arXiv:2305.04624v1 [math.SP] 8 May 2023

SPECTRAL PROPERTIES OF THE RHALY OPERATOR ON WEIGHTED NULL SEQUENCE SPACES AND ASSOCIATED OPERATOR IDEALS

ARNAB PATRA, JYOTI RANI, AND SANJAY KUMAR MAHTO

Abstract. In this article, a comprehensive study is made on the continuity, compactness, and spectrum of the lower triangular terraced matrix, introduced by H. C. Rhaly, Jr. [Houston J. Math. 15(1): 137-146, 1989], acting on the weighted null sequence spaces with bounded, strictly positive weights. Several spectral subdivisions such as point spectrum, residual spectrum, and continuous spectrum are also discussed. In addition, a new class of operator ideal $\chi^{(s)}_{c_0(r)}$ associated to the Rhaly operator on weighted c_0 space is defined using the concept of s-number and it is proved that under certain condition, $\chi^{(s)}_{c_0(r)}$ forms a quasi-Banach closed operator ideal.

1. Introduction

The linear map C which maps a sequence $\{x_n\}$ of real or complex numbers to its sequence of averages $\{\frac{x_1+\cdots+x_n}{n}\}$ is the well-known discrete Cesàro operator. Extensive study can be found in the literature on the continuity and spectrum of C defined over sequence spaces such as c_0, c, ℓ_p, bv_p $(1 \leq p < \infty)$, bv_0, etc [19, 1, 8, 13, 2, 17]. Recently Albanese et al. studied the spectral properties of Cesàro operator defined over the weighted sequence spaces [3, 4, 5]. Several generalised version of the Cesàro operator such as $p-$Cesàro operator [20, 7], discrete generalized Cesàro operator [21, 30, 31], $q-$Cesàro operator [28, 10, 11], etc. are also studied. For a sequence $a = \{a_n\}$ of real or complex numbers Rhaly [20] introduced the terraced matrix R_a also known as Rhaly matrix where

$$R_a = \begin{pmatrix} a_1 & 0 & 0 & 0 & \cdots \\ a_2 & a_2 & 0 & 0 & \cdots \\ a_3 & a_3 & a_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The operator represented by the above matrix is known as the Rhaly operator. The Cesàro operator C can be obtained by taking $a_n = \frac{1}{n}$. Rhaly [20] also considered the case $a_n = \frac{1}{n^p}$, $p \in \mathbb{R}$ which is the p-cesaro operator. Many researchers investigated the spectrum of Rhaly operators defined over several classical sequence spaces such as c_0 [25, 27, 24], ℓ_p [26], bv_0 [29], etc. To the best of our knowledge, no investigation is carried out so far on the spectral properties of Rhaly operators over the weighted sequence spaces. The present work is an attempt in this direction. The aim of this

2020 *Mathematics Subject Classification.* Primary 47A10, 47B37; Secondary 46B45.
Key words and phrases. Rhaly operators, Infinite matrices, Fine spectrum, Sequence spaces.

paper is to study the continuity, compactness and spectrum of the Rhaly operator defined over the weighted null sequence space $c_0(r)$ where $r = \{r_n\}$ is the weight vector. Also using the concept of s-number sequences, we introduce a new class of operator ideal $\chi^{(s)}_{c_0(r)}$ related to the Rhaly operator R_a acting on weighted c_0 spaces. It is demonstrated that under some assumptions on the sequence $a = \{a_n\}$, the class forms an quasi-Banach closed operator ideal. Moreover, an inclusion relation among operator ideals are also proved.

The remainder of this paper is organized as follows. Section 2 contains some notations and some useful results. Section 3 deals with the continuity and compactness and various spectral subdivision of Rhaly operator defined over weighted c_0 spaces. In section 4 we introduce a new class of operator ideal $\chi^{(s)}_{c_0(r)}$ and discussed few of its properties.

2. Preliminaries and Notations

Throughout the article, all the infinite sequences and matrices are indexed by the set of natural numbers $\mathbb{N}$. Let $T : X \to Y$ be a bounded linear operator where X and Y are complex Banach spaces. The range space of T and null space of T are denoted by $R(T)$ and $N(T)$ respectively. The Banach-adjoint of T, denoted by T^*, is a bounded linear operator $T^* : Y^* \to X^*$ which is defined by

$$(T^*\phi)(x) = \phi(Tx) \quad \text{for all } \phi \in Y^* \text{ and } x \in X,$$

where X^* and Y^* are the dual spaces of X and Y respectively. $\mathcal{B}(X, Y)$ and $\mathcal{K}(X, Y)$ denote the space of all bounded linear operators and the ideal of all compact operators form X into Y. If $X = Y$, then $\mathcal{B}(X, Y)$ and $\mathcal{K}(X, Y)$ are denoted by $\mathcal{B}(X)$ and $\mathcal{K}(X)$ respectively. For any $T \in \mathcal{B}(X)$, the operator norm is denoted by $\|T\|$ and the norm $\|.\|_\infty$ denotes the supremum norm on the sequence space c_0. For any operator $T \in \mathcal{B}(X)$, the resolvent set of T is the set of all complex numbers λ for which the operator $T - \lambda I$ has a bounded inverse in X where I is the identity operator in X. The resolvent set of T is denoted by $\rho(T, X)$. The complement of the resolvent set in the complex plane $\mathbb{C}$ is called the spectrum of T and it is denoted by $\sigma(T, X)$. The set of points $\lambda \in \mathbb{C}$ for which $N(T - \lambda I) \neq \{0\}$ is called the point spectrum of T and it is denoted by $\sigma_p(T, X)$. The set of points $\lambda \in \mathbb{C}$ for which $N(T - \lambda I) = \{0\}$, and $\overline{R(T - \lambda I)} = X$ but $R(T - \lambda I) \neq X$ is called the continuous spectrum of T and denoted by $\sigma_c(T, X)$, and the set of points $\lambda \in \mathbb{C}$ for which $N(T - \lambda I) = \{0\}$ and $\overline{R(T - \lambda I)} \neq X$ is called the residual spectrum of T and it is denoted by $\sigma_r(T, X)$. The three sets $\sigma_p(T, X), \sigma_c(T, X), \sigma_r(T, X)$ are disjoint and their union is the whole spectrum $\sigma(T, X)$.

Let $\mathbb{C}^{\mathbb{N}}$ denotes the space of all complex sequences. For an infinite positive real sequence $r = \{r_k\}$, the weighted null sequence space $c_0(r)$ is defined as

$$c_0(r) = \{\{x_k\} \in \mathbb{C}^{\mathbb{N}} : \lim_{k\to\infty} r_k x_k = 0\}$$

equipped with the norm $\|x\|_r = \sup_k |x_k| r_k$. If D_r is the diagonal matrix with i-th diagonal entry r_i then, D_r is an isometric isomorphism from $c_0(r)$ to c_0 since $x = \{x_k\} \in c_0(r)$ implies $D_r x = \{x_k r_k\} \in c_0$, and $\|x\|_r = \|D_r x\|_\infty$. Hence $\|x\|_r$ defines a norm on $c_0(r)$ and $(c_0(r), \|x\|_r)$ is a Banach space under this norm. The dual of $c_0(r)$ is linearly isometric with the weighted sequence space $\ell_1(r^{-1})$ where $r^{-1} = \{\frac{1}{r_n}\}$. Also it must be noted that if $\inf_k r_k > 0$ then $c_0(r) = c_0$ and the norms are equivalent. Therefore we are interested in the case $\inf_k r_k = 0$.

We record the following lemmas which are useful in this sequel.

Lemma 2.1. [22] The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in \mathcal{B}(c_0)$ if and only if the following conditions hold,

(i) the rows of A are in ℓ_1 and their ℓ_1 norms are bounded,
(ii) the columns of A are in c_0.

The operator norm of T is given by the supremum of the ℓ_1 norms of the rows.

Lemma 2.2. [6, Theorem 1] For two weight vectors $r = \{r_n\}$, $s = \{s_n\}$ and an infinite matrix $A = (a_{nk})$, the matrix $A \in \mathcal{B}(c_0(r), c_0(s))$ if and only if the following conditions hold,

(i) $\sup\limits_{n\in\mathbb{N}} s_n \sum\limits_{k=1}^{\infty} |\frac{a_{nk}}{r_k}| < \infty$,
(ii) $\lim\limits_{n\to\infty} s_k a_{nk} = 0 \ \forall\ k \in \mathbb{N}$.

Then the norm of A is given by $\|A\|_{r,s} = \sup\limits_{n\in\mathbb{N}} s_n \sum\limits_{k=1}^{\infty} |\frac{a_{nk}}{r_k}|$.

Lemma 2.3. [12, p. 59] The bounded linear operator $T : X \to Y$ has dense range if and only if T^* is one to one.

3. Boundedness and compactness of R_a

Let $r = \{r_n\}$ and $s = \{s_n\}$ are two bounded, strictly positive weight vectors for the sequence spaces $c_0(r)$ and $c_0(s)$ respectively. For any operator $T : c_0(r) \to c_0(s)$ the operator norm is denoted by $\|T\|_{r,s}$ and if T is defined from $c_0(s)$ to $c_0(s)$ then $\|T\|_{r,s} = \|T\|_s$. Following theorem is an immediate application of Lemma 2.2.

Theorem 3.1. The Rhaly operator $R_a : c_0(r) \to c_0(s)$ is bounded if and only if

$$\left\{ s_n a_n \sum_{k=1}^{n} \frac{1}{r_k} \right\}_{n\in\mathbb{N}} \in \ell_\infty,$$

and in this case, $\|R_a\|_{r,s} = \sup\limits_{n} s_n |a_n| \sum\limits_{k=1}^{n} \frac{1}{r_k}$.

Proof. It follows from Lemma 2.2 that the Rhaly operator R_a is a bounded linear operator from $c_0(r)$ to $c_0(s)$ if and only if

$$\sup_n s_n |a_n| \sum_{k=1}^{n} \frac{1}{r_k} < \infty \text{ and } \lim_{n\to\infty} a_n s_n = 0.$$

If $\sup\limits_{n} s_n |a_n| \sum\limits_{k=1}^{n} \frac{1}{r_k} < \infty$ then there exists a positive real number M such that,

$$s_n |a_n| \sum_{k=1}^{n} \frac{1}{r_k} < M \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.1}$$

Since $r = \{r_k\}$ is a bounded sequence of strictly positive real numbers, there exists a real number $M_1 > 0$ such that $\forall n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{1}{r_k} > n \frac{1}{M_1}.$$

Finally from (3.1) it follows that

$$0 < s_n \left|a_n\right| < \frac{MM_1}{n} \quad \text{for all} \quad n \in \mathbb{N},$$

and consequently $\lim_{n\to\infty} a_n s_n = 0$. Hence $\sup_n s_n \left|a_n\right| \sum\limits_{k=1}^{n} \frac{1}{r_k} < \infty$ implies $\lim_{n\to\infty} a_n s_n = 0$. Also the expression of $\left\|R_a\right\|_{r,s}$ is a direct consequence of Lemma 2.2. □

Theorem 3.2. The Rhaly operator $R_a : c_0(r) \to c_0(s)$ is compact if and only if

$$\left\{ s_n a_n \sum_{k=1}^{n} \frac{1}{r_k} \right\}_{n\in\mathbb{N}} \in c_0.$$

Proof. Let us consider the linear operator $T_{r,s} : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ defined as

$$T_{r,s}(x) = \left\{ a_n s_n \sum_{k=1}^{n} \frac{x_k}{r_k} \right\}_{n\in\mathbb{N}}.$$

Then $D_s R_a = T_{r,s} D_r$ and consequently $R_a : c_0(r) \to c_0(s)$ is a compact map if and only if $T_{r,s} \in \mathcal{K}(c_0)$. Let $\left\{ a_n s_n \sum\limits_{k=1}^{n} \frac{1}{r_k} \right\}_{n\in\mathbb{N}} \in c_0$. Define a sequence of operators $\{T_{r,s}^{(k)}\}_{k\in\mathbb{N}}$ where $T_{r,s}^{(k)} : c_0 \to c_0$ and

$$T_{r,s}^{(k)}(x) = \left\{ a_1 s_1 \frac{x_1}{r_1}, a_2 s_2 \left(\frac{x_1}{r_1} + \frac{x_2}{r_2} \right), \cdots, a_k s_k \sum_{i=1}^{k} \frac{x_i}{r_i}, 0, 0, \cdots \right\}.$$

For each $k \in \mathbb{N}$ the operator $T_{r,s}^{(k)}$ is a finite rank operator. Now

$$\left\| (T_{r,s} - T_{r,s}^{(k)})x \right\|_\infty = \sup_{n>k} \left|a_n\right| s_n \sum_{i=1}^{n} \frac{x_i}{r_i} \leq \left\|x\right\|_\infty \sup_{n>k} \left|a_n\right| s_n \sum_{i=1}^{n} \frac{1}{r_i}.$$

Hence

$$\left\| T_{r,s} - T_{r,s}^{(k)} \right\| \leq \sup_{n>k} \left|a_n\right| s_n \sum_{i=1}^{n} \frac{1}{r_i}.$$

Since $a_n s_n \sum\limits_{i=1}^{n} \frac{1}{r_i} \to 0$ as $n \to \infty$, by taking $k \to \infty$ on both side of the above relation we get

$$\left\| T_{r,s} - T_{r,s}^{(k)} \right\| \to 0.$$

Hence the sequence of operators $\{T_{r,s}^{(k)}\}$ converges to $T_{r,s}$ under the operator norm, and this implies $T_{r,s} \in \mathcal{K}(c_0)$.

For the converse part we adopt similar approach as in Proposition 2.2 [4]. Let $R_a : c_0(r) \to c_0(s)$ is a compact operator. This implies $T_{r,s} \in \mathcal{K}(c_0)$. This also implies $T_{r,s}(B_{c_0}[0,1])$ is a relatively compact set where $B_{c_0}[0,1]$ denotes the closed unit ball in c_0. Therefore using the result of [9, p. 15] there exists a sequence $y = \{y_n\} \in c_0$ such that

$$\left|(T_{r,s}(x))_n\right| \leq \left|y_n\right| \quad \text{for all} \quad n \in \mathbb{N},\ x \in B_{c_0}[0,1].$$

Since $y \in c_0$, for every $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that,

$$\left|(T_{r,s}(x))_n\right| < \epsilon \quad \text{for all} \quad n > n_0.$$

Let us consider for each $n \in \mathbb{N}$ the sequence $\{u^{(n)}\}$ with $\{u_k^{(n)}\}_{k\in\mathbb{N}} \in B_{c_0}[0,1]$ such that for $k \in \{1, 2, \cdots, n\}$, $u_k^{(n)} = 1$ and 0 otherwise. Then

$$\left|(T_{r,s}(u^{(n)}))_n\right| = s_n \left|a_n\right| \sum_{k=1}^{n} \frac{1}{r_k} < \epsilon \ \text{ for all } \ n > n_0.$$

Hence $\left\{a_n s_n \sum\limits_{k=1}^{n} \frac{1}{r_k}\right\}_n \in c_0$ and this proves the theorem. □

Remark 3.3. Few important remarks on the above result are mentioned below.

(a) If $r_n = s_n = 1$ for all $n \in \mathbb{N}$ then both the weighted sequence spaces $c_0(r)$ and $c_0(s)$ reduce to the classical sequence space c_0. In this setting, Theorem 3.1 yields, the Rhaly operator $R_a : c_0 \to c_0$ is bounded if and only if $\{na_n\} \in \ell_\infty$. This result is proved by Leibowitz [14] in Proposition 3.2.

(b) Since $|a_n| s_n \sum_{k=1}^{n} \frac{1}{s_k} \geq |a_n|$, $n \in \mathbb{N}$, a necessary condition for $R_a \in \mathcal{B}(c_0(s))$ and $R_a \in \mathcal{K}(c_0(s))$ is that $a_n \in \ell_\infty$ and $a_n \in c_0$ respectively.

Let $s = \{s_n\}$ be a decreasing, strictly positive sequence of real numbers. Then

$$s_n \left|a_n\right| \sum_{k=1}^{n} \frac{1}{s_k} \leq n \left|a_n\right|.$$

Therefore $R_a \in \mathcal{B}(c_0(s))$ if $\{na_n\} \in \ell_\infty$ and $R_a \in \mathcal{K}(c_0(s))$ if $\{na_n\} \in c_0$. Also let $\{e_n\}$ be a sequence in $c_0(r)$ such that the n-th entry is one and all other entries are zero. Then for $n \in \mathbb{N}$,

$$\left\|R_a e_n\right\|_s = \sup_{k\geq n} s_k \left|a_k\right| \geq a_n s_n = \left|a_n\right| \left\|e_n\right\|_s.$$

This gives the following corollary.

Corollary 3.4. Let $s = \{s_n\}$ be a bounded, decreasing, strictly positive sequence of real numbers. Then $R_a \in \mathcal{B}(c_0(s))$ if $\{na_n\} \in \ell_\infty$ and $R_a \in \mathcal{K}(c_0(s))$ if $\{na_n\} \in c_0$. Also

$$\sup_n \left|a_n\right| \leq \left\|R_a\right\|_s \leq \sup_n n \left|a_n\right|.$$

The following example implies the above corollary is merely a sufficient condition.

Example 3.5. Let for $n \in \mathbb{N}$, $a_n = \frac{1}{\log(n+1)}$ and $s_n = \frac{1}{2^n}$. Then $s_n \sum_{k=1}^{n} \frac{1}{s_k} = \frac{2^{n+1}-2}{2^n}$, $n \in \mathbb{N}$. Hence

$$a_n s_n \sum_{k=1}^{n} \frac{1}{s_k} = \frac{2^{n+1}-2}{2^n \log(n+1)} \to 0 \text{ as } n \to \infty.$$

Here, $\{s_n\}$ is a bounded strictly decreasing sequence and $\{na_n\}$ is unbounded but $R_a \in \mathcal{K}(c_0(s))$.

4. Spectral properties of R_a

From now onwards, let $\{a_n\}$ is a sequence of positive real numbers such that $\lim_{n\to\infty} na_n$ exists finitely and equal to $\chi \neq 0$ and $S = \{a_n : n \in \mathbb{N}\}$. The following lemma ([29, Lemma 2.7], [19, Lemma 7]) is useful in this sequel.

Lemma 4.1. Let $\lambda \in \mathbb{C} \setminus S$ and $\alpha = \Re(\frac{1}{\lambda})$. Then the following relation holds

$$\prod_{k=1}^{n} \left|1 - \frac{a_k}{\lambda}\right| \simeq \frac{1}{n^{\alpha\chi}},$$

where the notation $a_n \simeq b_n$ means the sequences $\left\{\frac{a_n}{b_n}\right\}$ and $\left\{\frac{b_n}{a_n}\right\}$ are bounded and for any $z \in \mathbb{C}$, $\Re(z)$ denotes the real part of z.

Using the similar concept as [19, Lemma 7], we prove a slightly modified result.

Lemma 4.2. For a fixed $m \in \mathbb{N}$, let $\lambda \in \mathbb{C} \setminus \{a_k : k = m+1, m+2, \cdots\}$ and $\alpha = \Re(\frac{1}{\lambda})$. Then the following relation holds

$$\prod_{k=m+1}^{n} \left|1 - \frac{a_k}{\lambda}\right| \simeq \frac{1}{n^{\alpha\chi}}.$$

Proof. Let $\frac{1}{\lambda} = \alpha + i\beta$ where α, $\beta \in \mathbb{R}$ and $\lim_{n\to\infty} na_n = \chi$ implies $a_n \simeq \frac{\chi}{n}$. Imitating the calculations of Lemma 7 in [19] and using the fact that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$, we have the following relation

$$\prod_{k=m+1}^{n} \left|1 - \frac{a_k}{\lambda}\right| \leq \mathcal{O}(1) \exp \sum_{k=m+1}^{n} \left(-\frac{\chi}{k}\alpha + \frac{(\alpha^2+\beta^2)}{2}\frac{\chi^2}{k^2}\right). \tag{4.1}$$

Now we use the following inequality

$$\sum_{k=m+1}^{n} \frac{1}{k} \geq \int_{m}^{n} \frac{1}{x+1} dx = \log(n+1) - \log(m+1) \geq \log n - \log(m+1).$$

Therefore from (4.1) we deduce that

$$\prod_{k=m+1}^{n} \left|1 - \frac{a_k}{\lambda}\right| \leq \frac{\mathcal{O}(1)}{n^{\chi\alpha}}.$$

Also note that

$$\sum_{k=m+1}^{n} \frac{1}{k} \leq \int_{m-1}^{n-1} \frac{1}{x+1} dx = \log n - \log m.$$

Using the above inequality, in a similar way it can be proved that

$$\prod_{k=m+1}^{n} \left|1 - \frac{a_k}{\lambda}\right|^{-1} \leq \mathcal{O}(1) n^{\chi\alpha}.$$

This proves the result. □

Next we derive the point spectrum of R_a.

Theorem 4.3. Let $s = \{s_n\}$ be a bounded, strictly positive sequence such that $R_a \in \mathcal{B}(c_0(s))$. Then

$$\sigma_p(R_a, c_0(s)) = \{\lambda \in S : \lim_{n\to\infty} a_n s_n n^{\alpha\chi} = 0\}$$

where $\alpha = \Re(\frac{1}{\lambda}) = \frac{1}{\lambda}$.

Proof. Let us consider $R_a x = \lambda x$ for some $\lambda \in \mathbb{C}$ and non-zero x. This gives the system of equations

$$\left.\begin{aligned} a_1 x_1 &= \lambda x_1 \\ a_2(x_1 + x_2) &= \lambda x_2 \\ &\vdots \\ a_k(x_1 + x_2 + \cdots + x_k) &= \lambda x_k \\ &\vdots \end{aligned}\right\}. \tag{4.2}$$

If $\lambda \notin S$ then from the above equations it follows that $x_k = 0$ for all $k \in \mathbb{N}$. Hence $\sigma_p(R_a, c_0(s)) \subseteq S$.. Any element λ of S belongs to $\sigma_p(R_a, c_0(s))$ if and only if $R_a x = \lambda x$ holds for some non-zero $x = \{x_n\} \in c_0(s)$. Simplifying $R_a x = \lambda x$ we get,

$$x_n = \prod_{j=m+1}^{n} \left(\frac{\lambda a_{j-1}^{-1}}{\lambda a_j^{-1} - 1} \right) x_m, \ n = m+1, m+2, \cdots, \tag{4.3}$$

where x_m is the first non-zero entry of $\{x_n\}$. Now we rearrange the right hand side of the above equation as follows;

$$\begin{aligned} x_n = \prod_{j=m+1}^{n} \left(\frac{\lambda a_{j-1}^{-1}}{\lambda a_j^{-1} - 1} \right) x_m = \prod_{j=m+1}^{n} \left(\frac{a_j a_{j-1}^{-1}}{1 - \frac{a_j}{\lambda}} \right) x_m &= \frac{\prod_{j=m+1}^{n} a_j a_{j-1}^{-1}}{\prod_{j=m+1}^{n} \left(1 - \frac{a_j}{\lambda}\right)} x_m \\ &= \frac{\frac{a_n}{a_m}}{\prod_{j=m+1}^{n} \left(1 - \frac{a_j}{\lambda}\right)} x_m. \end{aligned}$$

From Lemma 4.2 it can be deduced that $\lambda \in \sigma_p(R_a, c_0(s))$ if and only if

$$\lim_{n \to \infty} a_n s_n n^{\alpha\chi} = 0,$$

where $\alpha = \Re(\frac{1}{\lambda}) = \frac{1}{\lambda}$.

□

Remark 4.4. Here we mention few important observations related to the point spectrum.

(i) Since the weight vector $s = \{s_n\}$ is a bounded sequence, and $a_n \simeq \frac{\chi}{n}$, for any $\lambda \in S$ satisfies $\lambda > \chi$ we have

$$a_n n^{\alpha\chi} \simeq \chi n^{\alpha\chi - 1}$$

where $\alpha = \Re(\frac{1}{\lambda}) = \frac{1}{\lambda}$. This proves that $a_n s_n n^{\alpha\chi} \to 0$ as $n \to \infty$ and consequently $\lambda \in \sigma_p(R_a, c_0(s))$. In this regard we have the following inclusion relation

$$\{\lambda \in S : \lambda > \chi\} \subseteq \sigma_p(R_a, c_0(s)).$$

(ii) Let $\lambda \in \sigma_p(R_a, c_0(s))$. Then for all $\mu \in S$ such that $\lambda \leq \mu$ the following relation holds for all $n \in \mathbb{N}$

$$a_n s_n n^{\frac{\chi}{\mu}} \leq a_n s_n n^{\frac{\chi}{\lambda}}.$$

This shows that $\mu \in \sigma_p(R_a, c_0(s))$ for all $\lambda \leq \mu$.

(iii) If $a_n = \frac{1}{n}$ and $s_n = 1$ for all $n \in \mathbb{N}$ then $\chi = 1$. In this case $a_n s_n n^{\frac{\chi}{a_k}}$ does not converge to zero for any $k \in \mathbb{N}$ and hence $\sigma_p(R_a, c_0(s)) = \emptyset$. This reflects the case of the point spectrum of the Cesàro matrix over c_0 space which is obtained by Reade [19, Theorem 1].

Theorem 4.5. Let $s = \{s_n\}$ be a bounded strictly positive sequence such that $R_a \in \mathcal{B}(c_0(s))$. Then the following statements hold

(i) $0 \notin \sigma_p(R_a^*, c_0(s)^*)$,
(ii) $S \subseteq \sigma_p(R_a^*, c_0(s)^*)$,
(iii) Let $\lambda \in \mathbb{C} \setminus \overline{S}$. Then $\lambda \in \sigma_p(R_a^*, c_0(s)^*)$ if and only if

$$\sum_{n=1}^{\infty} \frac{1}{s_n n^{\alpha\chi}} < \infty,$$

where $\alpha = \Re(\frac{1}{\lambda})$.

Proof. (i) Let the relation $R_a^* x = 0$ holds for some x. This implies $\sum_{k=n}^{\infty} a_k x_k = 0$ for all $n \in \mathbb{N}$ and eventually we have $x_n = 0$ for all $n \in \mathbb{N}$. This shows that $0 \notin \sigma_p(R_a^*, c_0(s)^*)$.

(ii) Consider the equation $R_a^* x = \lambda x$ for some $\lambda \in \mathbb{C}$. This gives the following system

$$\left.\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \cdots &= \lambda x_1 \\ a_2x_2 + a_3x_3 + a_4x_4 + \cdots &= \lambda x_2 \\ a_3x_3 + a_4x_4 + \cdots &= \lambda x_3 \\ &\vdots \end{aligned}\right\}. \tag{4.4}$$

From the above system it follows that for all $n \in \mathbb{N}$ and $n \geq 2$

$$x_n = \prod_{j=1}^{n-1} \left(1 - \frac{a_j}{\lambda}\right) x_1. \tag{4.5}$$

If $\lambda = a_l$ for some $l \in \mathbb{N}$ then from the above equation, it follows

$$x_{l+1} = x_{l+2} = \cdots = 0.$$

Hence $\{x_k\} \in \ell_1(s^{-1})$. It follows that $S \subseteq \sigma_p(R_a^*, c_0(s)^*)$.

(iii) Now let $\lambda \in \mathbb{C} \setminus \overline{S}$. Then $R_a^* x = \lambda x$ implies the equation (4.5) for $\{x_n\}$. Then using Lemma 4.1 we get for $n \in \mathbb{N}$,

$$|x_n| = \left| \prod_{j=1}^{n-1} \left(1 - \frac{a_j}{\lambda}\right) x_1 \right| \simeq \frac{1}{n^{\alpha\chi}} |x_1|,$$

where $\alpha = \Re(\frac{1}{\lambda})$. Hence $\lambda \in \sigma_p(R_a^*, c_0(s)^*)$ if and only if $\sum_{n=1}^{\infty} \frac{1}{s_n n^{\alpha\chi}} < \infty$. □

Remark 4.6. The following facts are used to obtain the next lemma. For $\lambda \in \mathbb{C} \setminus \{0\}$ and a fixed $\chi > 0$ the following statements hold

(i) $\left|\lambda - \frac{\chi}{2}\right| < \frac{\chi}{2}$ if and only if $\Re\left(\frac{1}{\lambda}\right) > \frac{1}{\chi}$,
(ii) $\left|\lambda - \frac{\chi}{2}\right| = \frac{\chi}{2}$ if and only if $\Re\left(\frac{1}{\lambda}\right) = \frac{1}{\chi}$,
(iii) $\left|\lambda - \frac{\chi}{2}\right| > \frac{\chi}{2}$ if and only if $\Re\left(\frac{1}{\lambda}\right) < \frac{1}{\chi}$.

Lemma 4.7. Let $s = \{s_n\}$ be a bounded strictly positive sequence such that $R_a \in \mathcal{B}(c_0(s))$. Then

$$\sigma_p(R_a^*, c_0(s)^*) \subseteq \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{\chi}{2}\right| < \frac{\chi}{2}\right\} \cup S.$$

Proof. It is already proved that $a_n \in \sigma_p(R_a^*, c_0(s)^*)$ for all $n \in \mathbb{N}$. Let $\lambda \notin S$ such that $\lambda \in \sigma_p(R_a^*, c_0(s)^*)$. Then from Theorem 4.5 it follows that $\sum_{n=1}^{\infty} \frac{1}{s_n n^{\alpha\chi}} < \infty$, where $\alpha = \Re(\frac{1}{\lambda})$. Now

$$\frac{1}{s_n n^{\alpha\chi}} \geq ||s||_\infty^{-1} \frac{1}{n^{\alpha\chi}}.$$

holds for all $n \in \mathbb{N}$. Hence

$$\lambda \in \sigma_p(R_a^*, c_0(s)^*) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{\alpha\chi}} < \infty \Rightarrow \alpha\chi > 1,$$

and consequently λ satisfies $\left|\lambda - \frac{\chi}{2}\right| < \frac{\chi}{2}$. This proves the lemma. □

The next result provides an estimation of the spectrum of the operator R_a.

Theorem 4.8. Let $s = \{s_n\}$ be a bounded, decreasing sequence of positive real numbers such that $R_a \in \mathcal{B}(c_0(s))$, then

$$\sigma(R_a, c_0(s)) \subseteq \left\{\lambda \in \mathbb{C} \setminus \bar{S} : \left|\lambda - \frac{\chi}{2}\right| \leq \frac{\chi}{2}\right\} \cup \bar{S}.$$

Proof. From the inclusion relation (ii) in Theorem 4.5 we have, $S \subseteq \sigma_p(R_a^*, c_0(s)^*)$. Also it is known that for a Banach space X, $\sigma_p(T^*, X^*) \subseteq \sigma(T, X)$. Therefore we have $S \subseteq \sigma(R_a, c_0(s))$ with the fact that $\sigma(R_a, c_0(s))$ is a closed subset of $\mathbb{C}$. Hence $\overline{S} \subset \sigma(R_a, c_0(s))$. Again for $\lambda \notin \overline{S}$ the inverse $(R_a - \lambda I)^{-1}$ exists and has the matrix representation

$$(R_a - \lambda I)^{-1} = (b_{nk}) = \begin{cases} \dfrac{1}{a_n - \lambda}, & n = k \\ \dfrac{-a_n}{\lambda^2 \prod\limits_{j=k}^{n} (1 - \frac{a_j}{\lambda})}, & 1 \leq k < n \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

From Lemma 2.2 it follows that $(R_a - \lambda I)^{-1} = (b_{nk}) \in \mathcal{B}(c_0(s))$ if and only if both the conditions $\sup\limits_{n\in\mathbb{N}} s_n \sum\limits_{k=1}^{n} \frac{|b_{nk}|}{s_k} < \infty$, and $\lim\limits_{n\to\infty} s_n |b_{nk}| = 0$, $\forall k \in \mathbb{N}$ are satisfied. Let $\lambda \in \left\{\lambda \in \mathbb{C} \setminus \bar{S} : \left|\lambda - \frac{\chi}{2}\right| > \frac{\chi}{2}\right\}$. Then we have for $n \in \mathbb{N}$

$$\begin{aligned} s_n \sum_{k=1}^{n} \frac{|b_{nk}|}{s_k} &= s_n \sum_{k=1}^{n-1} \frac{|b_{nk}|}{s_k} + |b_{nn}| \\ &= s_n \sum_{k=1}^{n-1} \frac{a_n}{\left|\lambda^2 \prod\limits_{j=k}^{n} (1 - \frac{a_j}{\lambda})\right| s_k} + \left|\frac{1}{a_n - \lambda}\right|. \end{aligned} \tag{4.7}$$

Also, since $\lim_{n\to\infty} \frac{n^a}{(n+1)^a} = 1$ for all real a, Lemma 4.2 yields

$$\frac{D_1}{(n+1)^{\alpha\chi}} \leq \prod_{k=1}^{n} \left|1 - \frac{a_k}{\lambda}\right| \leq \frac{D_2}{(n+1)^{\alpha\chi}}, \tag{4.8}$$

for some positive constants D_1 and D_2 and Lemma 4.2 and equation (4.8) together imply

$$M_1 a_n \frac{n^{\alpha\chi}}{k^{\alpha\chi}} \leq \frac{a_n}{\prod\limits_{j=k}^{n} \left|1 - \frac{a_j}{\lambda}\right|} \leq M_2 a_n \frac{n^{\alpha\chi}}{k^{\alpha\chi}}, \tag{4.9}$$

for some positive constants M_1 and M_2. Therefore using the relation (4.9) in the equation (4.7) we obtain for $n \in \mathbb{N}$

$$s_n \sum_{k=1}^{n} \frac{|b_{nk}|}{s_k} \quad \leq \quad M_2 \frac{s_n}{|\lambda|^2} \sum_{k=1}^{n-1} \frac{a_n n^{\alpha\chi}}{s_k k^{\alpha\chi}} + \frac{1}{d_\lambda},$$

where $d_\lambda = \text{dist}\ (\lambda, \bar{S})$. Therefore we have

$$\sup_n a_n s_n n^{\alpha\chi} \sum_{k=1}^{n-1} \frac{1}{s_k k^{\alpha\chi}} < \infty \Rightarrow \sup_n s_n \sum_{k=1}^{n} \frac{|b_{nk}|}{s_k} < \infty.$$

As $\lambda \in \left\{\lambda \in \mathbb{C} \setminus \bar{S} : \left|\lambda - \frac{\chi}{2}\right| > \frac{\chi}{2}\right\}$, this is equivalent to $\alpha\chi < 1$. In this case for $n \in \mathbb{N}$,

$$a_n n^{\alpha\chi} \sum_{k=1}^{n-1} \frac{s_n}{s_k k^{\alpha\chi}} \quad \leq \quad (na_n) n^{\alpha\chi - 1} \sum_{k=1}^{n-1} \frac{1}{k^{\alpha\chi}} \leq D_1 n^{\alpha\chi - 1} \sum_{k=1}^{n-1} \frac{1}{k^{\alpha\chi}} \tag{4.10}$$

for some positive constant D_1 as $\{na_n\}$ is convergent sequence and $\{s_n\}$ is decreasing. Also

$$\sum_{k=1}^{n-1} \frac{1}{k^{\alpha\chi}} \leq \int\limits_0^{n-1} \frac{1}{x^{\alpha\chi}} dx = \lim_{\epsilon \to 0+} \int\limits_{\epsilon}^{n-1} \frac{1}{x^{\alpha\chi}} dx = \frac{(n-1)^{1-\alpha\chi}}{1 - \alpha\chi}.$$

The above inequality and (4.10) together imply

$$\sup_n a_n s_n n^{\alpha\chi} \sum_{k=1}^{n-1} \frac{1}{s_k k^{\alpha\chi}} < \infty.$$

Also from (4.9) we have for each $k \in \mathbb{N}$ and large n

$$0 \leq s_n |b_{nk}| = \frac{s_n a_n}{\left|\lambda^2 \prod\limits_{j=k}^{n} (1 - \frac{a_j}{\lambda})\right|} \leq M_2 a_n s_n \frac{n^{\alpha\chi}}{|\lambda|^2 k^{\alpha\chi}}.$$

Hence $\lim_{n\to\infty} a_n s_n n^{\alpha\chi} = 0 \Rightarrow \lim_{n\to\infty} s_n |b_{nk}| = 0$.

Again $0 \leq a_n s_n n^{\alpha\chi} = (na_n) s_n n^{\alpha\chi - 1} \leq D_2 n^{\alpha\chi - 1}$ holds for some positive constant D_2 as $\{na_n\}$ and $\{s_n\}$ are bounded. This also proves that $\lim\limits_{n\to\infty} a_n s_n n^{\alpha\chi} = 0$. Hence $(R_a - \lambda I)^{-1}$ exists and $(R_a - \lambda I)^{-1} \in \mathcal{B}(c_0(s))$ and consequently $\lambda \in \rho(R_a, c_0(s))$. Finally we have

$$\left\{\lambda \in \mathbb{C} \setminus \bar{S} : \left|\lambda - \frac{\chi}{2}\right| > \frac{\chi}{2}\right\} \subseteq \rho(R_a, c_0(s)).$$

This proves the result.

□

Now we summarize all the result on the spectrum and fine spectrum of the Rhaly operator over the sequence space $c_0(s)$. Consider the following sets

$$\begin{aligned} A_1 &= \left\{\lambda \in S : \lim_{n\to\infty} a_n s_n n^{\alpha\chi} = 0\right\}, \\ A_2 &= \left\{\lambda \in \mathbb{C} \setminus (S \cup \{0\}) : \sum_{n=1}^{\infty} \frac{1}{s_n n^{\alpha\chi}} < \infty\right\}, \end{aligned}$$

where $\alpha = \Re(\frac{1}{\lambda})$. In this setting we have the following result.

Theorem 4.9. Let $\{s_n\}$ be a bounded and strictly positive sequence such that $R_a \in \mathcal{B}(c_0(s))$ and $\lim_{n\to\infty} na_n = \chi \neq 0$ then the following statements hold

(i) $\sigma_p(R_a, c_0(s)) = A_1$,
(ii) $\sigma_p(R_a^*, c_0(s)^*) = A_2 \cup S$,
(iii) $\sigma_r(R_a, c_0(s)) = (A_2 \cup S) \setminus A_1 = A_2 \cup (S \setminus A_1)$.

In addition, if the sequence $\{s_n\}$ is a decreasing sequence then

(iv) $\sigma(R_a, c_0(s)) \subseteq \left\{\lambda \in \mathbb{C} \setminus \bar{S} : \left|\lambda - \frac{\chi}{2}\right| \leq \frac{\chi}{2}\right\} \cup \bar{S}$,
(v) $\{0\} \subseteq \sigma_c(R_a, c_0(s)) \subseteq (\left\{\lambda \in \mathbb{C} \setminus \bar{S} : \left|\lambda - \frac{\chi}{2}\right| \leq \frac{\chi}{2}\right\} \cup \bar{S}) \setminus (A_2 \cup S)$.

Proof. The relations (i) and (ii) are direct consequences of Theorem 4.3 and Theorem 4.5 respectively. Also from lemma 2.3 it follows that $\sigma_r(R_a, c_0(s)) = \sigma_p(R_a^*, c_0(s)^*) \setminus \sigma_p(R_a, c_0(s))$ which gives the expression for $\sigma_r(R_a, c_0(s))$ in (iii). The relation (iv) follows from Theorem 4.8. The first inclusion relation in (v) follows from Theorem 4.3 and Theorem 4.5(i) and the other inclusion relation in (v) follows from the fact that $\sigma_p(R_a, c_0(s))$, $\sigma_r(R_a, c_0(s))$, and $\sigma_c(R_a, c_0(s))$ forms a partition of $\sigma(R_a, c_0(s))$. □

5. Operator Ideal

The theory of operator ideals holds a distinctive place in functional analysis because of its numerous applications in spectrum theory, Banach space geometry, the theory of eigenvalue distributions, etc. Many researchers have developed classes of operator ideals that result from operators acting over sequence spaces [15, 16, 23]. In this section, we introduce a new class of operator ideal with the help of s-number sequences and discussed of its properties. Let X, Y, Z, X_0 and Y_0 are the Banach spaces, $\mathcal{B}$ denotes the class of all bounded linear operators between any pair of Banach spaces, X' denotes the dual of X, x' denotes the continuous linear functional on X. The sequences $r = \{r_n\}$, $t = \{t_n\}$ and $u = \{u_n\}$ are the weight vectors corresponding to sequence spaces $c_0(r)$, $c_0(t)$ and $c_0(u)$ respectively. Let $x' \in X'$ and $y \in Y$ then the mapping $(x' \otimes y) : X \to Y$ is defined by $(x' \otimes y)(x) = x'(x)y$, $x \in X$. Before proceeding, we list certain known definitions and results which are necessary for our results.

Definition 5.1. [18, p.79] A map $s : \mathcal{B} \to \mathbb{R}^{\mathbb{N}}$ which assigns every operator $\phi \in \mathcal{B}$ to a non-negative sequence $\{s_n(\phi)\}$ is called a s-number sequence if it satisfies the following properties:

(i) $\|\phi\| = s_1(\phi) \geq s_2(\phi) \geq \cdots \geq 0$ for $\phi \in \mathcal{B}(X, Y)$,
(ii) $s_{m+n-1}(\phi + \psi) \leq s_m(\phi) + s_n(\psi)$ for $\phi, \psi \in \mathcal{B}(X, Y)$,
(iii) $s_n(\zeta\phi\eta) \leq \|\zeta\| s_n(\phi) \|\eta\|$ for $\zeta \in \mathcal{B}(Y, Y_0)$, $\phi \in \mathcal{B}(X, Y)$, $\eta \in \mathcal{B}(X_0, X)$,
(iv) If $\text{rank}(\phi) < n$, then $s_n(\phi) = 0$,

(v) $s_n(I_2 : \ell_2^{(i)} \to \ell_2^{(i)}) = 1$, where I_2 denotes the identity operator on the i-dimensional Hilbert space $\ell_2^{(i)}$.

We call $s_n(\phi)$ the n-th s-number of the operator ϕ.

Definition 5.2. [18, p.25] For a subset $\mathcal{A}(X,Y)$ of $\mathcal{B}(X,Y)$, the class $\mathcal{A} = \cup_{X,Y}\mathcal{A}(X,Y)$ is said to be an operator ideal if each component $\mathcal{A}(X,Y)$ satisfies the following conditions:

(i) $x' \otimes y \in \mathcal{A}(X,Y)$ for $x' \in X'$ and $y \in Y$,
(ii) $\phi + \psi \in \mathcal{A}(X,Y)$ for $\phi, \psi \in \mathcal{A}(X,Y)$,
(iii) $\zeta\phi\eta \in \mathcal{A}(X_0,Y_0)$ for $\eta \in \mathcal{B}(X_0,X)$, $\phi \in \mathcal{A}(X,Y)$, $\zeta \in \mathcal{B}(Y,Y_0)$.

Definition 5.3. [18, p.25] A function $\Gamma : \mathcal{A} \to \mathbb{R}^+$ which assigns to every operator $\phi \in \mathcal{A}$ a non-negative number $\Gamma(\phi)$ is called a quasi-norm on the operator ideal $\mathcal{A}$ if it satisfies the following properties:

(i) $\Gamma(x' \otimes y) = \|x'\|\|y\|$ for $x' \in X'$ and $y \in Y$,
(ii) $\Gamma(\phi + \psi) \leq c(\Gamma(\phi) + \Gamma(\psi))$ for $\phi, \psi \in \mathcal{A}(X,Y)$ where $c \geq 1$ is a constant,
(iii) $\Gamma(\zeta\phi\eta) \leq \|\zeta\|\Gamma(\phi)\|\eta\|$ for $\eta \in \mathcal{B}(X_0,X), \phi \in \mathcal{A}(X,Y), \zeta \in \mathcal{B}(Y,Y_0)$.

Definition 5.4. [18, p.81] An s-number s is called multiplicative if

$$s_{m+n-1}(\phi\psi) \leq s_m(\phi)s_n(\psi) \text{ for } \psi \in \mathcal{L}(X,Y) \text{ and } \phi \in \mathcal{L}(Y,Z).$$

Definition 5.5. [18, p.26] An operator ideal $\mathcal{A}$ is closed if all components $\mathcal{A}(X,Y)$ are closed linear subsets of $\mathcal{B}(X,Y)$.

The following lemma is useful in this context.

Lemma 5.6. [18, p.80] If $\phi, \psi \in \mathcal{B}(X,Y)$ then $|s_i(\phi) - s_i(\psi)| \leq \|\phi - \psi\|,\ i \in \mathbb{N}$.

Now we define sequence space $\chi_{c_0(r)}$ associated to the Rhaly operator R_a as follows

$$\chi_{c_0(r)} = \{v \in \mathbb{C}^{\mathbb{N}} : R_a(v) \in c_0(r)\}.$$

Thus

$$\chi_{c_0(r)} = \left\{ v \in \mathbb{C}^{\mathbb{N}} : \lim_{i\to\infty} \left(\sum_{j=1}^{i} a_i v_j \right) r_i = 0 \right\}.$$

An operator $\phi \in \mathcal{B}(X,Y)$ is called s-type $\chi_{c_0(r)}$ operator if

$$\lim_{i\to\infty} \left(a_i \sum_{j=1}^{i} s_j(\phi) \right) r_i = 0$$

We denote the class of all s-type $\chi_{c_0(r)}$ operators by $\chi_{c_0(r)}^{(s)}$ i.e.,

$$\chi_{c_0(r)}^{(s)} = \left\{ \phi \in \mathcal{B} : \lim_{i\to\infty} \left(a_i \sum_{j=1}^{i} s_j(\phi) \right) r_i = 0 \right\}.$$

If $\phi \in \mathcal{B}(X,Y)$ then

$$\chi_{c_0(r)}^{(s)}(X,Y) = \left\{ \phi \in \mathcal{B}(X,Y) : \lim_{i\to\infty} \left(a_i \sum_{j=1}^{i} s_j(\phi) \right) r_i = 0 \right\}.$$

Lemma 5.7. If $r_n \leq t_n$ for all n then $\chi_{c_0(t)}^{(s)} \subseteq \chi_{c_0(r)}^{(s)}$.

Proof. Suppose $\phi \in \chi^{(s)}_{c_0(t)}$ then

$$\lim_{i\to\infty}\left(a_i\sum_{j=1}^{i}s_j(\phi)\right)t_i=0$$

Now,

$$\lim_{i\to\infty}\left(a_i\sum_{j=1}^{i}s_j(\phi)\right)r_i\leq\lim_{i\to\infty}\left(a_i\sum_{j=1}^{i}s_j(\phi)\right)t_i=0$$

Thus $\phi\in\chi^{(s)}_{c_0(r)}$. □

Theorem 5.8. The following statements hold,

(i) $\chi^{(s)}_{c_0(r)}$ is an operator ideal if $\{a_n\}\in c_0(r)$,
(ii) $\chi^{(s)}_{c_0(r)}$ is a closed operator ideal if $\{na_n\}\in c_0(r)$.

Proof. (i) Let X and Y are any two Banach spaces. Let $x'\in X'$ and $y\in Y$. Then $x'\otimes y$ is a rank one operator so $s_i(x'\otimes y)=0$ for all $i\geq 2$. Consider,

$$\begin{aligned}\lim_{i\to\infty}\left(a_i\sum_{j=1}^{i}s_j(x'\otimes y)\right)r_i&=\lim_{i\to\infty}a_is_1(x'\otimes y)r_i\\&=\|x'\otimes y\|\lim_{i\to\infty}a_ir_i\\&=0.\end{aligned}$$

Thus, $(x'\otimes y)\in\chi^{(s)}_{c_0(r)}(X,Y)$.
Suppose $\phi,\psi\in\chi^{(s)}_{c_0(r)}(X,Y)$. By using properties of s-number, we have

$$\begin{aligned}\lim_{i\to\infty}a_i\left(\sum_{j=1}^{i}s_j(\phi+\psi)\right)r_i&=\lim_{i\to\infty}\left(a_i\left(\sum_{j=1}^{i}s_{2j-1}(\phi+\psi)+\sum_{j=1}^{i}s_{2j}(\phi+\psi)\right)\right)r_i\\&\leq\lim_{i\to\infty}\left(a_i\left(\sum_{j=1}^{i}s_{2j-1}(\phi+\psi)+\sum_{j=1}^{i}s_{2j-1}(\phi+\psi)\right)\right)r_i\\&=2\lim_{l\to\infty}a_i\left(\sum_{j=1}^{i}s_{2j-1}(\phi+\psi)\right)r_i\\&\leq 2a_i\lim_{i\to\infty}\left(\sum_{j=1}^{i}(s_j(\phi)+s_j(\psi))\right)r_i\\&=2\lim_{i\to\infty}a_i\left(\sum_{j=1}^{i}s_j(\phi)\right)r_i+2\lim_{i\to\infty}a_ia_n\left(\sum_{j=1}^{i}s_j(\psi)\right)r_i\\&=0.\end{aligned}$$

This implies, $\lim_{i\to\infty} a_i \left(\sum_{j=1}^{i} s_j(\phi+\psi)\right) r_i = 0$. Thus, $\phi+\psi \in \chi^{(s)}_{c_0(r)}(X,Y)$.
Suppose $\zeta \in \mathcal{B}(Y,Y_0), \eta \in \mathcal{B}(X_0,X)$ and $\phi \in \chi^{(s)}_{c_0(r)}(X,Y)$. Then

$$\lim_{i\to\infty} a_i \left(\sum_{j=1}^{i} s_j(\zeta\phi\eta)\right) r_i = \|\zeta\|\|\eta\| \lim_{i\to\infty} a_i \left(\sum_{j=1}^{i} s_j(\phi)\right) r_i = 0.$$

Thus, $\zeta\phi\eta \in \chi^{(s)}_{c_0(r)}(X_0,Y_0)$. Hence $\chi^{(s)}_{c_0(r)}$ is an operator ideal.
(ii) Let $\{\phi_k\}$ be any sequence in $\chi^{(s)}_{c_0(r)}(X,Y)$ converging to $\phi \in \mathcal{B}(X,Y)$ with respect to the operator norm. Given $\epsilon > 0$, we can fix $k_0 \in \mathbb{N}$ such that

$$\|\phi - \phi_k\| \leq \epsilon \text{ for } k \geq k_0.$$

Now,

$$\begin{aligned}
\lim_{i\to\infty} \left(a_i \sum_{j=1}^{i} s_j(\phi)\right) r_i &= \lim_{i\to\infty} \left(a_i \sum_{j=1}^{i} s_j(\phi-\phi_k+\phi_k)\right) r_i \\
&\leq \lim_{i\to\infty} \left(a_i \sum_{j=1}^{i} s_1(\phi-\phi_k)\right) r_i + \lim_{i\to\infty} \left(a_i \sum_{j=1}^{i} s_j(\phi_k)\right) r_i \\
&= \|\phi-\phi_k\| \lim_{i\to\infty} i a_i r_i + \lim_{i\to\infty} \left(a_i \sum_{j=1}^{i} s_j(\phi_k)\right) r_i \\
&= 0.
\end{aligned}$$

Hence, $\phi \in \chi^{(s)}_{c_0(r)}(X,Y)$ as $\lim_{i\to\infty} \left(a_i \sum_{j=1}^{i} s_j(\phi)\right) r_i = 0$. The required result is proved.

□

Define a mapping $Q^{(s)} : \chi^{(s)}_{c_0(r)} \to \mathbb{R}^+$ where $\mathbb{R}^+$ is the set of all positive real numbers, by

$$Q^{(s)}(\phi) = \sup_i \left| a_i \sum_{j=1}^{i} s_j(\phi) \right| r_i \text{ where } \phi \in \chi^{(s)}_{c_0(r)}.$$

The next result proves that under certain condition $Q^{(s)}$ forms a quasi norm on $\chi^{(s)}_{c_0(r)}$.

Theorem 5.9. If $\sup_i |a_i| r_i = 1$ then the mapping $Q^{(s)}$ is a quasi norm on the set $\chi^{(s)}_{c_0(r)}$ and the operator ideal $\chi^{(s)}_{c_0(r)}$ is complete under the quasi-norm $Q^{(s)}$.

Proof. Let X and Y are any two Banach spaces and $x' \in X'$,$y \in Y$. Now

$$\begin{aligned}
Q^{(s)}(x'\otimes y) &= \sup_i \left| a_i \sum_{j=1}^{i} s_j(x'\otimes y) \right| r_i \\
&= \sup_i |a_i s_1(x'\otimes y)| \, r_i \\
&= \|x'\otimes y\| \sup_i a_i r_i \\
&= \|x'\otimes y\|.
\end{aligned}$$

Clearly $\|x' \otimes y\| = \|x'\|\|y\|$, so $Q^{(s)}(x' \otimes y) = \|x'\|\|y\|$.
Suppose $\phi, \psi \in \chi^{(s)}_{c_0(r)}(X, Y)$ and consider

$$\begin{aligned}
Q^{(s)}(\phi+\psi) =& \sup_i \left| a_i \sum_{j=1}^{i} s_j(\phi+\psi) \right| r_i \\
\leq& 2 \sup_i \left| a_i \sum_{j=1}^{i} (s_j(\phi) + s_j(\psi)) \right| r_i \\
=& 2 \sup_i \left| a_i \left(\sum_{j=1}^{i} s_j(\phi) \right) r_i + a_i \left(\sum_{j=1}^{i} s_j(\psi) \right) r_i \right| \\
\leq& 2 \sup_i \left| a_i \sum_{j=1}^{i} s_j(\phi) \right| r_i + 2 \sup_i \left| a_i \sum_{j=1}^{i} s_j(\psi) \right| r_i \\
\leq& 2 \left(Q^{(s)}(\phi) + Q^{(s)}(\psi) \right).
\end{aligned}$$

Thus,

$$Q^{(s)}(\phi+\psi) \leq 2 \left(Q^{(s)}(\phi) + Q^{(s)}(\psi) \right).$$

Let $\phi \in \chi^{(s)}_{c_0(r)}(X, Y)$ and $\zeta \in \mathcal{B}(Y, Y_0)$, $\eta \in \mathcal{B}(X_0, X)$.

$$\begin{aligned}
Q^{(s)}(\zeta\phi\eta) =& \sup_i \left| a_i \sum_{j=1}^{i} s_j(\zeta\phi\eta) \right| r_i \\
\leq& \|\zeta\|\|\eta\| \sup_i \left| a_i \left(\sum_{j=1}^{i} s_j(\phi) \right) \right| r_i.
\end{aligned}$$

This implies,

$$Q^{(s)}(\zeta\phi\eta) \leq \|\zeta\| Q^{(s)}(\phi) \|\eta\|.$$

Hence, $Q^{(s)}$ is quasi-norm on operator ideal $\chi^{(s)}_{c_0(r)}$.
Also, suppose $\phi \in \chi^{(s)}_{c_0(r)}(X, Y)$. We have

$$\begin{aligned}
Q^{(s)}(\phi) =& \sup_i \left| a_i \sum_{j=1}^{i} s_j(\phi) \right| r_i \\
\geq& \sup_i s_1(\phi) a_i r_i \\
=& \|\phi\| \sup_i a_i r_i \\
=& \|\phi\|.
\end{aligned}$$

This implies,

$$\|\phi\| \leq Q^{(s)}(\phi) \text{ for } \phi \in \chi^{(s)}_{c_0(r)}(X, Y). \tag{5.1}$$

□

Let $\{\phi_n\}$ be a Cauchy sequence in $\chi^{(s)}_{c_0(r)}(X,Y)$. Then for $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$Q^{(s)}(\phi_n - \phi_m) < \epsilon \text{ for all } n, m \geq k. \tag{5.2}$$

By using equation (5.1), we have

$$\|\phi_n - \phi_m\| < \epsilon \text{ for all } n, m \geq k.$$

This implies $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{B}(X,Y)$. Since $\mathcal{B}(X,Y)$ is a Banach space, there exists $\phi \in \mathcal{B}(X,Y)$ such that $\phi_n \to \phi$ as $n \to \infty$. Using Lemma 5.6, we have

$$|s_i(\phi_n - \phi_m) - s_i(\phi - \phi_m)| \leq \|\phi_n - \phi\|, \ i \in \mathbb{N}.$$

As $\phi_n \to \phi$, taking $n \to \infty$ and keeping m fixed we have

$$s_i(\phi_n - \phi_m) \to s_i(\phi - \phi_m), \ i \in \mathbb{N}. \tag{5.3}$$

Now, we shall show that $\phi_n \to \phi$ as $n \to \infty$ in $\chi^{(s)}_{c_0(r)}(X,Y)$. Again, from equation (5.2), we have

$$\sup_i \left| a_i \sum_{j=1}^{i} s_j(\phi_n - \phi_m) \right| r_i < \epsilon \text{ for all } m \geq k.$$

Keeping m fixed and letting $n \to \infty$, we obtain on using equation (5.3)

$$\sup_i \left| a_i \sum_{j=1}^{i} s_j(\phi - \phi_m) \right| r_i < \epsilon \text{ for all } n, m \geq k. \tag{5.4}$$

which gives $Q^{(s)}(\phi - \phi_m) < \epsilon$ for all $m \geq k$. This implies $\phi_m \to \phi$ under the quasi norm $Q^{(s)}$. Again,

$$\begin{aligned}
\sum_{j=1}^{i} s_j(\phi) = \sum_{j=1}^{i} s_{2j-1}(\phi) + \sum_{j=1}^{i} s_{2j}(\phi) \leq & 2 \sum_{j=1}^{i} s_{2j-1}(\phi) \\
= & 2 \sum_{j=1}^{i} s_{2j-1}(\phi - \phi_m + \phi_m) \\
\leq & 2 \left(\sum_{j=1}^{i} s_j(\phi - \phi_m) + \sum_{j=1}^{i} s_j(\phi_m) \right).
\end{aligned}$$

Thus,

$$\sum_{j=1}^{i} s_j(\phi) \leq 2 \left(\sum_{j=1}^{i} s_j(\phi - \phi_m) + \sum_{j=1}^{i} s_j(\phi_m) \right). \tag{5.5}$$

The above inequality implies that

$$\begin{aligned}
\left| a_i \sum_{j=1}^{i} s_j(\phi) \right| r_i \leq & 2 \left| a_i \left(\sum_{j=1}^{i} s_j(\phi - \phi_m) + \sum_{j=1}^{i} s_j(\phi_m) \right) \right| r_i \\
\leq & 2 \left| a_i \sum_{j=1}^{i} s_j(\phi - \phi_m) \right| r_i + 2 \left| a_i \sum_{j=1}^{i} s_j(\phi_m) \right| r_i.
\end{aligned}$$

By using relation (5.4), we obtain that

$$\left|a_i \sum_{j=1}^{i} s_j(\phi)\right| r_i \leq 2\epsilon + 2\left|a_i \sum_{j=1}^{i} s_j(\phi_m)\right| r_i \ \forall m \geq k.$$

Taking limit on both sides, we obtain that

$$\lim_{i\to\infty}\left|a_i \sum_{j=1}^{i} s_j(\phi)\right| r_i \leq 2\epsilon + 2\lim_{i\to\infty}\left|a_i \sum_{j=1}^{i} s_j(\phi_m)\right| r_i$$

Then,

$$\lim_{i\to\infty}\left|a_i \sum_{j=1}^{i} s_j(\phi)\right| r_i \leq 2\epsilon.$$

Thus, $\phi \in \chi^{(s)}_{c_0(r)}(X,Y)$, and this implies, $\chi^{(s)}_{c_0(r)}$ is complete under quasi-norm $Q^{(s)}$.

References

[1] Ali M Akhmedov and Feyzi Başar. On the fine spectrum of the Cesàro operator in c_0. *Math. J. Ibaraki Univ.*, 36:25–32, 2004.

[2] Ali M Akhmedov and Feyzi Başar. The fine spectra of the Cesàro operator C_1 over the sequence space bv_p, $(1 \leq p < \infty)$. *Math. J. Okayama Univ.*, 50:135–147, 2008.

[3] Angela A Albanese, José Bonet, and Werner J Ricker. Spectrum and compactness of the Cesàro operator on weighted ℓ_p spaces. *J. Aust. Math. Soc.*, 99(3):287–314, 2015.

[4] Angela A Albanese, José Bonet, and Werner J Ricker. Mean ergodicity and spectrum of the Cesàro operator on weighted c_0 spaces. *Positivity*, 20(4):761–803, 2016.

[5] Angela A Albanese, José Bonet, and Werner J Ricker. The Cesàro operator in weighted ℓ_1 spaces. *Math. Nachr.*, 291(7):1015–1048, 2018.

[6] Riddhick Birbonshi, Arnab Patra, and PD Srivastava. Infinite matrices bounded on weighted c_0 space. In *International Conference on Mathematics and Computing*, pages 331–338. Springer, 2017.

[7] C. Coşkun. The spectra and fine spectra for p-Cesaro operators. *Turkish J. Math.*, 21(2):207–212, 1997.

[8] Guillermo P Curbera and Werner J Ricker. Spectrum of the Cesàro operator in ℓ_p. *Arch. Math. (Basel)*, 100(3):267–271, 2013.

[9] Joseph Diestel. *Sequences and series in Banach spaces*, volume 92. Springer Science & Business Media, 2012.

[10] Nuh Durna and MERVE ESRA Turkay. The spectrum of q-cesaro matrices on c and its various spectral decomposition for $0 < q < 1$. *Oper. Matrices*, 15(3):795–813, 2021.

[11] Saad R El-Shabrawy. On q-cesàro operators: Boundedness, compactness and spectra. *Numer. Funct. Anal. Optim.*, 42(9):1019–1037, 2021.

[12] Israel Gohberg, Seymour Goldberg, and Marinus A. Kaashoek. *Classes of linear operators.* Birkhäuser, 1990.

[13] M. Gonzalez. The fine spectrum of the Cesàro operator in $\ell_p(1 < p < \infty)$. *Arch. Math. (Basel)*, 44(4):355–358, 1985.

[14] G. Leibowitz. Rhaly Matrices. *J. Math. Anal. Appl.*, 128(1):272–286, 1987.

[15] Amit Maji and PD Srivastava. On operator ideals using weighted cesaro sequence space. *Journal of the Egyptian Mathematical Society*, 22(3):446–452, 2014.

[16] Amit Maji and PD Srivastava. On some geometric properties of generalized musielak-orlicz sequence space and corresponding operator ideals. *Banach Journal of Mathematical Analysis*, 9(4):14–33, 2015.

[17] J. I. Okutoyi. On the spectrum of C_1 as an operator on bv_0. *J. Aust. Math. Soc.*, 48(1):79–86, 1990.

[18] Albrecht Pietsch. *Eigenvalues and s-Numbers.* Cambridge University Press, 1986.

[19] JB Reade. On the spectrum of the cesaro operator. *Bull. Lond. Math. Soc.*, 17(3):263–267, 1985.

[20] H. C. Rhaly. p - Cesàro matrices. *Houston J. Math.*, 15(1):137–146, 1989.
[21] HC Rhaly Jr. Discrete generalized cesaro operators. *Proc. Amer. Math. Soc.*, pages 405–409, 1982.
[22] A. Wilansky. *Summability through functional analysis*. North-Holland Mathematics Studies, 85, Amsterdam, Netherlands, 1984.
[23] Taja Yaying, Bipan Hazarika, and M Mursaleen. On sequence space derived by the domain of q-cesàro matrix in ℓ_p space and the associated operator ideal. *Journal of Mathematical Analysis and Applications*, 493(1):124453, 2021.
[24] M. Yildirim. On the spectrum and fine spectrum of the compact Rhally operators. *Indian J. Pure Appl. Math.*, 27(8):779–784, 1996.
[25] M Yildirim. On the spectrum of the rhaly operators on c_0 and c. *Indian J. Pure Appl. Math.*, 29(12):1301–1309, 1998.
[26] M Yildirim. On the spectrum of the Rhaly operators on ℓ_p. *Indian J. Pure Appl. Math*, 32(2):191–198, 2001.
[27] M Yildirim. The fine spectra of the rhaly operators on c_0. *Turkish J. Math.*, 26(3):273–282, 2002.
[28] Merve Esra Yildirim. The spectrum and fine spectrum of q-cesàro matrices with $0 < q < 1$ on c_0. *Numer. Funct. Anal. Optim*, 41(3):361–377, 2020.
[29] Mustafa Yildirim. On the spectrum of the Rhaly operators on bv_0. *Commun. Korean Math. Soc.*, 18(4):669–676, 2003.
[30] Mustafa Yildirim and N Durna. The spectrum and some subdivisions of the spectrum of discrete generalized cesàro operators on ℓ_p, $(1 < p < \infty)$. *J. Inequal. Appl.*, 2007:193, 2007.
[31] Mustafa Yildirim, Mohammad Mursaleen, and N Durna. The spectrum and fine spectrum of generalized rhaly-cesàro matrices on c_0 and c. *Oper. Matrices*, 12(4):955–975, 2018.

Department of mathematics, Indian Institute of Technology Bhilai, Chhattisgarh, India 492015
Email address: arnabp@iitbhilai.ac.in

Department of mathematics, Indian Institute of Technology Bhilai, Chhattisgarh, India 492015
Email address: jyotir@iitbhilai.ac.in

Department of mathematics, RNAR College Samastipur, LNMU Darbhanga, India
Email address: skmahto0777@gmail.com