

Semi-galois Categories IV: A deformed reciprocity law for Siegel modular functions

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Abstract

This paper is a sequel to our previous work [19], where we proved the “*modularity theorem*” for *algebraic Witt vectors* over imaginary quadratic fields. This theorem states that, in the case of imaginary quadratic fields K , the algebraic Witt vectors over K are precisely those generated by the *modular vectors* whose components are given by special values of deformation family of Fricke modular functions; arithmetically, this theorem implies certain congruences between special values of modular functions that are not necessarily galois conjugate. In order to take a closer look at this modularity theorem, the current paper extends it to the case of CM fields. The main results include (i) a construction of algebraic Witt vectors from special values of deformation family of Siegel modular functions on Siegel upper-half space $\mathfrak{H}_g$ given by ratios of theta functions, and (ii) a galois-theoretic characterization of which algebraic Witt vectors arise in this modular way, intending to exemplify a general galois-correspondence result which is also proved in this paper.

1 Introduction

The goal of this paper is to extend our previous work on the *modularity theorem for algebraic Witt vectors* over imaginary quadratic fields (Theorem 4.3.1, [19]) to the case of CM fields in particular. This theorem states that, if K is an imaginary quadratic field, the K -algebra $E_K = K \otimes_{O_K} W_{O_K}^a(O_{\bar{K}})$ of *algebraic Witt vectors* over K is identical to the K -algebra $M_K = K[\hat{f}_a \mid a \in \mathbb{Q}^2/\mathbb{Z}^2]$ generated by the *modular vectors* $\hat{f}_a$, whose components are given by special values of Fricke’s modular functions $f_a : \mathfrak{H} \rightarrow \mathbb{C}$ ($a \in \mathbb{Q}^2/\mathbb{Z}^2$), cf. §2. This theorem connects two objects (algebraic/analytic) of different nature; and technically this theorem is proved using *Shimura’s reciprocity law*, which relates special values of *galois-conjugate* modular functions (§6.8 [12]). But compared to Shimura’s reciprocity law, our modularity theorem is characterized by the fact that it relates, by congruences, special values of modular functions living at different levels that are *not necessarily galois conjugate*; in this sense, our modularity theorem strengthens/deforms Shimura’s reciprocity law in a way analogous to *Hida theory*, e.g. mod- p congruences between Fourier coefficients of non-galois-conjugate cusp forms [8]. In view of that Shimura’s reciprocity law is now known for more general Shimura varieties [9, 10], our modularity theorem would eventually be a facet of some more general theme. To pursue this prospect, the current paper provides further results of that general direction, working on the case of Siegel modular functions.

Our proof of the modularity theorem $E_K = M_K$ consisted of three major steps; the extension to CM field in this paper will basically follow this line. The first step is to see that we can identify the

algebraic Witt vectors $\xi \in E_K$ with locally constant, K^{ab} -valued, G_K -equivariant functions on the profinite Deligne-Ribet monoid DR_K (cf. Corollary 3.1.6 [19]), formally, $E_K \simeq \text{Hom}_{G_K}(DR_K, K^{ab})$; this identification is a consequence of the fact that the algebraic Witt vectors $\xi \in E_K$ generate all *galois objects* [17] of the semi-galois category $(\mathcal{C}_K, F_K)$ of certain Λ -rings whose fundamental monoid $\pi_1(\mathcal{C}_K, F_K)$ was known isomorphic to DR_K [3]; this identification itself holds for arbitrary number fields K and thus is used in the current paper too. The second step is to construct algebraic Witt vectors using special values of the family $\{f_a\}$ of Fricke's modular functions $f_a : \mathfrak{H} \rightarrow \mathbb{C}$ indexed by $a \in \mathbb{Q}^2/\mathbb{Z}^2$; thanks to the first step, the second step is reduced to constructing locally constant, K^{ab} -valued, G_K -equivariant functions on DR_K from these f_a 's, for which the adelic description of the Deligne-Ribet monoid DR_K by Yalkinoglu [21], $DR_K \simeq \hat{O}_K \times_{\hat{O}_K^\times} (\mathbb{A}_{K,f}^\times / K^\times)$, plays an auxiliary role. The third step is to prove that these algebraic Witt vectors constructed from f_a 's (namely the modular vectors $\hat{f}_a \in E_K$) actually generate E_K over K ; this last part is our key step, in which we utilized the *galois-orbit decompositions* of finite Deligne-Ribet monoids $DR_{\mathfrak{f}}$ for conductors $\mathfrak{f} \in I_K$ given as $DR_{\mathfrak{f}} = \coprod_{\mathfrak{d}|\mathfrak{f}} C_{\mathfrak{f}/\mathfrak{d}}$ [21] in order to separate components of modular vectors (= special values of Fricke functions f_a 's that may live at different levels), reducing it to Boolean-valued problems of judging prime ramifications (cf. the proof of Lemma 4.3.2 [19] in particular).

As mentioned above too, the current work also follows basically this line; but in the case of CM fields, major differences occur in the second and third steps. First, we shall replace Fricke modular functions $f_a : \mathfrak{H} \rightarrow \mathbb{C}$ with certain *ratios* $\theta_a^{k/l}$ of *theta functions*, which naturally define deformation families of modular functions on the Siegel upper-half space $\mathfrak{H}_g$; nevertheless, the basic idea behind these two families of modular functions is essentially the same, in that both families of functions (Fricke functions and ratios of theta functions) are those equally appearing in the construction of projective embedding of complex tori (resp. of dimension one and higher dimension) [12, 11], which will be a central step in our construction of modular vectors; essentially, their values (= components of modular vectors) both represent the coordinates of torsion points on elliptic curves and abelian varieties respectively. Our major subject in the second step is to give the right setting of necessary constructions and a proof that the ratios of theta functions naturally produce locally constant K^{ab} -valued G_K -equivariant functions on DR_K , hence, algebraic Witt vectors (i.e. *modular vectors*) over CM fields K (cf. §3).

Second, unlike the case of imaginary quadratic fields, these modular vectors do not generate all algebraic Witt vectors over CM fields; thus it should be of natural concern to characterize which algebraic Witt vectors are generated by our modular vectors. The subject of §4 is to discuss this problem, where we show that an algebraic Witt vector $\xi \in E_K$ is generated by the modular vectors if and only if it is periodic with respect to a certain monoid congruence $\sim_N$ of finite index on DR_K for some $N \geq 1$, where the congruences $\sim_N$ are given purely in an adelic term. Technically speaking, this result is a major result of the current paper and intended to exemplify a general *galois-theoretic correspondence* (mentioned in Remark 3.1.8 [19] without proof) between (i) Λ -subalgebras of E_K , (ii) profinite quotients of DR_K , and (iii) semi-galois full subcategories of $(\mathcal{C}_K, F_K)$; a proof of this general galois correspondence is also provided in §4.

It would be possible to extend the results of the current paper to more general situations, say, along the line of general Shimura varieties [9]; in this relation, compare our results to Yalkinoglu's [20] and Ha and Paugam's work [5]; in terms of general Shimura varieties the current work concerns the case of Siegel modular varieties (= Shimura varieties for general symplectic group) in particular. In this paper, however, we do not pursue such full generality yet; instead we shall focus on the most basic case where the common/different points of CM fields to the case of imaginary quadratic fields [19] become transparent.

Related works The current work is motivated by Yalkinoglu’s work [20] (2012) in particular. In that paper, Yalkinoglu constructed an arithmetic subalgebra of the Bost-Connes system for number fields containing CM fields using Siegel modular functions on Siegel modular varieties (§7 [20]); and if we restrict his arithmetic subalgebra (consisting of certain functions on a *BCM groupoid*; cf. §6 [20]) to the subspace of its objects (which is isomorphic to DR_K), we can in fact obtain a subalgebra of E_K under the identification $E_K = \text{Hom}_{G_K}(DR_K, K^{ab})$; in other words, up to this identification, his construction of arithmetic subalgebra can be also understood as a construction of algebraic Witt vectors over CM fields from special values of Siegel modular functions; this way, Yalkinoglu’s work [20] is relevant to our major goal in the current paper.

Comparing our construction to Yalkinoglu’s [20], however, highlights the above mentioned fact that our result can be regarded as a deformation of Shimura’s reciprocity law. On the one hand, the algebraic Witt vectors arising from Yalkinoglu’s construction, viewed as functions on DR_K , have their support only on the unit group $DR_K^\times = G_K^{ab} \subset DR_K$ (§7 [20]), while those of our construction have their support on the whole DR_K (cf. §3–§4). Technically speaking, it is due to this aspect that our algebraic Witt vectors can relate special values of Siegel modular functions living at different levels that are not necessarily galois conjugate, unlike Shimura’s reciprocity law only relates special values of galois-conjugate functions.

To be more specific, an algebraic Witt vector $\xi \in E_K$, viewed as a function on DR_K , is locally constant and thus factors through a finite quotient $DR_K \twoheadrightarrow DR_f$ for some conductor f ; this finite monoid DR_f has a natural action of its unit group $DR_f^\times$ isomorphic to the ray class group C_f , and then, has a C_f -orbit decomposition of the form $DR_f = \coprod_{\mathfrak{d}|f} C_{f/\mathfrak{d}}$; the G_K -equivariance of ξ together with K^{ab} -valuedness implies that ξ is a function $\coprod_{\mathfrak{d}} C_{f/\mathfrak{d}} \rightarrow K^{ab}$ such that, when restricted on each component $\xi|_{C_{f/\mathfrak{d}}} : C_{f/\mathfrak{d}} \rightarrow K^{ab}$, its value $\xi(\sigma)$ at $\sigma \in C_{f/\mathfrak{d}}$ is equal to $\xi(1)^\sigma$ and thus determined only by its value at the unit $1 \in C_{f/\mathfrak{d}}$. This way, the values of ξ within a single component $C_{f/\mathfrak{d}}$ are correlated by galois conjugate; but ξ being an algebraic Witt vector, the values of ξ on the whole $DR_f = \coprod_{\mathfrak{d}} C_{f/\mathfrak{d}}$ (i.e. those on different components too) must be correlated so that the distribution of the values of ξ on the whole $DR_f = \coprod_{\mathfrak{d}} C_{f/\mathfrak{d}}$ are *arithmetically smooth* (i.e. infinitely differentiable with respect to arithmetic derivatives) up to fixed denominator (i.e. ξ multiplied by some integer gets arithmetically smooth). In this sense, an algebraic Witt vector ξ is a data consisting, generally, not only of galois-conjugate values but connecting (or smoothly *interpolating*) non-galois conjugate values too, where “smooth” is in the sense of arithmetic derivative. Phrased in these specific terms, the algebraic Witt vectors arising from Yalkinoglu’s construction consist only of galois-conjugate values (= special values of galois conjugate Siegel modular functions) by his definition, and thus, in the above general sense, do not give interpolation of special values of non-galois-conjugate modular functions; on the other hand, our construction induces such interpolations in a natural way.¹

Technically speaking, it seems characteristic that, in our construction of algebraic Witt vectors (i.e. modular vectors) by special values of Siegel modular functions (living at different levels), moduli interpretations of Siegel modular varieties (or Siegel upper-half space $\mathfrak{H}_g$) play an intrinsic key role, which is also in a sharp contrast to Yalkinoglu’s work. In fact, as mentioned above, our modular vectors over CM fields (including imaginary quadratic fields) are constructed by choosing specific generators of the field $\mathfrak{F}$ of modular functions, namely, Fricke functions f_a ($a \in \mathbb{Q}^2/\mathbb{Z}^2$) and ratios

¹That said, it should be also recognized that Yalkinoglu’s work has his own goal that belongs to a natural research context on Bost-Connes systems, and for this goal, his construction was sufficient; our above argument is just to see that, reviewing Yalkinoglu’s work in the above specific way, our research subject (interpolations of special values of non-galois-conjugate Siegel modular functions) emerges naturally; solidifying this subject itself, as a sequel to our previous work [19], is also a major theme of the current work.

$\theta_a^{k/l}$ ($a \in \mathbb{Q}^{2g}/\mathbb{Z}^{2g}$) of theta functions (cf. §3); and these specific modular functions have natural moduli interpretation that they represent coordinates of torsion points of (projectively embedded) elliptic curves and abelian varieties. Based on this choice of generator, we can deform modular functions by deforming the indices $a \in \mathbb{Q}^{2g}/\mathbb{Z}^{2g}$ (cf. §3) of the generators (i.e. f_a and $\theta_a^{k/l}$), which correspond to descending the levels of torsion points of abelian varieties; our modular vectors are constructed by special values of thus deformed modular functions. In a sense, the reason why the values of our modular vectors $\xi : \coprod_{\mathfrak{d}|\mathfrak{d}} C_{\mathfrak{f}/\mathfrak{d}} \rightarrow K^{ab}$ can be correlated even among different components is that their values are controlled uniformly by this well-behaved deformation family $\theta_a^{k/l}$; in fact we can even characterize what algebraic Witt vectors are generated by our modular vectors, by geometry of abelian varieties, thanks to their geometric origin (cf. §4).

In the case of imaginary quadratic fields, the current work also overlaps with our own previous work [19]; in particular, we now have two classes of modular vectors, i.e. those by Fricke functions f_a and those by the ratios $\theta_a^{k/l}$ of theta functions. Although the latter cannot generate all algebraic Witt vectors over CM field of higher degree in general, we can see by our result in §4 that, at least in the case of imaginary quadratic fields, the latter too can generate all algebraic Witt vectors, as in the former case [19]; so, in view of the fact that the latter makes sense for any CM fields, modular vectors by the ratios of theta functions have a theoretical advantage/canonicity; in fact, our choice of ratios of theta functions for general CM case is a natural consequence of Mumford's general theory of projective embeddings of abelian varieties [11] and Shimura's theory of complex multiplication [13, 14]. Also as Iwao Kimura has told us in the context of our work [6] with Yasuhiro Ishitsuka on the decidability of integral Witt vectors over imaginary quadratic fields, numerical computations of special values of Fricke functions can be made fast by representing them by theta series [7]. In view of this, the modular vectors by ratios of theta functions may also have a computational advantage even for the case of imaginary quadratic fields only. Also, in the case of CM fields, Streng [15] has developed some computational results on Shimura's reciprocity law for Siegel modular functions, which is also relevant to the current work (§3–§4); in particular, Streng's result may be helpful for concrete numerical computations on our general theory of modular vectors developed below.

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2 Preliminaries

Throughout this paper, K denotes a number field, and O_K the ring of integers in K ; we denote by P_K the set of non-zero prime ideals of O_K , while I_K denotes the monoid of non-zero integral ideals of O_K ; $N\mathfrak{a}$ denotes the absolute norm of $\mathfrak{a} \in I_K$. Let $\mathbb{A}_K$ be the adele ring of K , $\mathbb{A}_{K,f}$ the finite adeles, and $\widehat{O}_K$ the finite integral adeles; the Artin map is denoted $[-] : \mathbb{A}_K^\times \rightarrow G_K^{ab}$, where G_K^{ab} is the galois group of the maximal abelian extension K^{ab} over K ; for a ring R with unit, $R^\times$ denotes the invertible elements in R . Note that this notation is different from that in [19], where we denoted R^* to mean the invertible elements of R ; in this paper, the superscript $*$ is reserved for the *reflex fields* E^* for CM fields with CM types (E, Φ) (cf. §3.2).

In this paper the reader is assumed to be familiar with the terminologies, notations, and main results in our previous works [16, 17, 18, 19]. In particular, $W_{O_K}^a(O_{\bar{K}})$ denotes the Λ -ring of integral

Witt vectors over O_K , while $E_K = K \otimes W_{O_K}^a(O_K)$ denotes the Λ -ring of algebraic Witt vectors over K . Also, $(\mathcal{C}_K, F_K)$ denotes the semi-galois category of finite etale Λ -rings over K having integral models, whose fundamental monoid $\pi_1(\mathcal{C}_K, F_K)$ is isomorphic to the Deligne-Ribet monoid DR_K [3].

2.1 Modular vectors

This subsection is devoted to an overview of our previous construction of modular vectors (§4 [19]), rephrasing it in terms of elliptic curves and their Kummer varieties in particular. This rephrase is to highlight the geometric ingredient of modular vectors, and to compare them to the case of CM fields given in §3, where elliptic curves (i.e. abelian varieties of dimension one) are replaced with abelian varieties of higher dimension (cf. §2.2).

2.1.1 Witt vectors as functions on DR_K

As outlined in §1, the construction of modular vectors is based on the fact that we can identify the algebraic Witt vectors $\xi \in E_K$ with locally constant K^{ab} -valued G_K -equivariant functions on the profinite Deligne-Ribet monoid DR_K , which holds for arbitrary number fields K (§3 [19]). Since we need this identification in this paper too, let us start with recalling some results on it:

Definition 2.1.1 (Deligne-Ribet monoid DR_K). The (*profinite*) *Deligne-Ribet monoid* DR_K is defined as the inverse limit of finite quotient monoids $I_K / \sim_{\mathfrak{f}} =: DR_{\mathfrak{f}}$ given for each $\mathfrak{f} \in I_K$, where $\sim_{\mathfrak{f}}$ is the monoid congruence of finite index on the monoid I_K defined by, for $\mathfrak{a}, \mathfrak{b} \in I_K$,

$$\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b} \iff \exists t \in K_+ \cap (1 + \mathfrak{f}\mathfrak{b}^{-1}), \mathfrak{a}\mathfrak{b}^{-1} = (t), \quad (2.1)$$

where K_+ denotes totally positive elements in K . Note that, for $\mathfrak{f} \mid \mathfrak{f}'$, we have a natural surjective monoid homomorphism $DR_{\mathfrak{f}'} \twoheadrightarrow DR_{\mathfrak{f}}$, with which $DR_{\mathfrak{f}}$'s constitute an inverse system of finite monoids; with this inverse system $\{DR_{\mathfrak{f}}\}_{\mathfrak{f} \in I_K}$ of *finite Deligne-Ribet monoids* $DR_{\mathfrak{f}}$ for conductors $\mathfrak{f} \in I_K$, the profinite Deligne-Ribet monoid DR_K is given by the inverse limit:

$$DR_K := \varprojlim_{\mathfrak{f} \in I_K} DR_{\mathfrak{f}} \quad (2.2)$$

By definition we have a canonical monoid homomorphism $I_K \rightarrow DR_K$, which is in fact injective; also, it is known [4] that the unit group $DR_{\mathfrak{f}}^\times$ of each $DR_{\mathfrak{f}}$ is isomorphic to the strict ray class group $C_{\mathfrak{f}}$ for the conductor $\mathfrak{f}$, namely $DR_{\mathfrak{f}}^\times \simeq C_{\mathfrak{f}}$; and hence, that $DR_K^\times \simeq \varprojlim_{\mathfrak{f}} C_{\mathfrak{f}} \simeq G_K^{ab}$, where the last isomorphism is the one given by class field theory. Therefore, the absolute galois group G_K naturally acts on DR_K (say, from the right) via the maximal abelian quotient $G_K \twoheadrightarrow G_K^{ab} \simeq DR_K^\times \subseteq DR_K$, with which DR_K forms a profinite right G_K -set. Thanks to this fact, it makes sense to consider locally constant K^{ab} -valued G_K -equivariant functions on DR_K , the set of which we shall denote by $\text{Hom}_{G_K}(DR_K, K^{ab})$.

This set $\text{Hom}_{G_K}(DR_K, K^{ab})$ has a natural K -algebra structure with respect to the point-wise operations, as well as the following operation $\psi_{\mathfrak{p}} : \text{Hom}_{G_K}(DR_K, K^{ab}) \rightarrow \text{Hom}_{G_K}(DR_K, K^{ab})$ for each $\mathfrak{p} \in P_K$, which is given by multiplication-by- $\mathfrak{p}$, regarding each $\mathfrak{p} \in P_K$ as an element in DR_K via the above-mentioned injection $I_K \hookrightarrow DR_K$:

$$\text{Hom}_{G_K}(DR_K, K^{ab}) \xrightarrow{\psi_{\mathfrak{p}}} \text{Hom}_{G_K}(DR_K, K^{ab}) \quad (2.3)$$

$$\xi(*) \longmapsto \xi(\mathfrak{p} \cdot *) \quad (2.4)$$

With these operators $\psi_{\mathfrak{p}}$, the K -algebra $\mathrm{Hom}_{G_K}(DR_K, K^{ab})$ forms a Λ -ring over K , which in fact is isomorphic to the Λ -ring E_K of algebraic Witt vectors in a canonical way:

Theorem 2.1.2 (Corollary 3.1.6, [19]). *We have a canonical isomorphism of Λ -rings over K :*

$$E_K \simeq \mathrm{Hom}_{G_K}(DR_K, K^{ab}). \quad (2.5)$$

Technically, the proof of this theorem is based on the two facts (cf. §3.2 [19]) that (i) each finite set $\Xi \subseteq E_K$ of algebraic Witt vectors generates a *galois object* X_Ξ of the semi-galois category $\mathcal{C}_K$, where $X_\Xi := K[\psi_{\mathfrak{a}}\xi \mid \xi \in \Xi, \mathfrak{a} \in I_K]$ is the Λ -ring generated over K by the orbit of Ξ under the action of $\psi_{\mathfrak{p}}$'s; and (ii) every galois object of $\mathcal{C}_K$ is isomorphic to X_Ξ for some finite $\Xi \subseteq E_K$. (See loc. cit. for more detail.)

In particular, the isomorphism from $\mathrm{Hom}_{G_K}(DR_K, K^{ab})$ to E_K is defined by restricting locally constant K^{ab} -valued G_K -equivariant functions $\xi : DR_K \rightarrow K^{ab}$ onto the submonoid $I_K \hookrightarrow DR_K$, inducing functions $\xi : I_K \rightarrow K^{ab}$, or in other words, vectors $\xi \in (K^{ab})^{I_K}$; the above theorem says that the vector $\xi \in (K^{ab})^{I_K}$ induced from $\xi \in \mathrm{Hom}_{G_K}(DR_K, K^{ab})$ always defines an algebraic Witt vector, namely $\xi \in E_K \subset (K^{ab})^{I_K}$ more than $\xi \in (K^{ab})^{I_K}$; and conversely, every algebraic Witt vector $\xi \in E_K$ arises uniquely in this way. In the sense of this correspondence, we can now identify algebraic Witt vectors $\xi \in E_K$ with locally constant K^{ab} -valued G_K -equivariant functions on the profinite monoid DR_K ; in particular, to construct algebraic Witt vectors $\xi \in E_K$ is equivalent to constructing such functions $\xi \in \mathrm{Hom}_{G_K}(DR_K, K^{ab})$.

2.1.2 Modular vectors

When K is an imaginary quadratic field, our previous work [19] constructed algebraic Witt vectors $\xi \in E_K$ from special values of the family $\{f_a\}$ of Fricke functions $f_a : \mathfrak{H} \rightarrow \mathbb{C}$, ($a \in \mathbb{Q}^2/\mathbb{Z}^2$), which are called *modular vectors*. In the rest of this subsection, we shall review this construction in terms of elliptic curves and their Kummer varieties, which is suitable for extension to abelian varieties of higher dimension (§3).

To this end, an adelic description of DR_K essentially by Yalkinoglu [21] plays an auxiliary role:

Proposition 2.1.3 (Proposition 8.2, Yalkinoglu [21]). *We have a natural isomorphism of profinite monoids:*

$$DR_K \simeq \widehat{O}_K \times_{\widehat{O}_K^\times} (\mathbb{A}_K^\times / \overline{K^\times \cdot K_\infty^{\times, \circ}}) \quad (2.6)$$

Here $K_\infty^{\times, \circ}$ denotes the connected component of the archimedean part $K_\infty^\times$ of the idele group $\mathbb{A}_K^\times$ containing the unit 1; the overline on $K^\times \cdot K_\infty^{\times, \circ}$ denotes the closure in $\mathbb{A}_K^\times$. Also the multiplicative group $\widehat{O}_K^\times$ acts on the product monoid $\widehat{O}_K \times (\mathbb{A}_K^\times / \overline{K^\times \cdot K_\infty^{\times, \circ}})$ by $u \cdot (\rho, s) := (\rho u, u^{-1}s)$, for which we take the quotient in (2.6). In particular, when K is imaginary quadratic, we have the following simpler form:

$$DR_K \simeq \widehat{O}_K \times_{\widehat{O}_K^\times} (\mathbb{A}_{K,f}^\times / K^\times) \quad (2.7)$$

Generally, for CM fields K , we can represent elements of DR_K as $[\rho, s]$ with $\rho \in \widehat{O}_K$ and $s \in \mathbb{A}_{K,f}^\times$.

The subject here is to review geometrically our previous construction in [19] of locally constant K^{ab} -valued G_K -equivariant functions on DR_K from *Fricke modular functions* $f_a : \mathfrak{H} \rightarrow \mathbb{C}$ indexed by $a \in \mathbb{Q}^2/\mathbb{Z}^2$. Technically the Fricke functions $f_a : \mathfrak{H} \rightarrow \mathbb{C}$ can be described explicitly but in a

case-by-case manner according as $K = \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$ or others (cf. §6.1 [12]); this is related to the fact that the automorphism groups of elliptic curves with complex multiplication in K differ for the respective cases (cf. §4.5 [12]). It is for this reason that our study of modular vectors was also done in a case-by-case manner (cf. e.g. §4.3 [19]).

Geometrically, we have a more uniform description of modular vectors via *Kummer varieties*, which is equal to the original one. The *Kummer variety* V_E of an elliptic curve E is the quotient of E by its automorphism group, $V_E = E/\text{Aut}(E)$, which is again an algebraic curve; and the quotient map $h : E \rightarrow V_E$ is called the *Weber function*; the Kummer varieties V_E are in fact isomorphic to the projective space $\mathbf{P}^1(\mathbb{C})$ in such a way that we have a natural geometric interpretation of the Fricke functions f_a : That is, for each non-zero $a = (a_1, a_2) \in \mathbb{Q}^2/\mathbb{Z}^2$ and $\tau \in \mathfrak{H}$, the special value $f_a(\tau)$ is (the x -coordinate of) the image of the torsion point $a_1\tau + a_2 \in \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ under the map $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) \rightarrow \mathbf{P}^1(\mathbb{C})$ given by the following composition:

$$\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) \xrightarrow{[\wp, \wp', 1]} E_\tau \xrightarrow{h} V_E \xrightarrow{\simeq} \mathbf{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}; \quad (2.8)$$

also, for $a = 0 \in \mathbb{Q}^2/\mathbb{Z}^2$, we put $f_0(\tau) = j(\tau)$.

With this geometric interpretation of the Fricke functions f_a , we now construct locally constant K^{ab} -valued G_K -equivariant functions $\hat{f}_a$ on DR_K , which provide our target modular vectors:

Definition 2.1.4 (modular vectors $\hat{f}_a$). For each $a = (a_1, a_2) \in \mathbb{Q}^2/\mathbb{Z}^2$, define $\hat{f}_a : DR_K \rightarrow \mathbb{C}$ by the following rule: Given each $[\rho, s] \in \hat{O}_K \times_{\hat{O}_K^\times} (\mathbb{A}_{K,f}^\times/K^\times) \simeq DR_K$, the value $\hat{f}_a(\rho, s)$ is the image of the torsion point $a_1\tau_K + a_2 \in K/O_K$ under the map $K/O_K \rightarrow \mathbf{P}^1(\mathbb{C})$ given by the composition:

$$\begin{array}{ccccccc} K/O_K & \xrightarrow{\rho} & K/O_K & \xrightarrow{s} & K/sO_K & \xrightarrow{\subseteq} & \mathbb{C}/sO_K \xrightarrow{\simeq} E_s \xrightarrow{h} \mathbf{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\ \mathbb{A}_{K,f}/\hat{O}_K & \xrightarrow{\rho} & \mathbb{A}_{K,f}/\hat{O}_K & \xrightarrow{s} & \mathbb{A}_{K,f}/s\hat{O}_K & & \end{array}$$

where $O_K = \mathbb{Z}\tau_K + \mathbb{Z}$ with $\tau_K \in \mathfrak{H}$ and $sO_K := s\hat{O}_K \cap K$ for each $s \in \mathbb{A}_{K,f}^\times$; and the isomorphism $\mathbb{C}/sO_K \simeq E_s$ is given by the Weierstrass $\wp$ -function $\wp(u, \Lambda_s)$ for the lattice $\Lambda_s = sO_K \subseteq \mathbb{C}$; also, if the value $\hat{f}_a(\rho, s)$ goes to ∞ by this map, we renormalize it as $j(\Lambda_s)$.

Proposition 2.1.5. *These functions $\hat{f}_a : DR_K \rightarrow \mathbb{C}$ are equal to those in [19], hence define locally constant K^{ab} -valued G_K -equivariant functions on DR_K , or equivalently, modular vectors $\hat{f}_a \in E_K$.*

Proof. The only slight difference from the original construction is that, in the above construction, we did not re-scale the lattice sO_K into the form $\mathbb{Z}\tau + \mathbb{Z}$ with $\tau \in \mathfrak{H}$; therefore, the lattice sO_K is of the form $sO_K = \mathbb{Z}w_1 + \mathbb{Z}w_2$ with $\tau_s = w_1/w_2 \in \mathfrak{H}$ in general. But this difference in scaling does not affect the resulting values because the complex torus $\mathbb{C}/sO_K$ is isomorphic to the rescaled one $\mathbb{C}/(\mathbb{Z}\tau_s + \mathbb{Z})$; and the Weber functions on elliptic curves are compatible with their isomorphisms (cf. §4.5 [12]). This also means that the above construction is independent of the choice of $s \in \mathbb{A}_{K,f}^\times$ that represents the class $[s] \in \mathbb{A}_{K,f}^\times/K^\times$, hence $\hat{f}_a : DR_K \rightarrow \mathbb{C}$ is well-defined. $\square$

2.2 Basic facts on CM fields

Now we prepare some basic results necessary to extend the above construction of modular vectors to the case of CM fields. In this process, as outlined in §1 too, we replace elliptic curves with abelian

varieties of higher dimension; accordingly, we need to recall projective embedding of complex tori of higher dimension by theta functions [11], which replaces projective embedding of one-dimensional complex tori $\mathbb{C}/(\mathbb{Z}w_1 + \mathbb{Z}w_2)$ by the Weierstrass $\wp$ -function $\wp$ (and $\wp'$). Starting from a few basic facts on CM fields, this subsection gathers some results related to projective embedding of complex tori with complex multiplication in CM fields; for more details on these topics, the reader is referred to e.g. [1, 9, 13, 14].

2.2.1 CM fields and Riemann forms

When K is an imaginary quadratic field, the complex tori $\mathbb{C}/sO_K \simeq K_\infty/sO_K$ are polarizable for any $s \in \mathbb{A}_{K,f}^\times$, which is not the case for complex tori of higher dimension in general. However, when K is a CM field (say, of degree $[K : \mathbb{Q}] = 2g$), the complex tori $\mathbb{C}^g/sO_K \simeq K_\infty/sO_K$ are always polarizable, hence, can be embedded into projective space as abelian varieties.

To be precise, let K be a CM field of degree $[K : \mathbb{Q}] = 2g$; also, fix a *CM type* $\Phi = \{\phi_1, \dots, \phi_g\}$. That is, $\phi_i : K \hookrightarrow \mathbb{C}$ are embeddings that are neither equal nor conjugate to each other; therefore, denoting by $\bar{\phi}_i : K \hookrightarrow \mathbb{C}$ the complex conjugate of ϕ_i , the set $\{\phi_1, \dots, \phi_g, \bar{\phi}_1, \dots, \bar{\phi}_g\}$ is precisely that of all embeddings $K \hookrightarrow \mathbb{C}$. We abusively use the same symbol $\Phi : K \rightarrow \mathbb{C}^g$ to denote also the map $\Phi : K \ni a \mapsto (\phi_1(a), \dots, \phi_g(a)) \in \mathbb{C}^g$; by this map Φ , the O_K -modules $sO_K := \widehat{sO_K} \cap K \subseteq K$ for $s \in \mathbb{A}_{K,f}^\times$ are embedded into $\mathbb{C}^g$ as lattices $\Lambda_s := \Phi(sO_K) \subseteq \mathbb{C}^g$, which then induce the complex tori $\mathbb{C}^g/\Lambda_s$. The first fact necessary in this paper is that the complex tori $\mathbb{C}^g/\Lambda_s$ are *polarizable* in the following sense, for any $s \in \mathbb{A}_{K,f}^\times$ of the CM field K :

Definition 2.2.1 (polarization on complex tori). For a lattice $\Lambda \subseteq \mathbb{C}^g$, the complex torus $\mathbb{C}^g/\Lambda$ is said to be *polarizable* if there exists an alternating $\mathbb{R}$ -bilinear form $\psi : \mathbb{C}^g \times \mathbb{C}^g \rightarrow \mathbb{R}$ such that:

1. $\psi(u, v) \in \mathbb{Z}$ for any $u, v \in \Lambda$;
2. $\psi(J(u), J(v)) = \psi(u, v)$ for any $u, v \in \mathbb{C}^g$;
3. $\psi(u, J(u)) > 0$ for any non-zero $u \in \mathbb{C}^g$;

where $J : \mathbb{C}^g \rightarrow \mathbb{C}^g$ is the standard complex structure. Such a form ψ is called a (non-degenerate) *Riemann form* on the complex torus $\mathbb{C}^g/\Lambda$.

The fact that the complex tori $\mathbb{C}^g/\Lambda_s$ are polarizable is based on a few structural properties of the CM field K (e.g. Theorem 4, §6.2 [13]): Being a CM field, K is by definition a totally imaginary quadratic extension of its maximal totally real subfield K_0 ; take a totally imaginary element $e \in K$ such that $-e^2$ is a totally positive element of K_0 and $\text{Im}(\phi_i(e)) > 0$ for all $\phi_i \in \Phi$, where $\text{Im}(z)$ denotes the imaginary part of $z \in \mathbb{C}$. With this e , define $\psi_e : \mathbb{C}^g \times \mathbb{C}^g \rightarrow \mathbb{R}$ by:

$$\psi_e(u, v) := \sum_{i=1}^g \phi_i(e) \cdot (u_i \bar{v}_i - \bar{u}_i v_i) \quad (2.9)$$

for $u = (u_1, \dots, u_g), v = (v_1, \dots, v_g) \in \mathbb{C}^g$. In particular, for each $a, b \in K$, we have:

$$\psi_e(\Phi(a), \Phi(b)) = \text{Tr}_{K/\mathbb{Q}}(e \cdot a \cdot b^\iota), \quad (2.10)$$

where $\text{Tr}_{K/\mathbb{Q}}$ denotes the trace over $\mathbb{Q}$; and $\iota \in \text{Gal}(K/K_0)$ denotes the non-trivial element of order 2. In particular, this means that, the O_K -module sO_K for any $s \in \mathbb{A}_{K,f}^\times$ being a fractional ideal of K , we have $n_s \cdot \psi_e(sO_K, sO_K) \subseteq \mathbb{Z}$ for some integer n_s ; and then, this $\psi := n_s \cdot \psi_e$ gives a Riemann form on the complex torus $\mathbb{C}^g/\Lambda_s$.

Remark 2.2.2 (symplectic basis and type). Given any Riemann form ψ on a complex torus $\mathbb{C}^g/\Lambda$ in general, the lattice Λ has a *symplectic basis*, which is a basis of the lattice Λ over $\mathbb{Z}$ by which the Riemann form $\psi : \mathbb{C}^g \times \mathbb{C}^g \rightarrow \mathbb{R}$ can be expressed by the matrix of the form:

$$\psi(u, v) = u \cdot \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} \cdot v^t \quad (2.11)$$

where $u, v \in \mathbb{C}^g$ in the left-hand side are re-represented in the right-hand side as real (row) vectors $u, v \in \mathbb{R}^{2g}$ with respect to the symplectic basis of Λ ; also $\delta \in M_g(\mathbb{Z})$ is a matrix of the form:

$$\delta = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_g \end{pmatrix} \quad (2.12)$$

with $0 < d_i$ and $d_i \mid d_{i+1}$; in the following we denote $\delta = \text{diag}[d_1, \dots, d_g]$. This δ is independent of the choice of symplectic basis of Λ for the Riemann form ψ ; thus it makes sense to say that the Riemann form ψ on the complex torus $\mathbb{C}^g/\Lambda$ is of *type* δ .

2.2.2 Projective embedding

By the above fact that the complex torus $\mathbb{C}^g/\Lambda_s$ is polarizable for any $s \in \mathbb{A}_{K,f}^\times$ of the CM field K with a fixed CM type Φ , it follows that the complex torus $\mathbb{C}^g/\Lambda_s$ can be isomorphically embedded as an abelian variety in a suitable projective space $\mathbf{P}^d(\mathbb{C})$. Such an embedding can be concretely provided by *theta functions* [11], which depends on a choice of Riemann form on the complex torus $\mathbb{C}^g/\Lambda_s$ and its type.

To see such embeddings and theta functions explicitly, take a lattice $\Lambda \subseteq \mathbb{C}^g$ such that we have a Riemann form ψ on the complex torus $\mathbb{C}^g/\Lambda$ of type $\delta = \text{diag}[d_1, \dots, d_g]$; then the lattice Λ has a symplectic basis $(\tau_1, \dots, \tau_g, \delta_1, \dots, \delta_g) =: (\tau, \delta) \in M_{g \times 2g}(\mathbb{C})$ so that $\Lambda = \mathbb{Z}^g \cdot \tau + \mathbb{Z}^g \cdot \delta$, where $\mathbb{Z}^g$ consists of row vectors and $\tau_i \in \mathbb{C}^g, \delta_i \in \mathbb{Z}^g$ are column vectors. Here the $g \times g$ matrix $\tau \in M_g(\mathbb{C})$ is in fact a member of the *Siegel upper-half space* $\mathfrak{H}_g$ defined by:

$$\mathfrak{H}_g := \left\{ \tau \in M_g(\mathbb{C}) \mid \tau^t = \tau, \text{Im}(\tau) > 0 \right\} \quad (2.13)$$

where $\text{Im}(\tau) > 0$ means that the imaginary part $\text{Im}(\tau) \in M_g(\mathbb{R})$ of τ is positive definite. We note conversely that, given $\tau \in \mathfrak{H}_g$, one obtains a lattice $\Lambda_\tau := \mathbb{Z}^g \tau + \mathbb{Z}^g \delta$ such that the complex torus $\mathbb{C}^g/\Lambda_\tau$ has a Riemann form ψ_τ of type δ , given by $\psi_\tau(u, v) := \text{Im}(u \cdot (\text{Im}(\tau))^{-1} \cdot v^t)$; then (τ, δ) gives a symplectic basis of Λ_τ for this Riemann form ψ_τ . Therefore, it follows that the complex tori $\mathbb{C}^g/\Lambda$ with Riemann forms ψ of type δ are precisely those of the form $\mathbb{C}^g/(\mathbb{Z}^g \tau + \mathbb{Z}^g \delta)$ for some $\tau \in \mathfrak{H}_g$ (cf. e.g. §8.1 [1]).

Now we are ready for projective embeddings of the complex tori $\mathbb{C}^g/(\mathbb{Z}^g \tau + \mathbb{Z}^g \delta)$ with Riemann forms ψ of type δ by the following *theta functions*:

Definition 2.2.3 (theta function θ^k). For each $k \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$, define $\theta^k : \mathbb{C}^g \times \mathfrak{H}_g \rightarrow \mathbb{C}$ as follows:

$$\theta^k(u; \tau) := \sum_{w \in k + \mathbb{Z}^g} \mathbf{e}\left(\frac{w^t \cdot \tau \cdot w}{2} + u \cdot w\right), \quad (2.14)$$

where $\mathbf{e}(t) := e^{2\pi it}$; also $u \in \mathbb{C}^g$ are row vectors, while $k \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$ and $w \in k + \mathbb{Z}^g$ are supposed to be column vectors.

The theta functions θ^k define projective embedding of complex tori of type $\delta = \text{diag}[d_1, \dots, d_g]$ when $3 \leq d_1$ in the following manner:

Definition 2.2.4 (projective embedding). Put $d := |\delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g| - 1 = d_1 d_2 \cdots d_g - 1$; then for each $\tau \in \mathfrak{H}_g$, we can define $\Theta_\tau : \mathbb{C}^g/(\mathbb{Z}^g\tau + \mathbb{Z}^g\delta) \rightarrow \mathbf{P}^d(\mathbb{C})$ by:

$$\mathbb{C}^g/(\mathbb{Z}^g\tau + \mathbb{Z}^g\delta) \xrightarrow{\Theta_\tau} \mathbf{P}^d(\mathbb{C}) \quad (2.15)$$

$$u \longmapsto [\theta^k(u; \tau)]_{k \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g} \quad (2.16)$$

where $[\theta^k]_{k \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g}$ denotes the homogeneous coordinate on the projective space $\mathbf{P}^d(\mathbb{C})$; by our assumption on δ , this map $\Theta_\tau : \mathbb{C}^g/(\mathbb{Z}^g\tau + \mathbb{Z}^g\delta) \rightarrow \mathbf{P}^d(\mathbb{C})$ defines a projective embedding, whose image $\Theta_\tau(\mathbb{C}^g/(\mathbb{Z}^g\tau + \mathbb{Z}^g\delta)) \subseteq \mathbf{P}^d(\mathbb{C})$ is an abelian variety and will be denoted $A_\tau \subseteq \mathbf{P}^d(\mathbb{C})$, thus $\Theta_\tau : \mathbb{C}^g/(\mathbb{Z}^g\tau + \mathbb{Z}^g\delta) \xrightarrow{\sim} A_\tau$.

2.2.3 Complex multiplication

In the case when K is an imaginary quadratic field, the special value $f(\tau)$ of a modular function on $\mathfrak{H}$ (rational over $\mathbb{Q}^{ab}$) at $\tau \in K \cap \mathfrak{H}$ is abelian over K ; Shimura's reciprocity law then shows that the galois conjugate $(f(\tau))^{[s]^{-1}}$ of this special value $f(\tau)$ for $s \in \mathbb{A}_{K,f}^\times$ is equal to the special value $f^{q_\tau(s)}(\tau)$ of a suitably defined galois conjugate modular function $f^{q_\tau(s)}$ (cf. Theorem 6.18 [12]). In terms of elliptic curves, this reciprocity law is proved using the following *main theorem of complex multiplication*:

Theorem 2.2.5 (main theorem of complex multiplication, Theorem 5.4 [12]). *Let $\mathfrak{a}$ be a fractional ideal of an imaginary quadratic field $K \subseteq \mathbb{C}$; and $\Theta : \mathbb{C}/\mathfrak{a} \rightarrow E_{\mathfrak{a}}$ an isomorphism of the complex torus $\mathbb{C}/\mathfrak{a}$ to an elliptic curve $E_{\mathfrak{a}}$. Then, for any automorphism $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ over K with $\sigma|_{K^{ab}} = [s]^{-1}$ for some $s \in \mathbb{A}_{K,f}^\times$, there exists a unique isomorphism $\Theta' : \mathbb{C}/s\mathfrak{a} \rightarrow E_{\mathfrak{a}}^\sigma$ such that the following diagram commutes:*

$$\begin{array}{ccccc} K/\mathfrak{a} & \longrightarrow & \mathbb{C}/\mathfrak{a} & \xrightarrow{\Theta} & E_{\mathfrak{a}} \\ s \downarrow & & & & \downarrow \sigma \\ K/s\mathfrak{a} & \longrightarrow & \mathbb{C}/s\mathfrak{a} & \xrightarrow{\Theta'} & E_{\mathfrak{a}}^\sigma \end{array} \quad (2.17)$$

In the case when K is a CM field of higher degree in general, a similar theorem still holds, where elliptic curves are replaced with abelian varieties of higher dimension; but in order to state it in the case of degree greater than 2, we need to consider not only K itself but also its *reflex field* K^* for a CM type of K :

Definition 2.2.6 (reflex field). Let K be a CM field of degree $[K : \mathbb{Q}] = 2g$ and $\Phi = \{\phi_1, \dots, \phi_g\}$ a CM type of K . Then the reflex field K^* of the CM type (K, Φ) is defined as follows:

$$K^* := \mathbb{Q}\left(\sum_{i=1}^g \phi_i(e) \mid e \in K\right). \quad (2.18)$$

Remark 2.2.7 (reflex norm). For the CM type (K, Φ) , the reflex field K^* is again a CM field; also, we have the *reflex norm* for (K, Φ) , which is a multiplicative monoid morphism $\eta : \mathbb{A}_{K^*} \rightarrow \mathbb{A}_K$; this morphism restricts to $\eta : \mathbb{A}_{K^*,f}^\times \rightarrow \mathbb{A}_{K,f}^\times$ and $\eta : \widehat{O}_{K^*} \rightarrow \widehat{O}_K$; for more details the reader is referred to e.g. §18.5 [13] or §9.4 [14].

Theorem 2.2.8 (main theorem of complex multiplication, Theorem 18.6 [13]). *Let $\mathfrak{a}$ be a fractional ideal of a CM field K ; and $\Theta : \mathbb{C}^g / \Phi(\mathfrak{a}) \rightarrow A_{\mathfrak{a}}$ an isomorphism of the complex torus $\mathbb{C}^g / \Phi(\mathfrak{a})$ to an abelian variety $A_{\mathfrak{a}}$ (§2.2.2). Then, for any automorphism $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ over K^* with $\sigma|_{(K^*)^{ab}} = [s]^{-1}$ for some $s \in \mathbb{A}_{K^*,f}^\times$, there exists a unique isomorphism $\Theta' : \mathbb{C}^g / \Phi(\eta(s)\mathfrak{a}) \rightarrow A_{\mathfrak{a}}^\sigma$ such that the following diagram commutes:*

$$\begin{array}{ccccc} E/\mathfrak{a} & \xrightarrow{\Phi} & \mathbb{C}^g / \Phi(\mathfrak{a}) & \xrightarrow{\Theta} & A_{\mathfrak{a}} \\ \eta(s) \downarrow & & & & \downarrow \sigma \\ E/\eta(s)\mathfrak{a} & \xrightarrow{\Phi} & \mathbb{C}^g / \Phi(\eta(s)\mathfrak{a}) & \xrightarrow{\Theta'} & A_{\mathfrak{a}}^\sigma \end{array} \quad (2.19)$$

3 Modular vectors over CM fields

The subject of this section is to construct *modular vectors* in the case when K is the reflex field of a CM type; to be compatible with notations in §1, let (E, Φ) be a CM type of degree $[E : \mathbb{Q}] = 2g$, and put $K = E^*$; by the identification discussed in §2.2.1, it suffices to construct locally constant K^{ab} -valued G_K -equivariant functions on DR_K . Thus, we divide this section into two sub-sections for (i) a construction of suitable functions on DR_K from modular functions on the Siegel upper-half space $\mathfrak{H}_g$ (§3.1), and (ii) a proof that they are indeed locally constant, K^{ab} -valued, and G_K -equivariant (§3.2).

3.1 Construction of modular vectors

Throughout this section, we shall fix a CM field E of degree $[E : \mathbb{Q}] = 2g$ together with a CM type $\Phi = \{\phi_1, \dots, \phi_g\}$ of E ; let $K = E^*$ be the reflex field of (E, Φ) . Also we fix a Riemann form on the complex torus $\mathbb{C}^g / \Phi(O_E)$, $\psi : \mathbb{C}^g \times \mathbb{C}^g \rightarrow \mathbb{R}$, so that the ring O_E of integers has a symplectic basis $(\tau_1, \dots, \tau_g, \delta_1, \dots, \delta_g)$ for ψ with which ψ is of type $\delta = \text{diag}[d_1, \dots, d_g]$ and $3 \leq d_i \mid d_{i+1}$. Such a Riemann form $\psi : \mathbb{C}^g \times \mathbb{C}^g \rightarrow \mathbb{R}$ always exists; the condition on its type δ is necessary to guarantee that the theta functions θ^k induce projective embeddings of complex tori $\mathbb{C}^g / (\mathbb{Z}^g \tau + \mathbb{Z}^g \delta)$ of type δ for each $\tau \in \mathfrak{H}_g$ (cf. §2.2.2); for a technical reason (cf. Proposition 8.10.2 [1]), we further suppose that $4 \leq d_1$ and $2 \mid d_1$ or $3 \mid d_1$.

In particular, by construction, the ring O_E of integers in E is mapped to a lattice $\Phi(O_E) \subseteq \mathbb{C}^g$ by $\Phi : E \rightarrow \mathbb{C}^g$, which can be written by a matrix $\tau_E \in \mathfrak{H}_g$ as in the following form:

$$\Phi(O_E) = \mathbb{Z}^g \tau_E + \mathbb{Z}^g \delta \quad (3.1)$$

Therefore, as discussed in §2.2.3, the complex torus $\mathbb{C}^g / \Phi(O_E) = \mathbb{C}^g / (\mathbb{Z}^g \tau_E + \mathbb{Z}^g \delta)$ is isomorphic to an abelian variety, which we shall denote $A_E \subseteq \mathbf{P}^d(\mathbb{C})$, through the following projective embedding $\Theta_E := \Theta_{\tau_E} : \mathbb{C}^g / \Phi(O_E) \hookrightarrow \mathbf{P}^d(\mathbb{C})$ given by (cf. Definition 2.2.4):

$$\mathbb{C}^g / \Phi(O_E) \xrightarrow{\Theta_E} \mathbf{P}^d(\mathbb{C}) \quad (3.2)$$

$$u \longmapsto [\theta^k(u; \tau_E)]_{k \in \delta^{-1}\mathbb{Z}^g / \mathbb{Z}^g}, \quad (3.3)$$

where $d = d_1 d_2 \cdots d_g - 1$. More generally, for any $s \in \mathbb{A}_{K,f}^\times$, the O_E -module $\eta(s)O_E := \eta(s)\widehat{O}_E \cap E$ is mapped to the lattice $\Phi(\eta(s)O_E) \subseteq \mathbb{C}^g$, where $\eta : \mathbb{A}_{K,f}^\times = \mathbb{A}_{E^*,f}^\times \rightarrow \mathbb{A}_{E,f}^\times$ denotes the reflex norm (cf. §2.2.3); in the following we denote this lattice $\Phi(\eta(s)O_E)$ by Λ_s for each $s \in \mathbb{A}_{K,f}^\times$, in particular, $\Lambda_E := \Lambda_1 = \Phi(O_E)$.

The main theorem of complex multiplication (Theorem 2.2.8) shows that, for each automorphism $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ over $K = E^*$ with $\sigma|_{K^{ab}} = [s]^{-1}$, we have a unique isomorphism $\Theta' : \mathbb{C}/\Lambda_s \rightarrow A_E^\sigma$ such that:

$$\begin{array}{ccccc} E/O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_E & \xrightarrow{\Theta_E} & A_E \\ \eta(s) \downarrow & & & & \downarrow \sigma \\ E/\eta(s)O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_s & \xrightarrow{\Theta'} & A_E^\sigma \end{array} \quad (3.4)$$

We will use these Θ' for our construction of modular vectors; but note that, while the isomorphism $\Theta_E : \mathbb{C}^g/\Lambda_E \rightarrow A_E$ is given explicitly by the theta functions $\theta^k(*; \tau_E)$ as above, the isomorphism $\Theta' : \mathbb{C}^g/\Lambda_s \rightarrow A_E^\sigma$ is not yet quite explicit for $s \in \mathbb{A}_{K,f}^\times$. Therefore, our task here is first to construct modular vectors over K using these abstract Θ' ; and then to describe the resulting modular vectors more explicitly.

Concerning this first task, notice that, while $\Theta' : \mathbb{C}^g/\Lambda_s \rightarrow A_E^\sigma$ itself depends on σ rather than $\sigma|_{K^{ab}} = [s]^{-1}$ (hence, is not actually determined only by s), the composition $\Theta' \circ \Phi : E/\eta(s)O_E \hookrightarrow \mathbb{C}^g/\Lambda_s \rightarrow A_E^\sigma \subseteq \mathbf{P}^d(\mathbb{C})$ does not because the image of $E/\eta(s)O_E$ are the torsion points of A_E^σ and the action of σ on the torsion points is determined by $[s]$. Therefore, it makes sense to denote this composition as $\Theta_s : E/\eta(s)O_E \rightarrow A_E^\sigma \subseteq \mathbf{P}^d(\mathbb{C})$ for each $s \in \mathbb{A}_{K,f}^\times$. With this in mind, we abusively denote as $A_E^\sigma = A_s$, and also $\Theta_s : \mathbb{C}^g/\Lambda_s \rightarrow A_s$.

Now we construct modular vectors using these (still abstract) isomorphisms $\Theta_s : \mathbb{C}^g/\Lambda_s \rightarrow A_s$. To this end, we note that each abelian variety A_s is a priori a subvariety of the ambient projective space $\mathbf{P}^d(\mathbb{C})$; thus the rational functions on $\mathbf{P}^d(\mathbb{C})$ uniformly restrict to each subvariety A_s . More explicitly, in terms of the homogeneous coordinates $[\theta^k]_{k \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g}$ of $\mathbf{P}^d(\mathbb{C})$, the rational functions on $\mathbf{P}^d(\mathbb{C})$ are of the form $f(\theta^k) = g(\theta^k)/h(\theta^k)$, where g, h are homogeneous polynomials of θ^k 's of the same degree; in particular, we mean by *rational functions (over K)* those $f(\theta^k) = g(\theta^k)/h(\theta^k)$ whose coefficients are in K . Also, for each $a = (a_1, a_2) \in \mathbb{Q}^{2g}/\mathbb{Z}^{2g}$, we say that a rational function $f(\theta^k)$ is *defined at a* if it is defined on the torsion point of A_E corresponding to $a_1\tau_E + a_2\delta \in E/O_E$ under the map $\Theta_E \circ \Phi : E/O_E \hookrightarrow \mathbb{C}^g/\Lambda_E \simeq A_E$; also we say that $f(\theta^k)$ is *defined for a* if $f(\theta^k)$ is defined at $a\eta(\rho)$ for any $\rho \in \widehat{O}_K$, where and in the following we identify $a = (a_1, a_2) \in \mathbb{Q}^{2g}/\mathbb{Z}^{2g}$ with $a_1\tau_E + a_2\delta \in E/O_E$.

Definition 3.1.1 (modular vector). Let $a \in N^{-1}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ and f a rational function defined for a . Then the *modular vector* $\widehat{f}_a : DR_K \rightarrow \mathbb{C}$ is defined by the following rule: Given each $[\rho, s] \in DR_K$, the value $\widehat{f}_a(\rho, s)$ is the image of the torsion point $a_1\tau_E + a_2\delta \in E/O_E$ (at the upper-left corner in (3.5)) under the map $E/O_E \rightarrow \mathbb{C}$ given by the following composition:

$$\begin{array}{ccccccc} E/O_E & \xrightarrow{\eta(\rho)} & E/O_E & \xrightarrow{\eta(s)} & E/\eta(s)O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_s \xrightarrow{\Theta_s} A_s \xrightarrow{f|_{A_s}} \mathbb{C} \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\ \mathbb{A}_{E,f}/\widehat{O}_E & \xrightarrow{\eta(\rho)} & \mathbb{A}_{E,f}/\widehat{O}_E & \xrightarrow{\eta(s)} & \mathbb{A}_{E,f}/\eta(s)\widehat{O}_E & & \end{array} \quad (3.5)$$

Remark 3.1.2 (rational functions over $\mathbb{Q}$). In the definition of modular vectors, we used rational functions f defined over K ; but it suffices for our purpose in §4 to consider only rational functions defined over $\mathbb{Q}$. In the latter case, nevertheless, the set of modular vectors itself does not form a K -algebra; in that case, we need to consider the K -algebra generated (over K) by modular vectors.

Lemma 3.1.3. *The above modular vectors $\hat{f}_a$ are well-defined as $\mathbb{C}$ -valued functions on DR_K .*

Proof. We need to check two points for the well-definedness of $\hat{f}_a$: The first is that the value $\hat{f}_a(\rho, s)$ is indeed finite; the second is that it depends only on the class $[\rho, s] \in DR_K$ rather than (ρ, s) . To see these, consider the following commutative diagram:

$$\begin{array}{ccccccc} E/O_E & \xrightarrow{\eta(\rho)} & E/O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_E & \xrightarrow{\Theta_E} & A_E \\ & & \eta(s) \downarrow & & & & \downarrow [s]^{-1} \\ & & E/\eta(s)O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_s & \xrightarrow{\Theta_s} & A_s \end{array} \quad (3.6)$$

Since the right-most vertical isomorphism $[s]^{-1} : A_E \rightarrow A_s$ (on the torsion points) depends only on the class $[s]$, it first follows that the value $\hat{f}_a(\rho, s)$ depends on $[s]$ rather than s itself; also, the torsion point $a_1\tau_E + a_2\delta \in N^{-1}O_E/O_E$ is mapped again to $N^{-1}O_E/O_E$ by $\eta(\rho) : E/O_E \rightarrow E/O_E$; therefore, the assumption that f is now defined for a implies that the value $\hat{f}_a(\rho, s)$ is finite. Finally the equality $\hat{f}_a(\rho u, u^{-1}s) = \hat{f}_a(\rho, s)$ for any $u \in \hat{O}_K^\times$ follows from the fact that the composition $\eta(u^{-1}s) \circ \eta(\rho u) : E/O_E \rightarrow E/\eta(u^{-1}s)O_E = E/\eta(s)O_E$ is equal to $\eta(s) \circ \eta(\rho) : E/O_E \rightarrow E/\eta(s)O_E$ since the reflex norm η is multiplicative. This completes the proof. $\square$

3.2 Basic properties

We prove here that the modular vectors $\hat{f}_a : DR_K \rightarrow \mathbb{C}$ are indeed locally constant K^{ab} -valued and G_K -equivariant, hence define algebraic Witt vectors $\hat{f}_a \in E_K$ over K . Moreover, as mentioned above too, the construction of modular vectors depends on the abstract isomorphisms $\Theta_s : \mathbb{C}^g/\Lambda_s \rightarrow A_s$; thus, we shall also give a more explicit description of modular vectors in terms of the theta functions $\theta^k : \mathbb{C}^g \times \mathfrak{H}_g \rightarrow \mathbb{C}$ of §2.2.2 at a concrete $\tau_s \in \mathfrak{H}_g$.

We start with proving the fact that the modular vectors define algebraic Witt vectors:

Proposition 3.2.1. *Take a rational function f and $a \in N^{-1}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ as in Definition 3.1.1; then the modular vector $\hat{f}_a : DR_K \rightarrow \mathbb{C}$ is locally constant K^{ab} -valued and G_K -equivariant.*

Proof. We first see that $\hat{f}_a : DR_K \rightarrow \mathbb{C}$ is K^{ab} -valued. To this end, note the following commutative diagram:

$$\begin{array}{ccccccc} E/O_E & \xrightarrow{\eta(\rho)} & E/O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_E & \xrightarrow{\Theta_E} & A_E \xrightarrow{f|_{A_E}} \mathbb{C} \\ & & \eta(s) \downarrow & & & & \downarrow \sigma \\ & & E/\eta(s)O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_s & \xrightarrow{\Theta_s} & A_s \xrightarrow{f|_{A_s}} \mathbb{C} \end{array} \quad (3.7)$$

where $\sigma|_{K^{ab}} = [s]^{-1}$. By definition of $\hat{f}_a$, this diagram shows that we have:

$$\hat{f}_a(\rho, s) = f(\theta^k(a'_1\tau_E + a'_2\delta; \tau_E))^\sigma \quad (3.8)$$

where $a' = (a'_1, a'_2) \in N^{-1}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ is defined as $a' = a \cdot \eta(\rho)$. Since $f(\theta^k) = g(\theta^k)/h(\theta^k)$ is now a rational function over K with homogeneous polynomials g, h of the same degree, it follows that this value $\widehat{f}_a(\rho, s)$ can be expressed as an algebraic combination of ratios of theta values in the following form for some $k, l \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$:

$$\left(\frac{\theta^k(a'_1\tau_E + a'_2\delta; \tau_E)}{\theta^l(a'_1\tau_E + a'_2\delta; \tau_E)} \right)^\sigma \quad (3.9)$$

In fact, these ratios are abelian over K since the following meromorphic function $\theta_a^{k/l}$ on $\mathfrak{H}_g$ defines a modular function on $\mathfrak{H}_g$ for all $a \in \mathbb{Q}^{2g}/\mathbb{Z}^{2g}$ and $k, l \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$ (cf. Theorem 27.13, [13]):

$$\theta_a^{k/l}(\tau) := \frac{\theta^k(a_1\tau + a_2\delta; \tau)}{\theta^l(a_1\tau + a_2\delta; \tau)} \quad (3.10)$$

Therefore, the action of σ on these values at τ_E is in fact equal to $[s]^{-1}$; and we eventually have:

$$\widehat{f}_a(\rho, s) = f(\theta^k(a'_1\tau_E + a'_2\delta; \tau_E))^{[s]^{-1}} \in K^{ab}. \quad (3.11)$$

From this expression, we also easily deduce that $\widehat{f}_a : DR_K \rightarrow K^{ab}$ is G_K -equivariant. (Recall that the action of G_K on DR_K factors through $G_K \rightarrow G_K^{ab}$, and the action of $[t] \in G_K^{ab}$ on $[\rho, s]$ is equal to $[\rho, t^{-1}s]$; see e.g. Lemma 4.2.3 [19].)

Finally, we see that $\widehat{f}_a : DR_K \rightarrow K^{ab}$ is locally constant; in fact, to be more specific, $\widehat{f}_a$ factors through the finite quotient $DR_K \twoheadrightarrow DR_{NO_K}$ for the conductor NO_K for $a \in N^{-1}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$. To see this, note first that the action of $\eta(\rho) \in \widehat{O}_E$ (by multiplication) on $N^{-1}O_E/O_E$ is determined by the class $[\rho] \in \widehat{O}_K/N\widehat{O}_K$ (hence also by the classes in $\widehat{O}_K/N'\widehat{O}_K$ for any $N \mid N'$); this follows from the fact that the reflex norm $\eta : \widehat{O}_K = \widehat{O}_{E^*} \rightarrow \widehat{O}_E$ maps $N\widehat{O}_K$ into $N\widehat{O}_E$ (cf. §18.5, [13]). Moreover, the values $\widehat{f}_a(\rho, s)$ belong at largest to the ray class field $K_{N'O_K}$ of the conductor $N'O_K$ for some N' such that $N \mid N'$ since these values are given by special values of $f(\theta^k)$ at N -torsion points of A_E , hence abelian over K in particular; thus the action of $[s]^{-1}$ on these values factors through the quotient $G_K^{ab} \twoheadrightarrow C_{N'O_K}$. Combining with the isomorphism $DR_{N'O_K} \simeq (O_K/N'O_K) \times_{(O_K/N'O_K)} C_{N'O_K}$ (cf. Proposition 7.2 [21]), we deduce that $\widehat{f}_a$ indeed factors through a finite quotient $DR_K \twoheadrightarrow DR_{N'O_K}$. This completes the proof. $\square$

As seen above, the proof of the fact that the modular vectors $\widehat{f}_a$ are locally constant K^{ab} -valued G_K -equivariant was possible even with the abstract isomorphisms $\Theta_s : \mathbb{C}^g/\Lambda_s \rightarrow A_s$. But our next task is to give a more explicit description of the values $\widehat{f}_a(\rho, s)$ by special values of ratios of theta functions $\theta^k(*; \tau_s)$ at an *explicit* $\tau_s \in \mathfrak{H}_g$ rather than the galois conjugates $\theta^k(*; \tau_E)^{[s]^{-1}}$ as given in the right hand side of (3.11); technically, this task is done by finding $\tau_s \in \mathfrak{H}_g$ such that the lattice $\Lambda_s = \Phi(\eta(s)O_E)$ is isomorphic to $\mathbb{Z}^g\tau_s + \mathbb{Z}^g\delta$.

To this end, note first that, for each $s \in \mathbb{A}_{K,f}^\times$, the $\mathbb{A}_{\mathbb{Q},f}$ -linear map $\mathbb{A}_{E,f} \ni t \mapsto \eta(s) \cdot t \in \mathbb{A}_{E,f}$ can be expressed as a $2g \times 2g$ matrix in $GL_{2g}(\mathbb{A}_{\mathbb{Q},f})$ with respect to the symplectic basis of O_E for the Riemann form ψ , which we abusively denote also as $\eta(s) \in GL_{2g}(\mathbb{A}_{\mathbb{Q},f})$; this adelic matrix $\eta(s)$ has the following standard matrix decomposition, with which we will construct the target $\tau_s \in \mathfrak{H}_g$:

Lemma 3.2.2 (cf. strong approximation theorem). *Put $\tilde{\eta}(s) \in GL_{2g}(\mathbb{A}_{\mathbb{Q},f})$ as follows:*

$$\tilde{\eta}(s) := \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}^{-1} \cdot \eta(s) \cdot \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \quad (3.12)$$

Then we have $\tilde{\eta}(s) = \tilde{u}_s \cdot \tilde{\alpha}_s$ with some $\tilde{u}_s \in GL_{2g}(\mathbb{A}_{\mathbb{Q},f})$ and $\tilde{\alpha}_s \in GSp_{2g}^+(\mathbb{Q})$ such that we have the equality $(\widehat{\mathbb{Z}}^g + \widehat{\mathbb{Z}}^g \delta) \cdot \tilde{u}_s = \widehat{\mathbb{Z}}^g + \widehat{\mathbb{Z}}^g \delta$.

Proof. First, by the basic property of the reflex norm (cf. Remark 2.2.7), the $\mathbb{A}_{\mathbb{Q},f}$ -linear map $\eta(s)$ in fact preserves the Riemann form $\psi = \psi_{\mathbb{A}_{\mathbb{Q},f}} : \mathbb{A}_{E,f} \times \mathbb{A}_{E,f} \rightarrow \mathbb{A}_{\mathbb{Q},f}$ up to a fixed scalar over $\mathbb{A}_{\mathbb{Q},f}$. That is, for any $u, v \in \mathbb{A}_{E,f}$:

$$\psi(u \cdot \eta(s), v \cdot \eta(s)) = \psi(u, v \cdot \eta(s) \eta(s)^t) \quad (3.13)$$

$$= \psi(u, v \cdot N_{K/\mathbb{Q}}(s)) \quad (3.14)$$

$$= N_{K/\mathbb{Q}}(s) \cdot \psi(u, v) \quad (3.15)$$

By definition of symplectic basis for the Riemann form ψ , this means that, in terms of the matrix $\eta(s) \in GL_{2g}(\mathbb{A}_{\mathbb{Q},f})$:

$$\eta(s) \cdot \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} \cdot \eta(s)^t = N_{K/\mathbb{Q}}(s) \cdot \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} \quad (3.16)$$

Note also that we have:

$$\begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \quad (3.17)$$

From this, we easily deduce $\tilde{\eta}(s) \in GSp_{2g}(\mathbb{A}_{\mathbb{Q},f})$. The decomposition in the claim is a consequence of the strong approximation theorem for $GSp_{2g}(\mathbb{A}_{\mathbb{Q},f})$ (cf. Lemma 8.3 and Lemma 9.5 [14]). $\square$

For each $s \in \mathbb{A}_{K,f}^\times$, let u_s and α_s be the similar transformations of $\tilde{u}_s$ and $\tilde{\alpha}_s$ as the above $\eta(s)$ of $\tilde{\eta}(s)$, whence $\eta(s) = u_s \cdot \alpha_s$ and $(\widehat{\mathbb{Z}}^g + \widehat{\mathbb{Z}}^g \delta) u_s = \widehat{\mathbb{Z}}^g + \widehat{\mathbb{Z}}^g \delta$; also, partition α_s into the following form:

$$\alpha_s = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix}. \quad (3.18)$$

We then define $\tau_s := \alpha_s(\tau_E)$ as $\tau_s := (a_s \tau_E + b_s \delta) \cdot (c_s \tau_E + d_s \delta)^{-1} \delta$, which is indeed a member of $\mathfrak{H}_g$ by $\tau_E \in \mathfrak{H}_g$ and $\tilde{\alpha}_s \in GSp_{2g}^+(\mathbb{Q})$; we see that this $\tau_s \in \mathfrak{H}_g$ is our target one.

Proposition 3.2.3 (see also Theorem 2.4, Streng [15]). *With notations as in Definition 3.1.1, we have the following equation:*

$$\widehat{f}_a(\rho, s) = f(\theta^k(a''_1 \tau_s + a''_2 \delta; \tau_s)). \quad (3.19)$$

where $\tau_s = \alpha_s(\tau_E)$ and $a'' = (a''_1, a''_2) \in \mathbb{Q}^{2g}/\mathbb{Z}^{2g}$ is defined as the image of $a \in \mathbb{Q}^{2g}/\mathbb{Z}^{2g}$ under the following map:

$$\mathbb{Q}^{2g}/\mathbb{Z}^{2g} \xrightarrow{\eta(\rho)} \mathbb{Q}^{2g}/\mathbb{Z}^{2g} \xrightarrow{u_s} \mathbb{Q}^{2g}/\mathbb{Z}^{2g}. \quad (3.20)$$

That is, $a'' = a \cdot \eta(\rho) u_s \in N^{-1} \mathbb{Z}^{2g}/\mathbb{Z}^{2g}$.

Proof. Again recall the following diagram defining the value of $\widehat{f}_a(\rho, s)$:

$$\begin{array}{ccccccc} E/O_E & \xrightarrow{\eta(\rho)} & E/O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_E & \xrightarrow{\Theta_E} & A_E \xrightarrow{f|_{A_E}} \mathbb{C} \\ & & \eta(s) \downarrow & & & & \downarrow \sigma \\ & & E/\eta(s)O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_s & \xrightarrow{\Theta_s} & A_s \xrightarrow{f|_{A_s}} \mathbb{C} \end{array} \quad (3.21)$$

Here the lattice $\Lambda_s = \Phi(\eta(s)O_E)$ is equal to $(\mathbb{Z}^g, \mathbb{Z}^g) \cdot u_s \alpha_s \cdot (\tau_E, \delta)^t = (\mathbb{Z}^g, \mathbb{Z}^g) \cdot \alpha_s \cdot (\tau_E, \delta)^t$ by our construction (Lemma 3.2.2), which is further isomorphic to $(\mathbb{Z}^g, \mathbb{Z}^g) \cdot (\tau_s, \delta)^t = \mathbb{Z}^g \tau_s + \mathbb{Z}^g \delta$ by the right action of $\chi_s := (c_s \tau_E + d_s \delta)^{-1} \delta$. With this in mind, we consider yet another diagram:

$$\begin{array}{ccccccc}
E/O_E & \xrightarrow{\eta(\rho)} & E/O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_E & \xrightarrow{\Theta_E} & A_E \xrightarrow{f|_{A_E}} \mathbb{C} \\
& & \eta(s) \downarrow & & & & \downarrow [s]^{-1} \quad \downarrow \sigma \\
& & E/\eta(s)O_E & \xrightarrow[\Phi]{} & \mathbb{C}^g/\Lambda_s & \xrightarrow[\simeq]{\chi_s} & \mathbb{C}^g/(\mathbb{Z}^g \tau_s + \mathbb{Z}^g \delta) \xrightarrow[\simeq]{\Theta_{\tau_s}} A_{\tau_s} \xrightarrow[f|_{A_{\tau_s}}]{} \mathbb{C}
\end{array} \tag{3.22}$$

By the defining property of $\Theta_s : \mathbb{C}^g/\Lambda_s \rightarrow A_s$, in order to prove (3.19), it suffices to prove that this diagram commutes.

We see that this commutativity follows from the fact that the ratios $\theta_a^{k/l}$ (3.10) of homogeneous coordinates $[\theta^k]$ at torsion points of $A_E \subseteq \mathbf{P}^d(\mathbb{C})$ define modular functions on $\mathfrak{H}_g$ (cf. Theorem 27.13 [13]), together with Shimura's reciprocity law (Theorem 9.6 [14]). That is, to be more specific, first note that since $f(\theta^k) = g(\theta^k)/h(\theta^k)$ is a rational function of homogeneous degree 0, it can be in fact composed of the ratios θ^k/θ^l of theta functions θ^k ; thus, $f(\theta^k(a_1\tau + a_2\delta; \tau))$ also defines a modular function on $\mathfrak{H}_g$ for each fixed $a \in \mathbb{Q}^{2g}/\mathbb{Z}^{2g}$, which we may denote as $f(\theta_a^{k/l}(\tau))$. Moreover, since f is defined over K and σ fixes K , we have $f(\theta_a^{k/l}(\tau))^\sigma = f((\theta_a^{k/l}(\tau))^{[s]^{-1}})$. Then applying Shimura's reciprocity law for these modular functions $\theta_a^{k/l}$ and $[s] \in G_K^{ab}$, we deduce that the middle rectangle commutes; in fact see Lemma 8.6 and Theorem 8.10 [14]² for a careful comparison of the definition of galois conjugate $(\theta_a^{k/l})^{\eta(s)}$ of the modular functions $\theta_a^{k/l}$ with our composition of the above maps; in particular we have $(\theta_a^{k/l})^{\eta(s)}(\tau_E) = \theta_{au_s}^{k/l}(\alpha_s(\tau_E)) = \theta_{au_s}^{k/l}(\tau_s)$. This completes the proof. $\square$

4 Galois correspondence

Unlike the case of imaginary quadratic fields, the modular vectors constructed above do not generate all algebraic Witt vectors over a CM field K of higher degree; therefore, it becomes reasonable to ask what algebraic Witt vectors are generated by our modular vectors. Denoting by M_K the set of modular vectors, we can see by definition that M_K is in fact a subalgebra of E_K over K ; moreover, it can be easily seen that M_K is a Λ -subalgebra of E_K (i.e. closed under the action of $\psi_{\mathfrak{p}}$'s), which follows from the same argument as Lemma 4.3.5 [19]. Therefore, the *general galois correspondence* mentioned in Remark 3.1.8 [19] implies that we have a dual characterization of modular vectors in terms of profinite quotient of DR_K .

This section is devoted to this problem and divided into two subsections. In the first subsection (§4.1), we give a proof of the general galois correspondence mentioned in Remark 3.1.8 [19] without proof, which claims a canonical bijective correspondence between (i) Λ -subalgebras of E_K , (ii) profinite quotients of DR_K , and (iii) semi-galois full subcategories of $\mathcal{C}_K$. This general correspondence, however, is not quite satisfactory in itself in that it just tells us only an existence of such a quotient but not its explicit structure. To remedy this issue, the second subsection (§4.2) is devoted to an explicit characterization of modular vectors, where we give an adelic description of the profinite

²Note that we use the lattice $\mathbb{Z}^g + \mathbb{Z}^g \delta$ rather than $\mathbb{Z}^g + \mathbb{Z}^g$; accordingly, we use matrices transformed by $\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}$ when we apply these theorems.

quotient of DR_K that corresponds to the Λ -subalgebra of modular vectors $M_K \subseteq E_K$ under our general galois correspondence.

4.1 General galois correspondence

In this subsection, we allow K to be an arbitrary number field; our goal here is to prove the following general galois correspondence mentioned in Remark 3.1.8 [19]:

Proposition 4.1.1 (galois correspondence). *There exists a canonical bijective correspondence between the following three classes of objects:*

1. Λ -subalgebras of E_K ;
2. profinite quotients of DR_K ; and
3. semi-galois full subcategories of $\mathcal{C}_K$.

Note that the correspondence between the second and the third is just a direct consequence of the general duality theorem between profinite monoids and semi-galois categories [16, 17] together with the fact that the fundamental monoid $\pi_1(\mathcal{C}_K, F_K)$ of $\mathcal{C}_K$ is isomorphic to DR_K [2, 3]; therefore, in order to prove this proposition, it suffices to show a bijective correspondence between the first and the second only. For this proof, however, the concept of *galois objects* in the semi-galois category $\mathcal{C}_K$ plays an auxiliary key role.

We start with the correspondence from the first to the second; and to this end, we need to recall a few basic concepts (with notations slightly different from those in [19]).

Definition 4.1.2 (monoid congruence on I_K). Let $\Xi \subseteq E_K$ be a finite subset of E_K . Then, define the monoid congruence $\mathfrak{a} \sim_\Xi \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in I_K$ as follows:

$$\mathfrak{a} \sim_\Xi \mathfrak{b} \iff \forall \xi \in \Xi, \psi_{\mathfrak{a}} \xi = \psi_{\mathfrak{b}} \xi. \quad (4.1)$$

Also, denote by D_Ξ the finite quotient monoid $D_\Xi := I_K / \sim_\Xi$.

Concerning this monoid congruence, note that, for two finite subsets $\Xi, \Xi' \subseteq E_K$ with $\Xi \subseteq \Xi'$, we have a canonical surjective monoid homomorphism $r_{\Xi}^{\Xi'} : D_{\Xi'} \rightarrow D_\Xi$; therefore, with respect to the inclusion relation $\Xi \subseteq \Xi'$ of finite subsets of E_K , we have an inverse system $\{D_\Xi, r_{\Xi}^{\Xi'}\}$ of finite monoids D_Ξ . The fact that E_K is the union of the galois objects X_Ξ of $\mathcal{C}_K$ (§3.1 [19]) implies in particular that DR_K is isomorphic to the inverse limit of this system $\{D_\Xi, r_{\Xi}^{\Xi'}\}$ when $\Xi \subseteq E_K$ runs over all finite subsets of E_K ; thus by restricting the range where $\Xi \subseteq E_K$ are allowed to run, we naturally obtain a quotient of DR_K :

Definition 4.1.3 (dual to Λ -subalgebra of E_K). Let $E \subseteq E_K$ be a Λ -subalgebra of E_K ; then define D_E as the following profinite monoid:

$$D_E := \varprojlim_{\leftarrow \Xi \subseteq E} D_\Xi. \quad (4.2)$$

Conversely, given a profinite quotient monoid $DR_K \twoheadrightarrow D$, we construct a Λ -subalgebra $E_D \subseteq E_K$ so that this construction $D \mapsto E_D$ gives the inverse of the above construction $E \mapsto D_E$. To this end, recall (§2.1) that we can identify E_K with the Λ -ring $\text{Hom}_{G_K}(DR_K, K^{ab})$ of locally constant K^{ab} -valued G_K -equivariant functions of DR_K ; under this identification, we can naturally describe the target correspondence $D \mapsto E_D$ as follows:

Definition 4.1.4 (dual to quotient of DR_K). Let $DR_K \twoheadrightarrow D$ be a profinite quotient monoid of DR_K ; then we define a Λ -subalgebra $E_D \subseteq E_K$ as follows:

$$E_D := \left\{ \xi \in \text{Hom}_{G_K}(DR_K, K^{ab}) \mid \xi \text{ factors through } DR_K \twoheadrightarrow D \right\}. \quad (4.3)$$

Lemma 4.1.5. *For any Λ -subalgebra $E \subseteq E_K$ we have the identity $E = E_{D_E}$.*

Proof. Given $\xi \in E$, the finite monoid $D_\xi := D_{\{\xi\}}$ is by definition a quotient of D_E . Since ξ , when viewed as a function on DR_K , factors through D_ξ , it follows that ξ also factors through D_E , hence $\xi \in E_{D_E}$, concluding $E \subseteq E_{D_E}$.

Conversely, given $\xi \in E_{D_E}$ that is viewed as a function on DR_K , the definition of E_{D_E} implies that $\xi : DR_K \rightarrow K^{ab}$ factors through $DR_K \twoheadrightarrow D_E$; therefore, D_ξ is a finite quotient of D_E . Then, by definition of D_E , this also implies that there exists a finite subset $\Xi \subseteq E$ such that D_ξ is a finite quotient of D_Ξ . Since $\text{Hom}_{G_K}(D_\Xi, K^{ab})$ is equal to the galois object $X_\Xi = K[I_K \Xi]$ (§2.1), we obtain $\xi \in X_\Xi$. But since E is now a Λ -subalgebra thus closed under ψ_p 's, we also have $X_\Xi \subseteq E$, hence $\xi \in X_\Xi \subseteq E$, concluding $E_{D_E} \subseteq E$. This completes the proof. $\square$

Lemma 4.1.6. *For any profinite quotient monoid $DR_K \twoheadrightarrow D$, we have a canonical isomorphism $D \simeq D_{E_D}$.*

Proof. First assume that two ideals $\mathfrak{a}, \mathfrak{b} \in I_K \hookrightarrow DR_K$ go into the same element in D under the quotient $DR_K \twoheadrightarrow D$; we show that $\mathfrak{a}, \mathfrak{b}$ go to the same element in D_{E_D} as well under the quotient $DR_K \twoheadrightarrow D_{E_D}$. Now the assumption implies that the operators $\psi_{\mathfrak{a}}, \psi_{\mathfrak{b}}$ define the same action on $E_D = \text{Hom}_{G_K}(D, K^{ab})$ by definition (cf. §2.1); therefore, so they do on $X_\Xi \subseteq E_D$ for any finite subset $\Xi \subseteq E_D$; in particular, we have $\psi_{\mathfrak{a}}\xi = \psi_{\mathfrak{b}}\xi$ for any $\xi \in \Xi \subseteq X_\Xi$, which means that $\mathfrak{a} \sim_\Xi \mathfrak{b}$. Now since Ξ is any finite subset of E_D , this eventually deduces that $\mathfrak{a}, \mathfrak{b}$ are mapped to the same element in D_{E_D} .

Conversely, assume that $\mathfrak{a}, \mathfrak{b}$ are mapped onto two different elements in D , whence we have a finite quotient $D \twoheadrightarrow H$ so that $\mathfrak{a}, \mathfrak{b}$ are still different in H as well; then $\psi_{\mathfrak{a}}$ and $\psi_{\mathfrak{b}}$ define different endomorphisms on the galois object $X_H := \text{Hom}_{G_K}(H, K^{ab})$ in $\mathcal{C}_K$, thus we have $\psi_{\mathfrak{a}}\xi \neq \psi_{\mathfrak{b}}\xi$ for some $\xi \in X_H$. Since X_H is now a Λ -subalgebra of E_D , this means that $\mathfrak{a}$ and $\mathfrak{b}$ are not congruent with respect to $\sim_{\{\xi\}}$ for the singleton $\{\xi\} \subseteq E_D$; that is, $\mathfrak{a} \neq \mathfrak{b}$ in the finite monoid $D_{\{\xi\}}$. Again, by definition of $D_{E_D} = \lim_{\Xi \subseteq E_D} D_\Xi$, this finite monoid $D_{\{\xi\}}$ is a quotient of D_{E_D} . Since $\mathfrak{a} \neq \mathfrak{b} \in D_\xi$, it follows that $\mathfrak{a} \neq \mathfrak{b} \in DR_{E_D}$ too.

The above arguments show that the two quotients $DR_K \twoheadrightarrow D$ and $DR_K \twoheadrightarrow D_{E_D}$ have the same kernel on the discrete submonoid I_K ; but since I_K is dense in DR_K , it follows that they have the same kernel on DR_K as well. Thus they must be isomorphic. $\square$

Basically, we use the above galois correspondence in the following form:

Corollary 4.1.7 (dual characterization). *Let $E \subseteq E_K$ be a Λ -subalgebra such that E corresponds to a profinite quotient $DR_K \twoheadrightarrow D$ under the above galois correspondence. Then $\xi \in E_K$ is a member of E if and only if D_ξ is a quotient of D .*

4.2 Characterization of modular vectors

Since the modular vectors constitute a Λ -subalgebra M_K of E_K , the general galois correspondence (Proposition 4.1.1) guarantees that we have a profinite quotient $DR_K \twoheadrightarrow D_{M_K}$ so that an algebraic

Witt vector $\xi \in E_K$ is modular if and only if D_ξ is a finite quotient of D_{M_K} ; that said, nevertheless, the general galois correspondence gives us only a somewhat tautological description of D_{M_K} , i.e. $D_{M_K} = \lim_{\Xi \subseteq M_K} D_\Xi$. In view of this issue, this subsection gives in a step-by-step manner a purely adelic description of this profinite monoid D_{M_K} , which is intrinsic for K and independent of modular vectors $\Xi \subseteq M_K$.

As the first step, we define finite quotients $DR_K \twoheadrightarrow D_N$ for $N \geq 1$ using projective embeddings of complex tori so that D_N 's constitute an inverse system of finite monoids whose inverse limit is isomorphic to our target profinite monoid D_{M_K} (Lemma 4.2.3); this description of D_{M_K} is not yet intrinsic for K but we utilize it to study the structure of D_{M_K} via geometry of abelian varieties. Based on this, as the second step, we describe the structure of these finite monoids D_N in a purely adelic term, which provides us a K -intrinsic description of $D_{M_K} \simeq \lim_N D_N$ and concludes our dual (K -intrinsic) characterization of modular vectors (cf. Theorem 4.2.6).

Definition 4.2.1 (congruence $\sim_N$). For $[\rho, s], [\lambda, t] \in DR_K$, we define $[\rho, s] \sim_N [\lambda, t]$ if and only if the following two parallel arrows $E/O_E \rightarrow \mathbf{P}^d(\mathbb{C})$ coincide on the subset $N^{-1}O_E/O_E \subseteq E/O_E$:

$$\begin{array}{ccccc}
 & & E/\eta(s)O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_s & \xrightarrow{\Theta_s} & A_s & \xrightarrow{\subseteq} & \mathbf{P}^d(\mathbb{C}) \\
 & \nearrow \eta(s) & & & & & & & \\
 E/O_E & \xrightarrow[\eta(\lambda)]{\eta(\rho)} & E/O_E & & & & & & \\
 & \searrow \eta(t) & & & & & & & \\
 & & E/\eta(t)O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_t & \xrightarrow{\Theta_t} & A_t & \xrightarrow{\subseteq} & \mathbf{P}^d(\mathbb{C})
 \end{array} \tag{4.4}$$

Lemma 4.2.2. *The congruence $\sim_N$ is a monoid congruence on DR_K of finite order.*

Proof. First to see that $\sim_N$ is a monoid congruence, it suffices to prove that $[\rho, s] \sim_N [\lambda, t]$ implies $[\rho\nu, sr] \sim_N [\lambda\nu, tr]$ for any $[\nu, r] \in DR_K$. But this follows from the following larger commutative diagram, where we used Shimura's reciprocity law (cf. §2.2.3) for the fractional ideals $\mathfrak{a} = \eta(s)O_E$ and $\mathfrak{a} = \eta(t)O_E$ in particular; note also that $\eta(\nu) : E/O_E \rightarrow E/O_E$ maps the subset $N^{-1}O_E/O_E$ to itself:

$$\begin{array}{ccccccc}
 & & E/\eta(sr)O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_{sr} & \xrightarrow{\Theta_{sr}} & A_{sr} \\
 & \nearrow \eta(sr) & \uparrow \eta(r) & & & & \uparrow [r]^{-1} \\
 & & E/\eta(s)O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_s & \xrightarrow{\Theta_s} & A_s \\
 & \nearrow \eta(s) & & & & & \downarrow [r]^{-1} \\
 E/O_E & \xrightarrow[\eta(\lambda)]{\eta(\rho)} & E/O_E & & & & \mathbf{P}^d(\mathbb{C}) \xrightarrow{[r]^{-1}} \mathbf{P}^d(\mathbb{C}) \\
 & \searrow \eta(t) & & & & & \downarrow [r]^{-1} \\
 & & E/\eta(t)O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_t & \xrightarrow{\Theta_t} & A_t \\
 & \searrow \eta(tr) & \downarrow \eta(r) & & & & \downarrow [r]^{-1} \\
 & & E/\eta(tr)O_E & \xrightarrow{\Phi} & \mathbb{C}^g/\Lambda_{tr} & \xrightarrow{\Theta_{tr}} & A_{tr}
 \end{array} \tag{4.5}$$

Also, the fact that the monoid congruence $\sim_N$ is of finite index follows from the same argument in the last part of the proof of Proposition 3.2.1. This completes the proof. $\square$

For $N \mid N'$, it is obvious from the definition that $[\rho, s] \sim_{N'} [\lambda, t]$ implies $[\rho, s] \sim_N [\lambda, t]$; therefore, writing $D_N := DR_K / \sim_N$, we have a surjective homomorphism $r_N^{N'} : D_{N'} \twoheadrightarrow D_N$ of finite monoids for $N \mid N'$. This way we obtain an inverse system $\{D_N, r_N^{N'}\}$ of finite monoids D_N , whose inverse limit is in fact isomorphic to D_{M_K} .

Lemma 4.2.3. *We have a canonical isomorphism of profinite monoids, $D_{M_K} \simeq \lim_{N \geq 1} D_N$.*

Proof. Let $X_N := \{\hat{f}_a \mid a \in N^{-1}O_E/O_E, f \text{ is rational over } K \text{ and defined for } a\} \subseteq M_K$; since M_K is the union of X_N 's, it suffices to prove that X_N is the galois object of $\mathcal{C}_K$ dual to D_N . First, by the fact that all $\hat{f}_a$ in X_N as functions on DR_K factor through DR_{NO_K} , it follows that X_N is finite etale over K ; also, by the same argument of Lemma 4.3.5 [19], it can be readily seen that X_N 's are closed under ψ_p 's. From these we deduce that $X_N = X_{\Xi(N)}$ for some finite set $\Xi(N) \subseteq X_N \subseteq M_K$, hence X_N is a galois object of $\mathcal{C}_K$.

The rest task is to prove that this galois object X_N is dual to D_N ; since we already know that X_N is galois, this is equivalent to proving that for any $[\rho, s], [\lambda, t] \in DR_K$, we have $[\rho, s] \sim_N [\lambda, t]$ if and only if the equality $\hat{f}_a(\rho, s) = \hat{f}_a(\lambda, t)$ holds for all $a \in N^{-1}O_E/O_E$ and rational functions f over K defined for a .

The only-if part is almost trivial: notice from the definition that $[\rho, s] \sim_N [\lambda, t]$ means that each N -torsion point $a_1\tau_E + a_2\delta \in N^{-1}O_E/O_E$ is mapped to the same point in $\mathbf{P}^d(\mathbb{C})$ under the two maps $E/O_E \rightarrow \mathbf{P}^d(\mathbb{C})$ given in (4.4); therefore, their images of $a_1\tau_E + a_2\delta$ cannot be distinguished by any rational function f defined on such torsion points, which mean that we have $\hat{f}_a(\rho, s) = \hat{f}_a(\lambda, t)$ for any $a \in N^{-1}O_E/O_E$ and rational functions f over K defined for a . (Hence X_N is contained in the galois object $\widehat{\text{Hom}}_{G_K}(D_N, K^{ab})$ dual to D_N .)

Conversely, suppose $\hat{f}_a(\rho, s) = \hat{f}_a(\lambda, t)$ for any $a \in N^{-1}O_E/O_E$ and rational functions f over K defined for a ; this means that each of the N -torsion points $a_1\tau_E + a_2\delta \in N^{-1}O_E/O_E$ is mapped to the points, say $p_{\rho,s}(a), p_{\lambda,t}(a) \in \mathbf{P}^d(\mathbb{C})$, by the two maps in (4.4) respectively so that they cannot be distinguished by any such rational functions f . We shall prove that, if these points $p_{\rho,s}(a), p_{\lambda,t}(a)$ are distinct for some $a \in N^{-1}O_E/O_E$, they can be distinguished by some rational function f over K defined for a : Since we suppose $p_{\rho,s}(a) \neq p_{\lambda,t}(a) \in \mathbf{P}^d(\mathbb{C})$, we can take some rational function $f_1 : \mathbf{P}^d(\mathbb{C}) \rightarrow \mathbf{P}^1(\mathbb{C})$ defined over $\mathbb{Q}$ such that $f_1(p_{\rho,s}(a)) \neq f_1(p_{\lambda,t}(a)) \in \mathbf{P}^1(\mathbb{C})$. Then consider the finite subset $S \subseteq \mathbf{P}^1(\mathbb{C})$ consisting of $f_1(p_{\rho,s}(a'))$ and $f_1(p_{\lambda,t}(a'))$ for all $a' \in N^{-1}O_E/O_E$, which (of course) contains $f_1(p_{\rho,s}(a))$ and $f_1(p_{\lambda,t}(a))$. If S does not contain $\infty \in \mathbf{P}^1(\mathbb{C})$, the original f_1 already gives our desired one; otherwise, take a rational function $h : \mathbf{P}^1(\mathbb{C}) \rightarrow \mathbf{P}^1(\mathbb{C})$ over $\mathbb{Q}$ so that all the points in $S \subseteq \mathbf{P}^1(\mathbb{C})$ are mapped injectively into $\mathbf{P}^1(\mathbb{C}) \setminus \{\infty\}$; then $h \circ f_1 : \mathbf{P}^d(\mathbb{C}) \rightarrow \mathbf{P}^1(\mathbb{C})$ is our desired one. This completes the proof. $\square$

From this, it follows that an algebraic Witt vector $\xi \in E_K$ is modular (i.e. $\xi \in M_K$) if and only if the finite monoid D_ξ is a quotient of D_N for some $N \geq 1$. Recall that D_ξ is the quotient finite monoid $D_\xi := I_K / \sim_\xi$, where the congruence $\sim_\xi$ is defined as $\mathfrak{a} \sim_\xi \mathfrak{b}$ if and only if $\psi_{\mathfrak{a}}\xi = \psi_{\mathfrak{b}}\xi$, or in more elementary words, $\xi_{\mathfrak{a}\mathfrak{c}} = \xi_{\mathfrak{b}\mathfrak{c}}$ for any $\mathfrak{c} \in I_K$. That is, the congruence $\sim_\xi$ on I_K concerns only the pattern of whether or not the components $\xi_{\mathfrak{a}\mathfrak{c}}, \xi_{\mathfrak{b}\mathfrak{c}}$ coincide for all $\mathfrak{c}$, and does not concern their actual values. The real ingredient of this dual characterization is that whether $\xi \in E_K$ is realized as a modular vector $\xi = \hat{f}_a$ analytically can be determined only by such a pattern of the equality between the components $\xi_{\mathfrak{a}}$ of ξ .

Nevertheless, the definition of the congruence $\sim_N$ depends on theta functions Θ_s and thus the finite monoid $D_N = I_K / \sim_N$ is not yet intrinsic for K . The subject of the rest of this section is to remedy this issue, giving a purely adelic description of the structure of D_N , for which the following lemma is our key result. In what follows, for each subset $S \subseteq O_E/NO_E$, let us denote by $A_E[S]$ the N -torsion points on A_E that correspond to S via the isomorphism $A_E[N] \simeq O_E/NO_E$; in particular, for each $\rho \in \widehat{O}_K$, we denote by $S_\rho \subseteq O_E/NO_E$ the image of the map $\eta(\rho) : O_E/NO_E \rightarrow O_E/NO_E$.

Lemma 4.2.4. *For any $[\rho, s], [\lambda, t] \in DR_K$, we have $[\rho, s] \sim_N [\lambda, t]$ if and only if there exists some $w \in (O_E/NO_E)^\times$ satisfying the following two conditions:*

1. $\eta(\rho)w = \eta(\lambda)$ in O_E/NO_E ;
2. the action of $[s^{-1}t] \in G_K^{ab}$ on $A_E[S_\rho]$ is equal to that of w .

where the action of $w \in (O_E/NO_E)^\times$ on $A_E[N]$ is given by the multiplication $A_E[N] \simeq O_E/NO_E \ni a \mapsto a \cdot w \in O_E/NO_E \simeq A_E[N]$.

Proof. We first prove the if part; suppose that the above two conditions hold for $[\rho, s], [\lambda, t] \in DR_K$ and $w \in (O_E/NO_E)^\times$, from which we deduce $[\rho, s] \sim_N [\lambda, t]$. Since $\eta(\rho) \cdot w = \eta(\lambda) \in O_E/NO_E$, we have the following equalities for any $k, l \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$ and $a \in N^{-1}O_E/O_E$:

$$\widehat{\theta}_a^{k/l}(\lambda, t) = \left(\theta_{a\eta(\lambda)}^{k/l}(\tau_E) \right)^{[t]^{-1}} \quad (4.6)$$

$$= \left(\theta_{a\eta(\rho)w}^{k/l}(\tau_E) \right)^{[t]^{-1}} \quad (4.7)$$

where $\widehat{\theta}^{k/l}(\rho, s)$ is defined as in Definition 3.1.1. But now by $a\eta(\rho) \in S_\rho$ and by the second condition, this last one is equal to $\widehat{\theta}_a^{k/l}(\rho, s)$, thus:

$$\widehat{\theta}_a^{k/l}(\lambda, t) = \widehat{\theta}_a^{k/l}(\rho, s). \quad (4.8)$$

Since $a \in N^{-1}O_E/O_E$ and $k, l \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$ are arbitrary, this implies that $[\rho, s] \sim_N [\lambda, t]$.

Conversely, we prove the only-if part: suppose that we have $[\rho, s] \sim_N [\lambda, t]$, from which we shall construct $w \in (O_E/NO_E)^\times$ satisfying the above two conditions. To this end, we note that the first condition almost immediately implies the second one: in fact, if we have $\eta(\rho)w = \eta(\lambda)$, the above argument shows $(\theta_{a\eta(\rho)}^{k/l}(\tau_E))^{[s^{-1}t]} = (\theta_{a\eta(\rho)w}^{k/l}(\tau_E))$ for any $k, l \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$ and $a \in N^{-1}O_E/O_E$; this means that $[s^{-1}t] \in G_K^{ab}$ defines the same action on $A_E[S_\rho]$ as w , hence the second condition is satisfied. Therefore it suffices to prove that we have $w \in (O_E/NO_E)^\times$ satisfying the first condition.

We prove this by essentially the same idea of our argument in §4.3 [19]: Note that $[\rho, s] \sim_N [\lambda, t]$ implies the equality $(\eta(\rho)O_E, NO_E) = (\eta(\lambda)O_E, NO_E)$ of the greatest common divisors. Indeed, by the definition of the congruence $\sim_N$, the assumption $[\rho, s] \sim_N [\lambda, t]$ implies the following equalities for all $a \in N^{-1}O_E/O_E$ and $k, l \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$:

$$\theta_{a\eta(\rho)}^{k/l}(\tau_E) = \left(\theta_{a\eta(\lambda)}^{k/l}(\tau_E) \right)^{[st^{-1}]}. \quad (4.9)$$

In particular, applying this equality to the case of $a = 0$ and using a decomposition $\eta(st^{-1}) = u \cdot \alpha$ (cf. Lemma 3.2.2) together with Shimura's reciprocity law, we have:

$$\theta_0^{k/l}(\tau_E) = \theta_0^{k/l}(\alpha(\tau_E)), \quad (4.10)$$

for all $k, l \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$. With this in mind, suppose now that $a\eta(\lambda) = 0$ for some $a \in N^{-1}O_E/O_E$; then we can show $a\eta(\rho) = 0$ for this a as well. In fact, for any $k, l \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$:

$$\theta_{a\eta(\rho)}^{k/l}(\tau_E) = \theta_0^{k/l}(\alpha(\tau_E)) \quad (4.11)$$

$$= \theta_0^{k/l}(\tau_E); \quad (4.12)$$

thus the injectivity of $\Theta_{\tau_E} : \mathbb{C}^g/\Lambda_E \hookrightarrow \mathbf{P}^d(\mathbb{C})$ implies $a\eta(\rho) = 0$. The converse argument then shows that $a\eta(\rho) = 0$ if and only if $a\eta(\lambda) = 0$ for all $a \in O_E/NO_E$, which means that $(\eta(\rho)O_E, NO_E) = (\eta(\lambda)O_E, NO_E)$ as desired, and equivalently, there exists some $w \in (O_E/NO_E)^\times$ such that $\eta(\rho)w = \eta(\lambda)$ in O_E/NO_E , hence the claim. $\square$

Next we characterize, in terms of an adelic condition, when $[s^{-1}t] \in G_K^{ab}$ defines the same action on subsets of $A_E[N]$ as that of $w \in (O_E/NO_E)^\times$, following the ideas of Theorem 2.5, Streng [15] and Corollary 18.9, Shimura [12]. In what follows we suppose that our CM-type (E, Φ) is *primitive*, whence $\text{End}_{\mathbb{Q}}(A_E) \simeq E$ (cf. §8.2, [13]):

Lemma 4.2.5. *For each $S \subseteq (O_E/NO_E)$, the galois action of $[s]^{-1} \in G_K^{ab}$ on $A_E[S]$ is equal to that of $w \in (O_E/NO_E)^\times$ on $A_E[S]$ if and only if s is a member of the following subset $T_{S,w} \subseteq \mathbb{A}_{K,f}^\times$:*

$$T_{S,w} := \{s \in \mathbb{A}_{K,f}^\times \mid \exists r \in E^\times, rr^\iota = N(sO_K), rO_E = \eta(s)O_E, r \cdot a \cdot w = a \cdot \eta(s) \ (\forall a \in S)\}. \quad (4.13)$$

Proof. First, suppose that $s \in \mathbb{A}_{K,f}^\times$ defines the same action as $w \in (O_E/NO_E)^\times$ on $A_E[S]$. Then by definition, we have $(\theta_a^{k/l}(\tau_E))^{[s]^{-1}} = \theta_{aw}^{k/l}(\tau_E)$ for any $k, l \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$ and $a \in S_\rho$. Applying this to the case of $a = 0$ in particular, we obtain:

$$\theta_0^{k/l}(\alpha_s(\tau_E)) = \theta_0^{k/l}(\tau_E), \quad (4.14)$$

for any $k, l \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g$. By Proposition 8.10.2 [1] claiming that the theta null values preserve the isomorphism classes of abelian varieties, we see that A_{τ_E} and A_{τ_s} are isomorphic. Thus we obtain an isomorphism $h : \mathbb{C}^g/\Phi(O_E) \rightarrow \mathbb{C}^g/\Phi(\eta(s)O_E)$ of complex tori, preserving their polarizations, which is given by the multiplication of some $r \in E^\times$ such that $rO_E = \eta(s)O_E$ and $\Theta_E(\Phi(a')) = \Theta_s(\Phi(ra'))$ for all $a' \in E/O_E$. Now for any $a \in S \subseteq N^{-1}O_E/O_E$ in particular, the assumption on s implies the identities $\Theta_s(\Phi(raw)) = \Theta_E(\Phi(aw)) = \Theta_E(\Phi(a))^{[s]^{-1}} = \Theta_s(\Phi(a \cdot \eta(s)))$; therefore, by the injectivity of $\Theta_s : \mathbb{C}^g/\Phi(\eta(s)O_E) \rightarrow A_s$, we have $r \cdot a \cdot w = a \cdot \eta(s)$. Moreover, since the isomorphism h preserves the polarizations, it follows by Theorem 18.6 [13] that $rr^\iota = N(sO_K)$, hence $s \in T_\rho$. Conversely let $s \in T_{S,w}$ and take $r \in E^\times$ as in the definition of $T_{S,w}$. Then by definition, the multiplication by r defines an isomorphism $h : \mathbb{C}^g/\Phi(O_E) \rightarrow \mathbb{C}^g/\Phi(\eta(s)O_E)$ of complex tori in such a manner that we have the identities $\Theta_E(\Phi(aw)) = \Theta_s(\Phi(raw)) = \Theta_s(\Phi(a \cdot \eta(s))) = \Theta_E(\Phi(a))^{[s]^{-1}}$ for each $a \in S$, which means that the galois action of $[s]^{-1}$ on $A_E[S]$ is equal to that of w , as required. $\square$

Now we conclude our target adelic description of the finite monoids D_N :

Theorem 4.2.6 (adelic description of D_N). *For any $[\rho, s], [\lambda, t] \in DR_K$, we have $[\rho, s] \sim_N [\lambda, t]$ if and only if there exists some $w \in (O_E/NO_E)^\times$ satisfying the following two conditions:*

1. $\eta(\rho)w = \eta(\lambda)$ in $N^{-1}O_E/O_E$;
2. $s^{-1}t \in T_{\rho,w} = T_{\lambda,w}$.

where $T_{\rho,w}, T_{\lambda,w} \subseteq \mathbb{A}_{K,f}^\times$ denote $T_{S_\rho,w}, T_{S_\lambda,w}$ respectively, which coincide under the first condition.

Proof. This follows from Lemma 4.2.4 and Lemma 4.2.5. In particular, note that the first condition implies $S_\rho = S_\lambda$. $\square$

Remark 4.2.7. When E is imaginary quadratic, the above results readily imply yet another form of the modularity theorem (cf. Theorem 4.3.1 [19]); that is, the modular vectors constructed by ratios of theta functions also exhaust all algebraic Witt vectors as in the case of those constructed by Fricke functions. To see this it suffices to show that $[\rho, s] \sim_N [\lambda, t]$ if and only if $[\rho, s] = [\lambda, t] \in DR_{NO_K}$, in other words, D_N is isomorphic to DR_{NO_K} .

Since E is imaginary quadratic, note first that its reflex field $K = E^*$ coincides with E itself; also the reflex norm $\eta : \mathbb{A}_{K,f} \rightarrow \mathbb{A}_{E,f}$ is the identity. Then Theorem 4.2.6 says that $[\rho, s] \sim_N [\lambda, t]$ if and only if there exists some $u \in (O_K/NO_K)^\times$ such that $\rho u = \lambda$ in O_K/NO_K and $s^{-1}t \in T_{\rho,u}$; but as in the proof of Lemma 4.2.4, one can also see that $s^{-1}t \in T_{\rho,u}$ if and only if the actions of $[s]$ and $[ut]$ on the subset $A_K[S_\rho]$ of the N -torsion points on the elliptic curve A_K coincide, or equivalently, $[s] = [ut]$ in the ray class group $C_{\mathfrak{d}_\rho}$ for the conductor $\mathfrak{d}_\rho := NO_K/(\rho O_K, NO_K)$. That is, the finite monoid D_N is isomorphic to $(O_K/NO_K) \times_{(O_K/NO_K)^\times} C_{NO_K} = DR_{NO_K}$, hence the claim.

In this sense, Theorem 4.2.6 can be seen as a natural generalization of our previous modularity theorem for algebraic Witt vectors over imaginary quadratic fields (Theorem 4.3.1 [19]) to the case of (primitive) CM fields; in particular, we showed that the use of ratios of theta functions instead of Fricke functions enables this higher-dimensional extension in a natural way, and also highlights the geometric nature of the modularity theorem in the sense that we discussed in §3.

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