Hard congestion limit of the dissipative Aw-Rascle system with a polynomial offset function

Muhammed Ali Mehmood*

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Abstract

We study the Aw-Rascle system in a one-dimensional domain with periodic boundary conditions, where the offset function is replaced by the gradient of the function ρ_n^γ, where $\gamma \to \infty$. The resulting system resembles the 1D pressureless compressible Navier-Stokes system with a vanishing viscosity coefficient in the momentum equation and can be used to model traffic and suspension flows. We first prove the existence of a unique global-in-time classical solution for fixed n. Unlike the previous result for this system, we obtain global existence without needing to add any approximation terms to the system. This is by virtue of a $n-$uniform lower bound on the density which is attained by carrying out a maximum-principle argument on a suitable potential, $W_n = \rho_n^{-1}\partial_x w_n$. Then we prove the convergence to a weak solution of a hybrid free-congested system as $n \to \infty$, which is known as the hard-congestion model.

Keywords: Aw-Rascle system, maximal packing, weak solutions, hard-congestion.

MSC: 35Q35, 35B25, 76T20, 90B20.

1 Introduction

1.1 The Aw-Rascle system

This paper aims to study a singular limit pertaining to the following generalisation of the Aw-Rascle [3] and Zhang [22] system:

$$\begin{cases} \partial_t \rho_n + \partial_x(\rho_n u_n) = 0, & \text{in } \Omega \times [0, \infty), & \text{(1a)} \\ \partial_t(\rho_n w_n) + \partial_x(\rho_n w_n u_n) = 0, & \text{in } \Omega \times [0, \infty). & \text{(1b)} \end{cases}$$

We take $\Omega = \mathbb{T}$ to be the one-dimensional torus, which we identify with $[0, 1]$. This system is to be solved for ρ_n and u_n which represent the density and the (actual) velocity respectively. The quantity w_n that appears in the above system is known as the desired velocity and differs from the actual velocity u_n by a cost (or offset) function C_n, which typically depends on the density ρ_n. More precisely, w_n is defined by the relation

$$w_n = u_n + C_n(\rho_n). \tag{1c}$$

The cost function may be interpreted as a quantity which measures the difficulty of moving in a certain direction. In this paper, we consider the case where our cost function is the gradient of a function $p_n(\rho_n)$, i.e.

$$C_n(\rho_n) = \partial_x p_n(\rho_n),$$

where

$$p_n(\rho_n) = \rho_n^{\gamma_n}. \tag{2}$$

Here, $\{\gamma_n\}_{n=1}^\infty$ is a sequence of positive real numbers satisfying $\gamma_n \to \infty$ as $n \to \infty$. We note that the singularity of this function only appears when considering the limit as $n \to \infty$; in the region where $\rho_n > 1$, we have $p_n(\rho_n) \to +\infty$ as $n \to \infty$. Our goal is to determine the existence of a solution to the above system for n fixed before investigating the limiting behaviour of the solution as $n \to \infty$.

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*Imperial College London, London, United Kingdom; muhammed.mehmood21@imperial.ac.uk

1.2 Background and motivation

The derivation and mathematical analysis of traffic flow models has been an active area of research over the past hundred years, with some notable early works devoted to hydrodynamic models being [17, 11, 22, 13]. The Aw-Rascle model, which is given by (1a)-(1c) with a scalar offset $C(\rho) = \rho^\gamma$, $\gamma > 1$, is one of the most famous examples of such models. The derivation of this model from the 'Follow-The-Leader' microscopic model for one-lane traffic can be seen in [2]. Much analysis has been devoted to the Aw-Rascle model with different choices of offset functions and domains over the past twenty years. We refer to [7, 6, 12, 8, 16] for some recent examples. It is known that although the model proposed by Aw and Rascle in [3] was an improvement upon previous iterations of traffic flow models, the system still exhibits some non-physical behaviour. For instance, solutions are assumed to adhere to the maximal density constraint $\rho < \overline{\rho}$ initially but proceed to violate it in finite time. In order to overcome these unrealistic behaviours, Berthelin et al [4] suggested to study the asymptotic limit of the system (1a)-(1b) accompanied by a singular scalar offset function $p_n(\rho_n)$ instead of ρ^γ. A recent paper by Chaudhuri et al [7] followed this suggestion and studied the case where the offset function is equal to the gradient of a singular function, i.e.

$$w_\epsilon = u_\epsilon + \partial_x p_\epsilon(\rho_\epsilon), \quad p_\epsilon(\rho_\epsilon) = \epsilon \frac{\rho_\epsilon^\gamma}{(1 - \rho_\epsilon)^\beta},$$

in a one-dimensional domain with periodic boundary conditions. With this choice, the density is now prevented from surpassing a maximal threshold $\overline{\rho} \equiv 1$. The authors go on to study the asymptotic limit $\epsilon \to 0$ which is known as the 'hard-congestion limit', and establish the existence of solutions (ρ, u, π) to the limiting system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & \text{(3a)} \\ \partial_t(\rho u + \partial_x \pi) + \partial_x((\rho u + \partial_x \pi)u) = 0, & \text{(3b)} \\ 0 \leq \rho \leq 1, \ (1 - \rho)\pi = 0, \ \pi \geq 0, & \text{(3c)} \end{cases}$$

where π is the limit of some singular function of ρ_ϵ. This is known as the hard congestion model. The interest in this system, which is an example of a free-congested system, emerges from the observation that it repairs the aforementioned issues with the classical Aw-Rascle model. In particular, from (3c) we notice that the potential π obtained in the singular limit is zero except when the density is maximal, where it acts similar to a Dirac measure. This reflects an important characteristic of traffic flow, which is that drivers do not typically slow down unless there is congestion. The authors of [7] prove two main results for their system; the existence of a unique global strong solution for ϵ fixed, and the existence of a subsequence converging to a solution of the hard-congestion model as $\epsilon \to 0$.

Our paper builds upon the work of Chaudhuri et al [7] and attempts to prove analogous results for the case where the singular offset function is of the form $p_n(\rho_n) = \rho_n^{\gamma_n}$. One particular piece of motivation for this problem is that our form of offset function makes it easier to perform numerical simulations (such as in [20, 1, 21]) and investigate the behaviour of solutions than the model in [7], for example. Additionally, our results could be used in conjunction with those in [7] in order to analyse the Aw-Rascle system with a more complex offset function and/or in the multi-dimensional case. One example of a multi-dimensional study of the Aw-Rascle system can be seen in a recent paper by Chaudhuri, Gwiazda and Zatorska [6].

For our analysis it will be useful to note that if we fix n with this choice of cost function, the system (1a)-(1b) can be formally rewritten as the one-dimensional compressible pressureless Navier-Stokes equations

$$\begin{cases} \partial_t \rho_n + \partial_x(\rho_n u_n) = 0, & \text{in } \mathbb{T} \times [0, \infty), & \text{(4a)} \\ \partial_t(\rho_n u_n) + \partial_x(\rho_n u_n^2) - \partial_x(\lambda_n(\rho_n)\partial_x u_n) = 0, & \text{in } \mathbb{T} \times [0, \infty), & \text{(4b)} \end{cases}$$

where $\rho_n, u_n : \mathbb{T} \times [0, \infty) \to \mathbb{R}$ are to be found and

$$\lambda_n(\rho_n) = \rho_n^2 p_n'(\rho_n), \ p_n(\rho_n) = \rho_n^{\gamma_n}, \ \gamma_n \in (0, \infty).$$

The systems (1a)-(1b) and (4a)-(4b) are equivalent for sufficiently regular solutions, and in particular for the class of regular solutions which we will consider. Interestingly, a similar approximation of a two-phase

system was carried out by Lions and Masmoudi [19], where the authors consider a compressible Navier-Stokes system with a pressure $\pi = \rho^\gamma$ and study the limit $\gamma \to \infty$. The presence of a constant viscosity coefficient allows the authors of [19] to control the gradient of the velocity $\nabla \mathbf{u}$ which gives way to a crucial uniform bound on the singular pressure. Although our system is pressureless, we note that the potential π_n which is defined through the relation

$$\pi'_n(\rho_n) := \rho_n p'_n(\rho_n) = \gamma_n \rho_n^{\gamma_n}, \tag{5}$$

plays a similar role to the pressure in the classical compressible Navier-Stokes model. We need to control this term in order to obtain the switching relation (3c) in the limit system. However, the presence of a degenerate viscosity coefficient in our system means that we cannot bound the potential in the same way that the pressure was bounded by the authors of [19]. For this reason, we need to carry out an improved potential estimate.

1.3 Main results

In this paper we adopt the Bochner space notation $X_t Y_x := X(0, T; Y(\mathbb{T}))$ for appropriate function spaces X and Y. We first provide a precise definition of regular solutions to the system (4a)-(4b), which are also classical.

Definition 1.1 (Global regular solution). *Suppose $n \in \mathbb{Z}^+$ is fixed, $T > 0$ and p_n is given by (2). Assume further that $(\rho_n^0, u_n^0) \in H^4(\mathbb{T}) \times H^4(\mathbb{T})$ and $0 < \rho_n^0(\cdot)$. The pair (ρ_n, u_n) is called a regular solution to (4a)-(4b) on $[0, T]$ if*

$$\rho_n \in C(0, T; H^4(\mathbb{T})), \ \ u_n \in C(0, T; H^4(\mathbb{T})) \cap L^2(0, T; H^5(\mathbb{T})),$$

and (ρ_n, u_n) satisfy (4a)-(4b) in $\mathbb{T} \times [0, T]$. The pair (ρ_n, u_n) is known as a global regular solution to (4a)-(4b) if it is a regular solution on $[0, T]$ for any $T > 0$.

We will prove two main results. Firstly, we will assert the existence of a unique global regular solution in the following result:

Theorem 1.2 (Global existence of a unique regular solution for fixed n). *Assume $(\rho_n^0, u_n^0) \in H^4(\mathbb{T}) \times H^4(\mathbb{T})$ and that $n \in \mathbb{Z}^+$ is fixed. Further assume that $0 < \rho_n^0(x)$ for $x \in \mathbb{T}$. Then there exists a unique pair (ρ_n, u_n) with initial data (ρ_n^0, u_n^0) which is a global regular solution to (4a)-(4b) in the sense of Definition 1.1.*

Remark 1.3. *The H^4 regularity on the initial data mentioned in Definition 1.1 and Theorem 1.2 is required in order for us to obtain a lower bound on the density. Our strategy demands that on the level of local-in-time solutions the desired velocity $w_n \in C_t H_x^3$, which itself requires $\rho_n, u_n \in C_t H_x^4$. To work with such solutions using Theorem 2.1 (local existence of solutions), we must take $(\rho_n^0, u_n^0) \in H^4(\mathbb{T}) \times H^4(\mathbb{T})$.*

We then assert the existence of weak solutions to the limiting system in the following theorem:

Theorem 1.4 (Global existence of a weak solution to the hard-congestion model). *Assume $(\rho_n^0, u_n^0) \in H^4(\mathbb{T}) \times H^4(\mathbb{T})$ and the existence of constants $C, \alpha, \hat{\rho} > 0$ independent of n such that*

$$0 < \rho_n^0(x) \leq 1 + \frac{1}{\gamma_n}, \quad \forall x \in \mathbb{T}, \tag{6}$$

$$0 < \alpha \leq |\mathbb{T}|^{-1} \int_{\mathbb{T}} \rho_n^0(x) \ dx \leq \hat{\rho} < 1, \tag{7}$$

$$\|\sqrt{\rho_n^0} w_n^0\|_{L_x^2} + \left\| \frac{\partial_x w_n^0}{\sqrt{\rho_n^0}} \right\|_{L_x^2} \leq C. \tag{8}$$

Then, the solution (ρ_n, u_n) established in Theorem 1.2 satisfies the following uniform bounds for $\gamma_n > 1$ and $\tau \in (0, T]$:

$$\|w_n\|_{L_t^\infty W_x^{1, \frac{4}{3}}} \leq C, \tag{9}$$

$$\|\pi_n\|_{L_{t,x}^1} + \|\partial_x \pi_n\|_{L_{t,x}^2} \leq C, \tag{10}$$

$$\|\pi_n\|_{L^\infty([\tau, T]; L^1(\mathbb{T}))} \leq C. \tag{11}$$

where $C > 0$ is independent of n. Suppose additionally that

$$\rho_n^0 \rightharpoonup \rho^0 \text{ weakly in } L^2(\mathbb{T}), \tag{12}$$

$$\rho_n^0 w_n^0 \rightharpoonup \rho^0 w^0 \text{ weakly in } L^2(\mathbb{T}). \tag{13}$$

Then there exists a subsequence (ρ_n, w_n, π_n) of solutions to (4a)-(4b) with initial data (ρ_n^0, w_n^0) which converges to (ρ, w, π) solving:

$$\begin{cases} -\int_0^t \int_{\mathbb{T}} \rho \partial_t \phi \, dxds + \int_{\mathbb{T}} \rho(x,t)\phi(x,t) - \rho^0(x)\phi(x,0) \, dx + \int_0^t \int_{\mathbb{T}} \partial_x \pi \partial_x \phi \, dxds = \int_0^t \int_{\mathbb{T}} \rho w \partial_x \phi \, dxds, \ (14a) \\[2mm] -\int_0^t \int_{\mathbb{T}} \rho w \partial_t \phi \, dxds + \int_{\mathbb{T}} \rho w \phi(x,t) - \rho^0 w^0(x)\phi(x,0) \, dx + \int_0^t \int_{\mathbb{T}} (w \partial_x \pi - \rho w^2)\partial_x \phi \, dxds = 0. \qquad (14b) \\[2mm] 0 \le \rho \le 1, \ (1-\rho)\pi = 0, \ \pi \ge 0. \hspace{7.5cm} (14c) \end{cases}$$

with initial data (ρ^0, w^0). Additionally, our solution possesses the following regularity for $p \in [1, \infty)$ and $\tau \in (0, T)$:

$$\rho \in C_{weak}(0, T; L^p(\mathbb{T})) \cap L^\infty((0,T) \times \mathbb{T}), \tag{15}$$

$$w \in C(0, T; W^{1, \frac{4}{3}}(\mathbb{T})), \tag{16}$$

$$\pi \in \mathcal{M}((0,T) \times \mathbb{T}) \cap L^\infty([\tau, T]; L^1(\mathbb{T})) \cap L^2([\tau, T]; L^2(\mathbb{T})). \tag{17}$$

Remark 1.5. *The upper bound in assumption (6) and the bound on $\sqrt{\rho_n^0}w_n^0$ in (8) are required to obtain important uniform bounds from the additional energy estimate (Lemma 2.4), and also to obtain the bound (11). The second bound appearing in (8) is a crucial part of our argument since it allows us to uniformly bound $\partial_x w_n$ and even $\partial_t w_n$ in $L_{t,x}^p$ spaces. The lower bound on the mean value in (7) is used to apply the Poincaré inequality which gives us a uniform $L_t^\infty L_x^1$ bound on w_n. Lastly, the upper bound assumption in (7) is used to derive the bound (11).*

Remark 1.6. *If we additionally assume that there exist constants $c, C > 0$ independent of n such that*

$$0 < c \le \inf_{\mathbb{T}} \rho_n^0, \quad \operatorname*{ess\,sup}_{\mathbb{T}} \frac{\partial_x w_n^0}{\rho_n^0} \le C,$$

then as a consequence of the proof of Theorem 1.2 we can show that the sequence of densities ρ_n is uniformly bounded away from zero, i.e. $0 < c \le \rho_n$. As a result the density for the limiting system inherits the same lower bound.

Remark 1.7. *Note that the limiting system (14a)-(14c) is slightly different in appearance to that of [7], since our system is written in terms of the desired velocity w rather than the actual velocity u. One can also notice the assumptions on the initial data (6)-(8) are weaker than those appearing in [7], in the sense that we do not assume any control over $\partial_x \pi_n^0$ or the singular quantity $\lambda_n(\rho_n^0)\partial_x u_n^0$. This is made possible through our decision to work with the $w-$formulation rather than the $u-$formulation. It is also worthwhile to mention that if we define the function $u := w - \overline{\rho^{-1}\partial_x \pi}$ where $\overline{\rho^{-1}\partial_x \pi}$ is the weak limit of $\rho_n^{-1}\partial_x \pi_n$, the limiting system is equivalent to (1a)-(1b) (without the index n) in the distributional sense, accompanied by the condition (14c).*

The global existence result given by Theorem 1.2 is an improvement upon the work of [7] where the authors added an approximation term to the system in order to derive an (ϵ dependent) lower bound on the density. Our derivation of the lower bound involves identifying a suitable potential for the Aw-Rascle system $W_n := \partial_x w_n / \rho_n$ and carrying out a maximum-principle argument to deduce that the maximum of W_n is a decreasing function over time. This approach takes inspiration from [9], [5] and shows that an approximation term is not necessary in order to obtain a (uniform) lower bound on the density for a generalised Aw-Rascle system. In fact, this argument also gives us control over the quantity $\partial_x w_n$ which is essential for the proof

of Theorem 1.4. Another difference between our paper and previous works on the Aw-Rascle system [7, 6] is that we pass to the limit in the so-called 'w−formulation'

$$\begin{cases} \partial_t \rho_n + \partial_x(\rho_n w_n - \partial_x \pi_n) = 0, & \text{(18a)} \\ \partial_t(\rho_n w_n) + \partial_x(\rho_n w_n^2) - \partial_x(w_n \partial_x \pi_n) = 0, & \text{(18b)} \end{cases}$$

rather than the 'u−formulation' seen in (4a)-(4b). The existence of a lower bound on the density (obtained in the proof of Theorem 1.2) makes this formulation more convenient, since we may divide by ρ_n in the momentum equation (18b) and directly obtain energy estimates on $\partial_x w_n$. In [7] where the u−formulation was used instead, the authors were required to assume that the singular quantity $\lambda_n \partial_x u_n$ is uniformly bounded at time $t = 0$ to complete the limit passage. We do not need to make such an assumption by working with w_n. It is worthwhile to note that in the distributional sense, our limiting system in w−form is (compare this with (3a)-(3c)):

$$\begin{cases} \partial_t \rho + \partial_x(\rho w - \partial_x \pi) = 0, & \text{(19a)} \\ \partial_t(\rho w) + \partial_x(\rho w^2) - \partial_x(w \partial_x \pi) = 0, & \text{(19b)} \\ 0 \le \rho \le 1, \ (1 - \rho)\pi = 0, \ \pi \ge 0. & \text{(19c)} \end{cases}$$

1.4 Overview of the paper

The paper is comprised as follows. In Sections 2.1/2.2 we make note of a local existence result and three key energy estimates. These allow us to obtain an upper bound of the density after which we prove a 'blow-up' lemma in Section 2.4, which is analogous to what can be seen in Theorem 1.1 of [9]. The blow-up lemma tells us that provided the density is positive on a domain $\mathbb{T} \times [0, T)$, our local solution can be extended past time T. In Section 2.5 we use this result to carry out a maximum-principle argument on the potential $W_n := \partial_x w_n / \rho_n$ in order to show that the density ρ_n is bounded from below on $\mathbb{T} \times [0, T]$. This implies that our solution exists globally, thanks to the blow-up lemma.

The second half of the paper (Sections 3 and 4) is dedicated towards the limit passage. We take advantage of the transport equation satisfied by w_n and W_n to uniformly bound w_n in $L_t^\infty W_x^{1,p}$, which eventually leads to strong convergence on w_n. This is particularly useful when passing to the limit in the non-linear terms appearing in (18b). A key obstacle standing in our way at this point is acquiring a bound on π_n and $\partial_x \pi_n$. Using the same strategy as the authors of [7] which is to test the momentum equation with an antiderivative of the density will not work for us, since our assumptions on the initial data are considerably weaker, and so we have access to fewer uniform bounds. In particular, we have no $L_{t,x}^p$ estimates on the momentum $\rho_n u_n$ at this stage. This prevents us from bounding many of the terms which would appear in the momentum equation after testing. To overcome this, we take a very intricate choice of test function in the continuity equation which gives us a $L_{t,x}^1$ bound on π_n. This delicate estimate is what allows us to converge towards a weak solution even with our relatively weak assumptions on the initial data. The last uniform bound we collect is a slightly stronger local-in-time estimate for π_n that is needed to derive the switching relation $(1 - \rho)\pi = 0$ for the limiting system. The uniform bounds which we obtain for π_n are weaker than those appearing in [7], which is to be expected. Nonetheless we show that it is still possible to obtain a weak solution to the limiting system.

2 Existence of a unique global regular solution for fixed n

2.1 Local existence

The existence of a unique local regular solution for fixed n can be shown in a very similar way to Proposition B.1 in [9]. The full details of the proof are omitted from this paper.

Theorem 2.1 (Existence of a unique local regular solution). *Assume* $(\rho_0, u_0) \in H^k(\mathbb{T}) \times H^k(\mathbb{T})$, $k \ge 1$ *and that* $r_0 := \min_{x \in \mathbb{T}} \rho_0 > 0$. *Then there exists* $T_0 > 0$ *(depending solely on* r_0 *and the initial data) and a unique solution* (ρ, u) *to* (4) *for* $t \in [0, T_0]$ *with initial data* (ρ_0, u_0) *such that*

$$\rho \in C(0, T_0; H^k(\mathbb{T})), \ u \in C(0, T_0; H^k(\mathbb{T})) \cap L^2(0, T_0; H^{k+1}(\mathbb{T})).$$

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Additionally, we have that $\rho(x,t) \geq \frac{r_0}{2}$ *for each* $(x,t) \in \mathbb{T} \times [0,T_0]$.

Starting with some initial data $(\rho_n^0, u_n^0) \in H^4(\mathbb{T}) \times H^4(\mathbb{T})$, we may take $k = 4$ in Theorem 2.1 to obtain the existence of a solution (ρ_n, u_n) to (4a)-(4b) on $[0, T_0]$ for some $T_0 > 0$, where $n \in \mathbb{Z}^+$ is fixed. Let us now denote by T^* the maximal time of existence of our solution (ρ_n, u_n). The purpose of this section is show that our solution can be extended to one that is defined globally in time.

2.2 Energy estimates

In this subsection, we assume that (ρ_n, u_n) is a regular solution to (4) on some time interval $[0, T]$ and that $\rho_n \geq 0$ on $\mathbb{T} \times [0, T]$. Our aim is to establish three uniform in time energy estimates which we will need in order to extend our solution to one that is defined globally in time. Our first two estimates are classical; the first is a consequence of the conservation of mass.

Lemma 2.2 (Conservation of mass). *Assume that* (ρ_n, u_n) *is a regular solution to* (4a)-(4b) *on the time interval* $[0, T]$ *and additionally that* $\rho_n(\cdot, t) \geq 0$ *on this interval. Then,*

$$\|\rho_n(t)\|_{L_x^1} = \underbrace{\|\rho_n^0\|_{L_x^1}}_{=: E_n^0}, \tag{20}$$

for all $t \in [0, T]$.

The next energy estimate is derived from the momentum equation upon multiplying by u_n and integrating by parts.

Lemma 2.3 (Basic energy). *Assume that* (ρ_n, u_n) *is a regular solution to* (4) *on the time interval* $[0, T]$ *and additionally that* $\rho_n(\cdot, t) \geq 0$ *on this interval. Then,*

$$\|\sqrt{\rho_n}u_n\|_{L_t^\infty L_x^2}^2 + 2\|\sqrt{\lambda_n(\rho_n)}\partial_x u_n\|_{L_t^2 L_x^2}^2 = \underbrace{\|\sqrt{\rho_n^0}u_n^0\|_{L_x^2}^2}_{=: E_n^1} \tag{21}$$

for all $t \in [0, T]$.

The final estimate in this section provides us with a bound on $\partial_x p_n(\rho_n)$. Here, it is convenient to introduce the notation

$$H_n(\rho_n) := \frac{1}{\gamma_n + 1}\rho_n^{\gamma_n + 1}. \tag{22}$$

Lemma 2.4 (Additional energy). *Assume that* (ρ_n, u_n) *is a regular solution to* (4) *on the time interval* $[0, T]$ *and additionally that* $\rho_n(\cdot, t) \geq 0$ *on this interval. Then,*

$$\|\sqrt{\rho_n}w_n\|_{L_t^\infty L_x^2}^2 + \|H_n(\rho_n)\|_{L_t^\infty L_x^1} + \frac{1}{2}\|\sqrt{\rho_n}\partial_x p_n(\rho_n)\|_{L_t^2 L_x^2}^2 \leq (T + 2)\underbrace{\left(\|\sqrt{\rho_n^0}w_n^0\|_{L_x^2}^2 + \|H_n(\rho_n^0)\|_{L_x^1}\right)}_{=: E_n^2}. \tag{23}$$

Proof. For regular solutions in the sense of Definition 1.1, (4b) is equivalent to

$$\partial_t(\rho_n w_n) + \partial_x(\rho_n w_n u_n) = 0, \text{ in } \mathbb{T} \times [0, T].$$

Multiplying this equation by w_n and integrating by parts, it is straightforward to show that

$$\|\sqrt{\rho_n}w_n\|_{L_t^\infty L_x^2} \leq \|\sqrt{\rho_n^0}w_n^0\|_{L_x^2}. \tag{24}$$

Substituting $w_n = u_n + \partial_x p_n(\rho_n)$ into the mass equation, we get

$$\partial_t\rho_n + \partial_x(\rho_n w_n) - \partial_x(\rho_n \partial_x p_n(\rho_n)) = 0,$$

in $\mathbb{T} \times [0, T]$. Multiplying this equation by $H_n'(\rho_n) = p_n(\rho_n)$, integrating over $\mathbb{T} \times [0, t]$ where $t \in [0, T]$ and using integration by parts leads to

$$\int_\mathbb{T} H_n(\rho_n(t)) - H_n(\rho_n^0) \, dx + \|\sqrt{\rho_n}\partial_x p_n(\rho_n)\|_{L_t^2 L_x^2}^2 = \int_0^t \int_\mathbb{T} (\partial_x p_n(\rho_n))\rho_n w_n \, dxds. \tag{25}$$

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Using Young's inequality,

$$\int_0^t \int_{\mathbb{T}} (\partial_x p_n(\rho_n))\rho_n w_n \, dxds \leq \frac{1}{2}\|\sqrt{\rho_n}\partial_x p_n(\rho_n)\|^2_{L^2_t L^2_x} + \frac{1}{2}\|\sqrt{\rho_n}w_n\|^2_{L^2_t L^2_x}.$$

Thus, returning to (25) and taking the essential supremum over all $t \in [0, T]$, we have

$$\|H_n(\rho_n)\|_{L^\infty_t L^1_x} + \frac{1}{2}\|\sqrt{\rho_n}\partial_x p_n(\rho_n)\|^2_{L^2_t L^2_x} \leq \|H_n(\rho_n^0)\|_{L^1_x} + \frac{1}{2}\|\sqrt{\rho_n}w_n\|^2_{L^2_t L^2_x}.$$

Adding (24) to both sides of this inequality and using the estimate $\|\sqrt{\rho_n}w_n\|_{L^2_{t,x}} \leq T\|\sqrt{\rho_n}w_n\|_{L^\infty_t L^2_x}$ gives us the final result. $\qquad\square$

2.3 Estimating the density from above

We now obtain an upper bound for the density.

Lemma 2.5. *Suppose (ρ_n, u_n) is a regular solution to (4a)-(4b) on $[0, T]$ and additionally that $\rho_n \geq 0$ on $[0, T]$. Then,*

$$\rho_n(t, x) \leq (C(\gamma_n + 1)(T + 5)E_n)^{\frac{1}{\gamma_n + 1}} =: \overline{\rho_n}, \quad \text{in } \mathbb{T} \times [0, T],$$

where $C > 0$ is independent of n and

$$E_n := E_n^0 + E_n^1 + E_n^2 = \|\rho_n^0\|_{L^1_x} + \|\sqrt{\rho_n^0}u_n^0\|^2_{L^2_x} + (\|\sqrt{\rho_n^0}w_n^0\|^2_{L^2_x} + \|H_n(\rho_n^0)\|_{L^1_x}).$$

Proof. It follows from the energy estimates (21), (23) and the triangle inequality that

$$\|\sqrt{\rho_n}\partial_x p_n(\rho_n)\|^2_{L^\infty_t L^2_x} = \|\sqrt{\rho_n}(w_n - u_n)\|^2_{L^\infty_t L^2_x} \leq \left(\|\sqrt{\rho_n}w_n\|^2_{L^\infty_t L^2_x} + \|\sqrt{\rho_n}u_n\|^2_{L^\infty_t L^2_x}\right)$$
$$\leq (T + 3)E_n, \tag{26}$$

where $C > 0$ denotes an arbitrary constant independent of n and T. Using the definition of p_n, we infer that

$$\partial_x H_n = (\gamma_n)^{-1}\rho_n\partial_x p_n(\rho_n)$$

and so by virtue of Young's inequality and (26),

$$\|\partial_x H_n\|_{L^\infty_t L^1_x} \leq (2\gamma_n)^{-1} \operatorname*{ess\,sup}_{t \in [0,T]} \left(\int_{\mathbb{T}} \rho_n \, dx + \int_{\mathbb{T}} \rho_n(\partial_x p_n(\rho_n))^2 \, dx\right)$$
$$= (2\gamma_n)^{-1} \left(\|\rho_n\|_{L^\infty_t L^1_x} + \|\sqrt{\rho_n}\partial_x p_n(\rho_n)\|^2_{L^\infty_t L^2_x}\right)$$
$$\leq \frac{(T + 4)E_n}{2\gamma_n} \leq (T + 4)E_n \tag{27}$$

for n sufficiently large such that $\gamma_n > 1/2$. We also know from Lemma 2.4 that $\|H_n\|_{L^\infty_t L^1_x} \leq E_n$ and so one can deduce from (27) and the embedding $W_x^{1,1} \hookrightarrow L_x^\infty$ that

$$\|H_n\|_{L^\infty_{t,x}} \leq C(T + 5)E_n,$$

where $C > 0$ arises due to the aforementioned embedding and is a constant independent of n and T. Recalling the form of H_n from (22), this implies

$$\frac{1}{\gamma_n + 1}\rho_n^{\gamma_n + 1} \leq C(T + 5)E_n,$$

from which the result follows. $\qquad\square$

2.4 A blow-up lemma

We wish to prove the following result:

Lemma 2.6 (Criteria for blow-up of regular solutions). *Suppose (ρ_n, u_n) is a regular solution to (4a) - (4b) on $[0, T^*)$ with initial data $(\rho_n^0, u_n^0) \in H^k(\mathbb{T}) \times H^k(\mathbb{T})$ where $k \geq 2$. Then provided that*

$$\underline{\rho_n} := \inf_{t \in [0,T^*)} \min_{x \in \mathbb{T}} \rho_n(t, x) > 0, \tag{28}$$

we have that

$$\sup_{t \in [0,T^*)} \|\rho_n\|_{L^\infty(0,t;H^k)} + \sup_{t \in [0,T^*)} \|u_n\|_{L^\infty(0,t;H^k)} + \sup_{t \in [0,T^*)} \|u_n\|_{L^2(0,t;H^{k+1})} < +\infty, \tag{29}$$

and therefore the solution can be extended to a larger time interval $[0, T)$, where $T > T^*$. In other words, the solution does not lose regularity unless the density reaches 0 somewhere in the domain.

Remark 2.7. *The extension of the solution past T^* can be justified as follows. Thanks to (29), the pair (ρ_n, u_n) admits a limit as $t \nearrow T^*$ and the left-sided derivatives satisfy the system at $t = T^*$. Since we have $\rho_n(T^*, \cdot) > 0$, we can use the local existence result from Theorem 2.1 to obtain a unique solution on $[T^*, T^* + \epsilon)$ for some $\epsilon > 0$. The extension is then given by the concatenation of the solutions on $[0, T^*]$ and $[T^*, T^* + \epsilon)$.*

Our proof is done by induction with respect to the regularity parameter k. The base step corresponds to showing that

$$\sup_{t \in [0,T^*)} \|\rho_n\|_{L^\infty(0,t;H^2)} + \sup_{t \in [0,T^*)} \|u_n\|_{L^\infty(0,t;H^2)} + \sup_{t \in [0,T^*)} \|u_n\|_{L^2(0,t;H^3)} < +\infty$$

provided $(\rho_n^0, u_n^0) \in H^2(\mathbb{T}) \times H^2(\mathbb{T})$. This is the goal of the current subsection. We remark that this assumption on the initial data allows us to deduce from Theorem 2.1 (existence of local solutions) that our solution satisfies

$$\rho_n \in C(0, T^*; H^2(\mathbb{T})), \ u_n \in C(0, T^*; H^2(\mathbb{T})) \cap L^2(0, T^*; H^3(\mathbb{T})),$$

which allows us to justify the computations which will follow. The inductive part of the proof is deferred to Appendix A.

2.4.1 Analysis of the singular diffusion V_n

We first define the function

$$V_n := \lambda_n(\rho_n) \partial_x u_n$$

and establish some basic properties in the form of the next two propositions. The function V_n corresponds to the 'active potential' which was first introduced in [9]. The authors of [9] used the active potential to prove higher order regularity estimates for strong solutions to their system, which shares similarities with the system we are considering for n fixed. The authors of [7] also made use of this function to prove higher-order regularity estimates for strong solutions for the Aw-Rascle system with a singular pressure. We follow a similar procedure. Firstly, let us find the equation satisfied by V_n.

Proposition 2.8 (Equation for the singular diffusion). *Let $n \in \mathbb{N}^+$. Suppose that (ρ_n, u_n) is a regular solution with $k = 2$ (in the sense of Definition (1.1)) to (4a)-(4b) on $\mathbb{T} \times [0, T]$ with initial data $(\rho_n^0, u_n^0) \in H^2(\mathbb{T}) \times H^2(\mathbb{T})$ and $\rho_n > 0$ on $[0, T]$. Then, $V_n := \lambda_n(\rho_n) \partial_x u_n$ satisfies the following equation almost everywhere in $\mathbb{T} \times [0, T]$:*

$$\partial_t V_n + \left(u_n + \frac{\lambda_n(\rho_n)}{\rho_n^2} \partial_x \rho_n \right) \partial_x V_n - \frac{\lambda_n(\rho_n)}{\rho_n} \partial_x^2 V_n = -\frac{(\lambda_n'(\rho_n)\rho_n + \lambda_n(\rho_n))}{(\lambda_n(\rho_n))^2} V_n^2. \tag{30}$$

Proof. For the sake of brevity we refer the reader to Lemma 3.7 of [7] for a complete proof. $\quad\square$

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We now prove a first regularity estimate for V_n. In what follows $\epsilon_i > 0$ will be used to denote the constants arising from an application of Young's inequality where the index i is used to distinguish between the different applications of Young's inequality. We also recall the Sobolev inequality

$$\|u\|_{L^\infty(\mathbb{T})} \leq \|u\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\|\partial_x u\|_{L^2(\mathbb{T})}^{\frac{1}{2}} + \|u\|_{L^2(\mathbb{T})}, \ \forall \ u \in H^1(\mathbb{T}). \tag{31}$$

Under the assumptions of Proposition 2.8, we have the following result.

Proposition 2.9. V_n *satisfies*

$$\|V_n\|_{L_t^\infty L_x^2}^2 + C_3 \|\partial_x V_n\|_{L_{t,x}^2}^2$$

$$\leq T\|V_n(0)\|_{L_x^2}^2 \exp\left(2C_1 T + 2C_2\|V_n\|_{L_{t,x}^2}^2\right)\left(C_1 T + C_2\|V_n(0)\|_{L_x^2}^2\right) =: \mathcal{V}_1 \tag{32}$$

and

$$\|V_n\|_{L_{t,x}^2} \leq (\gamma_n(\overline{\rho_n})^{\gamma_n + 1} E_n)^{\frac{1}{2}}, \tag{33}$$

where C_3 is a positive constant depending on n and

$$C_1 \equiv C_1(\gamma_n, \underline{\rho_n}, \|R\|_{L_t^\infty L_x^2}, E_n), \ C_2 \equiv C_2(\gamma_n, \underline{\rho_n}), \ C_3 \equiv C(\gamma_n, \underline{\rho_n}).$$

Proof. Multiplying (30) by V_n and integrating by parts leads to

$$\frac{1}{2}\frac{d}{dt}\int_\mathbb{T}|V_n|^2 \ dx + \int_\mathbb{T}\frac{\lambda_n(\rho_n)}{\rho_n}(\partial_x V_n)^2 \ dx = -\int_\mathbb{T} V_n \partial_x V_n \frac{\partial_x \lambda_n(\rho_n)}{\rho_n} \ dx - \int_\mathbb{T} u_n V_n \partial_x V_n \ dx$$

$$-\int_\mathbb{T}\frac{\lambda_n'(\rho_n)\rho_n + \lambda_n(\rho_n)}{(\lambda_n(\rho_n))^2}V_n^3 \ dx =: \sum_{i=1}^{3} I_1.$$

Defining $R := \rho_n^{-1}\partial_x \lambda_n(\rho_n)$, a direct computation reveals that $R = \gamma_n(\gamma_n + 1)\partial_x p_n(\rho_n)$, and so by (26)

$$\|R\|_{L_t^\infty L_x^2} \leq \frac{\gamma_n(\gamma_n + 1)}{\sqrt{\underline{\rho_n}}}(T + 3)E_n,$$

where $E_n := E_n^0 + E_n^1 + E_n^2$. We now estimate $I_1 - I_3$. Using the Holder and Sobolev inequalities,

$$I_1 \leq \|R\|_{L_x^2}\|\partial_x V_n\|_{L_x^2}\|V_n\|_{L_x^\infty} \leq \|\partial_x V_n\|_{L_x^2}\left(\|V_n\|_{L_x^2}^{\frac{1}{2}}\|\partial_x V_n\|_{L_x^2}^{\frac{1}{2}} + \|V_n\|_{L_x^2}\right)\|R\|_{L_x^2}.$$

Applying Young's inequality twice then gives us

$$I_1 \leq \epsilon_1\|\partial_x V_n\|_{L_x^2}^2 + \frac{1}{4\epsilon_1}\|R\|_{L_x^2}^2\left(\|V_n\|_{L_x^2}^{\frac{1}{2}}\|\partial_x V_n\|_{L_x^2}^{\frac{1}{2}} + \|V_n\|_{L_x^2}\right)^2$$

$$\leq (\epsilon_1 + \epsilon_2)\|\partial_x V_n\|_{L_x^2}^2 + \frac{1}{16\epsilon_1^2\epsilon_2}\|R\|_{L_x^2}^4\|V_n\|_{L_x^2}^2 + \frac{1}{2\epsilon_1}\|V_n\|_{L_x^2}^2\|R\|_{L_x^2}^2.$$

Next,

$$I_2 \leq \left|\int_\mathbb{T} u_n V_n \partial_x V_n\right| \leq \|u_n\|_{L_x^\infty}\|V_n\|_{L_x^2}\|\partial_x V_n\|_{L_x^2} \leq \epsilon_3\|\partial_x V_n\|_{L_x^2}^2 + \frac{1}{4\epsilon_3}\|u_n\|_{L_x^\infty}^2\|V_n\|_{L_x^2}^2.$$

Due to the Sobolev inequality (31),

$$\|u_n\|_{L_x^\infty}^2 \leq \|u_n\|_{L_x^2}\|\partial_x u_n\|_{L_x^2} + \|u_n\|_{L_x^2}^2 \leq \frac{3}{2}\|u_n\|_{L_x^2}^2 + \frac{1}{2}\|\partial_x u_n\|_{L_x^2}^2$$

$$\leq \frac{3}{2\underline{\rho_n}}\|\sqrt{\rho_n}u_n\|_{L_x^2}^2 + \frac{1}{2}\left\|\frac{1}{\lambda_n(\rho_n)}\right\|_{L_{t,x}^\infty}^2\|V_n\|_{L_x^2}^2 \tag{34}$$

$$\leq \frac{3}{2\underline{\rho_n}}\|\sqrt{\rho_n}u_n\|_{L_x^2}^2 + \gamma_n^{-2}\underline{\rho_n}^{-2-2\gamma_n}\|V_n\|_{L_x^2}^2. \tag{35}$$

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Therefore,

$$I_2 \leq \epsilon_3 \|\partial_x V_n\|_{L_x^2}^2 + (\epsilon_3 \underline{\rho_n})^{-1} E_n \|V_n\|_{L_x^2}^2 + (4\gamma_n^{-1} \epsilon_3 \underline{\rho_n}^{-1-\gamma_n})^{-1} \|V_n\|_{L_x^2}^4.$$

Moving onto I_3, we first remark that

$$\frac{\lambda_n'(\rho_n)\rho_n + \lambda_n(\rho_n)}{(\lambda_n(\rho_n))^2} = \frac{\gamma_n + 2}{\gamma_n \rho_n^{\gamma_n+1}}.$$

Then using the Holder, Sobolev (31) and Young's inequalities,

$$I_3 \leq \frac{\gamma_n + 2}{\gamma_n \underline{\rho_n}^{\gamma_n+1}} \|V_n\|_{L_x^\infty} \|V_n\|_{L_x^2}^2 \leq \frac{\gamma_n + 2}{\gamma_n \underline{\rho_n}^{\gamma_n+1}} \left(\|V_n\|_{L_x^2}^{\frac{5}{2}} \|\partial_x V_n\|_{L_x^2}^{\frac{1}{2}} + \|V_n\|_{L_x^2}^3 \right)$$

$$\leq \frac{\gamma_n + 2}{\gamma_n \underline{\rho_n}^{\gamma_n+1}} \left(\epsilon_4^{-\frac{4}{3}} \|V_n\|_{L_x^2}^{\frac{10}{3}} + \epsilon_4^4 \|\partial_x V_n\|_{L_x^2}^2 + \|V_n\|_{L_x^2}^3 \right).$$

By two further applications of Young's inequality,

$$I_3 \leq \frac{\gamma_n + 2}{\gamma_n \underline{\rho_n}^{\gamma_n+1}} \left(\epsilon_4^{-\frac{4}{3}} \|V_n\|_{L_x^2}^2 + \epsilon_4^4 \|\partial_x V_n\|_{L_x^2}^2 + (\epsilon_4)^{-\frac{16}{3}} \|V_n\|_{L_x^2}^4 \right),$$

where we have used the following observation due to Young's inequality:

$$\|V_n\|^{\frac{10}{3}} = \|V_n\|^2 \|V_n\|^{\frac{4}{3}} \leq \frac{2}{3} \|V_n\|^3 + \frac{1}{3} \|V_n\|^4 \leq \frac{1}{3} \|V_n\|^2 + \frac{2}{3} \|V_n\|^4.$$

Assembling our estimates for $I_1 - I_3$, we have

$$\frac{1}{2} \frac{d}{dt} \|V_n\|_{L_x^2}^2 + \int_{\mathbb{T}} \frac{\lambda_n(\rho_n)}{\rho_n} (\partial_x V_n)^2 \, dx \leq (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4^4) \|\partial_x V_n\|_{L_x^2}^2 + \frac{1}{16\epsilon_1^2 \epsilon_2} \|R\|_{L_x^2}^4 \|V_n\|_{L_x^2}^2 + \frac{1}{2\epsilon_1} \|V_n\|_{L_x^2}^2 \|R\|_{L_x^2}^2$$

$$+ (\epsilon_3 \underline{\rho_n})^{-1} E_n \|V_n\|_{L_x^2}^2 + (4\gamma_n^2 \epsilon_3 \underline{\rho_n}^{-1-\gamma_n})^{-1} \|V_n\|_{L_x^2}^4$$

$$+ \frac{\gamma_n + 2}{\gamma_n \underline{\rho_n}^{\gamma_n+1}} \left(\|V_n\|_{L_x^2}^2 + \epsilon_4^4 \|\partial_x V_n\|_{L_x^2}^2 + (\epsilon_4)^{-\frac{16}{3}} \|V_n\|_{L_x^2}^4 \right).$$

Observing that $\rho_n^{-1} \lambda_n(\rho_n) = \gamma_n \rho_n^{\gamma_n} \geq \gamma_n \underline{\rho_n}^{\gamma_n}$ and simplifying the RHS of the above inequality, we have

$$\frac{1}{2} \frac{d}{dt} \|V_n\|_{L_x^2}^2 + \left(\gamma_n \underline{\rho_n}^{\gamma_n} - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4^4 \right) \|\partial_x V_n\|_{L_x^2}^2$$

$$\leq \|V_n\|_{L_x^2}^2 \left\{ \frac{\|R\|_{L_t^\infty L_x^2}^4}{16\epsilon_1^2 \epsilon_2} + \frac{\|R\|_{L_t^\infty L_x^2}^2}{2\epsilon_1} + \frac{E_n}{\epsilon_3 \underline{\rho_n}} + \frac{\gamma_n + 2}{\gamma_n \underline{\rho_n}^{\gamma_n+1}} + \left[\frac{(\gamma_n + 2)\epsilon_4^{-\frac{16}{3}}}{\gamma_n \underline{\rho_n}^{\gamma_n+1}} + \frac{E_n}{4\gamma_n \epsilon_3 \underline{\rho_n}^{\gamma_n-1}} \right] \|V_n\|_{L_x^2}^2 \right\}.$$

Choosing $\epsilon_i > 0$ small enough (possibly dependent on n), we have

$$\frac{1}{2} \frac{d}{dt} \|V_n\|_{L_x^2}^2 + C_3 \|\partial_x V_n\|_{L_x^2}^2 \leq \|V_n\|_{L_x^2}^2 \left\{ C_1 + C_2 \|V_n\|_{L_x^2}^2 \right\}, \tag{36}$$

where

$$C_1 := \frac{\|R\|_{L_t^\infty L_x^2}^4}{16\epsilon_1^2 \epsilon_2} + \frac{\|R\|_{L_t^\infty L_x^2}^2}{2\epsilon_1} + \frac{E_n}{\epsilon_3 \underline{\rho_n}} + \frac{\gamma_n + 2}{\gamma_n \underline{\rho_n}^{\gamma_n+1}}, \ C_2 := \frac{(\gamma_n + 2)\epsilon_4^{-\frac{16}{3}}}{\gamma_n \underline{\rho_n}^{\gamma_n+1}} + \frac{E_n}{4\gamma_n \epsilon_3 \underline{\rho_n}^{\gamma_n-1}}, \ C_3 := \frac{1}{5} \gamma_n \underline{\rho_n}^{\gamma_n}.$$

An application of Gronwall's inequality yields

$$\frac{1}{2} \|V_n(t)\|_{L_t^\infty L_x^2}^2 \leq \|V_n(0)\|_{L_x^2}^2 \exp\left(C_1 T + C_2 \|V_n\|_{L_{t,x}^2}^2 \right).$$

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Exploiting this inequality in (36) leads to

$$\frac{1}{2}\frac{d}{dt}\|V_n\|_{L_x^2}^2 + C_3\|\partial_x V_n\|_{L_x^2}^2$$

$$\leq \|V_n(0)\|_{L_x^2}^2 \exp\left(C_1 T + C_2\|V_n\|_{L_{t,x}^2}^2\right)\left\{C_1 T + C_2\|V_n(0)\|_{L_x^2}^2 \exp\left(C_1 T + C_2\|V_n\|_{L_{t,x}^2}^2\right)\right\}.$$

Integrating in time and simplifying leads to (32). To finish, we notice that due to (21),

$$\|V_n\|_{L_{t,x}^2}^2 = \|\lambda_n(\rho_n)^2(\partial_x u_n)^2\|_{L_{t,x}^1} \leq \gamma_n \overline{\rho_n}^{\gamma_n+1}\|\lambda_n(\rho_n)(\partial_x u_n)^2\|_{L_{t,x}^1} \leq \gamma_n \overline{\rho_n}^{\gamma_n+1} E_n.$$

$\square$

The previous result allows us to deduce some regularity estimates on u_n and ρ_n. In the remainder of this section we denote by M_k^n an arbitrary function which satisfies

$$M_k^n \equiv M_k^n(t, \|\rho_n\|_{L_t^\infty H_x^k}, \|u_n\|_{L_t^\infty H_x^k}, \|u_n\|_{L_t^2 H_x^{k+1}}, \gamma_n, E_n, \overline{\rho_n}, (\underline{\rho_n})^{-1}), \quad \sup_{t\in[0,T^*)} M_k^n < +\infty.$$

Corollary 2.10. *We have that*

1. $\partial_x u_n$ *is bounded in* $L_t^2 L_x^2$ *with*

$$\|\partial_x u_n\|_{L_t^2 L_x^2}^2 \leq \frac{E_n}{\gamma_n \underline{\rho_n}^{\gamma_n+1}}. \tag{37}$$

2. $\partial_x u_n$ *is bounded in* $L_t^\infty L_x^2$ *with*

$$\|\partial_x u_n\|_{L_t^\infty L_x^2} \leq \frac{\mathcal{V}_1}{\gamma_n \underline{\rho_n}^{\gamma_n+1}} \leq M_0^n. \tag{38}$$

3. $\partial_x \rho_n$ *is bounded in* $L_t^\infty L_x^2$ *with*

$$\|\partial_x \rho_n\|_{L_t^\infty L_x^2} \leq \gamma_n^{-1}\underline{\rho_n}^{\frac{1}{2}-\gamma_n} E_n \leq M_0^n. \tag{39}$$

4. $\partial_x^2 u_n$ *is bounded in* $L_t^2 L_x^2$ *with*

$$\|\partial_x^2 u_n\|_{L_t^2 L_x^2} \leq \frac{1}{\gamma_n \underline{\rho_n}^{\gamma_n+1}}\left(\|\partial_x V_n\|_{L_t^2 L_x^2} + \|\partial_x \lambda_n(\rho_n)\|_{L_t^\infty L_x^2}\|\partial_x u_n\|_{L_t^2 L_x^2}^{\frac{1}{2}}\left(\|\partial_x^2 u_n\|_{L_t^2 L_x^2}^{\frac{1}{2}} + \|\partial_x V_n\|_{L_t^2 L_x^2}\right)\right)$$

$$\leq M_0^n \tag{40}$$

Proof. The first bound follows from (21). For the second bound, we use the relationship $V_n = \lambda_n(\rho_n)\partial_x u_n$ to estimate

$$\|\partial_x u_n\|_{L_t^\infty L_x^2} \leq \left\|\frac{1}{\lambda_n(\rho_n)}\right\|_{L_{t,x}^\infty}\|V_n\|_{L_t^\infty L_x^2} \leq \frac{1}{\gamma_n \underline{\rho_n}^{\gamma_n+1}}\|V_n\|_{L_t^\infty L_x^2} \leq \frac{\mathcal{V}_1}{\gamma_n \underline{\rho_n}^{\gamma_n+1}}.$$

For the third estimate, we first note that from (21) and (23), we have

$$\|\sqrt{\rho_n}\partial_x p_n(\rho_n)\|_{L_t^\infty L_x^2} = \|\sqrt{\rho_n}(w_n - u_n)\|_{L_t^\infty L_x^2} \leq \|\sqrt{\rho_n}w_n\|_{L_t^\infty L_x^2} + \|\sqrt{\rho_n}u_n\|_{L_t^\infty L_x^2}$$

$$\leq \|\sqrt{\rho_n^0}u_n^0\|_{L_x^2}^2 + \|\sqrt{\rho_n^0}w_n^0\|_{L_x^2}^2 \leq E_n. \tag{41}$$

On the other hand,

$$\|\sqrt{\rho_n}\partial_x p_n(\rho_n)\|_{L_t^\infty L_x^2} = \|\gamma_n\rho_n^{\gamma_n-\frac{1}{2}}\partial_x \rho_n\|_{L_t^\infty L_x^2} \geq \gamma_n\underline{\rho_n}^{\gamma_n-\frac{1}{2}}\|\partial_x \rho_n\|_{L_t^\infty L_x^2}. \tag{42}$$

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The result follows directly from (41) and (42). In order to deduce the fourth bound, we differentiate $V_n = \lambda_n(\rho_n)\partial_x u_n$ to find

$$\partial_x^2 u_n = \frac{\partial_x V_n - \partial_x \lambda_n(\rho_n)\partial_x u_n}{\lambda_n(\rho_n)}.$$

Thus, we have

$$\|\partial_x^2 u_n\|_{L_t^2 L_x^2} \leq \left\|\frac{1}{\lambda_n(\rho_n)}\right\|_{L_{t,x}^\infty} \left(\|\partial_x V_n\|_{L_t^2 L_x^2} + \|\partial_x \lambda_n \partial_x u_n\|_{L_t^2 L_x^2}\right).$$

Noticing that $\|\partial_x \lambda_n \partial_x u_n\|_{L_t^2 L_x^2} \leq \|\partial_x \lambda_n(\rho_n)\|_{L_t^\infty L_x^2}\|\partial_x u_n\|_{L_t^2 L_x^\infty}$ allows us to further deduce using the Sobolev inequality (31) that

$$\|\partial_x^2 u_n\|_{L_t^2 L_x^2} \leq \frac{1}{\gamma_n \underline{\rho_n}^{\gamma_n+1}} \left(\|\partial_x V_n\|_{L_t^2 L_x^2} + \|\partial_x \lambda_n(\rho_n)\|_{L_t^\infty L_x^2}(\|\partial_x u_n\|_{L_t^2 L_x^2}^{\frac{1}{2}} + \|\partial_x^2 u_n\|_{L_t^2 L_x^2}^{\frac{1}{2}} + \|\partial_x u_n\|_{L_t^2 L_x^2})\right).$$

A Gronwall argument can then be used to bound this quantity by M_0^n. $\qquad\square$

Remark 2.11. *It follows from the energy estimates* (20), (21) *and* (23) *and the upper/lower bounds on the density that* $\|\rho_n\|_{L_t^\infty L_x^2} + \|u_n\|_{L_t^\infty L_x^2} + \|u_n\|_{L_t^2 H_x^1} \leq C$, *where C is independent of time. Combining this with Corollary 2.10, the estimates we have acquired up to this point can be summarised as:*

$$\|\rho_n\|_{L_t^\infty H_x^1} + \|u_n\|_{L_t^\infty H_x^1} + \|u_n\|_{L_t^2 H_x^2} \leq M_0^n. \tag{43}$$

2.4.2 Completing the base step of induction

To complete the base step we need to show that

$$\|\partial_x^2 \rho_n\|_{L_t^\infty L_x^2} + \|\partial_x^2 u_n\|_{L_t^\infty L_x^2} + \|\partial_x^3 u_n\|_{L_{t,x}^2} < M_1^n. \tag{44}$$

We now make note of a result which will allow us to deduce the remaining higher-order regularity estimates. We assume ρ_n, u_n are smooth for the sake of convenience.

Proposition 2.12. *(Higher order regularity formulae) Let $k \geq 2$, $\rho_n \in C^1(0,T;C^k(\mathbb{T}))$, $u_n \in C(0,T;C^{k+1}(\mathbb{T}))$ satisfy* (4a). *Then,*

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^k \rho_n\|_{L_x^2}^2 \leq C\left(\|\partial_x^k u_n\|_{L_x^2}\|\partial_x^k \rho_n\|_{L_x^2}^2 + \|\rho_n\|_{L_x^\infty}\|\partial_x^k \rho_n\|_{L_x^2}\|\partial_x^{k+1} u_n\|_{L_x^2}\right) \tag{45}$$

Additionally, for $l \geq 1$ and $\rho_n \in C(0,T;C^{l+1}(\mathbb{T}))$, $u_n \in C(0,T;C^l(\mathbb{T}))$, $V_n \in C^1(0,T;C^{l+1}(\mathbb{T}))$ satisfying (30), *we have*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_x^l V_n|^2 \, dx = -\int_{\mathbb{T}}\partial_x^l\left(\left(u_n + \frac{\lambda_n(\rho_n)}{\rho_n^2}\partial_x\rho_n\right)\partial_x V_n\right)\partial_x^l V_n \, dx + \int_{\mathbb{T}}\partial_x^l\left(\frac{\lambda_n(\rho_n)}{\rho_n}\partial_x^2 V_n\right)\partial_x^l V_n$$

$$-\int_{\mathbb{T}}\partial_x^l\left(\frac{\lambda_n'(\rho_n)\rho_n + \lambda_n(\rho_n)}{(\lambda_n(\rho_n))^2}V_n^2\right)\partial_x^l V_n \, dx. \tag{46}$$

Proof. The proof is identical to that of Lemma 3.10 in [7]. $\qquad\square$

Taking $l = 1$ in (46), one can integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_x V_n|^2 \, dx + \int_{\mathbb{T}}\frac{\lambda_n(\rho_n)}{\rho_n}|\partial_x^2 V_n|^2 \, dx$$

$$= \int_{\mathbb{T}}\left(u_n + \frac{\lambda_n(\rho_n)}{\rho_n^2}\partial_x\rho_n\right)\partial_x V_n\partial_x^2 V_n \, dx + \int_{\mathbb{T}}\frac{\lambda_n'(\rho_n)\rho_n + \lambda_n(\rho_n)}{(\lambda_n(\rho_n))^2}V_n^2\partial_x^2 V_n \, dx =: J_1 + J_2.$$

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Due to (43), we have

$$J_1 \leq \|u_n + \rho_n^{-2}\lambda_n(\rho_n)\partial_x\rho_n\|_{L_x^2}\|\partial_x V_n\|_{L_x^\infty}\|\partial_x^2 V_n\|_{L_x^2}$$

$$\leq M_1^n\|\partial_x V_n\|_{L_x^\infty}\|\partial_x^2 V_n\|_{L_x^2}.$$

Using the Sobolev inequality (31) and Young's inequality, we arrive at

$$J_1 \leq \epsilon\|\partial_x^2 V_n\|_{L_x^2}^2 + M_1^n\|\partial_x V_n\|_{L_x^2}^2 \leq \epsilon\|\partial_x^2 V_n\|_{L_x^2}^2 + M_1^n.$$

Meanwhile,

$$J_2 \leq M_1^n\|V_n\|_{L_x^4}^2\|\partial_x^2 V_n\|_{L_x^2} \leq M_1^n + \epsilon\|\partial_x^2 V_n\|_{L_x^2}^2,$$

using Young's inequality and Proposition 2.9. Therefore we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_x V_n|^2\ dx + \int_{\mathbb{T}}\frac{\lambda_n(\rho_n)}{\rho_n}|\partial_x^2 V_n|^2\ dx \leq 2\epsilon\|\partial_x^2 V_n\|_{L_x^2}^2 + M_1^n$$

Using the observation $\rho_n^{-1}\lambda_n(\rho_n) = \gamma_n\rho_n^{\gamma_n} \geq \gamma_n\underline{\rho_n}^{\gamma_n}$ and choosing ϵ small enough (depending on n), we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_x V_n|^2\ dx + C\|\partial_x^2 V_n\|_{L_x^2}^2 \leq M_1^n, \tag{47}$$

where $C > 0$ is a constant depending on n. Integrating in time gives us

$$\|\partial_x V_n\|_{L_t^\infty L_x^2} + \|\partial_x^2 V_n\|_{L_{t,x}^2} \leq M_1^n. \tag{48}$$

Next, from the relationship $V_n = \lambda_n\partial_x u_n$, we infer that

$$\partial_x^2 u_n = \lambda_n^{-1}\partial_x V_n + V_n\partial_x(\lambda_n^{-1}) \tag{49}$$

and so

$$\|\partial_x^2 u_n\|_{L_t^\infty L_x^2} \leq \|\lambda_n^{-1}\|_{L_{t,x}^\infty}\|\partial_x V_n\|_{L_t^\infty L_x^2} + \|V_n\|_{L_{t,x}^\infty}\|\partial_x(\lambda_n^{-1})\|_{L_t^\infty L_x^2} \leq M_1^n. \tag{50}$$

Differentiating (49) once more,

$$\partial_x^3 u_n = \partial_x^2 V_n\lambda_n^{-1} + 2\partial_x V_n\partial_x\lambda_n^{-1} + V_n\partial_x^2\lambda_n^{-1}$$

and again we estimate

$$\|\partial_x^3 u_n\|_{L_{t,x}^2} \leq \|\partial_x^2 V_n\|_{L_{t,x}^2}\|\lambda_n^{-1}\|_{L_{t,x}^\infty} + 2\|\partial_x V_n\|_{L_t^\infty L_x^2}\|\partial_x\lambda_n^{-1}\|_{L_t^2 L_x^\infty} + \|V_n\|_{L_{t,x}^\infty}\|\partial_x^2\lambda_n^{-1}\|_{L_{t,x}^2}. \tag{51}$$

Note that $\partial_x\lambda_n^{-1} = -(1 + \gamma_n^{-1})\rho_n^{-\gamma_n-2}\partial_x\rho_n$ and so $\|\partial_x\lambda_n^{-1}\|_{L_t^2 L_x^\infty} \leq M_1^n\|\partial_x\rho_n\|_{L_t^2 L_x^\infty}$. Using the Sobolev inequality (31) we can show that

$$\|\partial_x\rho_n\|_{L_t^2 L_x^\infty} \leq M_1^n + M_1^n\|\partial_x^2\rho_n\|_{L_{t,x}^2}.$$

By computing $\partial_x^2\lambda_n^{-1}$ explicitly one can also show that $\|\partial_x^2\lambda_n^{-1}\|_{L_{t,x}^2} \leq M_1^n + M_1^n\|\partial_x^2\rho_n\|_{L_{t,x}^2}$. Returning to (51), we have

$$\|\partial_x^3 u_n\|_{L_{t,x}^2} \leq M_1^n + M_1^n\|\partial_x^2\rho_n\|_{L_{t,x}^2}. \tag{52}$$

Taking $k = 2$ in (45), we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^2\rho_n\|_{L_x^2}^2 \leq C\left(\|\partial_x^2 u_n\|_{L_x^2}\|\partial_x^2\rho_n\|_{L_x^2}^2 + \|\rho_n\|_{L_x^\infty}\|\partial_x^2\rho_n\|_{L_x^2}\|\partial_x^3 u_n\|_{L_x^2}\right)$$

$$\leq M_1^n\|\partial_x^2\rho_n\|_{L_x^2},$$

using (52) and the bounds previously obtained. An application of Gronwall's inequality grants us $\|\partial_x^2\rho_n\|_{L_t^\infty L_x^2} \leq M_1^n$. In light of this, the inequality (52) finally gives us $\|\partial_x^3 u_n\|_{L_{t,x}^2} \leq M_1^n$. Additionally recalling the bound (50), we have shown that (44) holds true. Thus we have

$$\|\rho_n\|_{L_t^\infty H_x^2} + \|u_n\|_{L_t^\infty H_x^2} + \|u_n\|_{L_t^2 H_x^3} \leq M_1^n, \tag{53}$$

where M_1^n satisfies $\sup_{t\in[0,T^*)} M_1^n < +\infty$. Thus the base step is complete. The inductive step is deferred to the Appendix. With this, we conclude the proof of Lemma 2.6.

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2.5 Completing the proof of global existence

We are now ready to complete the proof of Theorem 1.2 and obtain the existence of a global unique regular solution.

Proof of Theorem 1.2. We have previously asserted the existence of a unique global regular solution (ρ_n, u_n) to (4a)-(4b) on $[0, T_0]$, for some $T_0 > 0$ in Theorem 2.1. The blow-up result we obtained in the previous subsection (Lemma 2.6) also tells us that our solution can in fact be extended past T_0 provided the density ρ_n does not hit 0 anywhere in $\mathbb{T} \times [0, T_0]$. Since we know that $\rho_n \geq \frac{r_0}{2} > 0$ on $\mathbb{T} \times [0, T_0]$, we can extend this solution and declare that (ρ_n, u_n) will possess a maximal interval of existence $[0, T^*)$, where $T^* > T_0$. For the sake of a contradiction, let us assume that T^* is finite. Two particular implications of this statement are that

$$\rho_n(\cdot, t) > 0 \text{ for each } t \in [0, T^*), \tag{54}$$

and

$$\inf_{t \in [0, T^*)} \min_{x \in \mathbb{T}} \rho_n(t, x) = 0. \tag{55}$$

Let us now recall the original form of the Aw-Rascle system on our maximal interval of existence

$$\begin{cases} \partial_t \rho_n + \partial_x(\rho_n u_n) = 0, & \text{in } \mathbb{T} \times [0, T^*), & (56a) \\ \partial_t(\rho_n w_n) + \partial_x(\rho_n w_n u_n) = 0, & \text{in } \mathbb{T} \times [0, T^*). & (56b) \end{cases}$$

Since we know that (ρ_n, u_n) possesses the same regularity as given in the statement of this theorem, we can proceed in a similar way to [5] and [9] and define

$$W_n = \frac{\partial_x w_n}{\rho_n}. \tag{57}$$

We wish to show that W_n is bounded from above uniformly in time. It will be seen that this is sufficient to obtain a lower bound on ρ_n, which will become clear when we derive the evolution equation for $1/\rho_n$ and apply a maximum-principle argument. Our W_n corresponds to the so-called 'effective pressure' used by Burtea and Haspot in [5] and the function 'X' by Constantin et al in Theorem 1.5, [9]. A straightforward computation reveals that

$$\rho_n^2 \partial_t W_n = \rho_n \partial_t \partial_x w_n - \partial_x w_n \partial_t \rho_n. \tag{58}$$

Exploiting (56a), dividing by $\rho_n > 0$ and differentiating in space, one can show that (56b) is equivalent to

$$\partial_t \partial_x w_n = -\partial_x u_n \partial_x w_n - u_n \partial_x^2 w_n. \tag{59}$$

Substituting (59) and (56a) into (58) and dividing by $\rho_n > 0$ gives

$$\rho_n \partial_t W_n = -\rho_n W_n \partial_x u_n - u_n \partial_x^2 w_n - \frac{\partial_x w_n \partial_t \rho_n}{\rho_n}$$

$$= -\rho_n \partial_x w_n W_n - u_n \partial_x \rho_n W_n - u_n \rho_n \partial_x W_n + W_n(\rho_n \partial_x u_n + u_n \partial_x \rho_n)$$

$$= -u_n \rho_n \partial_x W_n. \tag{60}$$

where we have used the facts that $\partial_x w_n = W_n \rho_n$ and $\partial_x^2 w_n = W_n \partial_x \rho_n + \rho_n \partial_x W_n$, which both follow from (57). Dividing by ρ_n leaves us with

$$\partial_t W_n + u_n \partial_x W_n = 0. \tag{61}$$

It is clear that w_n and W_n satisfy the same transport equation. Now, we note that $w_n \in C(0, T; H^3)$ due to the relation $w_n = u_n + \partial_x p_n(\rho_n)$ and the regularity $\rho_n, u_n \in C(0, T; H^4)$ for all $T < T^*$. This allows us to deduce that $W_n \in C(0, T; H^2) \subset C([0, T] \times \mathbb{T})$ for all $T < T^*$. The equation (61) then implies that $\partial_t W_n \in C(0, T; H^1) \subset C([0, T] \times \mathbb{T})$ and so $W_n \in C^1([0, T] \times \mathbb{T})$ for all $T < T^*$. We now introduce a maximum function $W_n^M : \mathbb{R}^+ \to \mathbb{R}$ associated to W_n. For each $t \in [0, T^*)$, it is true that $W_n(\cdot, t)$ attains a

maximum at some point $x_t \in \mathbb{T}$, thanks to the regularity of W_n on $\mathbb{T} \times [0, T^*)$. Thus, the map $t \to x_t$ is well-defined on $[0, T^*)$ and the function

$$W_n^M(t) := W_n(x_t, t) \equiv \max_{x \in \mathbb{T}} W_n(x, t) \tag{62}$$

is also well-defined and Lipschitz continuous. The Rademacher theorem implies that W_n^M is differentiable almost everywhere on $[0, T^*)$. We now wish to prove that

$$\left(W_n^M\right)'(t) = \partial_t W_n(x_t, t) \tag{63}$$

for all $t \in [0, T^*)$ where W_n^M is differentiable. Using the classical definition of the derivative, we have for all t where W_n^M is differentiable,

$$\left(W_n^M\right)'(t) = \lim_{h \searrow 0} \left[\frac{W_n^M(t+h) - W_n^M(t)}{h} \right] = \lim_{h \searrow 0} \left[\frac{W_n(x_{t+h}, t+h) - W_n(x_t, t))}{h} \right]$$

$$\geq \lim_{h \searrow 0} \left[\frac{W_n(x_t, t+h) - W_n(x_t, t)}{h} \right] = \partial_t W_n(x_t, t).$$

On the other hand, using an alternative definition of the derivative, we have

$$\left(W_n^M\right)'(t) = \lim_{h \searrow 0} \left[\frac{W_n^M(t) - W_n^M(t-h)}{h} \right] = \lim_{h \searrow 0} \left[\frac{W_n(x_t, t) - W_n(x_{t-h}, t-h)}{h} \right]$$

$$\leq \lim_{h \searrow 0} \left[\frac{W_n(x_t, t) - W_n(x_t, t-h)}{h} \right] = \partial_t W_n(x_t, t),$$

and so (63) holds true. Evaluating (61) along the points (x_t, t) which correspond to the maximum points of W_n, we have

$$\partial_t W_n^M(t) = -u_n(x_t, t) \underbrace{\partial_x W_n(x_t, t)}_{=0} = 0, \tag{64}$$

for all $t \in [0, T^*)$ where W_n^M is differentiable, since x_t is a maximum point of $W_n(\cdot, t)$. This tells us that

$$\max_{x \in \mathbb{T}} \frac{\partial_x w_n}{\rho_n}(x, t) = \max_{x \in \mathbb{T}} \frac{\partial_x w_n^0}{\rho_n^0}(x)$$

for any $t \in [0, T^*)$, and as a consequence

$$\left(\frac{\partial_x w_n}{\rho_n} \right)(x, t) \leq \max_{\mathbb{T}} \frac{\partial_x w_n^0}{\rho_n^0} =: \mathcal{M}_n^0, \ \forall \ (x, t) \in \mathbb{T} \times [0, T^*). \tag{65}$$

We note that since we are working with periodic boundary conditions, the quantity $\max_{\mathbb{T}} \partial_x w_n^0$ is non-negative and therefore so is $\mathcal{M}_n^0$ due to our assumption $\rho_n^0 > 0$ on the initial data. Next, we turn to the evolution equation satisfied by $1/\rho_n$ on $\mathbb{T} \times [0, T^*)$. Since $\rho_n \neq 0$ on this domain, the function $1/\rho_n$ possesses the same regularity as ρ_n and it is straightforward to verify using the continuity equation that

$$\partial_t \left(\frac{1}{\rho_n} \right) + u_n \partial_x \left(\frac{1}{\rho_n} \right) = \frac{1}{\rho_n} \partial_x u_n. \tag{66}$$

Using the definition of w_n, namely that $w_n = u_n + \partial_x p_n(\rho_n)$, this becomes

$$\partial_t \left(\frac{1}{\rho_n} \right) = -u_n \partial_x \left(\frac{1}{\rho_n} \right) + \frac{1}{\rho_n} \partial_x w_n - \frac{1}{\rho_n} \left[p_n''(\rho_n) |\partial_x \rho_n|^2 + p_n'(\rho_n) \partial_x^2 \rho_n \right]$$

$$= -u_n \partial_x \left(\frac{1}{\rho_n} \right) + \frac{1}{\rho_n} \partial_x w_n - \frac{p_n''(\rho_n)}{\rho_n} |\partial_x \rho_n|^2 + p_n'(\rho_n) \left[-2\rho_n^2 \partial_x \left(\frac{1}{\rho_n} \right)^2 + \rho_n \partial_x^2 \left(\frac{1}{\rho_n} \right) \right], \tag{67}$$

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where we have used the relations

$$\partial_x \rho_n = -\rho_n^2 \partial_x \left(\frac{1}{\rho_n} \right), \quad \partial_x^2 \rho_n = 2\rho_n^3 \partial_x \left(\frac{1}{\rho_n} \right)^2 - \rho_n^2 \partial_x^2 \left(\frac{1}{\rho_n} \right).$$

In a similar way to the previous argument involving W_n, we can define $P_n(x,t) := \frac{1}{\rho_n(x,t)}$ and the corresponding maximum function

$$P_n^M(t) := \max_{x \in \mathbb{T}} \frac{1}{\rho_n(x,t)} \text{ on } [0, T^*).$$

One can repeat the argument for W_n^M to conclude that P_n^M is differentiable a.e. on $[0, T^*)$. We then have from equation (67) that for all $t \in [0, T^*)$ where P_n^M is differentiable,

$$\partial_t P_n^M = -u_n \partial_x P_n^M + \rho_n^{-1} \partial_x w_n - P_n^M p_n''(\rho_n) |\rho_n^2 \partial_x P_n^M|^2 - 2p_n'(\rho_n) \rho_n^2 \left(\partial_x P_n^M \right)^2 + \rho_n p_n'(\rho_n) \partial_x^2 P_n^M. \quad (68)$$

Using the facts that $\rho_n(\cdot, t) > 0$ on $[0, T^*)$, $\partial_x P_n^M(t) = 0$ and $\partial_x^2 P_n^M \leq 0$, this equation implies the inequality

$$(\partial_t P_n^M)(t) \leq \frac{\partial_x w_n}{\rho_n}(x, t) \leq \mathcal{M}_n^0, \quad \text{for a.e. } t \in [0, T^*), \quad (69)$$

where we have made use of (65). Integrating in time gives us

$$\rho_n(x, t) \geq \frac{1}{\mathcal{M}_n^0 t + (\inf_{\mathbb{T}} \rho_n^0)^{-1}}. \quad (70)$$

This lower bound contradicts our assumption (55) which was that

$$\inf_{t \in [0, T^*)} \min_{x \in \mathbb{T}} \rho_n(t, x) = 0.$$

We conclude that $T^* = +\infty$ and thus our solution is global in time. The uniqueness follows from the fact that our local solution was shown to be unique. Indeed, suppose that we have two global regular solutions (ρ_n^1, u_n^1) and (ρ_n^2, u_n^2) coinciding on $[0, T_1]$ and further assume that they differ almost everywhere on some interval $[T_1, T_2]$, i.e. that $\|\rho_n^1 - \rho_n^2\|_{L^\infty(T_1, T; L^\infty)} > 0$ for all $T \in (T_1, T_2)$ (and analogously for u_n^i). Then taking the initial data $(\rho_n^0, u_n^0) := (\rho_n^1(T_1), u_n^1(T_1))$, we deduce from Theorem 2.1 that $(\rho_n^1, u_n^1) = (\rho_n^2, u_n^2)$ on an interval $[T_1, T_1 + \epsilon]$ for some $\epsilon > 0$, which contradicts our assumption. $\qquad \square$

Remark 2.13. *If we assume the existence of positive constants $C, \underline{\rho}^0$ independent of n such that* $\operatorname{ess\,sup}_{\mathbb{T}} \frac{\partial_x w_n^0}{\rho_n^0} \leq C$ *and* $\rho_n^0 \geq \underline{\rho}^0 > 0$, *then the bound (70) gives us the uniform (in n) estimate*

$$\rho_n(x, t) \geq \frac{1}{Ct + (\underline{\rho}^0)^{-1}}.$$

Remark 2.14. *Our proof does not rely on the precise form of the offset function p_n, but it does require that $p_n(r)$ is increasing for $r \in [0, \overline{\rho_n}]$. Thus we expect this result to hold for a wide class of offset functions. In particular, our argument works for the choice $p_\epsilon(\rho_\epsilon) = \epsilon \rho_\epsilon^\gamma (1 - \rho_\epsilon)^{-\beta}$ taken by Chaudhuri et al in [7].*

3   Uniform in n estimates

In this section we obtain uniform estimates on the global regular solution (ρ_n, u_n), whose existence was asserted in the previous section.

3.1   Bounds from the energy estimates

Assuming the condition (6) and the first bound of (8) on the initial data, we have the following uniform estimate by virtue of our energy estimates.

$$\|\rho_n\|_{L_t^\infty L_x^1} + \|\sqrt{\rho_n} w_n\|_{L_t^\infty L_x^2} + \|H_n(\rho_n)\|_{L_t^\infty L_x^1} + \|\sqrt{\rho_n} \partial_x p_n(\rho_n)\|_{L_t^2 L_x^2} \leq C. \quad (71)$$

Note that the assumption (6) implies that $\|H_n(\rho_n^0)\|_{L_x^1}$ is uniformly bounded in n. Thus the estimate from Lemma 2.4 is also independent of n.

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3.2   Bounding the desired velocity

The goal of this section is to obtain the uniform bound (9) on the sequence of desired velocities, w_n. In the remainder of the paper we assume the hypotheses (6)-(8) and denote by C a positive constant independent of n. Our first result is a bound on the spatial derivative of w_n.

Proposition 3.1. *Provided $\gamma_n \geq 1$ we have*

$$\|\partial_x w_n\|_{L_t^\infty L_x^{4/3}} \leq C. \tag{72}$$

Proof. It follows from (61) that the potential $W_n = \rho_n^{-1} \partial_x w_n$ satisfies

$$\partial_t(\rho_n W_n) + \partial_x(\rho_n W_n u_n) = 0, \tag{73}$$

due to the continuity equation. Thus (ρ_n, W_n, u_n) satisfies the original momentum equation (1b). Multiplying by W_n and integrating by parts over space and time leads to

$$\int_{\mathbb{T}} \frac{(\partial_x w_n)^2}{\rho_n}(x,t) \ dx = \int_{\mathbb{T}} \frac{(\partial_x w_n^0)^2}{\rho_n^0}(x) \ dx =: \mathcal{D}_n^0. \tag{74}$$

Noticing that by Young's inequality

$$|\partial_x w_n|^{4/3} = \left|\frac{\partial_x w_n}{\sqrt{\rho_n}}\right|^{4/3} |\sqrt{\rho_n}|^{4/3} \leq \frac{2}{3}\left|\frac{\partial_x w_n}{\sqrt{\rho_n}}\right|^2 + \frac{1}{3}|\rho_n|^2,$$

we infer from (74) that

$$\|\partial_x w_n\|_{L_t^\infty L_x^{4/3}}^{3/4} \leq \mathcal{D}_n^0 + \|\rho_n\|_{L_t^\infty L_x^2}^2. \tag{75}$$

Next, recall from (71) we have that

$$\|H_n(\rho_n)\|_{L_t^\infty L_x^1} = \frac{1}{\gamma_n + 1} \operatorname*{ess\,sup}_{t \in [0,T]} \int_{\mathbb{T}} \rho_n^{\gamma_n+1} \ dx = \frac{1}{\gamma_n + 1}\|\rho_n\|_{L_t^\infty L_x^{\gamma_n+1}}^{\gamma_n+1} \leq C. \tag{76}$$

Therefore the assumption $\gamma_n \geq 1$ guarantees that $\|\rho_n\|_{L_t^\infty L_x^2} \leq C$ (uniformly in n). Returning to (75), we can obtain the required bound. $\qquad\square$

Remark 3.2. *The exponent $4/3$ is taken for convenience. A similar argument can be used to show $\|\partial_x w_n\|_{L_t^\infty L_x^p} \leq C$ for any $p \in [1,2)$, assuming γ_n is sufficiently large.*

We now obtain a bound for the sequence of desired velocities w_n.

Proposition 3.3. *The desired velocity w_n satisfies*

$$\|w_n\|_{L_t^\infty W_x^{1,\frac{4}{3}}} \leq C.$$

Proof. Taking $r = \rho_n$ in the generalised Poincare inequality (see Proposition A.2 in the Appendix) gives us

$$\|w_n\|_{L_x^1} \leq C\left(\|\partial_x w_n\|_{L_x^1} + \int_{\mathbb{T}} \rho_n w_n \ dx\right). \tag{77}$$

Using Young's inequality and Proposition 3.4,

$$\|\rho_n w_n\|_{L_t^\infty L_x^1} \leq \|\sqrt{\rho_n} w_n\|_{L_t^\infty L_x^2} + \|\rho_n\|_{L_t^\infty L_x^1} \leq C.$$

Then taking the supremum over time in (77) and using Proposition 3.1 gives us $\|w_n\|_{L_t^\infty L_x^1} \leq C$. Combining this estimate with Proposition 3.1 leads to the result.

$\qquad\square$

We now make note of some bounds which will be used to prove the remaining estimates in this section.

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Corollary 3.4. *Provided $\gamma_n \geq 1$ we have*

$$\|\sqrt{\rho_n} u_n\|_{L^2_{t,x}} + \|\rho_n\|_{L^\infty_t L^2_x} \leq C, \tag{78}$$

$$\|w_n\|_{L^\infty_{t,x}} + \|\rho_n w_n\|_{L^2_{t,x}} \leq C, \tag{79}$$

Proof. The first estimate in (78) follows from (71) and the relationship $\sqrt{\rho_n} u_n = \sqrt{\rho_n} w_n - \sqrt{\rho_n}\partial_x p_n(\rho_n)$, while the second was obtained from (76). The $L^\infty_{t,x}$ bound on w_n follows from the embedding $W^{1,1}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. For the final bound, we have $\|\rho_n w_n\|_{L^2_{t,x}} \leq t\|w_n\|_{L^\infty_{t,x}}\|\rho_n\|_{L^\infty_t L^2_x} \leq C$, thanks to (78). $\square$

3.3 An improved estimate for the potential

We now obtain a uniform bound on the potential π_n which is defined by the relation (5). It will be useful to note that

$$\pi_n(\rho_n) = \frac{\gamma_n}{\gamma_n + 1}\rho_n^{\gamma_n+1}. \tag{80}$$

Lemma 3.5. *The potential π_n satisfies*

$$\|\partial_x \pi_n\|_{L^2_{t,x}} \leq C. \tag{81}$$

Proof. First note that the continuity equation can be expressed as

$$\partial_t \rho_n + \partial_x(\rho_n w_n) = \partial_x^2 \pi_n. \tag{82}$$

Multiplying by π_n and integrating over $[0,t] \times \mathbb{T}$ and by parts, we find

$$\frac{1}{\gamma_n + 2}\int_{\mathbb{T}} -\rho_n^{\gamma_n+2}(x,t) + (\rho_n^0)^{\gamma_n+2}(x) \ dx + \int_0^t \int_{\mathbb{T}} \rho_n w_n \partial_x \pi_n \ dxds = \int_0^t \int_{\mathbb{T}} |\partial_x \pi_n|^2 \ dxds.$$

Since $\rho_n > 0$ this reduces to

$$\int_0^t \int_{\mathbb{T}} |\partial_x \pi_n|^2 \ dxds \leq \int_0^t \int_{\mathbb{T}} \rho_n w_n \partial_x \pi_n \ dxds + \frac{1}{\gamma_n + 2}\int_{\mathbb{T}} (\rho_n^0)^{\gamma_n+2}(x) \ dx. \tag{83}$$

Note that the assumption (6) implies that the final term on the right hand-side is uniformly bounded in n. Additionally, using Young's inequality with the first term on the right hand-side leads to

$$\frac{1}{2}\int_0^t \int_{\mathbb{T}} |\partial_x \pi_n|^2 \ dxds \leq C\|\rho_n w_n\|_{L^2_{t,x}}^2 + C.$$

The result follows from (79). $\square$

Corollary 3.6. *We have*

$$\|\rho_n u_n\|_{L^2_{t,x}} \leq C, \tag{84}$$

$$\|\sigma\sqrt{\rho_n} u_n\|_{L^\infty_t L^2_x} \leq C, \quad \text{where } \sigma(t) = \min(1,t). \tag{85}$$

Proof. Since $\rho_n u_n = \rho_n w_n - \partial_x \pi_n$, (84) follows from (79) and (81). To acquire (85), we multiply the momentum equation (4b) by σu_n. Integrating by parts in space and time leads to

$$\int_{\mathbb{T}} \sigma \rho_n u_n^2(t,x) - \sigma \rho_n u_n^2(t,0) \ dx - \int_0^t \int_{\mathbb{T}} \mathbb{1}_{[0,1]}(s)\rho_n u_n^2 \ dxds + \int_0^t \int_{\mathbb{T}} \sigma \lambda_n (\rho_n(\partial_x u_n)^2 \ dxds$$

$$= \int_0^t \int_{\mathbb{T}} \sigma \rho_n u_n(\partial_t u_n + u_n \partial_x u_n) \ dxds.$$

The term on the right hand-side disappears once we integrate by parts and use the continuity equation. Also notice that $\sigma(0) = 0$. Therefore we have

$$\int_{\mathbb{T}} \sigma \rho_n u_n^2(t,x) \ dx \leq \frac{1}{2}\int_0^t \int_{\mathbb{T}} \mathbb{1}_{[0,1]}(s)\rho_n u_n^2 \ dxds \leq \|\sqrt{\rho_n} u_n\|_{L^2_{t,x}} \leq C,$$

thanks to (78). $\square$

Lemma 3.7. *Under the assumptions* (6)- (8), *the potential* π_n *satisfies*

$$\|\pi_n\|_{L^1_{t,x}} \leq C. \tag{86}$$

Proof. We begin by defining the function $\psi_n : \mathbb{T} \times [0,T] \to \mathbb{R}$ given by

$$\psi_n(x,t) = \int_0^x \left(\rho_n^0(y) - \langle \rho_n \rangle \right) \, dy - \int_0^t \rho_n u_n(x,s) \, ds, \tag{87}$$

where $\langle \rho_n \rangle := |\mathbb{T}|^{-1} \int_{\mathbb{T}} \rho_n(t,y) \, dy$. Assuming that $\mathbb{T} = [0,1]$, we can verify some key properties of ψ_n. Firstly, we can show that ψ_n is periodic in space. Indeed, we have $\psi_n(0,t) = -\int_0^t \rho_n u_n(0,s) \, ds$ while

$$\psi_n(1,t) = \int_{\mathbb{T}} \rho_n^0(y) - \langle \rho_n \rangle \, dy - \int_0^t \rho_n u_n(1,s) \, ds = -\int_0^t \rho_n u_n(1,s) \, ds$$

$$= -\int_0^t \rho_n u_n(0,s) \, ds = \psi_n(0,t).$$

Next, we can directly compute $\partial_x \psi_n$ to find

$$\partial_x \psi_n = \rho_n^0(x) - \langle \rho_n \rangle - \int_0^t \partial_x(\rho_n u_n)(x,s) \, ds$$

$$= \rho_n^0(x) - \langle \rho_n \rangle + \rho_n(x,t) - \rho_n(x,0) = \rho_n - \langle \rho_n \rangle,$$

where we have made use of the continuity equation. Then by virtue of (78) we have $\|\partial_x \psi_n\|_{L^\infty_t L^2_x} \leq C$. In order to obtain a bound on ψ_n, we note that $\|\rho_n^0 - \langle \rho_n \rangle\|_{L^\infty_{t,x}} \leq C$ from assumption (6) and so

$$\|\psi_n(t)\|_{L^2_x} \leq C + \int_{\mathbb{T}} \left(\int_0^t \rho_n u_n(x,s) \, ds \right)^2 \, dx$$

$$\leq C + \|\rho_n u_n\|^2_{L^2_{t,x}} \leq C,$$

due to Jensen's inequality and (84). Thus we also have $\|\psi_n\|_{L^\infty_t L^2_x} \leq C$. From the Sobolev inequality (31) we can now deduce that $\|\psi_n\|_{L^\infty_{t,x}} \leq C$. We also note that $\partial_t \psi_n = -\rho_n u_n$ since $\langle \rho_n \rangle$ is constant as a function of time (and space). Therefore from (84) we have $\|\partial_t \psi_n\|_{L^2_{t,x}} \leq C$. Next, we define the test function $\zeta_n : \mathbb{T} \times [0,T] \to \mathbb{R}$ as

$$\zeta_n(x,t) := \int_0^x \left(\psi_n(y,t) - \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} \psi_n(z,t) \, dz \right) dy. \tag{88}$$

Note that ζ_n is periodic in space. Using the properties of ψ_n it is also straightforward to verify that $\|\zeta_n\|_{L^\infty_{t,x}} + \|\partial_x \zeta_n\|_{L^\infty_{t,x}} \leq C$. We now obtain a bound on $\partial_t \zeta_n$. Note that

$$\partial_t \zeta_n(x,t) = \int_0^x \left(-\rho_n u_n(y,t) + |\mathbb{T}|^{-1} \int_{\mathbb{T}} \rho_n u_n(z,t) \, dz \right) dy$$

and so $|\partial_t \zeta_n|^2 \leq 2\|\rho_n u_n(t)\|^2_{L^2_x} \leq C$ thanks to Jensen's inequality and (84). Integrating over space and time in this inequality gives $\|\partial_t \zeta_n\|_{L^2_{t,x}} \leq C$. Note that $\partial_x^2 \zeta_n = \rho_n - \langle \rho_n \rangle$ by construction. Multiplying the continuity equation (82) by ζ_n and integrating by parts in space and time, we therefore have

$$-\int_0^t \int_{\mathbb{T}} \rho_n \partial_t \zeta_n \, dxds + \int_{\mathbb{T}} \zeta_n \rho_n(x,t) - \zeta_n \rho_n(x,0) \, dx - \int_0^t \int_{\mathbb{T}} \partial_x \zeta_n \rho_n w_n \, dxds$$

$$= \int_0^t \int_{\mathbb{T}} (\rho_n - \langle \rho_n \rangle) \pi_n \, dxds.$$

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The absolute value of the left side of the above equation can be bounded independently of n due to the properties of ζ_n already established as well as Corollary 3.4. Therefore we find

$$\left| \int_0^t \int_{\mathbb{T}} (\rho_n - \langle \rho_n \rangle) \pi_n \, dx ds \right| \leq C. \tag{89}$$

We now wish to extract a bound on the quantity $\|\pi_n\|_{L^1_{t,x}}$. To this end, we define

$$S_m := \frac{1 + \langle \rho_n \rangle}{2}$$

and consider the regions $\{\rho \leq S_m\}$ and $\{\rho > S_m\}$ separately. Firstly, we have that

$$\left| \int_0^t \int_{\mathbb{T}} \pi_n(x,t) \mathbb{1}_{\{\rho \leq S_m\}} \, dx ds \right| \leq C \tag{90}$$

by the observation that when $\rho_n < S_m < 1$, we have $\pi_n(\rho_n) = \frac{\gamma_n}{\gamma_n + 1} \rho_n^{\gamma_n + 1} \to 0$ as $n \to \infty$. This in particular implies that

$$\left| \int_0^t \int_{\mathbb{T}} (\rho_n - \langle \rho_n \rangle) \pi_n(x,t) \mathbb{1}_{\{\rho > S_m\}} \, dx ds \right|$$

$$\leq \left| \int_0^t \int_{\mathbb{T}} (\rho_n - \langle \rho_n \rangle) \pi_n(x,t) \mathbb{1}_{\{\rho \leq S_m\}} \, dx ds \right| + \left| \int_0^t \int_{\mathbb{T}} (\rho_n - \langle \rho_n \rangle) \pi_n(x,t) \, dx ds \right| \leq C,$$

due to (90) and (89). When $\rho_n \geq S_m$, the assumption $\langle \rho_n \rangle \leq \hat{\rho} < 1$ implies that $\rho_n - \langle \rho_n \rangle \geq \frac{1 - \langle \rho_n \rangle}{2} \geq \frac{1 - \hat{\rho}}{2} > 0$. Thus, we have

$$C \geq \left| \int_0^t \int_{\mathbb{T}} (\rho_n - \langle \rho_n \rangle) \pi_n(x,t) \mathbb{1}_{\{\rho > S_m\}} \, dx ds \right| \geq \frac{1 - \hat{\rho}}{2} \left| \int_0^t \int_{\mathbb{T}} \pi_n(x,t) \mathbb{1}_{\{\rho > S_m\}} \, dx ds \right|,$$

and so we conclude from this and (90) that for any $t \in [0, T]$,

$$\|\pi_n\|_{L^1_{t,x}} = \int_0^t \int_{\mathbb{T}} \pi_n(x,t) \, dx ds \leq C. \tag{91}$$

$\square$

We can improve this bound via a Hoff-type estimate [15, 14], which also takes inspiration from [5].

Corollary 3.8. *For any $\tau \in (0, T)$, we have*

$$\|\pi_n\|_{L^\infty([\tau, T]; L^1(\mathbb{T}))} \leq C. \tag{92}$$

Proof. We multiply the momentum equation (4b) by $\sigma(t)\psi_n$, where $\sigma(t) := \min(1, t)$. Integrating by parts in space and time and simplifying,

$$-\int_0^t \int_{\mathbb{T}} \mathbb{1}_{[0,1]}(s) \psi_n \rho_n u_n \, dx ds + \int_{\mathbb{T}} \sigma(t) \psi_n \rho_n u_n(x,t) \, dx + \langle \rho_n \rangle \int_0^t \int_{\mathbb{T}} \sigma(s) \rho_n u_n^2 \, dx ds$$

$$= -\int_0^t \int_{\mathbb{T}} \sigma(s) (\rho_n - \langle \rho_n \rangle) \lambda_n(\rho_n) \partial_x u_n \, dx ds. \tag{93}$$

Using Corollaries 3.4, 3.6 and the regularity of ψ_n we obtain

$$\left| \int_0^t \int_{\mathbb{T}} \sigma(s) (\rho_n - \langle \rho_n \rangle) \lambda_n(\rho_n) \partial_x u_n \, dx ds \right| \leq C. \tag{94}$$

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Next, we derive the evolution equation for $(\rho_n - \langle\rho_n\rangle)\pi_n$ which reads

$$\partial_t\left[(\rho_n - \langle\rho_n\rangle)\pi_n\right] + \partial_x\left[(\rho_n - \langle\rho_n\rangle)\pi_n u_n\right] + \left[\rho_n(\rho_n - \langle\rho_n\rangle)\pi_n' + \langle\rho_n\rangle\pi_n\right]\partial_x u_n = 0. \tag{95}$$

Multiplying by $\sigma(\cdot)$ and integrating over space and time,

$$\int_0^t\int_{\mathbb{T}}\sigma(s)\partial_s\left[(\rho_n - \langle\rho_n\rangle)\pi_n\right]\,dxds = -\int_0^t\int_{\mathbb{T}}\sigma(s)(\rho_n - \langle\rho_n\rangle)\lambda_n(\rho_n)\partial_x u_n\,dxds$$
$$-\langle\rho_n\rangle\int_0^t\int_{\mathbb{T}}\pi_n\partial_x u_n\,dxds. \tag{96}$$

The first term on the RHS is bounded independently of n thanks to (94). To bound the final term, we can integrate by parts to see that

$$\int_0^t\int_{\mathbb{T}}\pi_n\partial_x u_n\,dxds = -\int_0^t\int_{\mathbb{T}}u_n\partial_x\pi_n\,dxds = -\int_0^t\int_{\mathbb{T}}u_n(\rho_n w_n - \rho_n u_n)\,dxds \leq C,$$

thanks to Corollaries 3.4 and 3.6. Therefore returning to (96) we find

$$\int_0^t\int_{\mathbb{T}}\sigma(s)\partial_s\left[(\rho_n - \langle\rho_n\rangle)\pi_n\right]\,dxds \leq C.$$

Integrating by parts,

$$\sigma(t)\int_{\mathbb{T}}(\rho_n - \langle\rho_n\rangle)\pi_n(x,t)\,dx \leq C + \int_0^t\int_{\mathbb{T}}\mathbb{1}_{[0,1]}(s)(\rho_n - \langle\rho_n\rangle)\pi_n(x,s)\,dxds.$$

Recalling the bound (89) we have for all $t \in [0,T]$

$$\sigma(t)\int_{\mathbb{T}}(\rho_n - \langle\rho_n\rangle)\pi_n(x,t)\,dx \leq C.$$

Considering the regions $\{\rho \leq S_m\}$ and $\{\rho > S_m\}$ separately and repeating the same argument which was seen in the proof of Lemma 3.7, we get

$$\sigma(t)\int_{\mathbb{T}}\pi_n(x,t)\,dx \leq C, \tag{97}$$

from which the bound follows. □

4   The limit passage

The aim of this section is to complete the proof of Theorem 1.4.

Proof of Theorem 1.4. For n fixed, we have proved the existence of a unique regular solution (ρ_n, w_n) to the system (18a)-(18b) on $\mathbb{T} \times [0,T]$ with $w_n = u_n + \partial_x p_n$. Multiplying (18a)-(18b) by $\phi \in C_c^\infty([0,T] \times \mathbb{T})$ and integrating by parts, an integral formulation for the above system is given by

$$\begin{cases} -\int_0^t\int_{\mathbb{T}}\rho_n\partial_t\phi\,dxds + \int_{\mathbb{T}}\rho_n\phi(x,t) - \rho_n^0\phi(x,0)\,dx + \int_0^t\int_{\mathbb{T}}\partial_x\pi_n\partial_x\phi\,dxds = \int_0^t\int_{\mathbb{T}}\rho_n w_n\partial_x\phi\,dxds, & (98a) \\ -\int_0^t\int_{\mathbb{T}}\rho_n w_n\partial_t\phi\,dxds + \int_{\mathbb{T}}\rho_n w_n\phi(x,t) - \rho_n^0 w_n^0\phi(x,0)\,dx + \int_0^t\int_{\mathbb{T}}(w_n\partial_x\pi_n - \rho_n w_n^2)\partial_x\phi\,dxds = 0. & (98b) \end{cases}$$

The estimates obtained in the previous section imply that there exists a triple (ρ, w, π) such that, up to the extraction of a subsequence, we have

$$\rho_n \rightharpoonup^* \rho \text{ weakly-* in } L^\infty(0,T;L^2(\mathbb{T})), \tag{99}$$

$$w_n \rightharpoonup^* w \text{ weakly-* in } L^\infty((0,T) \times \mathbb{T}), \tag{100}$$

$$\pi_n \rightharpoonup^* \pi \text{ weakly-* in } \mathcal{M}((0,T) \times \mathbb{T}), \tag{101}$$

$$\partial_x\pi_n \rightharpoonup^* \partial_x\pi \text{ weakly in } L^2(0,T;L^2(\mathbb{T})), \tag{102}$$

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Thanks to the estimate (97), the bound $\|\partial_x(\sigma\pi_n)\|_{L^2_{t,x}} \le C$ and the embedding $W^{1,2}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, we also have

$$\sigma\pi_n \rightharpoonup \sigma\pi \quad \text{weakly-* in } L^2(0,T;L^\infty(\mathbb{T})), \tag{103}$$

where again $\sigma(t) = \min(1,t)$. The key tool we will need in order to complete the limit passage is the strong convergence of w_n. Since $\|u_n\partial_x w_n\|_{L^1_{t,x}} \le \|\sqrt{\rho_n}u_n\|_{L^2_{t,x}}\|\rho_n^{-1/2}\partial_x w_n\|_{L^2_{t,x}} \le C$, the momentum equation $\partial_t w_n + u_n\partial_x w_n = 0$ implies that $\|\partial_t w_n\|_{L^1_{t,x}} \le C$. This observation along with the chain of embeddings $W^{1,\frac{4}{3}} \hookrightarrow\hookrightarrow L^\infty \hookrightarrow L^1$ allows us to use the Aubin-Lions lemma to deduce that

$$w_n \to w \quad \text{strongly in } L^\infty((0,T)\times\mathbb{T}). \tag{104}$$

Consequently, the following convergences hold:

$$\rho_n w_n \rightharpoonup^* \rho w \quad \text{weakly-* in } L^\infty(0,T;L^2(\mathbb{T})), \tag{105}$$

$$\rho_n w_n^2 \rightharpoonup^* \rho w^2 \quad \text{weakly-* in } L^\infty(0,T;L^2(\mathbb{T})), \tag{106}$$

thanks to (99). Due to the hypotheses (12) and (13), we have $\rho_n^0 \rightharpoonup \rho^0$ and $\rho_n^0 w_n^0 \rightharpoonup \rho^0 w^0$ weakly in $L^1(\mathbb{T})$. Therefore as $n \to \infty$ we have for all $t \in [0,T]$,

$$\int_{\mathbb{T}} \rho_n\phi(x,t) - \rho_n^0\phi(x,0) \ dx \to \int_{\mathbb{T}} \rho\phi(x,t) - \rho^0\phi(x,0) \ dx, \tag{107}$$

$$\int_{\mathbb{T}} \rho_n w_n\phi(x,t) - \rho_n^0 w_n^0\phi(x,0) \ dx \to \int_{\mathbb{T}} \rho w\phi(x,t) - \rho^0 w^0\phi(x,0) \ dx. \tag{108}$$

The convergences (99)-(108) are sufficient to pass to the limit in (98a) and (98b) to obtain (14a) and (14b) respectively. It remains to verify (14c). Due to the assumption (6) it is easy to see that the limit ρ satisfies $0 \le \rho$. To show $\rho \le 1$, we proceed in a similar way to that which can be seen in Theorem 4.1. of [19].

Proposition 4.1. *The density ρ satisfies $0 \le \rho \le 1$.*

Proof. For $1 < p < \infty$ such that $\gamma_n > p$, the embedding of L^p spaces implies that

$$\|\rho_n\|_{L^\infty_t L^p_x} \le \|\rho_n\|_{L^\infty_t L^1_x}^{\frac{a_n}{p}}\|\rho_n\|_{L^\infty_t L^{\gamma_n+1}_x}^{\frac{1-a_n}{p}}$$

where $a_n = \frac{p-\gamma_n}{1-\gamma_n} \to 1$ as $n \to \infty$. We now recall the bound

$$\|\rho_n\|_{L^\infty_t L^{\gamma_n+1}_x} \le (C(\gamma_n+1))^{\frac{1}{\gamma_n+1}}$$

which follows from (76). Thus, we have

$$\|\rho_n\|_{L^\infty_t L^p_x} \le C^{\frac{a_n}{p}}(C(\gamma_n+1))^{\frac{1-a_n}{\gamma_n+1}} =: C^{\frac{a_n}{p}}\tilde{C}_n,$$

and it is easy to see that $\tilde{C}_n \to 1$ as $n \to \infty$. Using Fatou's lemma,

$$\|\rho\|_{L^\infty_t L^p_x} \le \liminf_{n\to\infty}\|\rho_n\|_{L^\infty_t L^p_x} \le C^{\frac{1}{p}}.$$

Taking $p \to \infty$ in this inequality rewards us with

$$\|\rho\|_{L^\infty_{t,x}} \le \liminf_{p\to\infty}\|\rho\|_{L^\infty_t L^p_x} \le 1.$$

$\square$

Next, we estimate $\partial_t \rho_n$ and adopt the notation for the duality bracket $\langle \cdot, \cdot \rangle_* := \langle \cdot, \cdot \rangle_{(W_0^{1,2})^* \times W_0^{1,2}}$. For an arbitrary $f \in W_0^{1,2}(\mathbb{T})$, we have

$$\langle \partial_t \rho_n(t), f \rangle_* = \langle -\partial_x(\rho_n(t) u_n(t)), f \rangle_* = (\rho_n(t) u_n(t), \partial_x f)_{L_x^2}$$

$$\leq \|\rho_n u_n(t)\|_{L_x^2} \|f\|_{W_0^{1,2}}.$$

It follows from this that

$$\|\partial_t \rho_n\|_{L_t^2 W_x^{-1,2}} \leq \|\rho_n u_n\|_{L_{t,x}^2} \leq C. \tag{109}$$

We also know $\|\partial_x(\sigma \pi_n)\|_{L_{t,x}^2} \leq C$ and that $\sigma \pi_n, \rho_n$ converge weakly in $L_{t,x}^2$ to $\sigma \pi, \rho$ respectively. We can therefore invoke a standard compensated compactness argument (see Lemma 5.1 of [18]) to get

$$(1 - \rho_n)\sigma \pi_n \longrightarrow (1 - \rho)\sigma \pi, \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{T}). \tag{110}$$

On the other hand, we have

$$(1 - \rho_n)\sigma \pi_n = \frac{\gamma_n}{\gamma_n + 1}(1 - \rho_n)\sigma \rho_n^{\gamma_n + 1}$$

$$= \frac{\gamma_n}{\gamma_n + 1} \left((1 - \rho_n)\sigma \rho_n^{\gamma_n + 1} \mathbb{1}_{\{(x,t) \; : \; 0 < \rho_n(t,x) \leq 1\}} + (1 - \rho_n)\sigma \rho_n^{\gamma_n + 1} \mathbb{1}_{\{(x,t) \; : \; \rho_n(t,x) > 1\}} \right)$$

$$=: A_n + B_n.$$

It is straightforward to see that $A_n \to 0$ a.e. as $n \to \infty$. Thanks to Lemma 3.7, we can justify that

$$\|B_n\|_{L_{t,x}^1} = \int_0^T \int_{\mathbb{T}} |(1 - \rho_n)\sigma \pi_n| \mathbb{1}_{\{(x,t) \; : \; \rho_n(t,x) > 1\}} \; dx ds$$

$$\leq \|1 - \rho_n\|_{L_t^\infty L_x^1} \|\pi_n\|_{L_t^1 L_x^\infty} \cdot \mu \left(\{\rho_n > 1\} \right)$$

$$\leq C \cdot \mu \left(\{\rho_n > 1\} \right) \longrightarrow 0,$$

as $n \to \infty$, where μ represents the Lebesgue measure. Note that $\|\pi_n\|_{L_t^1 L_x^\infty} \leq C$ follows from Lemmas 3.5, 3.7 and the embedding $W^{1,1}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. Therefore we have that

$$(1 - \rho_n)\sigma \pi_n \to 0 \text{ strongly in } L^1(0, T; L^1(\mathbb{T})). \tag{111}$$

Together with (110), this allows us to deduce that

$$(1 - \rho)\sigma \pi = 0 \text{ a.e. in } \mathbb{T} \times [0, T]. \tag{112}$$

Since $\sigma(t) = \min(1, t)$ vanishes only on a set of measure zero (i.e. on $\{t = 0\} \times \mathbb{T}$), we in fact have

$$(1 - \rho)\pi = 0 \text{ a.e. in } \mathbb{T} \times [0, T]. \tag{113}$$

To conclude the proof of Theorem 1.4, we need to verify that our solution possesses the regularity specified in (15)-(17). We now check that for any $\phi \in \mathcal{D}(\mathbb{T})$, the function

$$t \mapsto \int_{\mathbb{T}} \rho(x, t)\phi(x) \; dx$$

is absolutely continuous on $[0, T]$. From (109) and the weak formulation of the continuity equation we have

$$\frac{d}{dt} \int_{\mathbb{T}} \rho \phi \; dx = \int_{\mathbb{T}} \rho u \phi' \; dx \in L^1(0, T).$$

Therefore we are able to say that

$$\rho \in C_{weak}(0, T; L^p(\mathbb{T}))$$

for $p \in [1, \infty)$. We finally observe that since $\|\partial_t w_n\|_{L_{t,x}^1} + \|w_n\|_{L_t^\infty W_x^{1,\frac{4}{3}}} \leq C$ we indeed have $w \in C(0, T; W^{1,\frac{4}{3}}(\mathbb{T})) \hookrightarrow C([0, T] \times \mathbb{T})$. This completes the proof. $\square$

A   Completing the blow-up lemma

In Section 2, we showed that

$$\sup_{t\in[0,T^*)} \|\rho_n\|_{L^\infty(0,t;H^2)} + \sup_{t\in[0,T^*)} \|u_n\|_{L^\infty(0,t;H^2)} + \sup_{t\in[0,T^*)} \|u_n\|_{L^2(0,t;H^3)} < +\infty. \tag{114}$$

In other words, we completed the base step of induction for Lemma 2.6. Recall that for a given integer $k \geq 0$ we also introduced the notation

$$M_k^n \equiv M_k^n(t, \|\rho_n\|_{L_t^\infty H_x^k}, \|u_n\|_{L_t^\infty H_x^k}, \|u_n\|_{L_t^2 H_x^{k+1}}, \gamma_n, E_n, \overline{\rho_n}, (\underline{\rho_n})^{-1}),$$

to denote an arbitrary function that is increasing with respect to the specified arguments, and which also satisfies $\sup_{t\in[0,T^*)} M_k^n < +\infty$. The inductive step corresponds to the following proposition:

Proposition A.1. *Suppose (ρ_n, u_n) is a regular solution to (4a)-(4b) on $[0, T^*)$ with initial data $(\rho_n^0, u_n^0) \in H^k(\mathbb{T}) \times H^k(\mathbb{T})$, where $n \in \mathbb{N}^+$ is fixed and $k > 2$. We assume that*

$$\|\rho_n\|_{L_t^\infty H_x^{k-1}} + \|u_n\|_{L_t^\infty H_x^{k-1}} + \|u_n\|_{L_t^2 H_x^k} \leq M_{k-2}^n, \tag{115}$$

Then,

$$\|\rho_n\|_{L_t^\infty H_x^k} + \|u_n\|_{L_t^\infty H_x^k} + \|u_n\|_{L_t^2 H_x^{k+1}} \leq M_{k-1}^n. \tag{116}$$

Proof. Taking $l = k - 1$ in (46), we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_x^{k-1}V_n|^2\,dx = -\int_{\mathbb{T}}\partial_x^{k-1}\left((u_n + \frac{\lambda_n(\rho_n)}{\rho_n^2}\partial_x\rho_n)\partial_xV_n\right)\partial_x^{k-1}V_n\,dx + \int_{\mathbb{T}}\partial_x^{k-1}\left(\frac{\lambda_n(\rho_n)}{\rho_n}\partial_x^2V_n\right)\partial_x^{k-1}V_n\,dx$$

$$-\int_{\mathbb{T}}\partial_x^{k-1}\left(\frac{\lambda_n'(\rho_n)\rho_n + \lambda_n(\rho_n)}{(\lambda_n(\rho_n))^2}V_n^2\right)\partial_x^{k-1}V_n\,dx.$$

Integrating by parts and introducing commutator notation,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_x^{k-1}V_n|^2\,dx + \int_{\mathbb{T}}\frac{\lambda_n(\rho_n)}{\rho_n}|\partial_x^kV_n|^2\,dx$$

$$= \int_{\mathbb{T}}\left(u_n + \frac{\lambda_n(\rho_n)}{\rho_n^2}\partial_x\rho_n\right)\partial_x^kV_n\,\partial_x^{k-1}V_n\,dx + \int_{\mathbb{T}}\left[\partial_x^{k-2},\ u_n + \frac{\lambda_n(\rho_n)}{\rho_n^2}\partial_x\rho_n\right]\partial_xV_n\,\partial_x^kV_n\,dx$$

$$-\int_{\mathbb{T}}\left[\partial_x^{k-2},\frac{\lambda_n(\rho_n)}{\rho_n}\right]\partial_x^2V_n\,\partial_x^kV_n\,dx - \int_{\mathbb{T}}\left(\frac{\lambda_n'(\rho_n)\rho_n + \lambda_n(\rho_n)}{(\lambda_n(\rho_n))^2}V_n\right)|\partial_x^{k-1}V_n|^2\,dx$$

$$-\int_{\mathbb{T}}\left[\partial_x^{k-1},\frac{\lambda_n'(\rho_n)\rho_n + \lambda_n(\rho_n)}{(\lambda_n(\rho_n))^2}V_n\right]V_n\partial_x^{k-1}V_n\,dx =: \sum_{n=1}^5 I_n. \tag{117}$$

We now estimate $I_1 - I_5$. Integrating by parts,

$$I_1 \leq \left|\int_{\mathbb{T}}(\partial_x^{k-1}V_n)^2(\partial_xu_n + \partial_x(\frac{\lambda(\rho_n)}{\rho_n^2}\partial_x\rho_n))\,dx\right|$$

$$\leq \|\partial_x^{k-1}V_n\|_{L_x^2}^2\left\{\|\partial_xu_n\|_{L_x^\infty} + \|\partial_x(\rho_n^{-2}\lambda_n(\rho_n))\partial_x\rho_n\|_{L_x^\infty} + \|\rho_n^{-2}\lambda(\rho_n)\partial_x^2\rho_n\|_{L_x^\infty}\right\}$$

$$\leq M_{k-1}^n\|\partial_x^{k-1}V_n\|_{L_x^2}^2.$$

Noticing that $\frac{\lambda_n(\rho_n)}{\rho_n^2}\partial_x\rho_n = \partial_x p_n(\rho_n)$ and using the well-known Kato-Ponce commutator estimates [10],

$$I_2 = \int_{\mathbb{T}}\left[\partial_x^{k-2}, u_n + \partial_x p_n(\rho_n)\right]\partial_xV_n\,\partial_x^{k-1}V_n\,dx \leq \left\|\left[\partial_x^{k-2}, u_n + \partial_x p_n(\rho_n)\right]\partial_xV_n\right\|_{L_x^2}\|\partial_x^kV_n\|_{L_x^2}$$

$$\leq \left(\|\partial_x^{k-2}(u_n + \partial_x p_n(\rho_n))\|_{L_x^2}\|\partial_xV_n\|_{L_x^\infty} + \|\partial_x(u_n + \partial_x p_n(\rho_n))\|_{L_x^\infty}\left\|\partial_x^{k-1}V_n\right\|_{L_x^2}\right)\|\partial_x^kV_n\|_{L_x^2}$$

$$\leq M_{k-1}^n + M_{k-1}^n\|\partial_x^{k-1}V_n\|_{L_x^2}^2 + M_{k-1}^n\|\partial_x^{k-1}\rho_n\|_{L_x^2}^2 + \epsilon\|\partial_x^kV_n\|_{L_x^2}^2$$

$$\leq M_{k-1}^n + M_{k-1}^n\|\partial_x^{k-1}V_n\|_{L_x^2}^2 + \epsilon_1\|\partial_x^kV_n\|_{L_x^2}^2.$$

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Next,

$$I_3 \leq \left| \int_{\mathbb{T}} \left[\partial_x^{k-2}, \frac{\lambda_n(\rho_n)}{\rho_n} \right] \partial_x^2 V_n \; \partial_x^k V_n \; dx \right| \leq \left\| \left[\partial_x^{k-2}, \frac{\lambda(\rho_n)}{\rho_n} \right] \partial_x^2 V_n \right\|_{L_x^2} \|\partial_x^k V_n\|_{L_x^2}$$

$$\leq \left(\left\| \partial_x^{k-2}(\rho_n^{-1}\lambda(\rho_n)) \right\|_{L_x^2} \|\partial_x^2 V_n\|_{L_x^\infty} + \|\partial_x(\rho_n^{-1}\lambda(\rho_n))\|_{L_x^\infty} \|\partial_x^{k-1} V_n\|_{L_x^2} \right) \|\partial_x^k V_n\|_{L_x^2}$$

$$\leq M_{k-1}^n + \epsilon_2 \|\partial_x^k V_n\|_{L_x^2}^2 + M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2}^2,$$

It is also straightforward to verify that

$$I_4 \leq \left| \int_{\mathbb{T}} \left(\frac{\lambda_n'(\rho_n)\rho_n + \lambda_n(\rho_n)}{(\lambda_n(\rho_n))^2} V_n \right) |\partial_x^{k-1} V_n|^2 \; dx \right| \leq M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2}^2.$$

Making use of commutator estimates once more, we have

$$I_5 \leq \left| \int_{\mathbb{T}} \left[\partial_x^{k-1}, \frac{\lambda_n'(\rho_n)\rho_n + \lambda_n(\rho_n)}{(\lambda_n(\rho_n))^2} V_n \right] V_n \partial_x^{k-1} V_n \; dx \right| = (\gamma_n + 2) \left| \int_{\mathbb{T}} \left[\partial_x^{k-1}, \; V_n(\lambda_n(\rho_n))^{-1} \right] V_n \partial_x^{k-1} V_n \right|$$

$$\leq (\gamma_n + 2) \| \left[\partial_x^{k-1}, \; V_n(\lambda_n(\rho_n))^{-1} \right] V_n \|_{L_x^2} \|\partial_x^{k-1} V_n\|_{L_x^2}$$

$$\leq (\gamma_n + 2) \left(\|\partial_x^{k-1}(V_n(\lambda_n(\rho_n))^{-1})\|_{L_x^2} \|V_n\|_{L_x^\infty} + \|\partial_x(V_n(\lambda_n(\rho_n))^{-1})\|_{L_x^\infty} \|\partial_x^{k-2} V_n\|_{L_x^2} \right) \|\partial_x^{k-1} V_n\|_{L_x^2}. \quad (118)$$

We now note that

$$\partial_x^{k-1}(V_n(\lambda_n(\rho_n))^{-1}) = \left[\partial_x^{k-1}, V_n \right] (\lambda_n(\rho_n))^{-1} + \partial_x^{k-1}(\lambda_n(\rho_n)^{-1})V_n,$$

and moreover it is straightforward to deduce that

$$\| \left[\partial_x^{k-1}, V_n \right] (\lambda_n(\rho_n))^{-1} \|_{L_x^2} \leq M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2} + M_{k-1}^n \|\partial_x^{k-1}(\lambda_n(\rho_n))^{-1}\|_{L_x^2}$$

$$\leq M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2} + M_{k-1}^n.$$

Thus, we have

$$\left\| \partial_x^{k-1} \left(V_n(\lambda_n(\rho_n))^{-1} \right) \right\|_{L_x^2} \leq M_{k-1}^n + M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2}.$$

Going back to (118), we then have

$$|I_5| \leq M_{k-1}^n + M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2} + M_{k-1}^n \|\partial_x^{k-2} V_n\|_{L_x^2}$$

$$\leq M_{k-1}^n + M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2}^2.$$

Returning to (117) and using our newfound estimates for $I_1 - I_5$, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} |\partial_x^{k-1} V_n|^2 \; dx + \int_{\mathbb{T}} \frac{\lambda_n(\rho_n)}{\rho_n} |\partial_x^k V_n|^2 \; dx$$

$$\leq M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2}^2 + M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2}^2$$

$$+ (\epsilon_1 + \epsilon_2) \|\partial_x^k V_n\|_{L_x^2}^2 + M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2}^2 + M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2}^2 + M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2}^2 + M_{k-1}^n$$

$$\leq M_{k-1}^n + (\epsilon_1 + \epsilon_2) \|\partial_x^k V_n\|_{L_x^2}^2 + M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2}^2.$$

Thanks to the existence of the lower bound for ρ_n on $[0, T^*)$, we have that $\rho_n^{-1}\lambda_n(\rho_n) = \gamma_n \rho_n^{\gamma_n} \geq \gamma_n \underline{\rho_n}^{\gamma_n}$ and thus

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} |\partial_x^{k-1} V_n|^2 \; dx + C \int_{\mathbb{T}} |\partial_x^k V_n|^2 \; dx \leq M_{k-1}^n + M_{k-1}^n \|\partial_x^{k-1} V_n\|_{L_x^2}^2, \quad (119)$$

where $\epsilon_i > 0$ are chosen so that $C > 0$. This choice may depend on n. In particular, we have

$$\frac{d}{dt}\|\partial_x^{k-1}V_n\|_{L_x^2}^2 \leq M_{k-1}^n + M_{k-1}^n\|\partial_x^{k-1}V_n\|_{L_x^2}^2.$$

An application of Gronwall's inequality gives us

$$\|\partial_x^{k-1}V_n\|_{L_t^\infty L_x^2} + \|\partial_x^k V_n\|_{L_{t,x}^2} \leq M_{k-1}^n. \tag{120}$$

Next, we attempt to estimate $\partial_x^k u_n$ and $\partial_x^{k+1}u_n$. Using the expression $V_n = \lambda_n(\rho_n)\partial_x u_n$ we have

$$\partial_x^k u_n = \partial_x^{k-1}(\lambda_n^{-1}V_n) = \lambda_n^{-1}\partial_x^{k-1}V_n + \left[\partial_x^{k-1}, \lambda_n^{-1}\right]V_n.$$

Then using commutator estimates,

$$\|\partial_x^k u_n\|_{L_x^2} \leq \|\lambda_n^{-1}\|_{L_x^\infty}\|\partial_x^{k-1}V_n\|_{L_x^2} + \|\partial_x^{k-1}(\lambda_n^{-1})\|_{L_x^2}\|V_n\|_{L_x^\infty} + \|\partial_x\lambda_n^{-1}\|_{L_x^\infty}\|\partial_x^{k-2}V_n\|_{L_x^2}. \tag{121}$$

It is important to mention that by induction and the Sobolev inequality (31) one can show that for $j \geq 1$,

$$\|\partial_x^j\lambda_n(\rho_n)\|_{L_t^\infty L_x^2} + \|\partial_x^j\lambda_n(\rho_n)^{-1}\|_{L_t^\infty L_x^2} \leq M_{j-1}^n + M_{j-1}^n\|\partial_x^j\rho_n\|_{L_t^\infty L_x^2}. \tag{122}$$

Then going back to (121) and using (120) and the inductive hypothesis (115) ,

$$\|\partial_x^k u_n\|_{L_t^\infty L_x^2} \leq M_{k-1}^n + M_{k-1}^n\|\partial_x^{k-1}\rho_n\|_{L_t^\infty L_x^2} \leq M_{k-1}^n. \tag{123}$$

Let's turn our attention to $\partial_x^{k+1}u_n$. In a similar fashion,

$$\partial_x^{k+1}u_n = \lambda_n^{-1}\partial_x^k V_n + \left[\partial_x^k, \lambda_n^{-1}\right]V_n \tag{124}$$

and so repeating the commutator estimates in the previous argument leads to

$$\|\partial_x^{k+1}u_n\|_{L_x^2} \leq M_{k-1}^n\|\partial_x^k V_n\|_{L_x^2} + M_{k-1}^n\|\partial_x^k(\lambda_n)^{-1}\|_{L_x^2} + M_{k-1}^n\|\partial_x^{k-1}V_n\|_{L_x^2}.$$

Squaring and integrating in time,

$$\|\partial_x^{k+1}u_n\|_{L_{t,x}^2}^2 \leq M_{k-1}^n\|\partial_x^k V_n\|_{L_{t,x}^2} + M_{k-1}^n\|\partial_x^k\rho_n\|_{L_t^2 L_x^2} + M_{k-1}^n\|\partial_x^{k-1}V_n\|_{L_{t,x}^2}$$
$$\leq M_{k-1}^n + M_{k-1}^n\|\partial_x^k\rho_n\|_{L_t^2 L_x^2}. \tag{125}$$

Note that we are not yet able to bound $\|\rho_n\|_{L_t^2 H_x^k}$. We now recall (45), which reads:

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^k\rho_n\|_{L_x^2}^2 \leq C\left(\|\partial_x^k u_n\|_{L_x^2}\|\partial_x^k\rho_n\|_{L_x^2}^2 + \|\rho_n\|_{L_x^\infty}\|\partial_x^k\rho_n\|_{L_x^2}\|\partial_x^{k+1}u_n\|_{L_x^2}\right). \tag{126}$$

Thanks to (125) and (123), we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^k\rho_n\|_{L_x^2}^2 \leq M_{k-1}^n\|\partial_x^k\rho_n\|_{L_x^2}^2$$

and therefore Gronwall's inequality gives us

$$\|\partial_x^k\rho_n\|_{L_t^\infty L_x^2} \leq M_{k-1}^n. \tag{127}$$

Returning to (125) and using this estimate also yields

$$\|\partial_x^{k+1}u_n\|_{L_{t,x}^2} \leq M_{k-1}^n. \tag{128}$$

Adding (123), (127) and (125),

$$\|\partial_x^k u_n\|_{L_t^\infty L_x^2} + \|\partial_x^k\rho_n\|_{L_t^\infty L_x^2} + \|\partial_x^{k+1}u_n\|_{L_{t,x}^2} \leq M_{k-1}^n.$$

Combining this with the inductive hypothesis (115), we have shown that

$$\|\rho_n\|_{L_t^\infty H_x^k} + \|u_n\|_{L_t^\infty H_x^k} + \|u_n\|_{L_t^2 H_x^{k+1}} \leq M_{k-1}^n. \tag{129}$$

Note that in all of our estimates the notation M_{k-1}^n was used to denote a function of time satisfying $\sup_{t\in[0,T^*)} M_{k-1}^n < +\infty$. Therefore the right-hand side of (129) also satisfies this condition. The proof is now complete. $\square$

We conclude with a proof of a generalised Poincare inequality, similar to Proposition 7.2 of [7].

Proposition A.2. *There exists $C \equiv C(M, K) > 0$ such that*

$$\|u\|_{L_x^1} \leq C \left(\|\partial_x u\|_{L_x^1} + \int_{\mathbb{T}} r|u| \ dx \right) \tag{130}$$

for any $u \in W^{1,1}(\mathbb{T})$ and any non-negative $r \in L^2(\mathbb{T})$ satisfying

$$0 < M \leq \int_{\mathbb{T}} r \ dx < +\infty, \qquad \int_{\mathbb{T}} r^2 \ dx \leq K, \tag{131}$$

for some constants $M, K > 0$.

Proof. We proceed by contradiction. If the proposition is false, then there exist sequences $\{u_n\}_{n=1}^{\infty}, \{r_n\}_{n=1}^{\infty}$ such that $\|u_n\|_{L_x^1} = 1$ (after suitable renormalisation), each r_n satisfies (131) and

$$\|\partial_x u_n\|_{L_x^1} + \int_{\mathbb{T}} r_n|u_n| \ dx \leq \frac{1}{n}. \tag{132}$$

The assumption (131) implies that there exists $r \in L^2(\mathbb{T})$ such that up to a subsequence, $r_n \rightharpoonup r$ weakly in $L^2(\mathbb{T})$. It is also easy to see that $\|u_n\|_{W^{1,1}(\mathbb{T})} \leq 2$. Therefore due to the compact embedding $W^{1,1} \hookrightarrow\hookrightarrow L^{\infty}$ we have in particular that $u_n \to u$ strongly in $L^2(\mathbb{T})$. From (132) we infer that $\partial_x u_n \to 0$ strongly in $L^1(\mathbb{T})$. Thus we may argue that $u_n \to u$ strongly in $W^{1,1}(\mathbb{T})$ with $\partial_x u = 0$. Passing to the limit in the inequality appearing in (132) and using the weak and strong L^2 convergences of r_n and u_n respectively, we find that

$$\int_{\mathbb{T}} r \ dx = 0, \tag{133}$$

which contradicts (131). $\qquad\square$

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