

A RELATION BETWEEN TWO DIFFERENT FORMULATIONS OF THE BERRY'S CONJECTURE

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ABSTRACT. The Random Wave Conjecture of M. V. Berry is the heuristic that eigenfunctions of a classically chaotic system should behave like Gaussian random fields, in the large eigenvalue limit. In this work we collect some definitions and properties of Gaussian random fields, and show that the formulation of the Berry's conjecture proposed using local weak limits is equivalent to the one that is based on the Benjamini-Schramm convergence. Finally, we see that both these formulations of the Berry's property imply another property known as inverse localization that relates high energy eigenfunctions and solutions to the Euclidean Helmholtz equation.

1. INTRODUCTION

In his influential papers [8, 9] Berry gave a heuristic description of the behavior of high-energy wave-functions of quantum chaotic systems. He suggested that high-frequency eigenfunctions of the Laplacian in geometries where the geodesic flow is sufficiently chaotic should, in some sense, at the wavelength scale, behave like an isotropic Gaussian field Ψ_{Berry} with covariance function

$$(1.1) \quad \mathbb{E} [\Psi_{Berry}(x)\Psi_{Berry}(y)] = \int_{\mathbb{S}^{d-1}} e^{2\pi i(x-y)\theta} d\omega_{d-1}(\theta) = c_d \frac{J_\Lambda(|x-y|)}{|x-y|^\Lambda},$$

where $d\omega_{d-1}$ is the uniform measure in the $(d-1)$ -sphere, J_Λ is the Λ -th Bessel function of the first kind with $\Lambda = \frac{d-2}{2}$ and $c_d > 0$ is a constant such that we have $\mathbb{E} [|\Psi(x)|^2] = 1$.

This ambiguous comparison between a deterministic system and a stochastic field is known as the Random Wave Model (RWM). The RWM was first introduced in the study of chaotic quantum billiards on flat domains and a weaker version of it was proved in the context of random regular graphs [5]. The RWM has led to many conjectures concerning L^p norms, semi-classical measures or volume and topology of nodal domains of chaotic eigenfunctions. Several of these conjectures have been addressed numerically ([16], [3], [4], [6]) or experimentally ([18], [7], [20]).

However, there is no agreement on how Berry's conjecture should be formulated rigorously because the idea of a sequence of deterministic objects having a random limit can be interpreted in different ways. The reader can for instance refer to [27], [28], [24] and [19] for different mathematical perspectives on Berry's conjecture. Some of these conjectures focus only on the values of the eigenfunctions, viewing $(M, d\text{Vol}_M)$ as a probability space and each eigenfunction as a random variable.

Key words and phrases. Random Wave Conjecture, Berry Field, Inverse Localization, Random Field, Benjamini-Schramm convergence, Gaussian Field, Eigenfunction, Ergodicity.

These formulations do not provide any insight into the properties of the nodal set, number of nodal domains and other related questions. Some other formulations, as the one in [10], have led to groundbreaking results concerning the nodal sets of monochromatic waves satisfying the RWM (see, for instance, [26]).

In [21], [22] and [1] one can find two different formulations of the Random Wave Conjecture that takes into account both the shape of the eigenfunction and its distribution of values. Although both formulations are based on different notions (the one proposed in [21] is related to what the author calls *local weak limits*, defined using a covering by charts of the manifold, and the one from [1] makes heavy use of the Benjamini-Schramm convergence), the idea behind is similar: for a random $x \in M$, the eigenfunction $\psi_n(\text{Exp}_x(\cdot/\sqrt{\lambda_n}))$ should behave like a random monochromatic wave as long as the geodesic flow on (M, g) is chaotic. In this work we show that both formulations of the Berry conjecture are equivalent in the context of a compact manifold M . As a consequence of the equivalence in compact manifolds, we are able to ensure that the local weak limit formulation does not depend on the choice of charts. Note that, doing the necessary identifications, in the graph theory literature it can be seen that Benjamini-Schramm limit and local weak limit are the same notions.

In the final section, we will introduce a related notion, the one of the inverse localization property, and show that, with any of these formulations, the Berry property implies the inverse localization. Roughly speaking, we say that a compact manifold M satisfies the inverse localization property if we can approximate an arbitrary solution of the Helmholtz equation on \mathbb{R}^d using high energy eigenfunctions of the manifold. For the time being, all the known examples of manifolds satisfying this property do have a high multiplicity, that seems to be key to construct the approximating eigenfunction. However, the version of the inverse localization that we get from the Berry property is in some sense stronger: there exists a sequence of eigenfunctions associated to different eigenvalues such that any solution to Helmholtz equation can be approximated by one of these. Notice that the sequence does not depend on the solution to Helmholtz equation chosen and that for any eigenvalue we are considering just one eigenfunction associated to it, so the degeneracy of eigenvalues plays no role here.

Organization of the paper. In Section 2 the properties asked to M are specified and some needed definitions are given. In Section 3, local weak limit and BS convergence are defined and the two different formulations of the Berry property are stated. The main theorem of the paper is as follows.

Theorem 1.1. *If M is a compact, connected Riemannian manifold covered by a finite family $\{U_m\}_{m=1}^{m_{\max}}$ of open subsets with some extra conditions (specified at the beginning of Section 2), then Berry's property in the local weak limit form (given in definition 3.2) is equivalent to Berry's property in BS form (as defined in 3.9).*

Sections 4 and 5 are devoted to prove both implications of this theorem. Finally, in Section 6, the inverse localization property is introduced and it is proved that the Berry property implies the former. To conclude, some comments are made around this idea.

2. STARTING DEFINITIONS AND NOTATION. GAUSSIAN FIELDS

In all what follows, we consider a compact connected Riemannian manifold of dimension d without boundary, (M, g) . dx is the volume measure on M and we will denote by Δ the Laplace-Beltrami operator on the manifold. An easy application of the classical spectral theorem for compact manifolds ensures that there exists $(\psi_n)_n$ an orthonormal basis of $L^2(M)$ that consists of functions on M such that $\Delta\psi_n + \lambda_n\psi_n = 0$ and $\|\psi_n\|_2^2 = \text{Vol}(M)$. Assume that the eigenvalues are ordered in non-decreasing order. We may furthermore suppose that we have a finite family $\{U_m\}_{m=1}^{m_{\max}}$ of open subsets with the following properties:

- $M \subset \bigcup_{m=1}^{m_{\max}} \overline{U_m}$,
- for any $m_1 \neq m_2$ we have $U_{m_1} \cap U_{m_2} = \emptyset$ and
- for any m , there exists a family of vector fields on U_m , say (V_1^m, \dots, V_d^m) , forming an orthonormal frame of the tangent bundle TU_m .

For each $x \in M$, we will denote by $\exp_x : T_x M \rightarrow M$ the exponential map at x induced by the metric g on M . Moreover, given x and y in M , we will denote by $\text{dist}(x, y)$ the Riemannian distance between x and y .

Unless otherwise stated, the spaces $C^\infty(M)$ and $C^\infty(\mathbb{R}^d)$ will be equipped with the topology of uniform convergence of derivatives on compact sets. We will also define in $C^\infty(\mathbb{R}^d)$ a distance d given by the Fréchet structure of $C^\infty(\mathbb{R}^d)$. This means that we first define the class of semi-norms

$$(2.1) \quad \|f\|_{k,n} = \sup \left\{ \left| f^{(k)}(x) \right| : x \in B(0, n) \right\}$$

and then the distance given by

$$(2.2) \quad d(f, g) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{2^{-k-n} \|f - g\|_{k,n}}{1 + \|f - g\|_{k,n}}.$$

With this distance, we recover the smooth topology in $C^\infty(\mathbb{R}^d)$ and we can use it to define balls $B(f, \epsilon)$. Notice that $C^\infty(\mathbb{R}^d)$ with this distance is a locally compact metric space. Moreover, when we speak of probability measures on these spaces, we will assume that they are equipped with the Borel σ -algebra.

Our next goal is to introduce the field Ψ_{Berry} . It is a stationary Gaussian field on \mathbb{R}^d whose spectral measure is the uniform measure on the unit sphere \mathbb{S}^d . We first recall some definitions:

Definition 2.1 (Smooth random field). A smooth random field (on \mathbb{R}^d) is a map X from a probability space (Ω, \mathcal{B}, P) to $C^\infty(\mathbb{R}^d)$ that is measurable, where $C^\infty(\mathbb{R}^d)$ is considered with the topology of convergence of all derivatives over all compact sets. Notice that for any $n \in \mathbb{N}$ and any $t_1, \dots, t_n \in \mathbb{R}^d$, the vector $(X(t_1), X(t_2), \dots, X(t_n))$ is a random vector.

Definition 2.2 (Gaussian random field). A Gaussian random field is a (smooth) random field where all the finite dimensional distributions $F_{t_1, \dots, t_k}(x_1, \dots, x_k) = P(X(t_1) \leq x_1, \dots, X(t_k) \leq x_k)$ are multivariate normal distributions for any choice of k and $\{t_1, \dots, t_k\}$. Since multivariate normal distributions are completely specified by expectations and covariances, to determine a Gaussian random field X_t it suffices to specify $m(t) := \mathbb{E}\{X_t\}$ and $C(t, s) := \text{Cov}\{X_t, X_s\}$ in an appropriate way. If $m \equiv 0$ we say that the Gaussian field is centered.

Definition 2.3 (Stationary and isotropic random field). We say that a Gaussian random field is stationary if $C(t, s)$ depends only on $t - s$ and $m(t) = m$ is constant. An isotropic Gaussian random field is a stationary Gaussian random field whose covariance function depends on the distance alone, i.e. $C(t, s) = C(\tau)$ where $\tau = \text{dist}(t, s)$.

Definition 2.4 (Law and equivalence of random fields). For a random field X , we define its law as the probability measure $\mu_X : \mathcal{B}(C^\infty(\mathbb{R}^d)) \rightarrow \mathbb{R}$ given by $\mu_X = PX^{-1} : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R}$, such that for any Borel set $A \in \mathcal{B}(C^\infty(\mathbb{R}^d))$, $\mu_X(A) = P(X^{-1}(A))$. Here P is the probability we have in the probability space (Ω, \mathcal{B}, P) where X is defined.

We say that two random fields X^1 and X^2 are equivalent if they have the same law, i.e. $\mu_{X^1} = \mu_{X^2}$. In the sequel we will always identify fields which are equivalent and we will speak indifferently of the field and its law.

As explained for instance in Section A.11 of [23] there is a bijection between smooth centered Gaussian fields on \mathbb{R}^d and positive definite functions defined on $\mathbb{R}^d \times \mathbb{R}^d$. Also, recall that, by Bochner's theorem [11, Section 2.1.11], for any finite Borel complex measure μ on \mathbb{R}^d , its Fourier transform $\hat{\mu}$ can be used to define a continuous positive definite function $K(x, y) = \hat{\mu}(x - y)$. If, in addition, μ is compactly supported, its Fourier transform is of class C^∞ , and gives rise to a unique smooth Gaussian field X on \mathbb{R}^d (up to law equivalence). In this case, we call μ the spectral measure of X .

Definition 2.5 (Ψ_{Berry} , the random monochromatic wave). We call random isotropic monochromatic wave, and denote by Ψ_{Berry} , the unique stationary Gaussian field on \mathbb{R}^d whose spectral measure is the uniform measure on the unit sphere \mathbb{S}^d . This field $\Psi_{Berry} : \mathbb{R}^d \rightarrow \mathbb{R}$ is uniquely defined as the centered stationary Gaussian random field, with covariance function

$$(2.3) \quad \mathbb{E}[\Psi_{Berry}(x)\Psi_{Berry}(y)] = \int_{\mathbb{S}^{d-1}} e^{2\pi i(x-y)\theta} d\omega_{d-1}(\theta),$$

where $d\omega_{d-1}$ is the uniform measure on \mathbb{S}^{d-1} .

Let us consider the space $FP := \{f \in C^\infty(\mathbb{R}^d), s.t. -\Delta f = f\}$. Then it is easy to check that Ψ_{Berry} is almost surely an element of FP . This way, if $A \subset FP$ is a Borel set, $P(\Psi_{Berry} \in A)$ is well-defined and

$$(2.4) \quad \begin{aligned} \mu_{Berry} : \mathcal{B}(FP) &\rightarrow \mathbb{R}_0^+ \\ A &\mapsto P(\Psi_{Berry} \in A), \end{aligned}$$

is a Borel measure on $FP \subset C^\infty(\mathbb{R}^d)$. In other words, the probability measure μ_{Berry} defined on $C^\infty(\mathbb{R}^d)$ is supported on FP . In what follows we will work with this measure μ_{Berry} and with Ψ_{Berry} indistinguishably.

3. STATEMENT OF THE CONJECTURES

We first consider the formulation of the conjecture proposed in [22]:

Recall that we have a finite family $\{U_m\}_{m=1}^{m_{\max}}$ of open subsets on which there exists a family of vector fields (V_1^m, \dots, V_d^m) forming an orthonormal frame of the tangent bundle. For a given $x \in U_m$, we define the function $\text{Exp}_x : \mathbb{R}^d \rightarrow \mathbb{R}$

as $\text{Exp}_x(y) := \exp_x\left(\sum_{j=1}^d y_j V_j^m(x)\right)$. Let p be a random point in U_m chosen uniformly with respect to the volume measure dx . For each n , let $\phi_p^n \in C^\infty(\mathbb{R}^d)$ be the random field defined by $\phi_p^n(y) := \psi_n(\text{Exp}_p(y/\sqrt{\lambda_n}))$. It is a random element of $C^\infty(\mathbb{R}^d)$. Notice that the definition of ϕ_p^n depends on the chart U_m chosen.

Definition 3.1 (Convergence in law). Let μ_{Berry} be the probability measure on $C^\infty(\mathbb{R}^d)$ associated to Ψ_{Berry} . We say that $\phi_p^n(y)$ converges in law as a random field towards Ψ_{Berry} in the frame U if, for any continuous, bounded functional $F : C^\infty(\mathbb{R}^d) \rightarrow \mathbb{R}$, we have

$$(3.1) \quad \frac{1}{\text{Vol}(U)} \int_U F(\phi_p^n) dp \xrightarrow{n \rightarrow \infty} \mathbb{E}_{\mu_{\text{Berry}}}[F] = \int_{C^\infty(\mathbb{R}^d)} F d\mu_{\text{Berry}}.$$

Then the conjecture as formulated in [22] can be stated as follows.

Definition 3.2 (Formulation of Berry's conjecture in the local weak limit form). We have the Berry property in the local weak limit form if for any U_m , as $\lambda_n \rightarrow \infty$ when $n \rightarrow \infty$, the family $\phi_p^n(y)$ converges in law as a random field towards the Gaussian field Ψ_{Berry} .

Remark 3.3. The construction of local weak limits depends on the choice of the partition $\{U_m\}_{m=1}^{m_{\max}}$ and frames $\{V_j^m\}_{j=1}^d$, $m = 1, 2, \dots, m_{\max}$, and so definition 3.2 does depend *a priori* on them. However, if it holds for one choice of sets U_m and frames V_m , then it also holds for any other choice. This can be seen, for example, as a consequence of the equivalence between this formulation and the one using Benjamini-Schramm convergence that we prove here.

On the other hand, we consider the formulation of the conjecture introduced in [1]. For the d -dimensional Riemannian manifold $M = (M, g)$, let $M_n = (M, g_n)$ denote the rescaling of M by the factor $\sqrt{\lambda_n}$, i.e. we change only the metric by multiplying every distance by $\sqrt{\lambda_n}$. A property of this M_n is that if $\phi : M \rightarrow \mathbb{R}$ is an eigenfunction of the Laplacian on M associated to an eigenvalue λ , then the very same function is also an eigenfunction on M_n with eigenvalue $\lambda' = \lambda/\lambda_n$.

Just like it is done in [1], we start by considering the space \mathcal{M}^d of pointed, connected, complete Riemannian manifolds of dimension d up to pointed isometries, with its smooth topology. The reader should see [2, §A.1] for a precise definition of this topology, however the philosophy is as follows: two pointed manifolds (M, p) and (N, q) are close if there exist two compact subsets of M and N containing large neighborhoods of the base points p and q respectively that are diffeomorphic via a map ϕ , that is close in the C^∞ metric to an isometry.

Another way of seeing this is the following: a sequence of pointed Riemannian manifolds, let us say (M_n, p_n) , converges in the C^∞ metric towards (M, p) if for every radius $R > 0$, there exists a sequence of maps $f_n : B_M(p, R) \rightarrow M_n$ with $f_n(p) = p_n$ such that the Riemannian metric $f_n^* g_n$ on the metric ball $B_M(p, R)$ inside M , pulled back from the Riemannian metric g_n on M_n , converges to the restriction of the Riemannian metric g on M in C^∞ -topology. It can be proved that the smooth topology on \mathcal{M}^d is induced by a Polish topology, i.e. the space \mathcal{M}^d is separable and completely metrizable; see [2, §A.2] for a proof of this result. The space \mathcal{M}^d is not compact but Cheeger's compactness theorem can be used to show that the subspace consisting of pointed manifolds (M, p) with uniformly bounded geometry is a compact subspace; see [2, §A.1] for the proof.

The next step is to construct a measure associated to (M_n, p) . Pushing forward the normalized Riemannian volume measure under the following map one obtains a probability measure μ_M on \mathcal{M}^d .

$$(3.2) \quad M \rightarrow \mathcal{M}^d, p \mapsto (M, p)$$

We now recall a common notion of convergence for these measures and use it to state the definition of BS convergence.

Definition 3.4 (Convergence in the weak* topology). A sequence of probability measures (μ_n) on \mathcal{M}^d converge to μ in the weak* topology if $\int F d\mu_n \rightarrow \int F d\mu$ for every bounded, continuous function $F : \mathcal{M}^d \rightarrow \mathbf{R}$.

Beware that some authors refer to the topology of the previous definition as weak topology.

Definition 3.5 (BS-convergence of manifolds). A sequence (M_n) of compact connected complete Riemannian d -manifolds is convergent in the sense of Benjamini-Schramm, or just BS-converges, if the sequence μ_{M_n} converges in the weak* topology of the set of all probability measures on \mathcal{M}^d .

Following [1], let us now explain a particularity of the measures thus obtained, though we will not use it in the sequel.

Let $T^1\mathcal{M}^d$ be the space of isometry classes of unit tangent bundles (T_p^1M, p, v) , where $v \in T_p^1M$. The geodesic flows on each T_p^1M combine to give a continuous flow on $T^1\mathcal{M}^d$,

$$(3.3) \quad \sigma_t : T^1\mathcal{M}^d \rightarrow T^1\mathcal{M}^d$$

What is more, in each fiber T_p^1M of the following map it one can define a (Liouville) measure $\omega_{M,p}$ induced by the Riemannian metric on M .

$$(3.4) \quad T^1\mathcal{M}^d \rightarrow \mathcal{M}^d; (M, p, v) \mapsto (M, p)$$

Any measure μ on \mathcal{M}^d can then be lifted to a measure $\tilde{\mu}$ on $T^1\mathcal{M}^d$ defined by the equation $d\tilde{\mu} = \omega_{M,p} d\mu$. The Liouville measure on the unit tangent bundle of a Riemannian manifold is invariant under the geodesic flow and, similarly, the measure $\tilde{\mu}_M$ is invariant under the flow \mathbf{g}_t .

Using the same notation as in [2], we say that a measure μ on \mathcal{M}^d is unimodular if $\tilde{\mu}$ is invariant under (3.3). We also refer to [2] for other characterizations of unimodularity. Then we have just seen the following remark, that we will not use in the proofs.

Remark 3.6. The weak* limit of μ_{M_n} is a unimodular probability measure on \mathcal{M}^d .

In general, around a randomly chosen point, M_n has no reason to be similar to a given manifold for large n . Instead, the limiting object is a unimodular probability measure on \mathcal{M}^d that precisely encodes how the manifold, for large n , looks like near randomly chosen base points. This perspective is studied in great details in [2]. By work of Cheeger and Gromov (see, for example [25, Chapter 10]), the subset of \mathcal{M}^d consisting of pointed manifolds (M, p) with bounded geometry is compact. Here, bounded geometry means that the sectional curvatures of M , and all of their

derivatives, are uniformly bounded, and the injectivity radius at the base point p is bounded away from zero. Compact manifolds are examples of manifolds with bounded geometry. By the Riesz representation theorem and Alaoglu's theorem, this implies that the set of unimodular probability measures supported on manifolds with bounded geometry is weak* compact, since unimodularity is a weak* closed condition.

One can similarly define the BS-convergence of a sequence $\{M_n\}_{n=1}^\infty$ of manifolds with associated functions $\phi_n : M_n \rightarrow \mathbb{R}$. We follow again [1]. Consider the space

$$(3.5) \quad \mathcal{E}^d = \left\{ (M, p, \phi) \left| \begin{array}{l} M \text{ connected, complete } d\text{-manifold,} \\ p \in M, \phi : M \rightarrow \mathbf{R} \text{ smooth} \end{array} \right. \right\} / \begin{array}{l} \text{pointed} \\ \text{isometries} \end{array}$$

equipped with its smooth topology where $[M, p, \phi]$ is close to $[N, q, \psi]$ if there are compact subsets of M and N containing large radius neighborhoods of p and q respectively, that are diffeomorphic via a map D that is C^∞ -close to an isometry and that also satisfies that ϕ and $\psi \circ D$ are C^∞ close. As seen in [1, Proposition 8], the topological space \mathcal{E}^d has a compatible structure of a Polish space (i.e. a complete, separable metric space).

We can now equip the topological space \mathcal{E}^d with the σ -algebra \mathcal{B} , generated by its open sets, and define a probability measure on \mathcal{E}^d as a σ -additive function $\mathcal{B} \rightarrow [0, 1]$ that maps the whole \mathcal{E}^d to 1. Similarly to what we did before, we can recall the following notion of convergence:

Definition 3.7 (Convergence in the weak* topology). A sequence of probability measures μ_n on \mathcal{E}^d is said to converge in the weak* topology towards a probability measure μ if for each bounded, continuous real function F on \mathcal{E}^d we have

$$(3.6) \quad \lim_{n \rightarrow \infty} \mu_n(F) = \mu(F).$$

For any M and $\phi : M \rightarrow \mathbb{R}$, smooth map, we can consider the push forward of the normalized Riemannian volume measure under the map

$$(3.7) \quad M \rightarrow \mathcal{E}^d, p \mapsto [M, p, \phi]$$

and obtain a probability measure $\mu_{M, \phi}$ on \mathcal{E}^d . As we did with \mathcal{M}^d , we shall denote by $T^1\mathcal{E}^d$ the space of isometry classes of tangent bundles with a function (T^1M, p, v, ϕ) where $v \in T_p^1M$. Here again it comes equipped with a continuous (geodesic) flow and any measure μ on \mathcal{E}^d can be lifted to a measure $\tilde{\mu}$ on $T^1\mathcal{E}^d$ using the volume form on the fiber. However, the measure $\tilde{\mu}_{M_n, \phi_n}$ is not invariant under the geodesic flow unless ϕ_n is constant on M_n .

Definition 3.8 (BS-convergence of manifolds with functions). A sequence (M_n, ϕ_n) where each M_n is a compact connected complete Riemannian d -manifold and $\phi_n : M_n \rightarrow \mathbb{R}$ is smooth, is convergent in the sense of Benjamini-Schramm, or just BS-converges, if there exists a probability measure μ on \mathcal{E}^d such that the sequence μ_{M_n, ϕ_n} converges to μ in the weak* topology.

Definition 3.9 (Formulation of Berry's conjecture in BS form, [1]). Let M be a compact d -dimensional manifold and $M_n = (M, g_n)$ be given by the rescaling of M by $\sqrt{\lambda_n}$; and let $(\psi_n)_n$ be an orthonormal basis of $L^2(M)$ that consists of eigenfunctions associated to eigenvalues λ_n . Then, we have the Berry property

in BS form if (M_n, ψ_n) BS converges to the isotropic monochromatic Gaussian random field with eigenvalue 1, Ψ_{Berry} . This means that μ_{M_n, ψ_n} converges to the probability measure μ_{Berry} on the space of smooth functions on \mathbb{R}^d .

Let us try to write this otherwise. We recall from the introduction that the probability measure μ_{Berry} associated to the random field Ψ_{Berry} is supported on functions u such that $-\Delta u = u$. In other words, Ψ_{Berry} is almost surely an eigenfunction of eigenvalue 1 of the Laplacian and, in particular, Ψ_{Berry} is almost surely a smooth function on \mathbb{R}^d .

Moreover, notice that, as a Riemannian metric is infinitesimally Euclidean, it follows from the definition that, as $n \rightarrow \infty$, the sequence (M_n) BS-converges to \mathbb{R}^d . Precisely, because of the weak* compactness, μ_{M_n} converges to a unimodular measure on \mathcal{M}^d . By the rescaling, M_n is closer to \mathbb{R}^d whenever $n \rightarrow \infty$, so the limit towards which μ_{M_n} converges should be the Dirac measure at $[\mathbb{R}^d, 0] \in \mathcal{M}^d$. Note that \mathbb{R}^d being homogeneous, the limit measure does not depend on a particular choice of base point. We loosely say that (M_n) BS-converges towards \mathbb{R}^d . Also remember that if $\phi : M \rightarrow \mathbb{R}$ is an eigenfunction of the Laplacian on M with eigenvalue λ , then the very same function is an eigenfunction on M_n with eigenvalue $\lambda' = \lambda/\lambda_n$.

We consider now the sequence inside \mathcal{E}^d given by

$$(3.8) \quad \mathcal{S}_{M,p} = \left\{ [M_n, p, \psi_n] \left| \begin{array}{l} M_n = (M, g_n), \text{ rescaling of } M \text{ by } \sqrt{\lambda_n}; \\ p \in M; \psi_n \text{ eigenfunction associated to } \lambda_n \end{array} \right. \right\}_{n=1}^{\infty}$$

Let us pay attention to bounded, continuous functionals of two specific kinds:

- $F : \mathcal{E}^d \rightarrow \mathbb{R}$ depending only on $[M, p]$. Then F can be seen as a functional on \mathcal{M}^d and $\mu_{M_n, \psi_n}(F) \equiv \mu_{M_n}(F)$. As we know that μ_{M_n} converges to the Dirac measure at $[\mathbb{R}^d, 0] \in \mathcal{M}^d$, then $\lim_{n \rightarrow \infty} \mu_{M_n}(F) = F([\mathbb{R}^d, 0])$.
- $G : \mathcal{E}^d \rightarrow \mathbb{R}$ depending only on $\phi(\exp_p)$ in a small neighborhood of 0 in \mathbb{R}^d . In the case of the sequence we are considering this is precisely $G([M_n, p, \psi_n]) = G(\psi_n(\text{Exp}_p(\cdot/\sqrt{\lambda_n})))$, where we can use the orthonormal basis around p because $\lambda_n \rightarrow \infty$ and G can be seen as a functional on $C^\infty(\mathbb{R}^d)$. Notice that we are making an abuse of notation by using the same letter G for both a functional on \mathcal{E}^d and $C^\infty(\mathbb{R}^d)$. Then, using the exponential map Exp , μ_{M_n, ψ_n} induces a measure on $C^\infty(\mathbb{R}^d)$, call it ν_{M_n, ψ_n} whose limit (which we will assume that exists) is supported in functions satisfying the Helmholtz equation $\Delta\psi + \psi = 0$, denoted by FP . This can be seen in the following lemma.

Lemma 3.10. *The limit $\lim_{n \rightarrow \infty} \nu_{M_n, \psi_n}$ is a probability measure supported in FP .*

Proof. Let us proceed by contradiction and assume the opposite. Suppose that there exists a function $f \in C^\infty(\mathbb{R}^d)$ which is not in FP but it is in the support of the limit measure $\lim_{n \rightarrow \infty} \nu_{M_n, \psi_n}$, meaning that for any $\epsilon > 0$ it is true that $\lim_{n \rightarrow \infty} \nu_{M_n, \psi_n}(B(f, \epsilon)) > 0$. Here the ball B is constructed using the distance d given by the Fréchet structure of $C^\infty(\mathbb{R}^d)$ defined in (2.2), and has positive mass asymptotically.

Recall that $C^\infty(\mathbb{R}^d)$ with this distance is a locally compact metric space and that FP is closed in the topology of smooth convergence of derivatives on compact

sets. Then we can find a small $\epsilon_0 > 0$ such that $\text{dist}(\overline{B(f, \epsilon_0)}, FP) \geq c_{\epsilon_0} > 0$ and also

$$(3.9) \quad \liminf_{n \rightarrow \infty} \text{Vol}(\{p \in M : \phi_p^n \in B(f, \epsilon_0)\}) \geq c'_{\epsilon_0} > 0.$$

On the one hand, it can be easily found a lower positive bound for $d(\phi_p^n, -\Delta\phi_p^n)$. Just notice that the map $g \mapsto d(g, -\Delta g)$ is continuous and strictly positive on the ball $\overline{B(f, \epsilon_0)}$ (because it is not in FP). Therefore, it is bounded from below by some positive constant ϵ_1 . In particular, as far as we consider the points $p \in M$ such that $\phi_p^n \in B(f, \epsilon_0)$, we have $d(\phi_p^n, -\Delta\phi_p^n) \geq \epsilon_1 > 0$. Notice that this bound does not depend on n , for any n large enough.

On the other hand, it can be easily seen that for those points p with ϕ_p^n in $B(f, \epsilon_0)$, the set of ϕ_p^n is bounded in $C^\infty(\mathbb{R}^d)$ and so all the derivatives of ϕ_p^n are uniformly bounded in n . Also notice that

$$(3.10) \quad d(\Delta\phi_p^n, -\phi_p^n) = d(\Delta\phi_p^n, \Delta_{M_n}\phi_p^n) = d((\Delta - \Delta_{M_n})\phi_p^n, 0) = o_{n \rightarrow \infty}(1) \cdot d(\phi_p^n, 0).$$

In the second last equality, we have used that the distance is a series of semi-norms, so that $d(u, v) = d(u - v, 0)$. Finally, in the last part we have used that $\|(\Delta - \Delta_{M_n})\phi_p^n\|_{k,m} = C^{k,m}/\sqrt{\lambda_n}$. We get this expression thanks to the bound of all the derivatives of ϕ_p^n and the explicit form of Δ_{M_n} depending on the metric $g_n = \sqrt{\lambda_n} \cdot g$.

To conclude we recall that, in particular, the eigenfunctions ϕ_p^n are uniformly bounded in n for the points p we are considering. As a consequence, $d(\Delta\phi_p^n, -\phi_p^n) = o_{n \rightarrow \infty}(1)$. This contradicts the positive bound already found, and so we can assure that such an f does not exist, concluding that μ_{Berry} is supported in FP . \square

The set of linear combinations of products of these two kinds of functions is a subset of $C(\mathcal{E}^d)$ that separates points, i.e., given distinct points $\alpha, \beta \in \mathcal{E}^d$, there exists a linear combination of products of functions of this type such that $\sum_i F_i(\alpha)G_i(\alpha) \neq \sum_i F_i(\beta)G_i(\beta)$. By mean of the Stone–Weierstrass theorem, we get to know that this set is dense in the space of bounded, continuous functionals on \mathcal{E}^d . Therefore, to study the limit of a measure in the set \mathcal{E}^d , it is enough to study it over products of F and G . The sum can be taken outside of the limit using linearity of the integral.

Then we can conclude that any weak limit of μ_{M_n, ψ_n} is supported in

$$(3.11) \quad \{[\mathbb{R}^d, p, \psi] \in \mathcal{E}^d : p \in \mathbb{R}^d, \Delta\psi = -\psi\} = \{[\mathbb{R}^d, 0, \psi] \in \mathcal{E}^d : \Delta\psi = -\psi\}.$$

This set is equivalent to FP/\sim where $f, g \in FP$ satisfy $f \sim g$ if, and only if, there exists an origin preserving isometry S such that $f = g \circ S$. We therefore loosely identify such a weak limit with a random field on \mathbb{R}^d .

All considered, we can now reformulate the conjecture proposed in [1] as follows: μ_{M_n, ϕ_n} converges to μ_{Berry} in the weak* topology or, what is the same, for each bounded, continuous real functional F of $C^\infty(\mathbb{R}^d)$ which is invariant under origin preserving isometries, we have

$$(3.12) \quad \lim_{n \rightarrow \infty} \nu_{M_n, \phi_n}(F) = \mu_{Berry}(F) = \mathbb{E}_{\mu_{Berry}}[F] = \int_{C^\infty(\mathbb{R}^d)} F d\mu_{Berry}.$$

Here F is only defined in $C^\infty(\mathbb{R}^d)$ but as both μ_{Berry} and $\lim_{n \rightarrow \infty} \nu_{M_n, \phi_n}$ are supported in $\{[\mathbb{R}^d, p, \psi] \in \mathcal{E}^d : p \in \mathbb{R}^d, \Delta\psi = \psi\}$, we only care about these points. As we are only considering classes up to pointed isometries, we notice that, for any isometry S of \mathbb{R}^d that preserves the origin $S(0) = 0$, then it should be true that $F(\psi) = F(\psi \circ S)$, this is, we only consider functionals that are invariant up to origin preserving isometries.

In the next two Sections we prove both implications of this theorem. First, in Section 4 we show that the conjecture of definition 3.2 implies the conjecture given by 3.9. Then, in Section 5, we prove the reciprocal, that the conjecture of definition 3.9 implies the conjecture given by 3.2.

4. LOCAL WEAK LIMIT FORMULATION IMPLIES BS FORMULATION

We start by noticing the following: let $U \subset M$ be a Borel set of positive Lebesgue measure, and suppose that we have the Berry's conjecture with the local formulation on U . Then for any Borel set $U' \subset U$ of positive Lebesgue measure, we also have the conjecture on U' . A proof of this fact can be found in [21] for the case of a domain of $M = \mathbb{R}^d$ and can be easily modified for our case. This means that if we consider M covered by a family of open sets where the Berry property holds, then for any open subset with an orthonormal frame we would also have the Berry property in that subset.

We continue by considering the map

$$(4.1) \quad H^n : M \rightarrow \mathcal{E}^d, p \mapsto [(M, g_n), p, \psi_n],$$

and applying a change of variables together with the definition of μ_{M_n, ϕ_n} to write, for any bounded continuous function F in \mathcal{E}^d ,

$$(4.2) \quad \mu_{M_n, \psi_n}(F) = \int_M F(H^n(p)) d\mu_{Unif}(p) = \frac{1}{\text{Vol}(M)} \int_M F(H^n(p)) dp.$$

Let us now split this integral using the partition introduced at the beginning of Section 2:

$$(4.3) \quad \mu_{M_n, \psi_n}(F) = \frac{1}{\text{Vol}(M)} \sum_{m=1}^{m_{max}} \int_{U_m} F(H^n(p)) dp.$$

As said in the previous section, it is enough to study products of functions of the two mentioned types:

$$(4.4) \quad \mu_{M_n, \psi_n}(F \cdot G) = \frac{1}{\text{Vol}(M)} \sum_{m=1}^{m_{max}} \int_{U_m} F([M_n, p]) G(\phi_p^n) dp.$$

Then,

$$(4.5) \quad \frac{1}{\text{Vol}(M)} \sum_{m=1}^{m_{max}} \int_{U_m} F([M_n, p]) G(\phi_p^n) dp = \sum_{m=1}^{m_{max}} \frac{\text{Vol}(U_m)}{\text{Vol}(M)} \left(\frac{1}{\text{Vol}(U_m)} \int_{U_m} F([M_n, p]) G(\phi_p^n) dp \right).$$

Let us consider first the case when $F \equiv c$, some constant. Using, by hypothesis, that the Berry property is satisfied with the local weak limit formulation, we can

assert that

$$(4.6) \quad \lim_{n \rightarrow \infty} \nu_{M_n, \psi_n}(c \cdot G) = \sum_{m=1}^{m_{max}} \frac{\text{Vol}(U_m)}{\text{Vol}(M)} \lim_{n \rightarrow \infty} \left(\frac{1}{\text{Vol}(U_m)} \int_{U_m} c \cdot G(\phi_p^n) dp \right) =$$

$$(4.7) \quad \sum_{m=1}^{m_{max}} \frac{\text{Vol}(U_m)}{\text{Vol}(M)} \mathbb{E}_{\mu_{Berry}}[c \cdot G] = \mathbb{E}_{\mu_{Berry}}[c \cdot G] = \mu_{Berry}(c \cdot G).$$

The next step is to notice that, since F depends only on the manifold and G is a bounded functional, we have for any m ,

$$(4.8) \quad \left| \int_{U_m} F([M_n, p]) G(\phi_p^n) dp - \int_{U_m} F([\mathbb{R}^d, 0]) G(\phi_p^n) dp \right| \leq$$

$$(4.9) \quad C \cdot \sup_{p \in M} |F([M_n, p]) - F([\mathbb{R}^d, 0])| = C \cdot \epsilon_n \xrightarrow{n \rightarrow \infty} 0$$

Therefore,

$$(4.10) \quad \lim_{n \rightarrow \infty} |\mu_{M_n, \phi_n}(F \cdot G) - \mu_{Berry}(c \cdot G)| \leq \lim_{n \rightarrow \infty} C \cdot \epsilon_n |\mu_{M_n, \phi_n}(c \cdot G) - \mu_{Berry}(c \cdot G)| = 0,$$

and so

$$(4.11) \quad \lim_{n \rightarrow \infty} \mu_{M_n, \psi_n}(F \cdot G) = \lim_{n \rightarrow \infty} \mu_{M_n, \psi_n}(c \cdot G) = \lim_{n \rightarrow \infty} \nu_{M_n, \psi_n}(c \cdot G) = \mu_{Berry}(c \cdot G),$$

for any G bounded, continuous functional on $C^\infty(\mathbb{R}^d)$. $c \cdot G$ is also any bounded, continuous functional on $C^\infty(\mathbb{R}^d)$. Notice that here we have again made an abuse of notation by using the same letter G for a certain class of functionals on \mathcal{E}^d and for functionals on $C^\infty(\mathbb{R}^d)$. This concludes the proof of this implication.

5. BS FORMULATION IMPLIES LOCAL WEAK LIMIT FORMULATION

For the other implication we assume that $\lim_{n \rightarrow \infty} \nu_{M_n, \psi_n}(F) = \mu_{Berry}(F)$, for any bounded, continuous real functional F of $C^\infty(\mathbb{R}^d)$ which is invariant under origin preserving isometries. This means that

$$(5.1) \quad \lim_{n \rightarrow \infty} \langle \nu_{M_n, \phi_n}, F \rangle = \sum_{m=1}^{m_{max}} \frac{\text{Vol}(U_m)}{\text{Vol}(M)} \lim_{n \rightarrow \infty} \left(\frac{1}{\text{Vol}(U_m)} \int_{U_m} F(\phi_p^n) dp \right) = \langle \mu_{Berry}, F \rangle,$$

when tested against bounded, continuous real functionals F of $C^\infty(\mathbb{R}^d)$ which are invariant under origin preserving isometries.

First, we will see that, up to a subsequence, for any m , there exists a limit μ_m such that for any functional G (which is not necessarily invariant w.r.t. isometries) we have:

$$(5.2) \quad \exists \lim_{n \rightarrow \infty} \left(\frac{1}{\text{Vol}(U_m)} \int_{U_m} G(\phi_p^n) dp \right) = \mu_m(G).$$

To prove this let us define the probability measure on the space of $C^\infty(\mathbb{R}^d)$, μ_m^n given by

$$(5.3) \quad \langle \mu_m^n, G \rangle = \frac{1}{\text{Vol}(U_m)} \int_{U_m} G(\phi_p^n) dp,$$

for any G bounded, continuous functional; and recall Prokhorov's theorem for this particular case: the collection $\{\mu_m^n\}_{n=1}^\infty$ is tight if, and only if, its closure is sequentially compact in the space of probability measures on $C^\infty(\mathbb{R}^d)$, i.e., up to a subsequence, there exists the limit $\lim_{n \rightarrow \infty} \mu_m^n = \mu_m$.

The only thing that is left to obtain (5.2) is to check that $\{\mu_m^n\}_{n=1}^\infty$ is tight but this is essentially [21, Lemma 1], where these measures are shown to be tight in the bigger space of C^k functions. The aim now is to prove that this limit is equal to μ_{Berry} for any $1 \leq m \leq m_{\max}$.

As can be seen in [21], this limit μ_m is translation invariant. We recall the proof of this fact here, but first we need to introduce few definitions.

Let $y \in \mathbb{R}^d$. For any $f \in C^\infty(\mathbb{R}^d)$, we define $\tau_y f \in C^\infty(\mathbb{R}^d)$ by $(\tau_y f)(\cdot) = f(y + \cdot)$. If F is now a functional on $C^\infty(\mathbb{R}^d)$, we define $\tau_y F$ by $(\tau_y F)(f) = F(\tau_y f)$ for all $f \in C^\infty(\mathbb{R}^d)$. Finally, if μ is a measure on $C^\infty(\mathbb{R}^d)$, we define $\tau_y \mu$ by $\langle \tau_y \mu, F \rangle = \langle \mu, \tau_y F \rangle$ for all bounded, continuous functionals F .

Lemma 5.1. *Let μ_m be a limit as considered in this section. Then for any $y \in \mathbb{R}^d$, we have $\tau_y \mu_m = \mu_m$.*

Proof. Notice that $\exists \lim_{n \rightarrow \infty} \mu_m^n = \mu_m$. We now compute, for any $y \in \mathbb{R}^d$ and F bounded, continuous functional,

$$(5.4) \quad \langle \mu_m^n, \tau_y F \rangle = \frac{1}{\text{Vol}(U_m)} \int_{U_m} F(\tau_y \phi_p^n) dp = \frac{1}{\text{Vol}(U_m)} \int_{U_m} F\left(\psi_n\left(\text{Exp}_p\left(\frac{\cdot + y}{\sqrt{\lambda_n}}\right)\right)\right) dp =$$

$$(5.5) \quad \frac{1}{\text{Vol}(U_m)} \int_{U_m} F\left(\psi_n\left(\text{Exp}_{\text{Exp}_p(y/\sqrt{\lambda_n})}\left(\frac{\cdot}{\sqrt{\lambda_n}}\right)\right)\right) + \mathcal{O}\left(1/\sqrt{\lambda_n}\right) dp =$$

$$(5.6) \quad \frac{1}{\text{Vol}(U_m)} \int_{U_m} F(\phi_p^n) dp + \mathcal{O}\left(1/\sqrt{\lambda_n}\right) + \mathcal{O}\left(\text{Vol}\left(O_m \Delta (O_m - y/\sqrt{\lambda_n})\right)\right) =$$

$$(5.7) \quad \langle \mu_m^n, F \rangle + o_{n \rightarrow \infty}.$$

Here O_m is the Euclidean domain that goes to U_m by Exp_p , $O_m \Delta (O_m - y/\sqrt{\lambda_n})$ is the symmetric difference between O_m and $O_m - y/\sqrt{\lambda_n}$ and its volume goes to zero with n , since $\text{Vol}(O_m \cap (O_m - y/\sqrt{\lambda_n})) \xrightarrow{n \rightarrow \infty} 0$. In the second line we have used that fact that F is a bounded, continuous functional and we are using geodesic coordinates in a small neighborhood of the origin.

Taking the limit $n \rightarrow \infty$, we obtain $\mu_m(\tau_y F) = \mu_m(F)$ for any F , so that $\tau_y \mu_m = \mu_m$. \square

It is easy to check that $(FP, \mathcal{B}(FP), \mu_{Berry})$ is a probability space on which \mathbb{R}^d acts by the translations τ as measure-preserving transformations. By the Fomin-Grenander-Maruyama theorem [23, Section B], μ_{Berry} is ergodic for the action of the translations, which means that for every set $A \in \mathcal{B}(FP)$ satisfying $\mu_{Berry}((\tau_y A) \Delta A) = 0$, either $\mu_{Berry}(A) = 0$, or $\mu_{Berry}(A) = 1$.

It can also be proved, see for instance [12], that ergodic measures are the extreme points of the set of the action invariant measures. This means that μ_{Berry} can not be expressed as a strict convex combination of two different translation invariant probability measures. It can be generalized by induction to any finite-length convex

combination and it is also true for general integration as can be seen by the following lemma.

Lemma 5.2. *If μ is a Borel measure that is ergodic with respect to translations and we have a decomposition $\mu = \int_Y v_y dm(y)$, where Y is a measurable set, v_y are all translations invariant probability measures and the integral is defined using the measure dm of Y . Then $v_y = \mu$ for m -a.e. y .*

Proof. Let $\mathcal{T}(C^\infty)$ be the set of all continuous, bounded functionals in $C^\infty(\mathbb{R}^d)$. If both sets

$$(5.8) \quad \bigcup_{F \in \mathcal{T}(C^\infty)} \{y \in Y, \langle F, \mu \rangle < \langle F, v_y \rangle\}$$

and

$$(5.9) \quad \bigcup_{F \in \mathcal{T}(C^\infty)} \{y \in Y, \langle F, \mu \rangle > \langle F, v_y \rangle\}$$

have measure zero then we have finished. Assume otherwise. Without loss of generality, we have a continuous functional F such that

$$(5.10) \quad A = \{y \in Y, \langle F, \mu \rangle < \langle F, v_y \rangle\}$$

satisfies $0 < m(A)$. Notice that if $m(A) = 1$ then we can not have $\mu = \int_Y v_y dm(y)$, so $0 < m(A) < 1$. Now, we have that for all $G \in \mathcal{T}(C^\infty)$,

$$(5.11) \quad \langle G, \mu \rangle = m(A) \cdot \left\langle G, \frac{1}{m(A)} \int_A v_y dm(y) \right\rangle + m(A^c) \cdot \left\langle G, \frac{1}{m(A^c)} \int_{A^c} v_y dm(y) \right\rangle.$$

This gives a representation of μ as a convex sum of two translation invariant probability measures. By ergodicity of μ , each of the measures is in fact μ . But this is a contradiction since integrating the first measure against F is strictly bigger than $\langle F, \mu \rangle$. This concludes the proof. \square

Consider now the set $\Omega = \{\omega : \mathbb{R}^d \rightarrow \mathbb{R}^d, \omega \text{ is an isometry preserving the origin}\}$ and its Haar measure dm . For any bounded continuous functional G , we consider the following

$$(5.12) \quad \tilde{G} = \frac{1}{|\Omega|} \int_{\Omega} G(\omega(\cdot)) dm(\omega),$$

which is defined using the Haar measure for the integral.

Obviously, this functional is still bounded and continuous and now it is invariant by isometries that preserve the origin.

Similarly to what we did before, if μ is a measure on $C^\infty(\mathbb{R}^d)$, we define $\tilde{\mu}$ by $\langle \tilde{\mu}, G \rangle = \langle \mu, \tilde{G} \rangle$ for all bounded, continuous functionals G . This way, we have that, for all m and any bounded continuous functional G

$$(5.13) \quad \langle \tilde{\mu}_m, G \rangle = \langle \mu_m, \tilde{G} \rangle.$$

This means that evaluating $\tilde{\mu}_m$ against any functional is the same as evaluating μ_m against an isometry invariant functional. If we sum along m and use the hypothesis,

we get

$$(5.14) \quad \left\langle \sum_m \frac{\text{Vol}(U_m)}{\text{Vol}(M)} \widetilde{\mu}_m, G \right\rangle = \left\langle \sum_m \frac{\text{Vol}(U_m)}{\text{Vol}(M)} \mu_m, \widetilde{G} \right\rangle = \mu_{\text{Berry}}(\widetilde{G}) = \mu_{\text{Berry}}(G).$$

It is easy to check that $\widetilde{\mu}_m$ preserves the property of being translation invariant: $\langle \widetilde{\mu}_m, \tau_y G \rangle = \langle \mu_m, \tau_y \widetilde{G} \rangle = \langle \mu_m, \widetilde{G} \rangle = \langle \widetilde{\mu}_m, G \rangle$. Also notice that the measure μ_{Berry} is invariant with respect to isometries so that

$$(5.15) \quad \mu_{\text{Berry}}(\widetilde{G}) = \mu_{\text{Berry}} \left(\frac{1}{|\Omega|} \int_{\Omega} G(\omega(\cdot)) dm(\omega) \right) = \frac{1}{|\Omega|} \int_{\Omega} \mu_{\text{Berry}}(G) dm(\omega) = \mu_{\text{Berry}}(G).$$

Then, we have μ_{Berry} as a linear combination of translation invariant measures. By using Fomin-Grenander-Maruyama theorem, we conclude that any $\widetilde{\mu}_m = \mu_{\text{Berry}}$.

The last step is to consider

$$(5.16) \quad \begin{aligned} \langle \mu_{\text{Berry}}, G \rangle &= \langle \widetilde{\mu}_m, G \rangle = \frac{1}{|\Omega|} \int_{\Omega} \langle \mu_m, G(\omega(\cdot)) \rangle dm(\omega) = \\ &= \frac{1}{|\Omega|} \int_{\Omega} \langle \mu_m \circ \omega, G \rangle dm(\omega) = \left\langle \frac{1}{|\Omega|} \int_{\Omega} (\mu_m \circ \omega) dm(\omega), G \right\rangle, \end{aligned}$$

where $\mu_m \circ \omega$ stands for $\langle \mu_m \circ \omega, G \rangle = \langle \mu_m, G(\omega(\cdot)) \rangle$. In this calculation we have μ_{Berry} as a linear combination of translation invariant measure $\mu_m \circ \omega$. Using lemma 5.2, this ensures that, for almost all $\omega \in \Omega$, $\mu_m \circ \omega = \mu_{\text{Berry}}$. Notice that it is enough to have one ω such that $\mu_m \circ \omega = \mu_{\text{Berry}}$ because then we can compose with ω^{-1} and get, $\mu_m = \mu_{\text{Berry}} \circ \omega^{-1} = \mu_{\text{Berry}}$, because μ_{Berry} is invariant.

This is the same as saying that ϕ_p^n converges in law as a random field towards Ψ_{Berry} in U_m . To conclude, just notice that this being true for any m is the formulation of Berry's conjecture that we aimed to obtain.

6. THE INVERSE LOCALIZATION PROPERTY

A related notion to the random wave model is the idea of *inverse localization* introduced in [15] and related works of the same authors. The local behavior of an eigenfunction on any compact Riemannian manifold associated to large eigenvalues over length scales of order $1/\sqrt{\lambda}$ is given by a solution to the Helmholtz equation,

$$(6.1) \quad \Delta h + h = 0.$$

A well known partial converse due to Hörmander [17] is that, given any ball $B \subset \mathbb{R}^d$ and any fixed solution h to the Helmholtz equation on B , one can pick a sequence of *approximate* Laplace eigenfunctions on the manifold whose behavior on a ball of radius $1/\sqrt{\lambda}$ reproduces that of h modulo a small error, λ being the approximate frequency of the approximate eigenfunction. A slightly imprecise but very intriguing question is whether one can replace approximate eigenfunctions by *bone fide* eigenfunctions in this estimate. The inverse localization principle gives an answer for this question in certain particular cases.

This idea was first introduced in [14], where the authors proved the inverse localization property for Beltrami fields on the three-dimensional sphere and flat torus. To our best knowledge, the only known examples of manifold satisfying the inverse localization property are certain flat tori [13] and the round sphere \mathbb{S}^d

and all Riemannian quotients thereof [15]. In these works, the definition of inverse localization is the following.

Definition 6.1. (Inverse localization) A compact manifold M has the inverse localization property if for some $p \in M$, any $\epsilon > 0$, any $r \in \mathbb{N} \cup \{0\}$ and any $h : \mathbb{R}^d \rightarrow \mathbb{R}$ solution to the Helmholtz equation (i.e. $\Delta h + h = 0$ in the whole \mathbb{R}^d), there exists a sequence of eigenvalues $\lambda_k \xrightarrow[k \rightarrow \infty]{} \infty$ and a sequence of associated eigenfunctions ψ_k , $\Delta_M \psi_k + \lambda_k \psi_k = 0$, that for any k large enough satisfy

$$(6.2) \quad \left\| \psi_k \left(\text{Exp}_p \left(\frac{\cdot}{\sqrt{\lambda_k}} \right) \right) - h \right\|_{C^r(B)} < \epsilon.$$

Here $\phi_p^k := \psi_k \left(\text{Exp}_p \left(\frac{\cdot}{\sqrt{\lambda_k}} \right) \right)$ is the standard localization of ψ_k and B is the unit Euclidean ball on the Euclidean space.

In this definition, the point around which we localize is fixed beforehand and does not depend on h (in fact, in all the known examples we can approximate any solution to the Helmholtz equation around any fixed point of the manifold). Moreover, in these examples, there are no restrictions on the eigenfunctions that can be used to approximate. However, the idea of inverse localization is imprecise and the rigorous definition can vary in these and other aspects, for example, the norm and domain of the approximation. The Berry property considered here implies a slightly different version of this inverse localization idea defined as follows.

Definition 6.2. (Strong inverse localization) A compact manifold M has the strong inverse localization property if for any eigenvalue λ there exist an eigenfunction ψ_λ , $\Delta_M \psi_\lambda + \lambda \psi_\lambda = 0$, such that for any $r \in \mathbb{N} \cup \{0\}$ and any $h : \mathbb{R}^d \rightarrow \mathbb{R}$ solution to the Helmholtz equation (i.e. $\Delta h + h = 0$ in the whole \mathbb{R}^d), there exists a positive measure subset of the manifold $N \subset M$ which satisfies that

$$(6.3) \quad \lim_{\lambda \rightarrow \infty} \left\| \psi_\lambda \left(\text{Exp}_p \left(\frac{\cdot}{\sqrt{\lambda}} \right) \right) - h \right\|_{C^r(B)} = 0,$$

for any $p \in N$.

Remark 6.3. Notice that Definition 6.2 does not imply Definition 6.1 because the set N depends on h , while in the latter the point p is fixed. However, it is stronger in the following two aspects:

- While for definition 6.1 to be true it is enough to have the existence of a sequence of eigenvalues along which we can approximate, in definition 6.2 the limit needs to be true for all the sequences of eigenvalues going to infinity.
- In definition 6.2, the eigenfunction associated to each eigenvalue is always the same. This means that it is possible to approximate any h using the same sequence of eigenfunctions. The same is not true for definition 6.1, where different eigenfunctions associated to the same eigenvalue can be needed to approach different solutions to the Helmholtz equation.

We recall that, by the local weak limit formulation, if M has the Berry property it is true that for any bounded, continuous functional $G : C^\infty(\mathbb{R}^d) \rightarrow \mathbb{R}$ it holds that

$$(6.4) \quad \mathbb{E}_p(G(\phi_p^k)) \xrightarrow[k \rightarrow \infty]{} \mathbb{E}(G(\Psi_{Berry})).$$

We can then assert the following

Proposition 6.4. *If (M, g) satisfies the Berry's property, then it also exhibits strong inverse localization.*

Proof. Fix a small error $\epsilon > 0$. Let $\chi \in C^\infty(\mathbb{R}^d)$ be an even decreasing function that is equal to 1 on $(0, \epsilon/2)$ and is supported on $(0, \epsilon)$. Fix a monochromatic wave h , a natural number $r \in \mathbb{N}$ and take the functional F defined as

$$(6.5) \quad F(f) := \chi\left(\|f - h\|_{C^r(B)}\right)$$

for any $f \in C^\infty(\mathbb{R}^d)$. Explicitly, we have that

$$(6.6) \quad F(f) = \chi\left(\|f - h\|_{C^r(B)}\right) = \begin{cases} 1 & \text{if } \|f - h\|_{C^r(B)} < \epsilon/2 \\ \text{Smooth in the middle} & \\ 0 & \text{if } \|f - h\|_{C^r(B)} > \epsilon. \end{cases}$$

In particular,

$$(6.7) \quad F(\phi_p^k) = \chi\left(\|\phi_p^k - h\|_{C^r(B)}\right),$$

for any k . This non-linear functional is continuous and bounded. Since the Berry property holds by assumption, we have

$$(6.8) \quad \mathbb{E}_p(F(\phi_p^k)) \xrightarrow[k \rightarrow \infty]{} \mathbb{E}(F(\Psi_{Berry})).$$

Notice that

$$(6.9)$$

$$\mathbb{E}(F(\Psi_{Berry})) = \int_{C^r(B)} F(f) d\mu_{Berry}(f) = \int_{C^r(B)} \chi\left(\|f - h\|_{C^r(B)}\right) d\mu_{Berry}(f) =$$

$$(6.10)$$

$$\int_{\{f \in C^r, \|f - h\| < \epsilon/2\}} \chi(\|f - h\|) d\mu_{Berry}(f) + \int_{\{f \in C^r, \epsilon/2 \leq \|f - h\| \leq \epsilon\}} \chi(\|f - h\|) d\mu_{Berry}(f)$$

$$(6.11)$$

$$= \mu_{Berry}(\{f \in C^r(B), \|f - h\|_{C^r(B)} \leq \epsilon/2\}) + \int_{\{f \in C^r, \epsilon/2 \leq \|f - h\| \leq \epsilon\}} \chi(\|f - h\|) d\mu_{Berry}(f).$$

The next step is to show that the set of functions that are $\epsilon/2$ -close to h in $C^r(B)$ metric has positive Berry measure. This is standard and can be deduced from [23, Section A7]. A brief sketch of a proof is as follows.

First, recall that any solution to the Helmholtz equation can be expanded as a Bessel-Fourier series as

$$(6.12) \quad h(x) = \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} c_{lm} \frac{J_{l+\frac{d}{2}-1}(|x|)}{|x|^{\frac{d}{2}-1}} Y_{lm}\left(\frac{x}{|x|}\right).$$

Here J_ν denotes the Bessel function of order ν and $\{Y_{lm}(\xi)\}$ is a real-valued orthonormal basis of d -dimensional spherical harmonics; the order l means that the spherical harmonic is the restriction to the sphere \mathbb{S}^{d-1} of a homogeneous harmonic polynomial of degree l , and d_l is the multiplicity of this space.

On the other hand, it is known that the field Ψ_{Berry} can be written as

$$(6.13) \quad \Psi_{Berry}(x) = \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} a_{lm} \frac{J_{l+\frac{d}{2}-1}(|x|)}{|x|^{\frac{d}{2}-1}} Y_{lm} \left(\frac{x}{|x|} \right),$$

where a_{lm} are independent Gaussian variables. Since the topology we are considering in $C^\infty(\mathbb{R}^d)$ is precisely that of convergence of each coefficient in the expansion, we have the desired result because for any $l \in \mathbb{N}$ and any $1 \leq m \leq d_l$, $P(a_{lm} = c_{lm}) > 0$.

Using this, we can then infer that $\mathbb{E}(F(\Psi_{Berry})) > 0$. We conclude that $\mathbb{E}_p(F(\phi_p^k))$ converges to a positive number when $k \rightarrow \infty$, and hence for big enough k (and so, small enough $1/\sqrt{\lambda_k}$) there must be a positive measure set of points on M for which $F(\phi_p^k) > 0$. This clearly implies, by the definition of F , that for those points and large values of k , we have that $\|\phi_p^k - h\|_{C^r(B)} < \epsilon$. This can be made true for any ϵ , so we have proved that

$$(6.14) \quad \lim_{k \rightarrow \infty} \|\phi_p^k - h\|_{C^r(B)} = 0.$$

This is, of course, also true for any subsequence of λ_k so indeed

$$(6.15) \quad \lim_{\lambda \rightarrow \infty} \left\| \psi_\lambda \left(\text{Exp}_p \left(\frac{\cdot}{\sqrt{\lambda}} \right) \right) - h \right\|_{C^r(B)} = 0,$$

for a positive measure set of points p , which is precisely definition 6.2. \square

Regarding this result, there are a few remarks to be made.

- The set of admissible points for any $k \in \mathbb{N}$, say $N^k \subset M$, satisfies the stronger condition of being ‘‘asymptotically dense’’. Since the property

$$(6.16) \quad \mathbb{E}_p(F(\phi_p^k)) \xrightarrow[k \rightarrow \infty]{} \mathbb{E}(F(\Psi_{Berry})).$$

is also true if we restrict ourselves by choosing p in an arbitrary open set, say $p \in U \subset M$, it is clear that the set $\lim_{k \rightarrow \infty} U \cap N^k$ has positive measure for any arbitrary open set. This implies that for any $\epsilon > 0$ and any point $q \in M$, the set $\lim_{k \rightarrow \infty} B(q, \epsilon) \cap N^k \neq \emptyset$, and therefore $\lim_{k \rightarrow \infty} N^k$ is dense in M when taking the limit.

- Our main interest is in the following easy application:

Corollary 6.5. *If (M, g) does not exhibit strong inverse localization, then it does not admit Berry’s property. For example, as seen in [13], some tori (irrational ones) do not satisfy the inverse localization property (neither 6.1 nor 6.2) and, therefore, do not satisfy Berry’s property.*

- The inverse localization property can be applied to study nodal sets of eigenfunctions. Precisely, it is true the following result [13]:

Theorem 6.6. *If a manifold M does satisfy 6.1 or 6.2, given any natural N and a collection of compact embedded hypersurfaces Σ_j ($1 \leq j \leq N$) of \mathbb{R}^d that are not linked, any positive integer r and any ϵ , there exists some $R > 0$ such that for all large enough n there is an eigenfunction ψ_n with eigenvalue λ_n such that the function*

$$(6.17) \quad \psi_n \left(\text{Exp}_p \left(\frac{\cdot}{\sqrt{\lambda_n}} \right) \right)$$

has at least N nodal components of the form

$$(6.18) \quad \tilde{\Sigma}_j^n := \lambda_n^{-1/2} \Phi_n(c_j \Sigma_j + p_j)$$

and at least N nondegenerate local extrema in the ball of radius R . Here $c_j > 0$, $p_j \in \mathbb{R}^d$, and Φ_n is a diffeomorphism of \mathbb{R}^d which is close to the identity $\|\Phi_n - \text{id}\|_{C^r(\mathbb{R}^d)} < \varepsilon$.

Therefore, using this formulation of the Berry's conjecture we can gain some knowledge in the topology of nodal set of eigenfunctions of manifolds satisfying the RWM.

- Definition 3.2 does also make sense if we consider M to be a manifold with boundary and $(\psi_n)_n$ to be Dirichlet eigenfunctions, and can be found in [21]. The inverse localization can also be defined in that context.

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