

AMBARZUMYAN-TYPE THEOREM FOR VECTORIAL STURM-LIOUVILLE OPERATOR WITH IMPULSES

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ABSTRACT. We consider the vector-impulsive Sturm-Liouville problem with Neumann conditions. The Ambarzumyan's theorem for the problem is proved, which states that if the eigenvalues of the problem coincide with those of the zero potential, then the potential is zero.

1. INTRODUCTION

It is well known that the paper [1] of Ambarzumyan may be thought of as the first paper in the theory of inverse spectral problems associated with Sturm-Liouville operators. [1] stated the following theorem:

If q is a smooth real-valued function, and $\{n^2 : n = 0, 1, 2, \dots\}$ is the set of eigenvalues of the boundary value problem

$$-y'' + q(x)y = \lambda y, \quad x \in (0, \pi), \quad y'(0) = y'(\pi) = 0,$$

then $q(x) = 0$ on $[0, \pi]$.

This theorem is called Ambarzumyan's theorem, and has been generalized in many directions. Without a claim to completeness we mention here the papers [2-9, 11-13], etc. In particular, the most recent paper [13] extended the Ambarzumyan's theorem for the classical Sturm-Liouville operator to the scalar impulsive Sturm-Liouville operator with Neumann conditions. Our interest is to extend the result of the paper [13] to the vector case.

In this paper we consider the following vectorial Sturm-Liouville operator $L(Q)$ with impulses

$$lY := -Y'' + Q(x)Y = \lambda \rho(x)Y, \quad x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right), \quad (1.1)$$

with the jump conditions

$$Y\left(\frac{\pi}{2} + 0\right) = aY\left(\frac{\pi}{2} - 0\right), \quad Y'\left(\frac{\pi}{2} + 0\right) = a^{-1}Y'\left(\frac{\pi}{2} - 0\right), \quad (1.2)$$

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and Neumann conditions

$$Y'(0) = Y'(\pi) = 0, \quad (1.3)$$

where

$$\rho(x) = \begin{cases} 1, & 0 < x < \frac{\pi}{2}, \\ \alpha^2, & \frac{\pi}{2} < x < \pi, \end{cases} \quad 0 < \alpha < 1,$$

$a > 0$ and Q is an $N \times N$ real symmetric matrix-valued function and $Q \in L^2_{N \times N}(0, \pi)$, Y is a column vector-valued function with N components, N is a positive integer, λ is a complex parameter. Clearly, $Q \in L^2_{N \times N}(0, \pi)$ if and only if each entry of the matrix Q belongs to $L^2(0, \pi)$.

Note that Ambarzumyan's theorem for the problem (1.1)-(1.3) with $\rho \equiv 1$ and $a = 1$, for the continuous potential Q , has been obtained in [3]. To our best knowledge, Ambarzumyan's theorem for the problem (1.1)-(1.3) hasn't been studied.

It is easy to verify that $L(Q)$ is a self-adjoint eigenvalue problem. Denote the set of eigenvalues by $\sigma(Q) = \{\lambda_n(Q)\}_{n=0}^{\infty}$, which can be arranged in an ascending order as (counted with multiplicity)

$$\lambda_0(Q) \leq \lambda_1(Q) \leq \cdots \leq \lambda_n(Q) \leq \cdots \longrightarrow +\infty.$$

The main result in this paper is as follows.

Theorem 1.1. *If $\lambda_n(Q) = \lambda_n(0)$ for all $n = 0, 1, 2, \dots$, then $Q(x) = 0$ a.e. $x \in (0, \pi)$.*

2. ANALYSIS OF THE CHARACTERISTIC FUNCTION

In this section we analyze the characteristic function of the problem $L(Q)$, which plays a key role in the proof of Theorem 1.1.

Let $Y_1(x) = Y(x)|_{(0, \frac{\pi}{2})}$, $Y_2(x) = Y(\pi - x)|_{(0, \frac{\pi}{2})}$, $Q_1(x) = Q(x)|_{(0, \frac{\pi}{2})}$, $Q_2(x) = Q(\pi - x)|_{(0, \frac{\pi}{2})}$. Then the problem $L(Q)$ can be rewritten as

$$-Y_1'' + Q_1(x)Y_1 = \lambda Y_1, \quad x \in \left(0, \frac{\pi}{2}\right), \quad (2.1)$$

$$-Y_2'' + Q_2(x)Y_2 = \lambda \alpha^2 Y_2, \quad x \in \left(0, \frac{\pi}{2}\right), \quad (2.2)$$

with

$$aY_1\left(\frac{\pi}{2}\right) - Y_2\left(\frac{\pi}{2}\right) = 0, \quad a^{-1}Y_1'\left(\frac{\pi}{2}\right) + Y_2'\left(\frac{\pi}{2}\right) = 0, \quad (2.3)$$

and

$$Y_1'(0) = 0 = Y_2'(0). \quad (2.4)$$

Respectively, let $\Phi_1(x, \lambda)$ and $\Phi_2(x, \lambda)$ be the solutions of the matrix differential equations

$$-\Phi'' + Q_1(x)\Phi = \lambda\Phi$$

and

$$-\Phi'' + Q_2(x)\Phi = \lambda\alpha^2\Phi$$

satisfying the initial conditions

$$\Phi(0) = I, \quad \Phi'(0) = 0,$$

where I is the $N \times N$ identity matrix. Then any solution $Y_1(x, \lambda)$ of the equation (2.1) satisfying $Y_1'(0) = 0$ can be expressed as

$$Y_1(x, \lambda) = \Phi_1(x, \lambda)C_1(\lambda), \quad (2.5)$$

and any solution $Y_2(x, \lambda)$ of the equation (2.2) satisfying $Y_2'(0) = 0$ can also be expressed as

$$Y_2(x, \lambda) = \Phi_2(x, \lambda)C_2(\lambda), \quad (2.6)$$

where $C_j(\lambda)$ ($j = 1, 2$) is arbitrary $N \times 1$ constant vector only depending on λ .

Using (2.5) and (2.6) in (2.3) we obtain

$$\begin{cases} a\Phi_1(\frac{\pi}{2}, \lambda)C_1(\lambda) - \Phi_2(\frac{\pi}{2}, \lambda)C_2(\lambda) = 0, \\ a^{-1}\Phi_1'(\frac{\pi}{2}, \lambda)C_1(\lambda) + \Phi_2'(\frac{\pi}{2}, \lambda)C_2(\lambda) = 0. \end{cases}$$

If the above system has nonzero solutions with respect to variables $C_1(\lambda)$ and $C_2(\lambda)$, then the coefficient determinant of the system must be vanished. Thus, the characteristic equation of the problem $L(Q)$ is given as

$$\det \begin{pmatrix} a\Phi_1(\frac{\pi}{2}, \lambda) & -\Phi_2(\frac{\pi}{2}, \lambda) \\ a^{-1}\Phi_1'(\frac{\pi}{2}, \lambda) & \Phi_2'(\frac{\pi}{2}, \lambda) \end{pmatrix} = 0.$$

The determinant

$$\omega_Q(\lambda) := \det \begin{pmatrix} a\Phi_1(\frac{\pi}{2}, \lambda) & -\Phi_2(\frac{\pi}{2}, \lambda) \\ a^{-1}\Phi_1'(\frac{\pi}{2}, \lambda) & \Phi_2'(\frac{\pi}{2}, \lambda) \end{pmatrix} \quad (2.7)$$

is called the characteristic function of the problem $L(Q)$, and its zeros coincide with the eigenvalues of $L(Q)$.

We now derive the asymptotic of the characteristic function $\omega_Q(\lambda)$. In the following we use M_{ii} to represent the entry of matrix M at the i -st row and i -st column, and $tr M$ to represent the trace of matrix M . Note that (Ref. [10])

$$\begin{cases} \Phi_1(\frac{\pi}{2}, \lambda) = \cos\left(\frac{\sqrt{\lambda}\pi}{2}\right)I + \frac{\sin(\frac{\sqrt{\lambda}\pi}{2})}{\sqrt{\lambda}}[Q_1] + \frac{\Psi_1(\lambda)}{\sqrt{\lambda}}, \\ \Phi_1'(\frac{\pi}{2}, \lambda) = -\sqrt{\lambda}\sin\left(\frac{\sqrt{\lambda}\pi}{2}\right)I + \cos\left(\frac{\sqrt{\lambda}\pi}{2}\right)[Q_1] + \Psi_2(\lambda), \\ \Phi_2(\frac{\pi}{2}, \lambda) = \cos\left(\frac{\sqrt{\lambda}\pi\alpha}{2}\right)I + \frac{\sin(\frac{\sqrt{\lambda}\pi\alpha}{2})}{\sqrt{\lambda\alpha}}[Q_2] + \frac{\Psi_3(\lambda)}{\sqrt{\lambda}}, \\ \Phi_2'(\frac{\pi}{2}, \lambda) = -\sqrt{\lambda\alpha}\sin\left(\frac{\sqrt{\lambda}\pi\alpha}{2}\right)I + \cos\left(\frac{\sqrt{\lambda}\pi\alpha}{2}\right)[Q_2] + \Psi_4(\lambda), \end{cases} \quad (2.8)$$

where $\Psi_1, \Psi_2 \in \mathcal{L}_{N \times N}^{\frac{\pi}{2}}$, $\Psi_3, \Psi_4 \in \mathcal{L}_{N \times N}^{\frac{\alpha\pi}{2}}$ ($\mathcal{L}_{N \times N}^b$ is the class of matrix-valued entire functions of order $\frac{1}{2}$ and exponential type $\leq b$, belonging to $L_{N \times N}^2(\mathbb{R})$ for real λ), $[Q_j] = \frac{1}{2} \int_0^{\frac{\pi}{2}} Q_j(x) dx$, $j = 1, 2$.

Using (2.8) in (2.7) and with the help of the Laplace expansion of determinants, we have

$$\begin{aligned}
\omega_Q(\lambda) &= (\sqrt{\lambda})^N \det \begin{pmatrix} a\Phi_1(\frac{\pi}{2}, \lambda) & -\Phi_2(\frac{\pi}{2}, \lambda) \\ a^{-1} \frac{\Phi_1'(\frac{\pi}{2}, \lambda)}{\sqrt{\lambda}} & \frac{\Phi_2'(\frac{\pi}{2}, \lambda)}{\sqrt{\lambda}} \end{pmatrix} \\
&= (\sqrt{\lambda})^N \left\{ \prod_{i=1}^N \left[a \cos \frac{\sqrt{\lambda}\pi}{2} + a \frac{\sin \frac{\sqrt{\lambda}\pi}{2}}{\sqrt{\lambda}} [Q_1]_{ii} + \frac{[\Psi_1(\lambda)]_{ii}}{\sqrt{\lambda}} \right] \right. \\
&\quad \times \prod_{i=1}^N \left[-\alpha \sin \frac{\sqrt{\lambda}\pi\alpha}{2} + \frac{\cos \frac{\sqrt{\lambda}\pi\alpha}{2}}{\sqrt{\lambda}} [Q_2]_{ii} + \frac{[\Psi_4(\lambda)]_{ii}}{\sqrt{\lambda}} \right] \\
&\quad + (-1)^N \prod_{i=1}^N \left[-a^{-1} \sin \frac{\sqrt{\lambda}\pi}{2} + a^{-1} \frac{\cos \frac{\sqrt{\lambda}\pi}{2}}{\sqrt{\lambda}} [Q_1]_{ii} + \frac{[\Psi_2(\lambda)]_{ii}}{\sqrt{\lambda}} \right] \\
&\quad \left. \times \prod_{i=1}^N \left[-\cos \frac{\sqrt{\lambda}\pi\alpha}{2} - \frac{\sin \frac{\sqrt{\lambda}\pi\alpha}{2}}{\sqrt{\lambda}\alpha} [Q_2]_{ii} + \frac{[\Psi_3(\lambda)]_{ii}}{\sqrt{\lambda}} \right] + \frac{\psi(\lambda)}{\lambda} \right\} \\
&:= \omega_0(\lambda) + (\sqrt{\lambda})^{N-1} [G(\lambda) + \psi(\lambda)],
\end{aligned} \tag{2.9}$$

where $\psi \in \mathcal{L}^{\frac{N\pi(1+\alpha)}{2}}$ (\mathcal{L}^b is the class of scalar entire functions of order $\frac{1}{2}$ and exponential type $\leq b$, belonging to $L^2(\mathbb{R})$ for real λ),

$$\begin{aligned}
\omega_0(\lambda) &= (\sqrt{\lambda})^N \left[\left(-\frac{\alpha a}{2} \right)^N \left(\sin \frac{\sqrt{\lambda}\pi(\alpha+1)}{2} + \sin \frac{\sqrt{\lambda}\pi(\alpha-1)}{2} \right)^N \right. \\
&\quad \left. + \left(-\frac{a^{-1}}{2} \right)^N \left(\sin \frac{\sqrt{\lambda}\pi(\alpha+1)}{2} - \sin \frac{\sqrt{\lambda}\pi(\alpha-1)}{2} \right)^N \right], \\
G(\lambda) &= A(\lambda) \left(-\frac{\alpha a}{2} \right)^{N-1} \left(\sin \frac{\sqrt{\lambda}\pi(\alpha+1)}{2} + \sin \frac{\sqrt{\lambda}\pi(\alpha-1)}{2} \right)^{N-1} \\
&\quad + B(\lambda) \left(-\frac{a^{-1}}{2} \right)^{N-1} \left(\sin \frac{\sqrt{\lambda}\pi(\alpha+1)}{2} - \sin \frac{\sqrt{\lambda}\pi(\alpha-1)}{2} \right)^{N-1}
\end{aligned}$$

with

$$\begin{aligned}
A(\lambda) &= \frac{a \operatorname{tr}[Q_2] + \alpha a \operatorname{tr}[Q_1]}{2} \cos \frac{\sqrt{\lambda}\pi(\alpha+1)}{2} \\
&\quad + \frac{a \operatorname{tr}[Q_2] - \alpha a \operatorname{tr}[Q_1]}{2} \cos \frac{\sqrt{\lambda}\pi(\alpha-1)}{2},
\end{aligned}$$

$$B(\lambda) = \frac{\alpha a^{-1} \operatorname{tr}[Q_1] + a^{-1} \operatorname{tr}[Q_2]}{2\alpha} \cos \frac{\sqrt{\lambda}\pi(\alpha+1)}{2} + \frac{\alpha a^{-1} \operatorname{tr}[Q_1] - a^{-1} \operatorname{tr}[Q_2]}{2\alpha} \cos \frac{\sqrt{\lambda}\pi(\alpha-1)}{2}.$$

In fact, the first term $\omega_0(\lambda)$ is the characteristic function of the problem $L(0)$. By the Palay-Wiener's theorem we find that for $\lambda \in \mathbb{R}$,

$$\psi(\lambda) = o(1), \quad \lambda \rightarrow +\infty,$$

and for $\sqrt{\lambda} = i\kappa (\kappa > 0)$,

$$\psi(\lambda) = o\left(e^{\frac{\kappa N \pi(1+\alpha)}{2}}\right), \quad \kappa \rightarrow +\infty. \quad (2.10)$$

3. PROOF

In this section we give the proof of Theorem 1.1, for which we need the following lemmas.

Lemma 3.1. *The characteristic function $\omega_Q(\lambda)$ of the problem $L(Q)$ is uniquely determined by $\sigma(Q) = \{\lambda_n(Q)\}_{n=0}^{\infty}$ (counted with multiplicity).*

Proof. From (2.9), we observe that $\omega_Q(\lambda)$ is entire in λ of order $\frac{1}{2}$. Hence, by Hadamard's factorization theorem, $\omega_Q(\lambda)$ is uniquely determined up to a multiplicative constant $C(Q)$ by its zeros:

$$\omega_Q(\lambda) = C(Q) \lambda^m \prod_{\{n: \lambda_n(Q) \neq 0\}} \left(1 - \frac{\lambda}{\lambda_n(Q)}\right), \quad (3.1)$$

where m is the multiplicity of zero eigenvalues (If zero isn't an eigenvalue, then $m = 0$). Substituting $\sqrt{\lambda} = i\kappa (\kappa > 0)$ into (2.9) yields that as $\kappa \rightarrow +\infty$,

$$\begin{aligned} \omega_Q(-\kappa^2) &= \omega_0(-\kappa^2) + (i\kappa)^{N-1} [G(-\kappa^2) + \psi(-\kappa^2)] \\ &= \kappa^N e^{\frac{\kappa N \pi(1+\alpha)}{2}} \left[\left(\frac{\alpha a}{4}\right)^N + \left(\frac{a^{-1}}{4}\right)^N \right] [1 + o(1)], \end{aligned} \quad (3.2)$$

where $G(-\kappa^2) = O\left(e^{\frac{\kappa N \pi(1+\alpha)}{2}}\right)$ and (2.10) are used.

Hence, from (3.1) and (3.2), we obtain

$$\begin{aligned} C(Q) &= (-1)^m \left[\left(\frac{\alpha a}{4}\right)^N + \left(\frac{a^{-1}}{4}\right)^N \right] \\ &\quad \times \lim_{\kappa \rightarrow +\infty} \left\{ \kappa^{N-2m} e^{\frac{\kappa N \pi(1+\alpha)}{2}} \left[\prod_{\{n: \lambda_n(Q) \neq 0\}} \left(1 + \frac{\kappa^2}{\lambda_n(Q)}\right) \right]^{-1} \right\}. \end{aligned} \quad (3.3)$$

With the help of (3.1) and (3.3), the proof of Lemma 3.1 is complete. \square

Lemma 3.2. *If $\lambda_n(Q) = \lambda_n(0)$ for all $n = 0, 1, 2 \dots$, then*

$$\text{tr}[Q_1] = \text{tr}[Q_2] = 0.$$

Proof. Since $\lambda_n(Q) = \lambda_n(0)$ for all $n = 0, 1, 2 \dots$, Lemma 3.1 tells that $\omega_Q(\lambda) = \omega_0(\lambda)$. Thus, it follows from (2.9) that for $\lambda \in \mathbb{C}$,

$$G(\lambda) + \psi(\lambda) = 0,$$

that is,

$$\begin{aligned} & A(\lambda) \left(-\frac{\alpha a}{2}\right)^{N-1} \left(\sin \frac{\sqrt{\lambda}\pi(\alpha+1)}{2} + \sin \frac{\sqrt{\lambda}\pi(\alpha-1)}{2}\right)^{N-1} \\ & + B(\lambda) \left(-\frac{a^{-1}}{2}\right)^{N-1} \left(\sin \frac{\sqrt{\lambda}\pi(\alpha+1)}{2} - \sin \frac{\sqrt{\lambda}\pi(\alpha-1)}{2}\right)^{N-1} \\ & + \psi(\lambda) = 0. \end{aligned} \quad (3.4)$$

In particular, taking $\sqrt{\lambda} = 2n$ ($n \in \mathbb{N}$), then the equation (3.4) is transformed into

$$a \text{tr}[Q_2] \cos(n\pi\alpha) \left(2 \sin(n\pi\alpha)\right)^{N-1} + o(1) = 0, \quad n \rightarrow +\infty. \quad (3.5)$$

When the impulse parameter $\alpha \in (0, 1)$ is a irrational number, then (3.5) deduces $\text{tr}[Q_2] = 0$.

When the impulse parameter $\alpha \in (0, 1)$ is a rational number, denoting $\alpha = \frac{q}{p}$ (p, q are coprime natural number and $q < p$). Moreover, if $\alpha = \frac{q}{p} \neq \frac{1}{2}$, then taking $n = pk + 1$ ($k \in \mathbb{N}$) in (3.5) yields

$$a \text{tr}[Q_2] \cos \frac{q\pi}{p} \left(2 \sin \frac{q\pi}{p}\right)^{N-1} + o(1) = 0, \quad k \rightarrow +\infty,$$

which deduces $\text{tr}[Q_2] = 0$.

If $\alpha = \frac{q}{p} = \frac{1}{2}$, then taking $\sqrt{\lambda} = 8n + 1$ ($n \in \mathbb{N}$), the equation (3.4) is transformed into

$$a^{-1} \text{tr}[Q_2] + o(1) = 0, \quad n \rightarrow +\infty,$$

which yields $\text{tr}[Q_2] = 0$.

Substituting $\text{tr}[Q_2] = 0$ into the equation (3.4), we obtain

$$\begin{aligned} & \text{tr}[Q_1] \left\{ (\alpha a)^N \left(\sin \frac{\sqrt{\lambda}\pi\alpha}{2}\right)^N \left(\cos \frac{\sqrt{\lambda}\pi}{2}\right)^{N-1} \sin \frac{\sqrt{\lambda}\pi}{2} \right. \\ & \left. - a^{-N} \left(\cos \frac{\sqrt{\lambda}\pi\alpha}{2}\right)^N \left(\sin \frac{\sqrt{\lambda}\pi}{2}\right)^{N-1} \cos \frac{\sqrt{\lambda}\pi}{2} \right\} + \psi(\lambda) = 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \operatorname{tr}[Q_1] \left\{ (\alpha a)^N \frac{\left(\sin \frac{\sqrt{\lambda}\pi\alpha}{2} \right)^N \left(\cos \frac{\sqrt{\lambda}\pi}{2} \right)^{N-1} \sin \frac{\sqrt{\lambda}\pi}{2}}{e^{\frac{\tau N\pi(1+\alpha)}{2}}} \right. \\ \left. - a^{-N} \frac{\left(\cos \frac{\sqrt{\lambda}\pi\alpha}{2} \right)^N \left(\sin \frac{\sqrt{\lambda}\pi}{2} \right)^{N-1} \cos \frac{\sqrt{\lambda}\pi}{2}}{e^{\frac{\tau N\pi(1+\alpha)}{2}}} \right\} + \frac{\psi(\lambda)}{e^{\frac{\tau N\pi(1+\alpha)}{2}}} = 0, \end{aligned} \quad (3.6)$$

where $\tau = |\operatorname{Im}\sqrt{\lambda}|$.

In particular, taking $\sqrt{\lambda} = i\kappa$ ($\kappa > 0$) in (3.6) and with the help of (2.10), we get as $\kappa \rightarrow +\infty$,

$$\begin{aligned} \operatorname{tr}[Q_1] \left\{ (\alpha a)^N \left(\frac{e^{-\kappa\pi\alpha} - 1}{2i} \right)^N \left(\frac{e^{-\kappa\pi} + 1}{2} \right)^{N-1} \left(\frac{e^{-\kappa\pi} - 1}{2i} \right) \right. \\ \left. - a^{-N} \left(\frac{e^{-\kappa\pi\alpha} + 1}{2} \right)^N \left(\frac{e^{-\kappa\pi} - 1}{2i} \right)^{N-1} \left(\frac{e^{-\kappa\pi} + 1}{2} \right) \right\} + o(1) = 0. \end{aligned}$$

Letting $\kappa \rightarrow +\infty$ in the above equation yields

$$\operatorname{tr}[Q_1] [(\alpha a)^N + a^{-N}] = 0,$$

that is,

$$\operatorname{tr}[Q_1] = 0.$$

The proof of Lemma 3.2 is complete. \square

Introducing the Hilbert space $L_N^2(0, \pi) := \bigoplus_{i=1}^N L^2(0, \pi)$ with the inner product

$$(f, g) = \int_0^\pi g^\dagger(x) f(x) dx = \sum_{i=1}^N \int_0^\pi f_i(x) \overline{g_i(x)} dx,$$

where $f = (f_1, \dots, f_N)^t \in L_N^2(0, \pi)$, $g = (g_1, \dots, g_N)^t \in L_N^2(0, \pi)$, and g^t denotes the transpose of the vector g , and g^\dagger denotes the conjugate transpose of the vector g . Clearly, $f = (f_1, \dots, f_N)^t \in L_N^2(0, \pi)$ if and only if $f_i \in L^2(0, \pi)$, $i = 1, \dots, N$. The domain of self-adjoint operator $L(Q)$ is

$$\begin{aligned} D(L(Q)) = \left\{ Y \in L_N^2(0, \pi) \mid Y' \in AC_{N,loc} \left(\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \right), \right. \\ \left. lY \in L_N^2(0, \pi), Y \text{ satisfying (1.2) and (1.3)} \right\}. \end{aligned}$$

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. It is easy to see that the operator $L(0)$ is non-negative and $0 \in \sigma(0)$, so zero is its smallest eigenvalue, namely $\lambda_0(0) = 0$.

Let

$$Y_i(x) = \begin{cases} e_i, & 0 < x < \frac{\pi}{2}, \\ ae_i, & \frac{\pi}{2} < x < \pi, \end{cases}$$

where e_i is the $N \times 1$ unit vector, whose the i -st component is 1 ($i = 1, 2, \dots, N$).

By the variational principle, we obtain

$$\begin{aligned} 0 &= \lambda_0(0) = \lambda_0(Q) \\ &= \inf_{0 \neq Y \in D(L(Q))} \frac{\int_0^\pi [-Y^\dagger(x)Y''(x) + Y^\dagger(x)Q(x)Y(x)] dx}{\int_0^\pi [\rho(x)Y^\dagger(x)Y(x)] dx} \\ &\leq \frac{\int_0^\pi [-Y_i^\dagger(x)Y_i''(x) + Y_i^\dagger(x)Q(x)Y_i(x)] dx}{\int_0^\pi [\rho(x)Y_i^\dagger(x)Y_i(x)] dx} \\ &= \frac{\int_0^{\frac{\pi}{2}} Q_{1ii}(x) dx + a^2 \int_0^{\frac{\pi}{2}} Q_{2ii}(x) dx}{\frac{\pi}{2}(1 + \alpha^2 a)} \\ &= \frac{2[Q_1]_{ii} + 2a^2[Q_2]_{ii}}{\frac{\pi}{2}(1 + \alpha^2 a)}, \quad i = 1, 2, \dots, N. \end{aligned} \tag{3.7}$$

Note that M_{ii} represents the entry of matrix M at the i -st row and i -st column. Adding N inequalities in (3.7), together with Lemma 3.2, yields that

$$0 \leq \frac{2tr[Q_1] + 2a^2tr[Q_2]}{\frac{\pi}{2}(1 + \alpha^2 a)} = 0.$$

Therefore each of right-hand side in (3.7) vanishes. So $Y_i(x)$ ($i = 1, 2, \dots, N$) is the eigenfunction corresponding to the first eigenvalue 0 of $L(Q)$.

Substituting $Y_i(x)$ into equation (1.1), we have

$$Q(x)e_i = 0 \text{ a.e. } x \in (0, \pi), \quad i = 1, 2, \dots, N,$$

which are equivalent to

$$Q(x)I = 0 \text{ a.e. } x \in (0, \pi).$$

Thus $Q(x) = 0$ a.e. $x \in (0, \pi)$. The proof of Theorem 1.1 is complete. \square

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