{L^2(F)} \stackrel{\text{def}}{=} \int_F \|f(z)\|^2_{V_n^0} d\mu_{\mathbb{H}}(z) < \infty.$$

We denote the space of such functions by $C^\infty_\varphi\left(\mathbb{H}; V_n^0\right)$. The completion of $C^\infty_\varphi\left(\mathbb{H}; V_n^0\right)$ with respect to $\| \bullet \|_{L^2(F)}$ is denoted by $L^2_\varphi\left(\mathbb{H}; V_n^0\right)$; the isomorphism above extends to one between $L^2_{\text{new}}(X_\varphi)$ and $L^2_\varphi\left(\mathbb{H}; V_n^0\right)$.

Let $C^\infty_{c,\varphi}\left(\mathbb{H}; V_n^0\right)$ denote the subset of $C^\infty_\varphi\left(\mathbb{H}; V_n^0\right)$ consisting of functions which are compactly supported modulo Γ. We let $H^2_\varphi\left(\mathbb{H}; V_n^0\right)$ denote the completion of $C^\infty_{c,\varphi}\left(\mathbb{H}; V_n^0\right)$ with respect to the norm

$$\|f\|^2_{H^2_\varphi(\mathbb{H}; V_n^0)} \stackrel{\text{def}}{=} \|f\|^2_{L^2(F)} + \|\Delta f\|^2_{L^2(F)}.$$

We let $H^2(X_\varphi)$ denote the completion of $C^\infty_c(X_\varphi)$ with respect to the norm

$$\|f\|^2_{H^2(X_\varphi)} \stackrel{\text{def}}{=} \|f\|^2_{L^2(X_\varphi)} + \|\Delta f\|^2_{L^2(X_\varphi)}.$$

Viewing $H^2(X_\varphi)$ as a subspace of $L^2(X_\varphi)$, we let

$$H^2_{\text{new}}(X_\varphi) \stackrel{\text{def}}{=} H^2(X_\varphi) \cap L^2_{\text{new}}(X_\varphi).$$

There is an isometric isomorphism between $H^2_{\text{new}}(X_\varphi)$ and $H^2_\varphi\left(\mathbb{H}; V_n^0\right)$ that intertwines the two relevant Laplacian operators.

Unitary bundles

We identify the space of smooth global sections of $E_\phi \to X$, denoted $C^\infty(X; E_\phi)$, with the space $C^\infty_\phi(\mathbb{H}; \mathbb{C}^n)$ of smooth $\mathbb{C}^n$ valued functions on $\mathbb{H}$ which transform as

$$f(\gamma z) = \rho_\phi(\gamma) f(z).$$

We let $L^2(X; E_\phi)$ denote the completion of $C^\infty(X; E_\phi)$ with respect to the norm

$$\|f\|^2_{L^2(F)} \stackrel{\text{def}}{=} \int_F \|f(z)\|^2_{\mathbb{C}^n} d\mu(z).$$

Analogously to the case of covers, we define $C^\infty_{c,\phi}(\mathbb{H}; \mathbb{C}^n)$ to be the subset of $C^\infty_\phi(\mathbb{H}; \mathbb{C}^n)$ of functions which are compactly supported modulo Γ and $H^2(X; E_\phi)$ to be the completion of $C^\infty_{c,\phi}(\mathbb{H}; \mathbb{C}^n)$ with respect to the norm

$$\|f\|^2_{H^2_\phi(\mathbb{H};\mathbb{C}^n)} \stackrel{\text{def}}{=} \|f\|^2_{L^2(F)} + \|\Delta f\|^2_{L^2(F)}.$$

3 Random matrix theory

In this section we introduce the necessary random matrix theory results. Recall Γ is a free group on d generators $\gamma_1, \ldots, \gamma_d$. The wordlength $\mathrm{wl}(\gamma)$ is the length of γ as a reduced word in $\gamma_1, \ldots, \gamma_d, \gamma_1^{-1}, \ldots, \gamma_d^{-1}$. Let $\lambda : \Gamma \to \mathrm{End}\left(l^2(\Gamma)\right)$ denote the right regular representation of Γ. We rely heavily on recent extremely powerful results of Bordenave and Collins [BC23]. Firstly, for random unitaries, we need the following.

Theorem 3.1 ([BC23, Corollary 1.2] and [BC23, Lemma 7.4])**.** *Let $m \leqslant n^{\frac{1}{32d+160}}$ and $a_0, a_1, \ldots a_d \in M_m(\mathbb{C})$ with $a_0 = a_0^*$. Then there exists a constant $c_1 > 0$ such that for a random $\phi \in (\mathrm{Hom}(\Gamma, U(n)), \mathbb{P}_n)$, with probability at least $1 - \exp(-\sqrt{n})$,*

$$\begin{aligned}&\left\| a_0 \otimes \mathrm{Id}_{\mathbb{C}^n} + \sum_{i=1}^d \left(a_i \otimes \rho_\phi(\gamma_i) + a_i^* \otimes \rho_\phi\left(\gamma_i^{-1}\right)\right) \right\|_{\mathbb{C}^m \otimes \mathbb{C}^n} \\ \leqslant &\left\| a_0 \otimes \mathrm{Id}_{\ell^2(\Gamma)} + \sum_{i=1}^d \left(a_i \otimes \lambda(\gamma_i) + a_i^* \otimes \lambda\left(\gamma_i^{-1}\right)\right) \right\|_{\mathbb{C}^m \otimes \ell^2(\Gamma)} \left(1 + \frac{c_1}{n^{\frac{1}{32d+160}}}\right).\end{aligned}$$

We also need the following result for random permutations.

Theorem 3.2 ([BC23, Corollary 1.4])**.** *Let $m \leqslant n^{\sqrt{\log n}}$ and $a_0, a_1, \ldots a_d \in M_m(\mathbb{C})$ with $a_0 = a_0^*$. Then there exists a constant $c_2 > 0$ such that for a uniformly random $\varphi \in \mathrm{Hom}(\Gamma, S_n)$, with probability at least $1 - \frac{c_2}{\sqrt{n}}$,*

$$\begin{aligned}&\left\| a_0 \otimes \mathrm{Id}_{V_n^0} + \sum_{i=1}^d \left(a_i \otimes \rho_\varphi(\gamma_i) + a_i^* \otimes \rho_\varphi\left(\gamma_i^{-1}\right)\right) \right\|_{\mathbb{C}^m \otimes V_n^0} \\ \leqslant &\left\| a_0 \otimes \mathrm{Id}_{\ell^2(\Gamma)} + \sum_{i=1}^d \left(a_i \otimes \lambda(\gamma_i) + a_i^* \otimes \lambda\left(\gamma_i^{-1}\right)\right) \right\|_{\mathbb{C}^m \otimes \ell^2(\Gamma)} \left(1 + \frac{c_2}{(\log n)^{\frac{1}{4}}}\right).\end{aligned}$$

Theorem 3.1 and Theorem 3.2 both concern linear polynomials. The polynomial to which we shall want to apply Theorem 3.1 and Theorem 3.2 will not be linear so we need to apply a linearization procedure in order to access these bounds. The idea is that we can trade a polynomial of large degree for one of smaller degree at a cost of replacing $M_m(\mathbb{C})$ by $M_m(\mathbb{C}) \otimes M_k(\mathbb{C})$ for some k. This procedure is known as the linearization trick [Pi96], [HT05].

We use an effective linearization proved in [BC23, Section 8]. In [BC23, Section 8], the authors considered operators of the form $\sum_{g \in B_l} a_\gamma \otimes \rho_\phi(\gamma)$ where B_l is the ball of size l in the word metric of Γ with our fixed choice of generators (note that linear means $l \leqslant 1$ in this context). In our case, the operators we want to consider will be of the form $\sum_{g \in S} a_\gamma \otimes \rho_\phi(\gamma)$ where $S \subset B_l$ where $|S|$ is roughly of size l which shall give us a quantitative saving. This is only a minor adaptation to the arguments in [BC23, Section 8], however since this is a key point for our method, we include the details.

We say that a subset $S \subset \Gamma$ is symmetric if $g \in S$ implies $g^{-1} \in S$.

Lemma 3.3. *Let $l \geqslant 2$ be an even integer and let $S \subset B_l$. Consider $(a_g)_{g \in S}$ with $a_g \in M_m(\mathbb{C})$. Then there exists a symmetric set $S_1 \subset B_{\frac{l}{2}}$ with $|S_1| \leqslant 4|S|$, $(b_g)_{g \in S_1}$ with $b_g \in M_m(\mathbb{C}) \otimes M_{2|S_1|}(\mathbb{C})$ and $\theta \geqslant 0$ such that for any unitary representation (ρ, V) of Γ,*

$$\| \sum_{\gamma \in S} a_\gamma \otimes \rho(\gamma) \|_{\mathbb{C}^m \otimes V} = \| \sum_{\gamma \in S_1} b_\gamma \otimes \rho(\gamma) \|^2_{\mathbb{C}^m \otimes \mathbb{C}^{2|S_1|} \otimes V} - \theta,$$

where

$$\theta \leqslant 4|S| \| \sum_{\gamma \in S} a_\gamma \otimes \lambda(\gamma) \|_{\mathbb{C}^m \otimes l^2(\Gamma)}.$$

Proof. We consider a set $S_1 \subset B_{\frac{l}{2}}$ such that

$$S \subset \left\{ g^{-1}h \mid g, h \in S_1 \right\}.$$

We claim we can choose S_1 so that

$$|S_1| \leqslant 4|S|.$$

Indeed if $w \in S \cap B_{\frac{l}{2}}$, we can just take w and the identity to be in S_1. If $w \in S$ has word-length $> \frac{l}{2}$, then it can be written as $g^{-1}h$ for two words $g, h \in B_{\frac{l}{2}}$ and we add both words to S_1. We make S_1 symmetric by including the inverses of any word already added, at worst doubling the size of S_1.

Note that we can enlarge S to a symmetric set without changing the size of S_1, since S_1 is symmetric. After possibly replacing $M_m(\mathbb{C})$ with $M_m(\mathbb{C}) \otimes M_2(\mathbb{C})$ and enlarging S to a symmetric set, we can assume that the symmetry condition $a_\gamma = a^*_{\gamma^{-1}}$ holds, in particular $P \stackrel{\text{def}}{=} \sum_{\gamma \in S} a_\gamma \otimes \rho(\gamma)$ is self-adjoint e.g. [BC23, Proof of Theorem 1.1].

We now follow [BC23, Proof of Lemma 8.1]. Consider the element $\tilde{a} \in M_m(\mathbb{C}) \otimes M_{|S_1|}(\mathbb{C})$ defined by $(\tilde{a}_{g,h})_{g,h \in S_1}$,

$$\tilde{a}_{g,h} = \frac{1}{\# \left\{ (g', h') \in S_1 \times S_1 \mid (g')^{-1} h' = g^{-1}h \right\}} a_{g^{-1}h},$$

when $g^{-1}h \in S$ and $\tilde{a}_{g,h} = 0$ otherwise. Then

$$\sum_{\substack{g,h \in S_1 \\ g^{-1}h = w \in S}} \tilde{a}_{g,h} = a_w.$$

We have

$$\|\tilde{a}\|^2 \leqslant \| \sum_{g,h\in S_1} \tilde{a}_{g,h}\tilde{a}^*_{g,h}\| \leqslant \| \sum_{w\in S} a_w a_w^* \| \leqslant \| \sum_{w\in S} a_w \otimes \lambda\left(w\right) \|^2.$$

The operator $\tilde{a} + \|\tilde{a}\|\mathrm{Id}_{m|S_1|}$ is positive semi-definite and we let $\tilde{b} \in M_m\left(\mathbb{C}\right) \otimes M_{|S_1|}\left(\mathbb{C}\right)$ be its self-adjoint square root. For $g \in S_1$ we define

$$b_g \overset{\text{def}}{=} \tilde{b}\left(\mathrm{Id}_m \otimes e_{g,\emptyset}\right) \in M_m\left(\mathbb{C}\right) \otimes M_{|S_1|}\left(\mathbb{C}\right),$$

where $e_{g,h} \overset{\text{def}}{=} \delta_g \otimes \delta_h \in M_{B_{\frac{l}{2}}}\left(\mathbb{C}\right)$ and $\emptyset$ is the unit in Γ. Then defining

$$Q \overset{\text{def}}{=} \sum_{g\in S_1} b_g \otimes \rho(g),$$

we have

$$\begin{aligned} Q^*Q &= \sum_{g,h\in S_1} \left(\mathrm{Id}_m \otimes e_{\emptyset,g}\right) \tilde{b}^2 \left(\mathrm{Id}_m \otimes e_{h,\emptyset}\right) \otimes \rho(g^{-1}h) \\ &= \sum_{g,h\in S_1} e_{\emptyset,\emptyset} \otimes \left(\tilde{a}_{g,h} + \|\tilde{a}\|\mathbf{1}_{g=h}\mathrm{Id}_m\right) \otimes \rho(g^{-1}h) \\ &= e_{\emptyset,\emptyset} \otimes \left(\sum_{\gamma\in S} a_\gamma \otimes \rho\left(\gamma\right) + \theta \mathrm{Id}_{\mathbb{C}^m\otimes V} \right), \end{aligned}$$

where

$$\theta \leqslant |S_1| \, \| \sum_{\gamma\in S} a_\gamma \otimes \lambda\left(\gamma\right) \| \leqslant 4\,|S| \, \| \sum_{\gamma\in S} a_\gamma \otimes \lambda\left(\gamma\right) \|.$$

It follows that

$$\|Q\|^2 = || \sum_{\gamma\in S} a_\gamma \otimes \rho\left(\gamma\right) + \theta \mathrm{Id}_{\mathbb{C}^m\otimes V} \|_{\mathbb{C}^m\otimes V} = \| \sum_{\gamma\in S} a_\gamma \otimes \rho\left(\gamma\right) \|_{\mathbb{C}^m\otimes V} + \theta.$$

□

We can iterate this process to obtain the following, c.f. [BC23, Lemma 8.2].

Lemma 3.4. *Let $l \geqslant 2$ be an integer, $S \subset B_l$ and let $v = \lceil \log_2 l \rceil$. Then for each $k \in \{0, \ldots, v\}$ there is:*

- *An integer $n_k \geqslant 1$ with $n_v \leqslant 2l\,|S|^{\lceil \log_2 l \rceil}\, l^{(\lceil \log_2 l \rceil - 1)}$.*
- *A symmetric set $S_k \subset B_{2^{v-k}}$ with $S_0 = S$, $|S_k| \leqslant \min\left\{4^k\,|S|\,, |B_{2^{v-k}}|\right\}$.*
- *A set $\left(a_g^k\right)_{g\in S_k}$ with $a_g^k \in M_m\left(\mathbb{C}\right) \otimes M_{n_k}\left(\mathbb{C}\right)$.*
- *A constant $\theta_k \geqslant 0$ such that for $k \geqslant 1$,*

$$\theta_k \leqslant \| \sum_{\gamma\in S_{k-1}} a_\gamma^{k-1} \otimes \lambda\left(\gamma\right) \|_{\mathbb{C}^m\otimes\mathbb{C}^{n_{k-1}}\otimes l^2(\Gamma)} \, |S_k|\,,$$

such that for any unitary representation (ρ, V) *of* Γ,

$$\| \sum_{\gamma \in S_{k-1}} a_g^{k-1} \otimes \rho(\gamma) \|_{\mathbb{C}^m \otimes \mathbb{C}^{n_{k-1}} \otimes V} = \| \sum_{\gamma \in S_k} a_g^k \otimes \rho(\gamma) \|^2_{\mathbb{C}^m \otimes \mathbb{C}^{n_k} \otimes V} - \theta_k.$$

Proof. This is a straightforward consequence of iterating the procedure of Lemma 3.3. We have

$$n_v \leqslant \prod_{i=1}^{v} 2\,|S_i| \leqslant \prod_{i=1}^{v} 2 \cdot 4^i \cdot |S| = 2^v 4^{\frac{v(v-1)}{2}} |S|^v ,$$

where $v = \lceil \log_2 l \rceil$ which gives

$$n_v \leqslant 2l\,|S|^{\lceil \log_2 l \rceil}\, l^{(\lceil \log_2 l \rceil - 1)}.$$

□

As a consequence we obtain the following, c.f. [BC23, Lemma 8.3]

Lemma 3.5. *Let* $l \geqslant 2$ *be an integer,* $S \subset B_l$ *and set* $v = \lceil \log_2 l \rceil$. *Consider* $\left(a_g^v\right)_{g \in S_v}$ *as in Lemma 3.4 and denote* $a_0 = a_\emptyset^v$, $a_i = a_{\gamma_i}^v$ *for* $1 \leqslant i \leqslant 2d$. *Let* (ρ, V) *be any unitary representation of* Γ. *We have that for* $0 < \epsilon < 1$, *if*

$$2\epsilon l^2\,|S|^{\lceil \log_2 l \rceil}\, l^{(\lceil \log_2 l \rceil - 1)} < 1$$

and

$$\|a_0 \otimes \mathrm{Id}_V + \sum_{i=1}^{2d} a_i \otimes \rho(\gamma_i)\|_{\mathbb{C}^m \otimes \mathbb{C}^{n_v} \otimes V} \leqslant \|a_0 \otimes \mathrm{Id}_{l^2(\Gamma)} + \sum_{i=1}^{2d} a_i \otimes \lambda(\gamma_i)\|_{\mathbb{C}^m \otimes \mathbb{C}^{n_v} \otimes l^2(\Gamma)} (1+\epsilon),$$

then

$$\| \sum_{\gamma \in S} a_\gamma \otimes \rho(\gamma) \|_{\mathbb{C}^m \otimes V} \leqslant \| \sum_{\gamma \in S} a_\gamma \otimes \lambda(\gamma) \|_{\mathbb{C}^m \otimes l^2(\Gamma)} \left(1 + 2\epsilon l^2\,|S|^{\lceil \log_2 l \rceil}\, l^{(\lceil \log_2 l \rceil - 1)}\right).$$

Proof. For $k \in \{1, \ldots, v\}$, let $a_g^k \in M_m(\mathbb{C}) \otimes M_{n_k}(\mathbb{C})$ for $g \in S_k$ be as given by Lemma 3.4. For some $k \in \{1, \ldots, v\}$, assume that for some $0 < \epsilon_k < 1$,

$$\| \sum_{\gamma \in S_k} a_\gamma^k \otimes \rho(\gamma) \| \leqslant \| \sum_{\gamma \in S_k} a_\gamma^k \otimes \lambda(\gamma) \| (1+\epsilon_k).$$

Then by Lemma 3.4 applied twice,

$$\begin{aligned}
\| \sum_{\gamma \in S_{k-1}} a_\gamma^{k-1} \otimes \rho(\gamma) \| - \| \sum_{\gamma \in S_{k-1}} a_\gamma^{k-1} \otimes \lambda(\gamma) \| &= \| \sum_{\gamma \in S_k} a_\gamma^k \otimes \rho(\gamma) \|^2 - \| \sum_{\gamma \in S_k} a_\gamma^k \otimes \lambda(\gamma) \|^2 \\
&\leqslant \epsilon_k (1 + 2\epsilon_k) \left(\| \sum_{\gamma \in S_{k-1}} a_\gamma^{k-1} \otimes \lambda(\gamma) \| + \theta_k \right) \\
&\leqslant 4 \cdot 4^k\,|S|\,\epsilon_k \| \sum_{\gamma \in S_{k-1}} a_\gamma^{k-1} \otimes \lambda(\gamma) \|. \qquad (3.1)
\end{aligned}$$

By assumption, $\epsilon_v = \epsilon < 1$ and then by setting $\epsilon_{k-1} \overset{\text{def}}{=} 4 \cdot 4^k |S| \epsilon_k$ (recalling $\theta_k \leqslant 4^k |S| \| \sum_{\gamma \in S_{k-1}} a_\gamma^{k-1} \otimes \lambda(\gamma) \|$ from Lemma 3.4), By the definition of ϵ_{k-j} we see

$$\epsilon_0 = \epsilon \prod_{i=1}^{v} 4 \cdot 4^i |S| \leqslant 2\epsilon l^2 |S|^{\lceil \log_2 l \rceil} l^{(\lceil \log_2 l \rceil - 1)},$$

If one picks $2\epsilon l^2 |S|^{\lceil \log_2 l \rceil} l^{(\lceil \log_2 l \rceil - 1)} < 1$ then this ensures that $\epsilon_{k-j} < 1$ for $j = 1, \ldots, k$ and we can apply the inequality (3.1) inductively starting from $k = v$ to $k = 1$ provided that each subsequent $\epsilon_{k-j} < 1$. □

By Lemma 3.5, we obtain the following corollary from Theorem 3.1.

Corollary 3.6. *Let $m \geqslant 1$, $l \geqslant 2$ and let $S \subset B_l$ be a finite set such that*

$$2ml |S|^{\lceil \log_2 l \rceil} l^{(\lceil \log_2 l \rceil - 1)} \leqslant \exp\left(n^{\frac{1}{32d+160}}\right),$$

and

$$2c_1 l^2 |S|^{\lceil \log_2 l \rceil} l^{(\lceil \log_2 l \rceil - 1)} \leqslant n^{\frac{1}{32d+160}},$$

where c_1 is the constant in Theorem 3.6. Let $\gamma \mapsto a_\gamma \in M_m(\mathbb{C})$ be any map supported in S. For a random $\phi \in (\mathrm{Hom}(\Gamma, U(n)), \mathbb{P}_n)$, with probability at least $1 - \exp(-\sqrt{n})$ one has

$$\| \sum_{\gamma \in S} a_\gamma \otimes \rho_\phi(\gamma) \|_{\mathbb{C}^m \otimes \mathbb{C}^n} \leqslant \| \sum_{\gamma \in S} a_\gamma \otimes \lambda(\gamma) \|_{\mathbb{C}^m \otimes l^2(\Gamma)} \left(1 + c_1 \frac{2l^2 |S|^{\lceil \log_2 l \rceil} l^{(\lceil \log_2 l \rceil - 1)}}{n^{\frac{1}{32d+160}}}\right).$$

By applying Lemma 3.5 and Theorem 3.2 have an analogous corollary for permutation matrices.

Corollary 3.7. *Let m and l satisfy*

$$2ml |S|^{\lceil \log_2 l \rceil} l^{(\lceil \log_2 l \rceil - 1)} \leqslant n^{\sqrt{\log n}}.$$

Let $S \subset B_l$ be a finite set whose size satisfies

$$2c_2 l^2 |S|^{\lceil \log_2 l \rceil} l^{(\lceil \log_2 l \rceil - 1)} \leqslant (\log(n))^{\frac{1}{4}},$$

where c_2 is the constant in Theorem 3.7. Let $\gamma \mapsto a_\gamma \in M_m(\mathbb{C})$ be any map supported in S. For a uniformly random $\varphi \in \mathrm{Hom}(\Gamma, S_n)$, with probability at least $1 - \frac{c_2}{\sqrt{n}}$ one has

$$\| \sum_{\gamma \in S} a_\gamma \otimes \rho_\varphi(\gamma) \|_{\mathbb{C}^m \otimes V_n^0} \leqslant \| \sum_{\gamma \in S} a_\gamma \otimes \lambda(\gamma) \|_{\mathbb{C}^m \otimes l^2(\Gamma)} \left(1 + c_2 \frac{2l^2 |S|^{\lceil \log_2 l \rceil} l^{(\lceil \log_2 l \rceil - 1)}}{(\log(n))^{\frac{1}{4}}}\right).$$

4 Construction of the parametrix

Our parametrix construction is the same as in [HM23].

4.1 Cusp parametrix

In this subsection we introduce the cuspidal part of the parametrix.

Recall that we made the assumption that X has only one cusp to simplify notation. We identify the cusp $\mathcal{C}$ with

$$\mathcal{C} \stackrel{\text{def}}{=} (1,\infty) \times S^1$$

with the metric

$$\frac{dr^2 + dx^2}{r^2},$$

where $(r,x) \in (1,\infty) \times S^1$. For each $n \in \mathbb{N}$ we will define the cutoff functions $\chi^+_{\mathcal{C},n}, \chi^-_{\mathcal{C},n} : \mathcal{C} \to [0,1]$ to be functions that are identically zero in a neighborhood of $\{1\} \times S^1$, identically equal to 1 in a neighborhood of $\{\infty\} \times S^1$, such that

$$\chi^+_{\mathcal{C},n}\chi^-_{\mathcal{C},n} = \chi^-_{\mathcal{C},n}. \tag{4.1}$$

We extend $\chi^{\pm}_{\mathcal{C},n}$ by 0 to functions on X. Let $\kappa : \mathbb{N} \to (0,\infty)$ be some given function. Later on (Lemma 5.4) we shall pick specific functions $\kappa(n)$ for the case of covers and unitary bundles. As indicated by the subscript, the functions $\chi^+_{\mathcal{C},n}, \chi^-_{\mathcal{C},n}$ will depend on n through the function $\kappa(n)$.

Lemma 4.1. *Given $\kappa : \mathbb{N} \to (0,\infty)$, for each $n \in \mathbb{N}$ we can choose $\chi^{\pm}_{\mathcal{C},n}$ as above so that*

$$\|\nabla \chi^+_{\mathcal{C},n}\|_\infty, \|\Delta \chi^+_{\mathcal{C},n}\|_\infty \leq \frac{\kappa(n)}{30}.$$

Proof. One can find a $\tau_0 > 1$ and a smooth function $\chi^+_{\mathcal{C},0} : [0,\infty) \to [0,1]$ with $\chi^+_{\mathcal{C},0} \equiv 0$ for τ in $[0,1]$, $\chi^+_{\mathcal{C},0} \equiv 1$ for $\tau \geq \tau_0$ such that

$$\sup_{[0,\infty)} |(\chi^+_{\mathcal{C},0})'|, \sup_{[0,\infty)} |(\chi^+_{\mathcal{C},0})''| \leq 1.$$

Then defining

$$\chi^+_{\mathcal{C},n}(t) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{for } t \in [0,1] \\ \chi^+_{\mathcal{C},0}\left(\frac{\kappa(n)}{60}(t-1)+1\right) & \text{for } t \in (1,\infty) \end{cases},$$

we have

$$\sup_{[0,\infty)} |(\chi^+_{\mathcal{C},n})'|, \sup_{[0,\infty)} |(\chi^+_{\mathcal{C},n})''| \leq \frac{\kappa(n)}{60}.$$

Note that $\chi^+_{\mathcal{C},n}(\tau) \equiv 1$ for $\tau \geqslant \tau_n \stackrel{\text{def}}{=} \frac{60}{\kappa(n)}(\tau_0 - 1) + 1$. The calculation in [HM23, Lemma 4.1] gives

$$\|\nabla \chi^+_{\mathcal{C},n}\|_\infty = \sup_{[0,\infty)} |(\chi^+_{\mathcal{C},n})'| \leqslant \frac{\kappa(n)}{30},$$

and

$$\|\Delta \chi^+_{\mathcal{C},n}\|_\infty = \sup_{[0,\infty)} |(\chi^+_{\mathcal{C},n})'' - (\chi^+_{\mathcal{C},n})'| \leqslant \frac{\kappa(n)}{30}.$$

If one chooses $\chi^-_{\mathcal{C},n}$ to be a function with $\chi^-_{\mathcal{C}}(\tau) \equiv 0$ for $\tau \leq \tau_n$ and $\chi^-_{\mathcal{C}}(\tau) \equiv 1$ for $\tau \geq 2\tau_n$, (4.1) is satisfied and the lemma is proved. □

We obtain the operators

$$\chi^{\pm}_{\mathcal{C},n,\phi} : L^2\left(X; E_\phi\right) \to L^2\left(X; E_\phi\right),$$

in the unitary case by multiplication by $\chi^{\pm}_{\mathcal{C},n}$. For the case of covers, we lift $\chi^{\pm}_{\mathcal{C},n}$ to X_φ via the covering map to obtain a function $\chi^{\pm}_{\mathcal{C},n,\varphi}$ in $L^2\left(X_\varphi\right)$ and view $\chi^{\pm}_{\mathcal{C},n,\varphi}$ as a multiplication operator

$$\chi^{\pm}_{\mathcal{C},n,\varphi} : L^2\left(X_\varphi\right) \to L^2\left(X_\varphi\right)$$

We extend $\mathcal{C}$ to the parabolic cylinder

$$\tilde{\mathcal{C}} \stackrel{\text{def}}{=} (0,\infty) \times S^1,$$

with the same metric. Letting $\mathcal{C}_\varphi$ denote the subset of X_φ that covers $\mathcal{C}$, we let $\tilde{\mathcal{C}}_\varphi$ be the corresponding extension of C_φ. We consider the Laplacian

$$\Delta_{\tilde{\mathcal{C}}_\varphi} : H^2_{\text{new}}\left(\tilde{\mathcal{C}}_\varphi\right) \to L^2_{\text{new}}\left(\tilde{\mathcal{C}}_\varphi\right).$$

Given $\phi \in \text{Hom}\left(\mathbb{Z}; \text{U}\left(n\right)\right)$ we consider the associated unitary bundle $E_{\phi,\tilde{\mathcal{C}}} \to \tilde{\mathcal{C}}$ with the Laplacian

$$\Delta_{\phi,\tilde{\mathcal{C}}} : H^2\left(\tilde{\mathcal{C}}; E_{\phi,\tilde{\mathcal{C}}}\right) \to L^2\left(\tilde{\mathcal{C}}; E_{\phi,\tilde{\mathcal{C}}}\right).$$

By [HM23, Lemma 4.2] and [Za22, Lemma 3.1] (see also [DFP21, Proposition 4.16]), we have that the corresponding resolvents,

$$\begin{aligned} R_{\tilde{C},P,\varphi}\left(s\right) &\stackrel{\text{def}}{=} \left(\Delta_{\tilde{\mathcal{C}}_\varphi} - s(1-s)\right)^{-1} : L^2_{\text{new}}\left(\tilde{\mathcal{C}}_\varphi\right) \to H^2_{\text{new}}\left(\tilde{\mathcal{C}}_\varphi\right), \\ R_{\tilde{C},U,\phi}\left(s\right) &\stackrel{\text{def}}{=} \left(\Delta_{\phi,\tilde{\mathcal{C}}} - s(1-s)\right)^{-1} : L^2\left(\tilde{\mathcal{C}}; E_{\phi,\tilde{\mathcal{C}}}\right) \to H^2\left(\tilde{\mathcal{C}}; E_{\phi,\tilde{\mathcal{C}}}\right), \end{aligned}$$

exist as bounded operators for $\text{Re}\left(s\right) > \frac{1}{2}$ with

$$\|R_{\tilde{C},P,\varphi}\left(s\right)\|_{L^2_{\text{new}}(\tilde{\mathcal{C}}_\varphi)}, \|R_{\tilde{C},P,\varphi}\left(s\right)\|_{L^2_{\text{new}}(\tilde{\mathcal{C}}_\varphi)} \leqslant \frac{5}{4\kappa\left(n\right)},$$

and

$$\|R_{\tilde{C},U,\phi}\left(s\right)\|_{L^2(\tilde{\mathcal{C}};E_{\phi,\tilde{\mathcal{C}}})}, \|\Delta R_{\tilde{C},U,\phi}\left(s\right)\|_{L^2(\tilde{\mathcal{C}};E_{\phi,\tilde{\mathcal{C}}})} \leqslant \frac{5}{4\kappa\left(n\right)},$$

for $s \in \left[\frac{1}{2} + \sqrt{\kappa\left(n\right)}, 1\right]$. Precisely as in [HM23], we define the cusp parametrix for a cover X_φ by

$$\begin{aligned} \mathbb{M}^{\text{cusp}}_{P,\varphi}(s) &\stackrel{\text{def}}{=} \chi^{+}_{\mathcal{C},n,\varphi} R_{\tilde{C},\varphi}\left(s\right) \chi^{-}_{\mathcal{C},n,\varphi} \\ \mathbb{M}^{\text{cusp}}_{P,\varphi}(s) &: L^2_{\text{new}}\left(X_\varphi\right) \to H^2_{\text{new}}\left(X_\varphi\right). \end{aligned}$$

Here we view $\chi^{-}_{\mathcal{C},n,\varphi} : L^2_{\text{new}}\left(X_\varphi\right) \to L^2_{\text{new}}\left(\tilde{\mathcal{C}}_\varphi\right)$ and $\chi^{+}_{\mathcal{C},n,\varphi} : H^2_{\text{new}}\left(\tilde{\mathcal{C}}_\varphi\right) \to H^2_{\text{new}}\left(X_\varphi\right)$ in the natural way. As in [Za22], we analogously define the cusp parametrix for unitary bundles E_ϕ as

$$\begin{aligned} &\mathbb{M}^{\text{cusp}}_{U,\phi}(s) : L^2\left(X; E_\phi\right) \to L^2\left(X; E_\phi\right) \\ &\mathbb{M}^{\text{cusp}}_{U,\phi}(s) \stackrel{\text{def}}{=} \chi^{+}_{\mathcal{C},n,\phi} R_{\tilde{C},\phi}\left(s\right) \chi^{-}_{\mathcal{C},n,\phi}. \end{aligned} \tag{4.2}$$

We have

$$\begin{aligned}(\Delta - s(1-s))\mathbb{M}^{\text{cusp}}_{P,\varphi}(s) &= \chi^-_{\mathcal{C},n} + [\Delta, \chi^+_{\mathcal{C},n,\varphi}] R_{\tilde{C}}(s)\, \chi^-_{\mathcal{C},n} \\ &= \chi^-_{\mathcal{C},n,\varphi} + \mathbb{L}^{\text{cusp}}_{P,\varphi}(s)\end{aligned} \tag{4.3}$$

where

$$\mathbb{L}^{\text{cusp}}_{P,\varphi}(s) \stackrel{\text{def}}{=} [\Delta, \chi^+_{\mathcal{C},n,\varphi}] R_{\tilde{C},\varphi}(s)\, \chi^-_{\mathcal{C},n,\varphi}.$$

Similarly

$$(\Delta - s(1-s))\mathbb{M}^{\text{cusp}}_{U,\phi}(s) = \chi^-_{\mathcal{C},n} + \mathbb{L}^{\text{cusp}}_{U,\phi}(s),$$

where

$$\mathbb{L}^{\text{cusp}}_{U,\phi}(s) \stackrel{\text{def}}{=} [\Delta, \chi^+_{\mathcal{C},n,\phi}] R_{\tilde{C},\phi}(s)\, \chi^-_{\mathcal{C},n,\phi}.$$

By Lemma,4.1, it follows by [HM23, Lemma 4.3] (or [Za22, Lemma 3.2] for the unitary case) that for $s \in \left[\frac{1}{2} + \sqrt{\kappa(n)}, 1\right]$,

$$\|\mathbb{L}^{\text{cusp}}_{P,\varphi}(s)\|_{L^2_{\text{new}}(X_\varphi)},\ \|\mathbb{L}^{\text{cusp}}_{U,\phi}(s)\|_{L^2(X;E_\phi)} \leq \left(\|(\Delta\chi^+_{\mathcal{C},n})\|_\infty + 2\|\nabla\chi^+_{\mathcal{C},n}\|_\infty\right) \cdot \frac{5}{4\kappa(n)} \leqslant \frac{1}{8}. \tag{4.4}$$

This deterministic bound on the cusp parametrix will be sufficient for our purposes.

4.2 Operators on $\mathbb{H}$

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$, let

$$R_{\mathbb{H}}(s) : L^2(\mathbb{H}) \to L^2(\mathbb{H}), R_{\mathbb{H}}(s) \stackrel{\text{def}}{=} (\Delta_{\mathbb{H}} - s(1-s))^{-1},$$

be the resolvent on the upper half plane. Then $R_{\mathbb{H}}(s)$ is an integral operator with radial kernel $R_{\mathbb{H}}(s;r)$. Let $\chi_0 : \mathbb{R} \to [0,1]$ be a smooth function such that

$$\chi_0(t) = \begin{cases} 1 & \text{if } t \leqslant 0, \\ 0 & \text{if } t \geqslant 1. \end{cases}.$$

For $T > 0$, we define a smooth cutoff function χ_T by

$$\chi_T(t) \stackrel{\text{def}}{=} \chi_0(t - T).$$

We then define the operator $R^{(T)}_{\mathbb{H}}(s) : L^2(\mathbb{H}) \to L^2(\mathbb{H})$ to be the integral operator with radial kernel

$$R^{(T)}_{\mathbb{H}}(s;r) \stackrel{\text{def}}{=} \chi_T(r) R_{\mathbb{H}}(s;r).$$

It is proved in [HM23, Lemma 5.3] that for any $f \in C^\infty_c(\mathbb{H})$ and $s \in \left[\frac{1}{2}, 1\right]$, we have

1. $R^{(T)}_{\mathbb{H}}(s) f \in H^2(\mathbb{H})$.
2. $(\Delta - s(1-s))\, R^{(T)}_{\mathbb{H}}(s) f = f + \mathbb{L}^{(T)}_{\mathbb{H}}(s) f$ as equivalence classes of L^2 functions where $\mathbb{L}^{(T)}_{\mathbb{H}}(s)$ is defined to be the integral operator with radial kernel

$$\mathbb{L}^{(T)}_{\mathbb{H}}(s;r_0) \stackrel{\text{def}}{=} \left(-\frac{\partial^2}{\partial r^2}[\chi_T] - \frac{1}{\tanh r}\frac{\partial}{\partial r}[\chi_T]\right) R_{\mathbb{H}}(s;r_0) - 2\frac{\partial}{\partial r}[\chi_T]\frac{\partial R_{\mathbb{H}}}{\partial r}(s;r_0). \tag{4.5}$$

We recall some important properties of $\mathbb{L}_{\mathbb{H}}^{(T)}(s;r)$ from [HM23, Lemma 5.1].

Lemma 4.2. *We have*

1. *For $T>0$ and $s\in[\frac{1}{2},1]$, $\mathbb{L}^{(T)}(s;\bullet)$ is smooth and supported in $[T,T+1]$.*
2. *There is a constant $C>0$ such that for any $T>0$ and $s\in[\frac{1}{2},1]$ we have*
$$|\mathbb{L}_{\mathbb{H}}^{(T)}(s;r_0)|\leq Ce^{-sr_0}$$
3. *There is a constant $C>0$ such that for any $T>0$, $s\in\left[\frac{1}{2},1\right]$ and $r_0\in[T,T+1]$*
$$\left|\frac{\partial\mathbb{L}_{\mathbb{H}}^{(T)}}{\partial s}(s_0;r_0)\right|\leq C.$$

Our goal is to eventually use a Neumann series argument to invert $(\Delta-s(1-s)):H^2_{\text{new}}(X_\varphi)\to L^2_{\text{new}}(X_\varphi)$ which will require an estimate for the operator norm of $\mathbb{L}_{\mathbb{H}}^{(T)}(s)$. This is given by [HM23, Lemma 5.2].

Lemma 4.3 ([HM23, Lemma 5.2])**.** *There is a constant $C>0$ such that for any $T>0$ and $s\in[\frac{1}{2},1]$ the operator $\mathbb{L}_{\mathbb{H}}^{(T)}(s)$ extends to a bounded operator on $L^2(\mathbb{H})$ with operator norm*
$$\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\|_{L^2(\mathbb{H})}\leq CTe^{\left(\frac{1}{2}-s\right)T}.$$

We will need to ensure that, for example,
$$\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\|_{L^2(\mathbb{H})}<\frac{1}{5},$$
for $s\in\left[\frac{1}{2}+\sqrt{\kappa(n)},1\right]$. This means we have to take $T(n)$ such that,
$$Te^{-T\sqrt{\kappa(n)}}<\frac{1}{5}\tag{4.6}$$
for all sufficiently large n. We will eventually take $\kappa(n)=\frac{4(\log T)^2}{T^2}$ which ensures (4.6).

4.3 Interior parametrix

As in [HM23, Za22], we define,
$$R_{\mathbb{H},U,n}^{(T)}(s;x,y)\overset{\text{def}}{=}R_{\mathbb{H}}^{(T)}(s;x,y)\mathrm{Id}_n,\ R_{\mathbb{H},P,n}^{(T)}(s;x,y)\overset{\text{def}}{=}R_{\mathbb{H}}^{(T)}(s;x,y)\mathrm{Id}_{V_n^0},$$
$$\mathbb{L}_{\mathbb{H},U,n}^{(T)}(s;x,y)\overset{\text{def}}{=}\mathbb{L}_{\mathbb{H}}^{(T)}(s;x,y)\mathrm{Id}_n,\ \mathbb{L}_{\mathbb{H},P,n}^{(T)}(s;x,y)\overset{\text{def}}{=}\mathbb{L}_{\mathbb{H}}^{(T)}(s;x,y)\mathrm{Id}_{V_n^0},$$
and $R_{\mathbb{H},U,n}^{(T)}(s),R_{\mathbb{H},P,n}^{(T)}(s),\mathbb{L}_{\mathbb{H},U,n}^{(T)}(s),\mathbb{L}_{\mathbb{H},P,n}^{(T)}(s)$ as the corresponding integral operators. The relevant properties are summarized in the following Lemma.

Lemma 4.4 ([HM23, Lemma 5.5])**.** *For all $s\in[\frac{1}{2},1]$,*

1. *The integral operator $R_{\mathbb{H},P,n}^{(T)}(s)(1-\chi_{\mathcal{C},n}^-)$ is well-defined on $C_{c,\varphi}^\infty(\mathbb{H};V_n^0)$ and extends to a bounded operator*
$$R_{\mathbb{H},P,n}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^-\right):L^2_\varphi\left(\mathbb{H};V_n^0\right)\to H^2_\varphi\left(\mathbb{H};V_n^0\right).$$

2. *The integral operator* $\mathbb{L}^{(T)}_{\mathbb{H},P,n}(s)(1-\chi^-_{\mathcal{C},n})$ *is well-defined on* $C^\infty_{c,\phi}\left(\mathbb{H};V^0_n\right)$ *and and extends to a bounded operator on* $L^2_\varphi(\mathbb{H};V^0_n)$.

3. *We have*

$$[\Delta - s(1-s)]\, R^{(T)}_{\mathbb{H},P,\varphi}(s)(1-\chi^-_{\mathcal{C},n}) = (1-\chi^-_{\mathcal{C},n}) + \mathbb{L}^{(T)}_{\mathbb{H},P,n}(s)(1-\chi^-_{\mathcal{C},n}) \tag{4.7}$$

as an identity of operators on $L^2_\varphi(\mathbb{H};V^0_n)$.

The analogous statement holds for $R^{(T)}_{\mathbb{H},U,n}(s), \mathbb{L}^{(T)}_{\mathbb{H},U,n}(s)$. We define our interior parametrix for surfaces X_φ,

$$\mathbb{M}^{\text{int}}_{P,\varphi}(s) : L^2_{\text{new}}\left(X_\varphi\right) \to H^2_{\text{new}}\left(X_\varphi\right),$$

to be the operator corresponding under $L^2_{\text{new}}\left(X_\varphi\right) \cong L^2_\varphi\left(\mathbb{H};V^0_n\right)$ and $H^2_{\text{new}}\left(X_\varphi\right) \cong H^2_\varphi\left(\mathbb{H};V^0_n\right)$ to the integral operator $R^{(T)}_{\mathbb{H},U,n}(s)\left(1-\chi^-_{\mathcal{C},n}\right)$. We make the analogous definition for unitary bundles and use the notation

$$\mathbb{M}^{\text{int}}_{U,\phi}(s) : L^2\left(X;E_\phi\right) \to H^2\left(X;E_\phi\right).$$

Then by defining

$$\mathbb{M}_{P,\varphi}\left(s\right) = \mathbb{M}^{\text{int}}_{P,\varphi}(s) + \mathbb{M}^{\text{cusp}}_{P,\varphi}(s),$$

we obtain, using (4.3),

$$\begin{aligned}\left(\Delta_{X_\varphi} - s(1-s)\right)\mathbb{M}_{P,\varphi}\left(s\right) &= \left(1-\chi^-_{\mathcal{C},n,\varphi}\right) + \mathbb{L}^{\text{int}}_{P,\varphi}(s) + \chi^-_{\mathcal{C},n} + \chi^+_{\mathcal{C},n,\varphi} R_{\tilde{C},\varphi}\left(s\right)\chi^-_{\mathcal{C},n,\varphi} \\ &= 1 + \mathbb{M}^{\text{int}}_{P,\varphi}(s) + \mathbb{M}^{\text{cusp}}_{P,\varphi}(s).\end{aligned} \tag{4.8}$$

We make analogous definition for the case of unitaries with the notation $\mathbb{M}_{U,\phi}\left(s\right)$ and (4.8) holds in this context.

5 Probabilistic bounds on operator norms

In this section we prove the probabilistic estimates needed for the proofs of Theorem 1.2 and Theorem 1.4.

5.1 Preliminaries

Throughout this subsection, let $\kappa : \mathbb{N} \to (0,\infty)$ be given and let $\chi^\pm_{\mathcal{C},n}$ be chosen as to satisfy the conclusion of Lemma 4.1. Eventually we will take $\kappa\left(n\right) = \frac{64(32d+160)(\log\log n)^2}{\log n}$ for random unitaries, where d is the rank of the free group Γ, and $\kappa\left(n\right) = \frac{4\cdot 24^2(\log\log\log n)^2}{\log\log n}$ for random covers. The purpose of this subsection is to ensure that our random operators $\mathbb{M}^{\text{int}}(s)$ are of the correct form as to apply Corollary 3.6 and Corollary 3.7.

Let $f \in C^\infty_\phi\left(\mathbb{H};\mathbb{C}^n\right)$ with $\|f\|^2_{L^2(F)} < \infty$. We have

$$\begin{aligned}\mathbb{L}^{(T)}_{\mathbb{H},U,n}(s)\left(1-\chi^-_{\mathcal{C},n}\right)[f](x) &= \int_{y\in\mathbb{H}} \mathbb{L}^{(T)}_{\mathbb{H},U,n}\left(s;x,y\right)\left(1-\chi^-_{\mathcal{C},n}(y)\right) f(y)d\mathbb{H}(y) \\ &= \sum_{\gamma\in\Gamma}\int_{y\in F} \mathbb{L}^{(T)}_{\mathbb{H},U,n}\left(s;\gamma x,y\right)\rho_\phi\left(\gamma^{-1}\right)\left(1-\chi^-_{\mathcal{C},n}\left(y\right)\right) f(y)d\mathbb{H}(y).\end{aligned} \tag{5.1}$$

We have an isomorphism of Hilbert spaces

$$L^2_\phi(\mathbb{H};\mathbb{C}^n) \cong L^2(F)\otimes\mathbb{C}^n;$$
$$f \mapsto \sum_{e_i}\langle f|_F, e_i\rangle_{\mathbb{C}^n}\otimes e_i.$$

Conjugating by this isomorphism,

$$\mathbb{L}^{(T)}_{\mathbb{H},U,n}(s)\left(1-\chi^-_{\mathcal{C},n}\right) \cong \mathcal{L}_{U,n,\phi}(s) \stackrel{\text{def}}{=} \sum_{\gamma\in\Gamma} a^{(T)}_{\gamma,n}(s)\otimes\rho_\phi\left(\gamma^{-1}\right),$$

where

$$a^{(T)}_{\gamma,n}(s) : L^2(F)\to L^2(F)$$
$$a^{(T)}_{\gamma,n}(s)[f](x) \stackrel{\text{def}}{=} \int_{y\in F}\mathbb{L}^{(T)}_{\mathbb{H}}(s;\gamma x,y)\left(1-\chi^-_{\mathcal{C},n}(y)\right)d\mathbb{H}(y).$$

Note that for any $n\in\mathbb{N}$,$T>1$, $s\in\left[\frac{1}{2},1\right]$ and $\gamma\in\Gamma$, the operator $a^{(T)}_{\gamma,n}(s)$ is an Hilbert-Schmidt operator whose Hilbert-Schmidt norm can be bounded by a constant which only depends on X. Indeed by Lemma 4.2, we have

$$\int_{x,y\in F}\left|\mathbb{L}^{(T)}_{\mathbb{H}}(s;\gamma x,y)\left(1-\chi^-_{\mathcal{C},n}(y)\right)\right|^2 d\mathbb{H}(x)d\mathbb{H}(y) \leqslant C\mathrm{Vol}(X)^2.$$

In precisely the same way, for the case of covers, we have

$$L^2_\phi\left(\mathbb{H};V^0_n\right) \cong L^2(F)\otimes V^0_n$$

and

$$\mathbb{L}^{(T)}_{\mathbb{H},P,n}(s)\left(1-\chi^-_{\mathcal{C},n}\right) \cong \mathcal{L}_{P,n,\varphi}(s) \stackrel{\text{def}}{=} \sum_{\gamma\in\Gamma} a^{(T)}_{\gamma,n}(s)\otimes\rho_\varphi\left(\gamma^{-1}\right).$$

It is crucial that the map $\gamma\mapsto a^{(T)}_{\gamma,n}(s)$ has finite support whose size we can control.

Lemma 5.1. *Given n and $T>0$, there is a finite set $S(T)\subset\Gamma$ which contains the support of the map $\gamma\mapsto a^{(T)}_{\gamma,n}(s)$ for any any $s>\frac{1}{2}$. There is a constant $C>0$ such that*

$$|S(T)| \leqslant C\kappa(n)^2 e^{2T}, \tag{5.2}$$

and if $\gamma\in S(T)$ then its word-length $\mathrm{wl}(\gamma)$ *satisfies*

$$\mathrm{wl}(\gamma)\leqslant C\kappa(n)^2 e^{2T}. \tag{5.3}$$

Proof. We define

$$K_n \stackrel{\text{def}}{=} \mathrm{Supp}\left(1-\chi^-_{\mathcal{C},n}\right)\subset F.$$

Recall from (2.1) that $H(L)$ is the region of the fundamental domain F with $y\geqslant L$. By the definition of $\chi^-_{\mathcal{C},n}$ (Section 4.1), we have

$$K_n \subset F\backslash H\left(\frac{C}{\kappa(n)}\right),$$

for some constant. We have that

$$F\backslash H\left(\frac{C}{\kappa\left(n\right)}\right)=\left(F\backslash H\left(1\right)\right)\bigsqcup\left(H\left(1\right)\backslash H\left(\frac{C}{\kappa\left(n\right)}\right)\right).$$

The diameter of $\left(F\backslash H\left(1\right)\right)$ is bounded by a constant depending only on X. The diameter of $H\left(1\right)\backslash H\left(\frac{C}{\kappa(n)}\right)$ is bounded above by $\log\left(\frac{C}{\kappa(n)}\right)+2$. It follows that

$$\operatorname{diam}\left(K_n\right)\leqslant C+\log\left(\frac{1}{\kappa\left(n\right)}\right).$$

Then for $x\in F$, by Lemma 4.2, the expression

$$\mathbb{L}_{\mathbb{H}}^{(T)}\left(s;\gamma x,y\right)\left(1-\chi_{\mathcal{C},n}^{-}\left(y\right)\right)$$

is non-zero only when $y\in K_n$ and $d\left(\gamma x,y\right)\leqslant T+1$. Recall that F is a Dirichlet domain about some point w, we can assume $w\in K_n$. Then

$$\begin{aligned}d\left(\gamma x,w\right)&\leqslant d\left(\gamma x,y\right)+d\left(w,y\right)\\&\leqslant T+1+\operatorname{diam}\left(K_n\right).\end{aligned}$$

Then since F is a Dirichlet domain about w,

$$\begin{aligned}d\left(\gamma w,w\right)&\leqslant d\left(\gamma w,\gamma x\right)+d\left(\gamma x,w\right)=d\left(w,x\right)+d\left(\gamma x,w\right)\leqslant 2d\left(\gamma x,w\right)\\&\leqslant 2\left(C+\log\left(\frac{1}{\kappa\left(n\right)}\right)+T\right).\end{aligned}$$

Then we can employ a lattice point count to deduce that

$$\begin{aligned}\left|S\left(T\right)\right|&\leqslant\#\left\{\gamma\in\Gamma\mid d\left(\gamma w,w\right)\leqslant C+2\log\kappa\left(n\right)+2T\right\}\\&\leqslant C\exp\left(2\left(C+\log\left(\frac{1}{\kappa\left(n\right)}\right)+T\right)\right)\leqslant C\frac{e^{2T}}{\kappa\left(n\right)^2},\end{aligned}$$

proving (5.2).

We now show that property (5.3) holds. We assumed that F is a Dirichlet domain for Γ, we can also assume that F is such that the set of side pairings $\left\{h_1,\ldots,h_k,h_1^{-1},\ldots,h_k^{-1}\right\}$ for F contain our choice of generators $\gamma_1,\ldots,\gamma_d$ and their inverses. We let $\overline{\mathrm{wl}}\left(\gamma\right)$ denote the minimal length of γ as a word in $\left\{h_1,\ldots,h_k,h_1^{-1},\ldots,h_k^{-1}\right\}$. Since any h_i or its inverse h_i^{-1} is a finite word in $\gamma_1,\ldots,\gamma_d,\gamma_1^{-1},\ldots,\gamma_d^{-1}$ it follows that there is a constant $C>0$ with

$$\mathrm{wl}\left(\gamma\right)\leqslant C\overline{\mathrm{wl}}\left(\gamma\right).$$

We now set about bounding

$$\sup_{\gamma\in S(T)}\overline{\mathrm{wl}}\left(\gamma\right).$$

By the previous argument, if $\gamma\in S\left(T\right)$ then

$$\gamma F\cap B\left(w,\operatorname{diam}\left(K_n\right)+T+1\right)\neq\emptyset. \tag{5.4}$$

We claim that if γ satisfies (5.4) and $\overline{\mathrm{wl}}\left(\gamma\right)\geqslant 1$, then there is a γ' with $\overline{\mathrm{wl}}\left(\gamma\right)=\overline{\mathrm{wl}}\left(\gamma'\right)-1$ which satisfies (5.4). The case $\overline{\mathrm{wl}}\left(\gamma\right)=1$ is clear since $w\in F$. For $l>1$ let Γ_l denote the elements of

Γ with $\overline{\mathrm{wl}}(\gamma) = l$. Since $\left\{h_1, \ldots, h_k, h_1^{-1}, \ldots, h_k^{-1}\right\}$ are side pairings for the Dirichlet domain F, we see that see that

$$\bigcup_{\gamma\in\Gamma} \gamma F \backslash \left(\bigcup_{\gamma\in\Gamma_l} \gamma F\right) = \left(\bigcup_{i<l} \bigcup_{\gamma\in\Gamma_i} \gamma F\right)^{\circ} \sqcup \left(\bigcup_{i>l} \bigcup_{\gamma\in\Gamma_i} \gamma F\right)^{\circ}.$$

is disconnected. Here U°denotes the interior of U. Therefore if there claim were not true, then one could find an $l > 1$ with

$$\bigcup_{\gamma\in\Gamma_l} \gamma F \cap B\left(w, \operatorname{diam}\left(K_n\right) + T + 1\right) \neq \emptyset, \tag{5.5}$$

such that

$$\bigcup_{\gamma\in\Gamma_{l-1}} F \cap B\left(w, \operatorname{diam}\left(K_n\right) + T + 1\right) = \emptyset,$$

in particular,

$$B\left(w, \operatorname{diam}\left(K_n\right) + T + 1\right) \subset \bigcup_{\gamma\in\Gamma} \gamma F \backslash \left(\bigcup_{\gamma\in\Gamma_{l-1}} \gamma F\right).$$

Then since the ball of radius r in the hyperbolic plane is connected and the identity in Γ satisfies (5.4),

$$B\left(w, \operatorname{diam}\left(K_n\right) + T + 1\right) \subset \left(\bigcup_{i<l-1} \bigcup_{\gamma\in\Gamma_i} \gamma F\right)^{\circ}.$$

This gives a contradiction to (5.5) and the claim follows. It follows that if γ satisfies (5.4) then $\overline{\mathrm{wl}}(\gamma)$ is bounded above by the number of $\gamma \in \Gamma$ which satisfy (5.4). Then by the argument that led to (5.1),

$$\sup_{\gamma\in S(T)} \mathrm{wl}(\gamma) \leqslant C\#\left\{\gamma \in \Gamma \mid \gamma F \cap B\left(w, \operatorname{diam}\left(K_n\right) + T + 1\right) \neq \emptyset\right\} \leqslant C\frac{e^{2T(n)}}{\kappa\left(n\right)^2},$$

and the claim is proved. □

Currently, our operators $\sum_{\gamma\in S} a_{\gamma,n}^{(T)}(s) \otimes \rho_\phi\left(\gamma^{-1}\right)$ whose norm we wish to bound are almost of the form of Corollary 3.6 except $a_{\gamma,n}^{(T)}(s) : L^2(F) \to L^2(F)$ is not a matrix. However each $a_{\gamma,n}^{(T)}(s)$ is compact so can be approximated by finite rank operators. We need an effective version of this whilst having control over the rank in terms of the error.

Lemma 5.2. *Let $s \in \left[\frac{1}{2}, 1\right]$ be given. For every $n \in \mathbb{N}$ and $T > 1$, there exists a finite dimensional subspace $W \subset L^2(X)$ with $|W| \leqslant C\left(S\left(T\right)\right)^3$ for some constant C and finite rank operators $b_{\gamma,n}^{(T)} : W \to W$ for each $\gamma \in S\left(T\right)$ such that*

$$\|b_{\gamma,n}^{(T)}(s) - a_{\gamma,n}^{(T)}(s)\|_{L^2(F)} \leqslant \frac{1}{20|S(T)|},$$

Proof. Let $\gamma \in S\left(T\right)$, then since $a_{\gamma}^{(T)}(s)$ is compact, it has a singular value decomposition

$$a_{\gamma,n}^{(T)}(s) = \sum_{i\in\mathbb{N}} s_n\left(a_{\gamma,n}^{(T)}(s)\right) \langle \cdot, e_i\rangle f_i,$$

where $\{e_i\}_{i\in\mathbb{N}}$ and $\{f_i\}_{i\in\mathbb{N}}$ are orthonormal systems in $L^2(F)$ and $\{s_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of non-negative real numbers [RS81, Theorem VI.17]. The singular values $\{s_n(A)\}_{n\in\mathbb{N}}$ of a compact operator A on an Hilbert space $\mathcal{H}$ are the eigenvalues of $|A| = \sqrt{AA^*}$ and if A is Hilbert-Schmidt then $\|A\|_{\text{H.S.}}^2 = \sum_{j\in\mathbb{N}} s_j(A)^2$.

By defining

$$b_{\gamma,n}^{(T)}(s) \stackrel{\text{def}}{=} \sum_{i=1}^{r} s_i\left(a_{\gamma}^{(T)}(s)\right)\langle\cdot, e_i\rangle f_i,$$

we see that $b_{\gamma}^{T}(s) : W_\gamma \to W_\gamma$ where $|W_\gamma| \leqslant 2r$ and

$$\|b_{\gamma,n}^{(T)}(s) - a_{\gamma,n}^{(T)}(s)\| \leqslant s_{r+1}(A).$$

We want r to be such that

$$s_{r+1}\left(a_{\gamma,n}^{(T)}(s)\right) \leqslant \frac{1}{20|S(T)|}. \tag{5.6}$$

We have

$$\sum_{i=1}^{\infty} s_i\left(a_{\gamma,n}^{(T)}(s)\right)^2 = \|a_{\gamma,n}^{(T)}(s)\|_{\text{H.S}}^2 \leqslant C,$$

Then

$$\begin{aligned} 0 \leqslant \sum_{i=r+1}^{\infty} s_i\left(a_{\gamma,n}^{(T)}(s)\right)^2 &= \|a_{\gamma,n}^{(T)}(s)\|_{\text{H.S}}^2 - \sum_{i=1}^{r} s_i\left(a_{\gamma,n}^{(T)}(s)\right)^2 \\ &\leqslant C - rs_r\left(a_{\gamma,n}^{(T)}(s)\right)^2. \end{aligned}$$

In particular,

$$s_r\left(a_{\gamma,n}^{(T)}(s)\right) \leqslant \sqrt{\frac{C}{r}}.$$

Taking $r \geqslant 400 \cdot C \cdot S(T)^2$ guarantees that (5.6) is satisfied. Then $|W_\gamma| \leqslant CS(T)^2$ for each $\gamma \in S(T)$ and taking

$$W = \bigcup_{\gamma\in S(T)} W_\gamma,$$

gives the conclusion. □

Finally we prove a simple deviations bound.

Lemma 5.3. *There exists a constant $c_3 > 0$ depending only on X such that for any $T > 1$, any $\gamma \in S(T)$ and $s_1, s_2 \in \left[\frac{1}{2}, 1\right]$,*

$$\|a_{\gamma,n}^{(T)}(s_1) - a_{\gamma,n}^{(T)}(s_2)\|_{L^2(F)} \leq c_3|s_1 - s_2|.$$

Proof. The operator

$$a_{\gamma,n}^{(T)}(s_1) - a_{\gamma,n}^{(T)}(s_2)$$

is an integral operator with kernel

$$\left(\mathbb{L}_{\mathbb{H}}^{(T)}(s;\gamma x, y) - \mathbb{L}_{\mathbb{H}}^{(T)}(s;\gamma x, y)\right)\left(1 - \chi_{\mathcal{C},n}^{-}(y)\right).$$

We have for any $T>1$, $\gamma\in S(T)$, by Lemma 4.4,

$$\begin{aligned}\left|\mathbb{L}_{\mathbb{H}}^{(T)}(s;\gamma x,y)-\mathbb{L}_{\mathbb{H}}^{(T)}(s;\gamma x,y)\right| &\leqslant \sup_{s\in\left[\frac{1}{2},1\right]}\left|\frac{\partial}{\partial s}\mathbb{L}_{\mathbb{H}}^{(T)}(s;\gamma x,y)\right||s_1-s_2| \\ &\leqslant C|s_1-s_2|.\end{aligned}$$

Then we see

$$\|a_\gamma^{(T)}(s_1)-a_\gamma^{(T)}(s_2)\|_{L^2(F)}\leqslant\|a_\gamma^{(T)}(s_1)-a_\gamma^{(T)}(s_2)\|_{\text{H.S.}}\leqslant c_3|s_1-s_2|,$$

for some constant $c_3>0$. □

5.2 Random operator bounds

We are now in a position to apply the results of Section 3 to our random operators $\mathcal{L}_{U,n,\phi}(s)$ and $\mathcal{L}_{P,n,\varphi}(s)$.

Lemma 5.4. *With notations as above,*

- *Taking* $T=\frac{\sqrt{\log n}}{4\sqrt{32d+160}}$ *and* $\kappa(n)=\frac{64(32d+160)(\log\log n)^2}{\log n}$, *we have that with probability tending to* 1 *as* $n\to\infty$,
$$\sup_{s\in\left[\frac{1}{2}+\sqrt{\kappa(n)},1\right]}\|\mathcal{L}_{U,n,\phi}(s)\|_{L^2(F)\otimes\mathbb{C}^n}<\frac{3}{5}.$$
- *Taking* $T=\frac{\sqrt{\log\log n}}{24}$ *and* $\kappa(n)=\frac{4\cdot 24^2(\log\log\log n)^2}{\log\log n}$, *we have that with probability tending to* 1 *as* $n\to\infty$
$$\sup_{s\in\left[\frac{1}{2}+\sqrt{\kappa(n)},1\right]}\|\mathcal{L}_{P,n,\varphi}(s)\|_{L^2(F)\otimes V_n^0}<\frac{3}{5}.$$

Proof. We first treat the unitary case. Let $T=\frac{\sqrt{\log n}}{4\sqrt{32d+160}}$, $\kappa(n)=\frac{64(32d+160)(\log\log n)^2}{\log n}$ and let $s\in\left[\frac{1}{2}+\sqrt{\kappa(n)},1\right]$ be fixed. Then by Lemma 5.2, there exists a finite dimensional subspace $W\subset L^2(X)$ with $m=|W|\leqslant C\frac{e^{3T}}{\kappa(n)^3}$ and operators $b_\gamma^{(T)}:W\to W$ for each $\gamma\in S(T)$ such that

$$\|b_{\gamma,n}^{(T)}(s)-a_{\gamma,n}^{(T)}(s)\|_{L^2(F)}\leqslant\frac{1}{20|S(T)|}.$$

It follows that

$$\|\mathcal{L}_{U,n,\phi}(s)-\sum_{\gamma\in S(T)}b_{\gamma,n}^{(T)}(s)\otimes\rho_\phi(\gamma)\|_{L^2(X)\otimes\mathbb{C}^n}\leqslant\frac{1}{20}. \tag{5.7}$$

We now apply Corollary 3.6 to $\sum_{\gamma\in S(T)}b_{\gamma,n}^{(T)}(s)\otimes\rho_\phi(\gamma)$.

By Lemma 5.1, we have that $S(T)\subset B_l$ where $l\leqslant C\frac{e^{2T}}{\kappa(n)^2}$ and $|S(T)|\leqslant C\frac{e^{2T}}{\kappa(n)^2}$. Recall we made the choices $T=\frac{\sqrt{\log n}}{4\sqrt{32d+160}}$, $\kappa(n)=\frac{64(32d+160)(\log\log n)^2}{\log n}$. We now check that the assumptions of Corollary 3.6 are satisfied.

$$\begin{aligned}\log\left(2ml|S|^{\lceil\log_2 l\rceil}l^{(\lceil\log_2 l\rceil-1)}\right)&\leqslant C+\log m+\log l+\log l\log|S|+(\log l)^2\\ &\leqslant C+5T-5\log\kappa(n)+2(2T-2\log\kappa(n)+\log C)^2\\ &\ll T^2\ll\log n.\end{aligned}$$

It follows that for some constant C,

$$2ml\left|S\right|^{\lceil\log_2 l\rceil}l^{(\lceil\log_2 l\rceil-1)}<n^C<\exp\left(n^{\frac{1}{32d+160}}\right),$$

for n large enough and the first assumption of Corollary 3.6 holds. We remark that we didn't need to use the precise choice of constants in T and κ to check the first condition. To check the second condition, we observe that

$$\log\left(l^2\left|S\right|^{\lceil\log_2 l\rceil}l^{(\lceil\log_2 l\rceil-1)}\right)\leqslant 9\left(T-\log\kappa\left(n\right)\right)^2<12T^2<\frac{1}{32d+160}\log n, \tag{5.8}$$

for sufficiently large n, and by exponentiating (5.8)

$$\frac{l^2\left|S\right|^{\lceil\log_2 l\rceil}l^{(\lceil\log_2 l\rceil-1)}}{n^{\frac{1}{32d+160}}}\to 0,$$

as $n\to\infty$. We are now in the position to apply Corollary 3.6. We learn that with probability at least

$$1-\exp\left(-\sqrt{n}\right),$$

we have

$$\begin{aligned}\|\sum_{\gamma\in S}b_{\gamma,n}^{(T)}(s)\otimes\rho_\varphi\left(\gamma\right)\|_{\mathbb{C}^m\otimes\mathbb{C}^n}&\leqslant\|\sum_{\gamma\in S}b_{\gamma,n}^{(T)}(s)\otimes\lambda\left(\gamma\right)\|_{\mathbb{C}^m\otimes l^2(\Gamma)}\left(1+\frac{l^2\left|S\right|^{\lceil\log_2 l\rceil}l^{\frac{3}{2}(\lceil\log_2 l\rceil-1)}}{n^{\frac{1}{32d+160}}}\right)\\&=\|\sum_{\gamma\in S}b_{\gamma,n}^{(T)}(s)\otimes\lambda\left(\gamma\right)\|_{\mathbb{C}^m\otimes l^2(\Gamma)}\left(1+o\left(1\right)\right).\end{aligned}$$

We have an isometric linear isomorphism

$$\begin{aligned}L^2\left(F\right)\otimes\ell^2\left(\Gamma\right)&\cong L^2\left(\mathbb{H}\right),\\f\otimes\delta_\gamma&\mapsto f\circ\gamma^{-1},\end{aligned}\tag{5.9}$$

(with $f\circ\gamma^{-1}$ extended by zero from a function on γF). Indeed, let $L_F^2\left(\mathbb{H}\right)$ denote the square-integrable functions on $\mathbb{H}$ which are supported on finitely many Γ-translates of F. Then any $g\in L_F^2\left(\mathbb{H}\right)$ can be written as

$$g=\sum_{i\in I}g\mid_{\gamma_iF}=\sum_{i\in I}g_i\circ\gamma_i^{-1},$$

where $g_i\in L^2\left(F\right)$ and $I\subset\Gamma$ is a finite set, then the inverse to the map (5.9) is given by

$$g\mapsto\sum_{i\in I}g_i\otimes\delta_{\gamma_i}. \tag{5.10}$$

One can check that the map (5.10) is an isometric isomorphism of $L_F^2\left(\mathbb{H}\right)$ onto its image, which is dense in $L^2\left(F\right)\otimes l^2\left(\Gamma\right)$. Since $L_F^2\left(\mathbb{H}\right)$ is dense in $L^2\left(\mathbb{H}\right)$, the map (5.10) extends to an isometric isomorphism between $L^2\left(\mathbb{H}\right)$ and $L^2\left(F\right)\otimes l^2\left(\Gamma\right)$.

Under this isomorphism, the operator $\sum_{\gamma\in S}a_{\gamma,n}^{(T)}(s)\otimes\lambda\left(\gamma^{-1}\right)$ is conjugated to

$$\mathbb{L}_{\mathbb{H}}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right):L^2(\mathbb{H})\to L^2(\mathbb{H})$$

from Section 4.2. Since $\left(1-\chi_{\mathcal{C},n}^{-}\right)$ is valued in $[0,1]$, multiplication by it has operator norm ≤ 1 on $L^2(\mathbb{H})$, we see that

$$\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right)\|_{L^2(\mathbb{H})} \leqslant \|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\|_{L^2(\mathbb{H})} < CT\left(n\right)e^{-T(n)\left(\frac{1}{2}-s\right)}.$$

Since $s \in \left[\frac{1}{2}+\sqrt{\kappa\left(n\right)},1\right]$ and $\kappa\left(n\right)=\frac{4(\log T(n))^2}{T(n)^2}$, we have

$$T\left(n\right)e^{-T(n)\left(\frac{1}{2}-s\right)} \leqslant T\left(n\right)e^{-2\log(T(n))} = o\left(1\right).$$

Then we have

$$\|\sum_{\gamma\in S} a_{\gamma,n}^{(T)}(s)\otimes\lambda(\gamma^{-1})\|_{L^2(F)\otimes l^2(\Gamma)} < \frac{1}{10}, \tag{5.11}$$

for sufficiently large n. By the argument that led to (5.7), we see

$$\|\sum_{\gamma\in S} b_{\gamma,n}^{(T)}\otimes\lambda\left(\gamma^{-1}\right)-\sum_{\gamma\in S} a_{\gamma,n}^{(T)}(s)\otimes\lambda(\gamma^{-1})\|_{\mathbb{C}^m\otimes l^2(\Gamma)} < \frac{1}{20}. \tag{5.12}$$

Then by (5.7), (5.11) and (5.12), for our fixed choice of s,

$$\|\mathcal{L}_{U,n,\phi}(s)\|_{L^2(F)\otimes\mathbb{C}^n} < \frac{2}{5},$$

with probability at least

$$1-\exp\left(-\sqrt{n}\right).$$

We now use a finite net argument to control all $s \in \left[\frac{1}{2}+\sqrt{\kappa\left(n\right)},1\right]$ uniformly. Let $\mathcal{Y}$ be a finite set of points in $\left[\frac{1}{2}+\sqrt{\kappa\left(n\right)},1\right]$ so that each point of $\left[\frac{1}{2}+\sqrt{\kappa\left(n\right)},1\right]$ is of distance at most

$$\frac{1}{5|S\left(T\right)|c_3},$$

from the finite set $\mathcal{Y}$, where c_3 is the constant in Lemma 5.3. We can pick $\mathcal{Y}$ so that $|\mathcal{Y}| \leqslant 5c_3\left|S\left(T\right)\right| \leqslant C\frac{e^{2T}}{\kappa(n)^2}$. Then by applying an intersection bound, the probability that

$$\|\mathcal{L}_{U,n,\phi}(s)\|_{L^2(F)\otimes\mathbb{C}^n} < \frac{2}{5}$$

for *every point* $s \in \mathcal{Y}$ is bounded below by

$$1-C\exp\left(-\sqrt{n}\right)\left|S\left(T\right)\right| \geqslant 1-n\exp\left(-\sqrt{n}\right), \tag{5.13}$$

which tends to 1 as $n\to\infty$ and

$$\sup_{s\in\mathcal{Y}}\|\mathcal{L}_{U,n,\phi}(s)\|_{L^2(F)\otimes\mathbb{C}^n} \leqslant \frac{2}{5},$$

a.a.s. Finally, for $s_1, s_2 \in \left[\frac{1}{2}+\sqrt{\kappa\left(n\right)},1\right]$,

$$\mathcal{L}_{U,n,\phi}(s_1)-\mathcal{L}_{U,n,\phi}(s_2) = \sum_{\gamma\in S(T)}\left[a_{\gamma}^{(T)}(s_1)-a_{\gamma}^{(T)}(s_2)\right]\otimes\rho_{\phi}\left(\gamma^{-1}\right). \tag{5.14}$$

Then by Lemma 5.3, for some constant $c_3 > 0$ we have

$$\|a_\gamma^{(T)}(s_1) - a_\gamma^{(T)}(s_2)\|_{L^2(F)} \leq c_3|s_1 - s_2|,$$

for all $\gamma \in S(T)$ and $s_1, s_2 \in [s_0, 1]$. We see that,

$$\|\mathcal{L}_{U,n,\phi}(s_1) - \mathcal{L}_{U,n,\phi}(s_2)\|_{L^2(F)\otimes\mathbb{C}^n} \leq |S(T)|\, c_3|s_1 - s_2|.$$

Then by the choice of $\mathcal{Y}$, it follows that

$$\sup_{s\in\mathcal{Y}} \|\mathcal{L}_{U,n,\phi}(s)\| \leqslant \frac{2}{5} \implies \sup_{s\in\left[\frac{1}{2}+\sqrt{\kappa(n)},1\right]} \|\mathcal{L}_{U,n,\phi}(s)\| \leqslant \frac{3}{5}.$$

Since the prior happens with probability tending to 1 as $n \to \infty$, the first claim is proved.

The argument in the case of random covers is similar, one just needs to verify that the choices of $\kappa(n)$ and T allow the same conclusions as in the unitary case. We want to apply Corollary 3.7, leading us to require that

$$2ml\,|S|^{\lceil \log_2 l\rceil}\, l^{(\lceil \log_2 l\rceil - 1)} \leqslant n^{\sqrt{\log n}},$$

and

$$l^2\,|S|^{\lceil \log_2 l\rceil}\, l^{(\lceil \log_2 l\rceil - 1)} \leqslant (\log(n))^{\frac{1}{4}}.$$

Since $m \leqslant C\frac{e^{3T}}{\kappa(n)^3}$ and $l, |S| \leqslant C\frac{e^{2T}}{\kappa(n)^2}$, c.f. Lemma 5.1 and Lemma 5.2, it is a simple calculation to check both inequalities are satisfied if one takes $T = \frac{\sqrt{\log\log n}}{24}$ and $\kappa(n) = \frac{4\cdot 24^2(\log\log\log n)^2}{\log\log n}$. Finally, we just need that

$$\frac{1}{\sqrt{n}} \cdot |S(T)| \to 0,$$

in order to apply the same intersection bound argument (5.13), which holds by our assumptions on T and κ. □

6 Proofs of Theorem 1.2 and Theorem 1.4

It is now straightforward to conclude Theorem 1.2 and Theorem 1.4. Recall, for the case of unitary bundles, that

$$\mathbb{M}_{U,\phi}(s) \stackrel{\text{def}}{=} \mathbb{M}_{U,\phi}^{\text{int}}(s) + \mathbb{M}_{U,\phi}^{\text{cusp}}(s),$$

then $\mathbb{M}_{U,\phi}(s) : L^2(X; E_\phi) \to H^2(X; E_\phi)$ is a bounded operator and

$$\left(\Delta_{X_\phi} - s(1-s)\right)\mathbb{M}_{U,\phi}(s) = 1 + \mathbb{L}_{U,\phi}^{\text{int}}(s) + \mathbb{L}_{U,\phi}^{\text{cusp}}(s), \tag{6.1}$$

by Section 4.3. We proved in Lemma 5.4 that there is a constant c_4 (whose precise value can be read off in Lemma 5.4) such that a.a.s.

$$\|\mathbb{L}_{U,\phi}^{\text{int}}(s)\| \leqslant \frac{3}{5},$$

for all $s \in \left[\frac{1}{2} + \sqrt{c_4}\frac{\log\log n}{\sqrt{\log n}}, 1\right]$. Then since by (4.4)

$$\|\mathbb{L}_{U,\phi}^{\text{cusp}}(s)\| \leqslant \frac{1}{8},$$

we have a.a.s.

$$\sup_{s\in\left[\frac{1}{2}+\sqrt{c_4}\frac{\log\log n}{\sqrt{\log n}},1\right]} \|\mathbb{L}_{\phi}^{\text{int}}(s)+\mathbb{L}_{\phi}^{\text{cusp}}(s)\| \leqslant \frac{4}{5}.$$

This implies that a.a.s.

$$\mathbb{M}_{U,\phi}(s)\left(1+\mathbb{L}_{U,\phi}^{\text{int}}(s)+\mathbb{L}_{U,\phi}^{\text{cusp}}(s\right)^{-1}$$

exists as a bounded operator $L^2(X;E_\phi)\to H^2(X;E_\phi)$ for every $s\in\left[\frac{1}{2}+\sqrt{c_4}\frac{\log\log n}{\sqrt{\log n}},1\right]$. Then by (6.1), we have that a.a.s.

$$\inf\operatorname{spec}(\Delta_\phi)\geqslant\frac{1}{4}-c_4\frac{(\log\log n)^2}{\log n}.$$

To conclude Theorem 1.2, we apply precisely the same argument, using Lemma 5.4, to conclude that there is a constant $c_5>0$ such that a.a.s

$$\mathbb{M}_{P,\varphi}(s)\left(1+\mathbb{L}_{P,\varphi}^{\text{int}}(s)+\mathbb{L}_{P,\varphi}^{\text{cusp}}(s)\right)^{-1},$$

exists as a bounded operator $L^2_{\text{new}}(X_\varphi)\to H^2_{\text{new}}(X_\varphi)$ for all $s\in\left[\frac{1}{2}+\sqrt{c_5}\frac{\log\log\log n}{\sqrt{\log\log n}},1\right]$. Then

$$\operatorname{spec}(\Delta_{X_n})\cap\left[0,\frac{1}{4}-c_5\frac{(\log\log\log n)^2}{\log\log n}\right)=\operatorname{spec}(\Delta_X)\cap\left[0,\frac{1}{4}-c_5\frac{(\log\log\log n)^2}{\log\log n}\right),$$

a.a.s.

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