Stochastic Differential Equations Driven by G-Brownian Motion with Mean Reflections

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Abstract

In this paper, we study the mean reflected stochastic differential equations driven by G-Brownian motion, where the constraint depends on the distribution of the solution rather than on its paths. Well-posedness is achieved by first investigating the Skorokhod problem with mean reflection under the G-expectation. Two approaches to constructing the solution are introduced, both offering insights into desired properties and aiding in the application of the contraction mapping method. Additionally, a new technique is proposed to prove the first propagation of chaos result for mean reflected G-SDEs, overcoming challenges posed by the nonlinearity of G-expectation and the non-deterministic nature of the quadratic variation of G-Brownian motion.

Key words: G-expectation, reflected SDEs, mean reflection, particle system
MSC-classification: 60G65, 60H10

1 Introduction

We firstly give the background in Subsection 1.1 and then state our contributions in Subsection 1.2, followed with the organization of the paper in Subsection 1.3.

1.1 Background

Stochastic differential equations (SDEs) with reflecting boundaries, commonly referred to as reflected SDEs, were introduced by Skorokhod in the 1960s (Skorokhod, 1961). Later, El Karoui et al. (1997a) introduced the reflected backward SDE (BSDE), where the first component of the solution is constrained to remain above a specified continuous process, known as the obstacle. Reflected SDEs and reflected BSDEs are intimately linked to various fields including optimal stopping problems (see, e.g., Cheng and Riedel (2013)), pricing for American options (see, e.g., El Karoui et al. (1997b)), and the obstacle problem for partial differential equations (PDEs) (see, e.g., Bally et al. (2002)). Hence, they have attracted a great deal of attention in the probability community, such as Chaleyat-Maurel and El Karoui (1978); Tanaka (1979); Lions and Sznitman (1984); Ma and Zhang (2005); Burdzy et al. (2009); Slaby (2010); Hamadene and Zhang (2010); Ning and Wu (2021, 2023); Ning et al. (2024) and the references therein, providing a comprehensive overview of this theory. In all the aforementioned papers, the constraints depend on the paths of the solution.

Over the past decade, Bouchard et al. (2015) pioneered the modeling of BSDEs with mean reflection, where the terminal condition constrains the distribution of the BSDE at terminal time. Mean

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reflected BSDEs (MRBSDEs) were formally introduced by Briand et al. (2018). Subsequently, the forward version was proposed in Briand et al. (2020a), focusing on the following type of mean reflected SDEs (MRSDEs): for $t \in [0, T]$,

$$\begin{cases} X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + A_t, \\ \mathsf{E}[h(X_t)] \geq 0 \quad \text{and} \quad \int_0^T \mathsf{E}[h(X_t)]dA_t = 0, \end{cases} \tag{1.1}$$

where $b, \sigma, h : \mathbb{R} \to \mathbb{R}$ are given Lipschitz functions and W is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Here, the compensating reflection component depends on the distribution of the solution. The authors established both the existence and uniqueness of the solution. MRSDEs and MRBSDEs have attracted warm interests in the probability community, which include, but are not limited to, the following: quadratic MRBSDEs (Hibon et al., 2018), MRSDEs with jumps (Briand et al., 2020b), large deviation principle for the MRSDEs with jumps (Li, 2018), MRSDEs with two constraints (Falkowski and Słomiński, 2021), multi-dimensional MRBSDEs (Qu and Wang, 2023), and the well-posedness of MRBSDEs with different reflection restrictions (Falkowski and Słomiński, 2022; Li, 2023).

All the above mean reflected problems were considered in the classical probability space until Liu and Wang (2019) studied the mean reflected BSDE driven by G-Brownian motion (G-BSDEs). The introduction of G-Brownian motion and G-expectation was a significant development in the field of stochastic analysis (Peng, 2007, 2008, 2019). The nonlinear G-expectation theory was motivated by the consideration of Knightian uncertainty, especially volatility uncertainty, and the stochastic interpretation of fully nonlinear PDEs. Roughly speaking, the G-expectation can be seen as an upper expectation taking over a non-dominated family of probability measures. Under this framework, G-Brownian motion and the associated G-Itô's calculus were established. Moreover, Gao (2009) obtained the well-posedness of SDEs driven by G-Brownian motion. For the reflected case, Lin (2013) first investigated the scalar-valued SDE driven by G-Brownian motion (G-SDEs) whose solution is required to be above a prescribed G-Itô's process. Then, Lin and Soumana Hima (2019) considered the reflected G-SDEs in non-convex domains. For the non-reflected case, Sun et al. (2023) studied the G-SDEs whose coefficients may depend on the distribution of the solution. For the reflected G-BSDEs, Li et al. (2018b); Li and Peng (2020); Li and Song (2021) deal with the lower obstacle case, the upper obstacle case and the double obstacles case, respectively. Recently, Li and Ning (2024a) investigated doubly reflected G-BSDEs and established their connection to fully nonlinear PDEs with double obstacles.

Natural questions arise regarding whether mean reflected G-SDEs can be well-defined and under what conditions. Additionally, since the reflection process depends on the position's law, it exhibits nonlinearity in the McKean–Vlasov terminology. Consequently, it appears worthwhile to investigate whether such a system can be viewed as the asymptotic counterpart of a mean field Skorokhod problem, namely, as the asymptotic dynamics of an interacting particle system (IPS) reflected in a mean field. As the IPS dimension tends to infinity, the system converges towards a limit where any finite subset of particles becomes asymptotically independent of each other. This phenomenon, commonly known as the propagation of chaos (POC), is further elaborated in the classical work by Sznitman (1991). The POC for MRSDEs was established in Briand et al. (2020a) and then generalized to the backward framework in Briand and Hibon (2021). These two POC results specifically address single reflection cases, while the POC result for MRBSDEs with double reflection is provided in Li and Ning (2024b). However, so far, there is no established POC for both mean reflected G-SDEs. Therefore, in this paper, we will also explore the POC after establishing the well-posedness of mean reflected G-SDEs.

1.2 Our contributions

In this paper, we introduce the G-SDE with mean reflection in the following form: for $t \in [0,T]$,

$$\begin{cases} X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t h(x, X_s)d\langle B\rangle_s + \int_0^t \sigma(s, X_s)dB_s + A_t, \\ \widehat{\mathbb{E}}[l(t, X_t)] \geq 0 \quad \text{and} \quad \int_0^T \widehat{\mathbb{E}}[l(t, X_t)]dA_t = 0, \end{cases} \tag{1.2}$$

where B represents the G-Brownian motion, $\widehat{\mathbb{E}}$ denotes the G-expectation, and the functions b, h, σ, and l satisfy certain regularity conditions, which will be elucidated later along with a review of the G-expectation theory. The solution to the G-SDE (1.2) is a pair of processes (X, A), where A is a nondecreasing deterministic function, behaving in a minimal way such that the Skorokhod condition is satisfied. The well-posedness of a unique solution to the G-SDE (1.2) is established in Theorem 4.1, with the assumptions enforced on the coefficient functions being comparable to those of the non-reflected case studied by Gao (2009).

Theorem 4.1 is achieved by first investigating the Skorokhod problem with mean reflection under the G-expectation, which is rigorously defined in Definition 3.1 and practically illustrated in Example 3.3. A crucial result is the existence and uniqueness of the solution to that problem in Theorem 3.2. Two approaches to constructing the solution are presented. The first approach relies on an intermediate result from Liu and Wang (2019), outlined in Proposition 3.4, although further stochastic analysis is evidently required in this context. The second approach, requiring an additional assumption to establish Theorem 3.2, is denoted as Theorem 3.6. However, this method establishes a link between the Skorokhod problem with mean reflection and the deterministic Skorokhod problem, leading to the acquisition of desired properties as illustrated in Proposition 3.10 and subsequently in Corollary 3.11. By combining Theorem 3.2, both construction methods, and the attained properties through a contraction mapping argument, we establish the existence and uniqueness of solutions for (1.2). Following this, the desired moment estimate of the solution to mean reflected G-SDEs is attained in Proposition 4.2, ensuring the continuity of the compensating term A. However, to prove the POC, a more refined result is required. In essence, this necessitates the compensating term to be Lipschitz continuous when the loss function l is sufficiently smooth, a condition satisfied as shown in Proposition 4.3.

In this paper, we creatively develop a new technique to prove the first POC result for mean reflected G-SDEs. So far, in the classical probability space setting, all three existing POC results in the mean reflected context (Briand et al. (2020a) on MRSDEs, Briand and Hibon (2021) on MRBSDEs, and Li and Ning (2024b) on MRBSDEs with double reflection) have relied on controlling the Wasserstein distance of the empirical measure of independent and identically distributed samples to the true distribution. However, within the G-expectation framework, we must deal with a family of distributions and hence a family of Wasserstein distances, rendering this approach invalid. We propose a new method utilizing technical stochastic analysis. However, two challenges persist in the G-expectation case. Firstly, due to the nonlinearity of the G-expectation, we typically cannot change the order of G-expectation and integral. Secondly, the non-deterministic nature of the quadratic variation of G-Brownian motion leads to some unexpected estimates (see Remark 5.4 for details). Consequently, we only consider the case where the coefficient function $h \equiv 0$ and the loss function l is linear. Furthermore, we provide a sufficient condition for the "Fubini theorem" under G-expectation, allowing the order of G-expectation and integral to be interchanged. Based on these results, with the assistance of Proposition 4.3, by appropriately designing the IPS as a family of interacting G-SDEs with oblique reflection, we achieve the POC for the mean reflected G-SDE (1.2).

1.3 Organization of the paper

The paper is structured as follows: Section 2 provides a review of fundamental notations and results in the G-framework. The mean reflected Skorokhod problem is defined and examined in Section 3. Section 4 focuses on establishing the well-posedness of the mean reflected G-SDE (1.2). Lastly, in Section 5, we present an approximation for mean reflected G-SDEs using an IPS system. Throughout the paper, the letter C, with or without subscripts, will denote a positive constant whose value may change for different usage.

2 Preliminaries

We review some fundamental notions and results of G-expectation and G-stochastic calculus. The readers may refer to Peng (2007, 2008, 2019) for more details. For simplicity, we only consider the one-dimensional G-Brownian motion, noting that the results still hold for the multidimensional case.

Let $\Omega_T = C_0([0,T];\mathbb{R})$, the space of real-valued continuous functions starting from the origin, i.e., $\omega_0 = 0$ for any $\omega \in \Omega_T$, be endowed with the supremum norm. Let $\mathcal{B}(\Omega_T)$ be the Borel set and B be the canonical process. Set

$$L_{ip}(\Omega_T) = \Big\{\varphi(B_{t_1}, ..., B_{t_n}) : \ n \in \mathbb{N}, \ t_1, \cdots, t_n \in [0,T], \ \varphi \in C_{b,Lip}(\mathbb{R}^n)\Big\},$$

where $C_{b,Lip}(\mathbb{R}^n)$ denotes the set of all bounded Lipschitz functions on $\mathbb{R}^n$. We fix a sublinear and monotone function $G : \mathbb{R} \to \mathbb{R}$ defined by

$$G(a) = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-), \tag{2.1}$$

where $0 < \underline{\sigma}^2 < \overline{\sigma}^2$. The associated G-expectation on $(\Omega_T, L_{ip}(\Omega_T))$ can be constructed in the following way. Given that $\xi \in L_{ip}(\Omega_T)$ can be represented as

$$\xi = \varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_n}),$$

set for $t \in [t_{k-1}, t_k)$ with $k = 1, \cdots, n$,

$$\widehat{\mathbb{E}}_t[\varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_n})] = u_k(t, B_t; B_{t_1}, \cdots, B_{t_{k-1}}),$$

where $u_k(t, x; x_1, \cdots, x_{k-1})$ is a function of (t,x) parameterized by $(x_1, \cdots, x_{k-1})$ such that it solves the following fully nonlinear PDE defined on $[t_{k-1}, t_k) \times \mathbb{R}$:

$$\partial_t u_k + G(\partial_x^2 u_k) = 0,$$

whose terminal conditions are given by

$$\begin{cases} u_k(t_k, x; x_1, \cdots, x_{k-1}) = u_{k+1}(t_k, x; x_1, \cdots, x_{k-1}, x), & k < n, \\ u_n(t_n, x; x_1, \cdots, x_{n-1}) = \varphi(x_1, \cdots, x_{n-1}, x). \end{cases}$$

Hence, the G-expectation of ξ is $\widehat{\mathbb{E}}_0[\xi]$, denoted as $\widehat{\mathbb{E}}[\xi]$ for simplicity. The triple $(\Omega_T, L_{ip}(\Omega_T), \widehat{\mathbb{E}})$ is called the G-expectation space and the process B is called the G-Brownian motion.

For $\xi \in L_{ip}(\Omega_T)$ and $p \geq 1$, we define

$$\|\xi\|_{L_G^p} = (\widehat{\mathbb{E}}|\xi|^p])^{1/p}.$$

The completion of $L_{ip}(\Omega_T)$ under this norm is denote by $L_G^p(\Omega)$. For all $t \in [0,T]$, $\widehat{\mathbb{E}}_t[\cdot]$ is a continuous mapping on $L_{ip}(\Omega_T)$ w.r.t the norm $\|\cdot\|_{L_G^1}$. Hence, the conditional G-expectation $\widehat{\mathbb{E}}_t[\cdot]$ can be extended continuously to the completion $L_G^1(\Omega_T)$. Furthermore, Denis et al. (2011) proved that the G-expectation has the following representation.

Theorem 2.1 (Denis et al. (2011)) *There exists a weakly compact set $\mathcal{P}$ of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that*

$$\widehat{\mathbb{E}}[\xi] = \sup_{\mathsf{P}\in\mathcal{P}} \mathsf{E}^{\mathsf{P}}[\xi], \qquad \forall \xi \in L_G^1(\Omega_T).$$

We call $\mathcal{P}$ a set that represents $\widehat{\mathbb{E}}$.

For $\mathcal{P}$ being a weakly compact set that represents $\widehat{\mathbb{E}}$, we define the following two Choquet capacities

$$V(A) = \sup_{\mathsf{P}\in\mathcal{P}} \mathsf{P}(A) \quad \text{and} \quad v(A) = \inf_{\mathsf{P}\in\mathcal{P}} \mathsf{P}(A), \qquad \forall A \in \mathcal{B}(\Omega_T).$$

A set $A \in \mathcal{B}(\Omega_T)$ is called polar if $V(A) = 0$. A property holds "*quasi-surely*" (q.s.) if it holds outside a polar set. In this paper, we do not distinguish two random variables X and Y if $X = Y$, q.s.. The following proposition can be seen as the strict comparison property for the G-expectation.

Proposition 2.2 (Li and Lin (2017)) *Let $X, Y \in L_G^1(\Omega_T)$ with $X \le Y$, q.s.. The following properties hold:*

(i) If $v(X < Y) > 0$, then $\widehat{\mathbb{E}}[X] < \widehat{\mathbb{E}}[Y]$;

(ii) If $\widehat{\mathbb{E}}[X] < \widehat{\mathbb{E}}[Y]$, then $V(X < Y) > 0$.

The lemma below will be utilized in constructing the solution in the next section.

Lemma 2.3 (Liu and Wang (2019)) *Suppose that $X \in L_G^p(\Omega_T)$ with some $p \ge 1$. Then, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for all set $O \in \mathcal{B}(\Omega_T)$ with $V(O) \le \delta$, we have*

$$\sup_{t\in[0,T]} \widehat{\mathbb{E}}\Big[\big|\widehat{\mathbb{E}}_t[X]\big|^p \mathbb{1}_O\Big] \le \varepsilon.$$

The following result can be regarded as the monotone convergence theorem under the G-expectation.

Lemma 2.4 (Denis et al. (2011)) *Suppose $\{X_n\}_{n\in\mathbb{N}}$ and X are $\mathcal{B}(\Omega_T)$-measurable.*

(1) If $X_n \uparrow X$ q.s. and $\mathsf{E}^{\mathsf{P}}[X_1^-] < \infty$ for all $\mathsf{P} \in \mathcal{P}$, then $\widehat{\mathbb{E}}[X_n] \uparrow \widehat{\mathbb{E}}[X]$.

(2) If $\{X_n\}_{n\in\mathbb{N}} \subset L_G^1(\Omega_T)$ satisfies $X_n \downarrow X$ q.s., then $\widehat{\mathbb{E}}[X_n] \downarrow \widehat{\mathbb{E}}[X]$.

We need the following norms and spaces to specify the regularity conditions imposed on the parameter functions.

Definition 2.5 *Let $M_G^0(0,T)$ be the collection of processes such that*

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbb{1}_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$ for a given partition $\{t_0, \cdots, t_N\}$ of $[0,T]$. For each $p \ge 1$ and $\eta \in M_G^0(0,T)$, denote

$$\|\eta\|_{H_G^p} = \left\{ \widehat{\mathbb{E}}\bigg(\int_0^T |\eta_s|^2 ds \bigg)^{p/2} \right\}^{1/p} \quad \text{and} \quad \|\eta\|_{M_G^p} = \left\{ \widehat{\mathbb{E}}\bigg(\int_0^T |\eta_s|^p ds \bigg) \right\}^{1/p}.$$

Let $H_G^p(0,T)$ and $M_G^p(0,T)$ be the completions of $M_G^0(0,T)$ under the norms $\|\cdot\|_{H_G^p}$ and $\|\cdot\|_{M_G^p}$, respectively.

Denote by $\langle B\rangle$ the quadratic variation process of the G-Brownian motion B. For two processes $\xi \in M_G^1(0,T)$ and $\eta \in M_G^2(0,T)$, the G-Itô integrals $(\int_0^t \xi_s d\langle B\rangle_s)_{0\le t\le T}$ and $(\int_0^t \eta_s dB_s)_{0\le t\le T}$ are well defined, see Li and Peng (2011) and Peng (2019). The subsequent proposition can be interpreted as the Burkholder–Davis–Gundy (BDG) inequality within the G-expectation framework.

Proposition 2.6 (Peng (2019)) *If $\eta \in H_G^\alpha(0,T)$ with $\alpha \ge 1$ and $p \in (0,\alpha]$, then we have*

$$\underline{\sigma}^p c\widehat{\mathbb{E}}_t\bigg(\int_t^T |\eta_s|^2 ds\bigg)^{p/2} \le \widehat{\mathbb{E}}_t\bigg[\sup_{u\in[t,T]}\bigg|\int_t^u \eta_s dB_s\bigg|^p\bigg] \le \bar{\sigma}^p C\widehat{\mathbb{E}}_t\bigg(\int_t^T |\eta_s|^2 ds\bigg)^{p/2},$$

where $0<c<C<\infty$ are constants depending on p and T.

Let

$$S_G^0(0,T) = \Big\{h(t, B_{t_1\wedge t},\dots,B_{t_n\wedge t}) : t_1,\dots,t_n \in [0,T],\ h \in C_{b,Lip}(\mathbb{R}^{n+1})\Big\}.$$

For $p \ge 1$ and $\eta \in S_G^0(0,T)$, set

$$\|\eta\|_{S_G^p} = \bigg\{\widehat{\mathbb{E}} \sup_{t\in[0,T]} |\eta_t|^p\bigg\}^{1/p}.$$

Denote by $S_G^p(0,T)$ the completion of $S_G^0(0,T)$ under the norm $\|\cdot\|_{S_G^p}$. Li et al. (2018a) proved the following uniform continuity property for the processes in $S_G^p(0,T)$.

Proposition 2.7 (Li et al. (2018a)) *For $Y \in S_G^p(0,T)$ with $p \ge 1$, we have, by setting $Y_s = Y_T$ for $s > T$,*

$$\limsup_{\varepsilon\to 0}\widehat{\mathbb{E}}\bigg[\sup_{t\in[0,T]}\sup_{s\in[t,t+\varepsilon]}|Y_t - Y_s|^p\bigg] = 0.$$

3 The Skorokhod problem with mean reflection

In this section, we study the Skorokhod problem with mean reflection under the G-expectation. In Subsection 3.1, we rigorously define this problem in Definition 3.1, present our primary result in Theorem 3.2, and then illustrate with a concrete example in financial mathematics. In Subsection 3.2, we give the proof of Theorem 3.2. An alternative way to construct the solution is provided in Subsection 3.3. However, this approach necessitates an additional assumption (H_l') to attain Theorem 3.2, thus designated as Theorem 3.6. The benefit of this alternative lies in its establishment of a connection between the Skorokhod problem with mean reflection and the deterministic Skorokhod problem. Both methods of construction yield intermediary results vital for establishing the well-posedness of the mean reflected G-SDE (1.2) in Section 4.

3.1 Definition and an illustration

The assumptions below encapsulate the properties of the running loss function l and the original process S under consideration.

(H_l) The function $l : \Omega_T \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(1) $l(t,x)$ is uniformly continuous w.r.t. t and x, uniformly in ω.

(2) For any $t \in [0,T]$, $l(t,x)$ is strictly increasing in x, q.s..

(3) For any $(t,x) \in [0,T]\times\mathbb{R}$, $l(t,x) \in L_G^1(\Omega_T)$ and $\widehat{\mathbb{E}}[\lim_{x\uparrow\infty} l(t,x)] > 0$.

(4) For any $(t, x) \in [0, T] \times \mathbb{R}$, $|l(t, x)| \leq \kappa(1 + |x|)$ for some $\kappa > 0$, q.s..

(H_S) There exists some $p \geq 1$ such that $S \in S_G^p(0, T)$ and $\widehat{\mathbb{E}}[l(0, S_0)] \geq 0$.

Define $C[0, T]$ as the set of all real-valued deterministic continuous functions on $[0, T]$, and define $I[0, T]$ as the subset of $C[0, T]$ consisting of non-decreasing functions with initial value 0. We now proceed to provide the definition of the solution.

Definition 3.1 *Considering (l, S) satisfying (H_l) and (H_S), we define a pair of processes $(X, A) \in S_G^p(0, T) \times I[0, T]$ as a solution to the Skorokhod problem with mean reflection associated with (l, S), denoted as $\mathbb{SP}(l, S)$, if for $t \in [0, T]$,*

(a) $X_t = S_t + A_t$,

(b) $\widehat{\mathbb{E}}[l(t, X_t)] \geq 0$,

(c) $\int_0^T \widehat{\mathbb{E}}[l(t, X_t)] dA_t = 0$.

Now, we present the main result of this section whose proof is provided in Subsection 3.2.

Theorem 3.2 *Under Assumptions (H_l) and (H_S), there exists a unique solution $(X, A) \in S_G^p(0, T) \times I[0, T]$ to the Skorokhod problem $\mathbb{SP}(l, S)$.*

We illustrate the solution to the Skorokhod problem $\mathbb{SP}(l, S)$ with a concrete example below.

Example 3.3 *Let l be a function satisfying (H_l). For a fixed $t \in [0, T]$, we define a map $\rho_t : L_G^1(\Omega_t) \to \mathbb{R}$ as*

$$\rho_t(X) = \inf\Big\{x \in \mathbb{R} : \widehat{\mathbb{E}}[l(t, x + X)] \geq 0\Big\}.$$

It is easy to check that ρ_t is nonincreasing and translation invariant. That is

- *If $X, Y \in L_G^1(\Omega_t)$ with $X \leq Y$, then $\rho_t(X) \geq \rho_t(Y)$;*
- *$\rho_t(X + m) = \rho_t(X) - m$, for $X \in L_G^1(\Omega_t)$ and $m \in \mathbb{R}$.*

Therefore, ρ_t can be regarded as a static risk measure. The risk position X is called acceptable at time t if $\rho_t(X) \leq 0$. In fact, suppose that $\rho_t(X) > 0$, the value $\rho_t(X)$ can be regarded as the amount of money to be added by an agent in order to make the risk position X acceptable at time t. The readers may refer to Artzner et al. (1999) for the background of risk measures.

Consider an agent who wants to hold a stock evolving according to

$$S_t = S_0 + \int_0^t \mu(S_s) ds + \int_0^t \sigma(S_s) dB_s,$$

where $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions. Given the dynamic risk measure $\{\rho_t\}_{t \in [0,T]}$, one can ask how to make sure that the risk position S_t remains acceptable at each time t. In order to satisfy this constraint, the agent need to add some cash in it. We denote by A_t the cumulative amount of cash needed to be added and X_t be the associated value process. Then, we have

$$X_t = S_0 + \int_0^t \mu(S_s) ds + \int_0^t \sigma(S_s) dB_s + A_t.$$

Clearly, the agent would like to cover the risk in a minimal way, which leads to the condition

$$\int_0^T \widehat{\mathbb{E}}[l(t, X_t)] dA_t = 0.$$

That is, (X, A) is the solution of a Skorokhod problem $\mathbb{SP}(l, S)$.

3.2 First construction of solutions

The first method to establish the existence of the Skorokhod problem $\mathbb{SP}(l,S)$ relies on the following two propositions.

Proposition 3.4 (Liu and Wang (2019)) *Let (H_l) hold and $X \in L_G^1(\Omega_T)$. Then*

(i) for each $(t,x) \in [0,T] \times \mathbb{R}$, $l(t,x+X) \in L_G^1(\Omega_T)$,

(ii) the map $x \to l(t,x+X)$ is continuous under the norm $\|\cdot\|_{L_G^1}$; in particular, $x \to \widehat{\mathbb{E}}[l(t,x+X)]$ is continuous and strictly increasing.

Proposition 3.5 *Let (H_l) hold and $S \in S_G^p(0,T)$ where $p \geq 1$. Then the map $t \to l(t,S_t)$ is continuous under the norm $\|\cdot\|_{L_G^1}$; in particular, $t \to \widehat{\mathbb{E}}[l(t,S_t)]$ is continuous.*

Proof. By Assumption (H_l), for any $\varepsilon > 0$, there exists a constant $\delta > 0$, such that $|l(t,x)-l(s,y)| \leq \varepsilon$ for any $|t-s|+|x-y| \leq \delta$. It is easy to check that, for any $|s-t| \leq \delta$,

$$\begin{aligned}\widehat{\mathbb{E}}|l(t,S_t)-l(s,S_s)| \leq& \widehat{\mathbb{E}}|l(t,S_t)-l(s,S_t)| + \widehat{\mathbb{E}}\Big[|l(s,S_t)-l(s,S_s)|\mathbb{1}_{\{|S_t-S_s|>\delta\}}\Big] \\ &+ \widehat{\mathbb{E}}\Big[|l(s,S_t)-l(s,S_s)|\mathbb{1}_{\{|S_t-S_s|\leq\delta\}}\Big] \\ \leq& 2\varepsilon + C\widehat{\mathbb{E}}\Big[\Big(1+\sup_{t\in[0,T]}|S_t|\Big)\mathbb{1}_{\{|S_t-S_s|>\delta\}}\Big].\end{aligned}$$

By Proposition 2.7 and Markov's inequality, we have

$$\lim_{s\to t} V(\{|S_t-S_s|>\delta\}) = 0.$$

Then by Lemma 2.3,

$$\limsup_{s\to t} \widehat{\mathbb{E}}|l(t,S_t)-l(s,S_s)| \leq 2\varepsilon.$$

Since ε can be chosen arbitrarily small, the proof is complete. ∎

Proof of Theorem 3.2. The proof proceeds in two steps, where we prove existence in the first step and then uniqueness in the second step.

Step 1. By Proposition 3.4, $\widehat{\mathbb{E}}[l(t,x+X)]$ is well-defined for $X \in L_G^1(\Omega_T)$. In order to solve the Skorokhod problem $\mathbb{SP}(l,S)$, we need to use the operator $L_t : L_G^1(\Omega_T) \to [0,\infty)$ defined as follows:

$$L_t(X) = \inf\Big\{x \geq 0 : \widehat{\mathbb{E}}[l(t,x+X)] \geq 0\Big\}. \tag{3.1}$$

Under Assumption (H_l), the operator L_t is well-defined since by Lemma 2.4,

$$\lim_{x\to\infty} \widehat{\mathbb{E}}[l(t,x+X)] = \widehat{\mathbb{E}} \lim_{x\to\infty} l(t,x+X) = \widehat{\mathbb{E}} \lim_{x\to\infty} l(t,x) > 0.$$

It follows from Propositions 3.4 and 3.5 that the map $x \to \widehat{\mathbb{E}}[l(t,x+S_t)]$ is continuous and strictly increasing, and the map $t \to \widehat{\mathbb{E}}[l(t,x+S_t)]$ is continuous. We first prove that the map $t \to L_t(S_t)$ is continuous. First, suppose that $\widehat{\mathbb{E}}[l(t,S_t)] > 0$, which by the definition of $L_t(S_t)$ yields that $L_t(S_t) = 0$. Note that

$$\lim_{s\to t} \widehat{\mathbb{E}}[l(s,X_s)] = \widehat{\mathbb{E}}[l(t,X_t)] > 0.$$

Then, if $|s-t|$ is small enough, we have $\widehat{\mathbb{E}}[l(s,X_s)] > 0$ and consequently, $L_s(X_s) = 0$. Second, suppose that $\widehat{\mathbb{E}}[l(t,S_t)] \leq 0$. For any $\varepsilon > 0$, we have

$$\lim_{s\to t} \widehat{\mathbb{E}}[l(s, L_t(S_t) - \varepsilon + S_s)] = \widehat{\mathbb{E}}[l(t, L_t(S_t) - \varepsilon + S_t)] < \widehat{\mathbb{E}}[l(t, L_t(S_t) + S_t)] = 0$$
$$\text{and} \quad 0 < \widehat{\mathbb{E}}[l(t, L_t(S_t) + \varepsilon + S_t)] = \lim_{s\to t} \widehat{\mathbb{E}}[l(s, L_t(S_t) + \varepsilon + S_s)],$$

where we have used Proposition 2.2 in the inequality. Then, if $|s-t|$ is small enough, we have

$$\widehat{\mathbb{E}}[l(s, L_t(S_t) - \varepsilon + S_s)] < 0 < \widehat{\mathbb{E}}[l(s, L_t(S_t) + \varepsilon + S_s)],$$

which implies that $|L_s(S_s) - L_t(S_t)| \leq \varepsilon$. Therefore, the map $t \to L_t(S_t)$ is continuous. Define the function A by setting

$$A_t = \sup_{s\in[0,t]} L_s(S_s),$$

and then define

$$X_t = S_t + A_t.$$

We are going to show that (X,A) is the solution to the Skorokhod problem $\mathbb{SP}(l,S)$. In fact, it is clear that $(X,A) \in S_G^p(0,T) \times I[0,T]$ and

$$\widehat{\mathbb{E}}[l(t,X_t)] = \widehat{\mathbb{E}}[l(t,S_t + A_t)] \geq \widehat{\mathbb{E}}[l(t, S_t + L_t(S_t))] \geq 0.$$

By the definition of A, we have $A_t = L_t(S_t)$, dA_t-a.e. and $\mathbb{1}_{\{L_t(S_t)=0\}} = 0$ dA_t-a.e. Noting that $\widehat{\mathbb{E}}[l(t, S_t + L_t(S_t))] = 0$ when $L_t(S_t) > 0$, we finally have

$$\int_0^T \widehat{\mathbb{E}}[l(t,X_t)]dA_t = \int_0^T \widehat{\mathbb{E}}[l(t, S_t + L_t(S_t))]dA_t = \int_0^T \widehat{\mathbb{E}}[l(t, S_t + L_t(S_t))]\mathbb{1}_{\{L_t(S_t)>0\}}dA_t = 0.$$

Step 2. We now prove uniqueness. Suppose that (X^1,A^1) and (X^2,A^2) are two solutions to the Skorokhod problem $\mathbb{SP}(l,S)$. Suppose that there exists some $t \in (0,T)$, such that $A_t^1 < A_t^2$. Set

$$\tau = \sup\Big\{u \leq t : A_u^1 = A_u^2\Big\}.$$

It is easy to check that for $u \in (\tau, t]$, $A_u^1 < A_u^2$. Due to the strict increasing property of l and Proposition 2.2, for any $u \in (\tau, t]$, we have

$$0 \leq \widehat{\mathbb{E}}[l(u, S_u + A_u^1)] < \widehat{\mathbb{E}}[l(u, S_u + A_u^2)].$$

The flat-off condition (c) in Definition 3.1 implies that $dA^2 = 0$ on the interval $[\tau, t]$. It follows that

$$A_\tau^2 = A_t^2 > A_t^1 \geq A_\tau^1,$$

which contradicts the definition of τ. The proof is complete. ■

3.3 Second construction of solutions

In this subsection, we offer an alternative construction for the solution to the Skorokhod problem with mean reflection. The advantage of this approach lies in its ability to establish a connection between the Skorokhod problem with mean reflection and the deterministic Skorokhod problem. However, a drawback is that it requires an additional assumption (H_l') to achieve Theorem 3.2, hence labeled as Theorem 3.6.

Theorem 3.6 *Suppose Assumption (H_l) and (H_S) hold, as well as the following condition:*

(H_l') There exist an increasing function $F : [0, \infty) \to [0, \infty)$ with $F(0) = 0$ and two constants $0 < c_l < C_l$, such that

(1) For any $t, s \in [0, T]$, $x \in \mathbb{R}$

$$|l(t, x) - l(s, x)| \leq F(|t - s|).$$

(2) For any $t \in [0, T]$ and $x, y \in \mathbb{R}$,

$$c_l|x - y| \leq |l(t, x) - l(t, y)| \leq C_l|x - y|. \tag{3.2}$$

Then there exists a unique solution $(X, A) \in S_G^p(0, T) \times I[0, T]$ to the Skorokhod problem $\mathbb{SP}(l, S)$.

Both assumptions (H_l) and (H_l') impose regularity conditions on the function l. We delineate the comparison in the following remark.

Remark 3.7 *Clearly, Assumption (H_l') yields that l is uniformly continuous in (t, x), which is (H_l) (1). Next, suppose (H_l') (2) holds true, and then for any $X \in L_G^1(\Omega_T)$, $L_t(X)$ is well-defined if we consider (H_l) but omit $\widehat{\mathbb{E}}[\lim_{x\to\infty} l(t, x)] > 0$ in (H_l) (3). In fact, for any $x \geq 0$, (H_l') (2) implies that*

$$l(t, x + X) - l(t, X) \geq c_l x.$$

It follows that

$$\lim_{x\to\infty} \widehat{\mathbb{E}}[l(t, x + X)] \geq \lim_{x\to\infty} (\widehat{\mathbb{E}}[l(t, X)] + c_l x) = \infty. \tag{3.3}$$

Furthermore, under Assumption (H_l) and (H_l') (2), Lemma 3.12 in Liu and Wang (2019) indicates that for any $X, Y \in L_G^1(\Omega_T)$, we have

$$|L_t(X) - L_t(Y)| \leq \frac{C_l}{c_l} \widehat{\mathbb{E}}[|X - Y|], \text{ for any } t \in [0, T].$$

For any $t \in [0, T]$ and $Y \in L_G^1(\Omega_T)$, recalling that Proposition 3.4 ensures that $l(t, Y - \widehat{\mathbb{E}}[Y] + z) \in L_G^1(\Omega_T)$ for each fixed $z \in \mathbb{R}$, we define a mapping $H(t, \cdot, Y) : \mathbb{R} \to \mathbb{R}$ by

$$H(t, z, Y) = \widehat{\mathbb{E}}[l(t, Y - \widehat{\mathbb{E}}[Y] + z)]. \tag{3.4}$$

Lemma 3.8 *Suppose that l satisfies (H_l) and (H_l'). Then, for any $t \in [0, T]$ and $Y \in L_G^1(\Omega_T)$, $H(t, \cdot, Y)$ is strictly increasing, continuous and*

$$\lim_{z\to-\infty} H(t, z, Y) = -\infty, \quad \lim_{z\to+\infty} H(t, z, Y) = +\infty.$$

Proof. For any $z, z' \in \mathbb{R}$, it is easy to check that

$$\begin{aligned}&\left|\widehat{\mathbb{E}}[l(t, Y - \widehat{\mathbb{E}}[Y] + z)] - \widehat{\mathbb{E}}[l(t, Y - \widehat{\mathbb{E}}[Y] + z')]\right| \\ \le& \widehat{\mathbb{E}}\Big[\Big|l(t, Y - \widehat{\mathbb{E}}[Y] + z) - l(t, Y - \widehat{\mathbb{E}}[Y] + z')\Big|\Big] \le C_l|z - z'|.\end{aligned}$$

Hence, $H(t, \cdot, Y)$ is continuous. Suppose that $z < z'$. Proposition 3.4 (ii) yields that

$$H(t, z, Y) = \widehat{\mathbb{E}}[l(t, Y - \widehat{\mathbb{E}}[Y] + z)] < \widehat{\mathbb{E}}[l(t, Y - \widehat{\mathbb{E}}[Y] + z')] = H(t, z', Y),$$

which implies that $H(t, \cdot, Y)$ is strictly increasing. The last assertion can be proved similarly with (3.3). The proof is complete. ■

By Lemma 3.8, we may define the inverse map $H^{-1}(t, \cdot, Y) : \mathbb{R} \to \mathbb{R}$. In fact, for any $z \in \mathbb{R}$,

$$H^{-1}(t, z, Y) = \bar{z} \Longleftrightarrow \widehat{\mathbb{E}}[l(t, Y - \widehat{\mathbb{E}}[Y] + \bar{z})] = z.$$

Lemma 3.9 *Suppose that l satisfies (H_l) and (H_l') and $Y \in S_G^p(0,T)$ with $p \geq 1$. If $\bar{z} = \{\bar{z}_t\}_{t\in[0,T]} \in C[0,T]$, then $z = \{H(t, \bar{z}_t, Y_t)\}_{t\in[0,T]} \in C[0,T]$. Similarly, if $z = \{z_t\}_{t\in[0,T]} \in C[0,T]$, then $\bar{z} = \{H^{-1}(t, z_t, Y_t)\}_{t\in[0,T]} \in C[0,T]$.*

Proof. For any $\bar{z} \in C[0,T]$ and any $s, t \in [0,T]$, it is easy to check that

$$|z_t - z_s| \leq F(|t - s|) + C_l\Big\{|\bar{z}_t - \bar{z}_s| + 2\widehat{\mathbb{E}}|Y_t - Y_s|\Big\}.$$

By Proposition 2.7, we have $z \in C[0,T]$. It remains to prove the second assertion. Given a sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$ with $\lim_{n\to n} t_n = t$, we first claim that the sequence $\{\bar{z}_{t_n}\}$ is bounded. Otherwise, there exists a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$, such that $\lim_{k\to\infty} \bar{z}_{t_{n_k}} = \infty$. By Proposition 2.7 and the Lipschitz property of l in (t, x), we have

$$\lim_{k\to\infty} \left|\widehat{\mathbb{E}}[l(t, Y_t - \widehat{\mathbb{E}}[Y_t] + \bar{z}_{t_{n_k}})] - \widehat{\mathbb{E}}[l(t_{n_k}, Y_{t_{n_k}} - \widehat{\mathbb{E}}[Y_{t_{n_k}}] + \bar{z}_{t_{n_k}})]\right| = 0.$$

Lemma 3.8 indicates that

$$\lim_{k\to\infty} \widehat{\mathbb{E}}[l(t, Y_t - \widehat{\mathbb{E}}[Y_t] + \bar{z}_{t_{n_k}})] = \infty.$$

Hence, we deduce that

$$z_t = \lim_{k\to\infty} z_{t_{n_k}} = \lim_{k\to\infty} \widehat{\mathbb{E}}[l(t_{n_k}, Y_{t_{n_k}} - \widehat{\mathbb{E}}[Y_{t_{n_k}}] + \bar{z}_{t_{n_k}})] = \infty,$$

which is a contradiction. To show that $\bar{z}_{t_n} \to \bar{z}_t$, it suffices to prove that for any subsequence $\{n'\} \subseteq \mathbb{N}$, one can choose a subsequence $\{n''\} \subseteq \{n'\}$ such that $\bar{z}_{t_{n''}} \to \bar{z}_t$. Since $\{\bar{z}_{t_{n'}}\}$ is bounded, there exists a subsequence $\{n''\} \subseteq n'$ such that $\bar{z}_{t_{n''}} \to z''$. By the definition of $\bar{z}$, we have

$$\widehat{\mathbb{E}}[l(t_{n''}, Y_{t_{n''}} - \widehat{\mathbb{E}}[Y_{t_{n''}}] + \bar{z}_{t_{n''}})] = z_{t_{n''}}.$$

Letting n'' goes to infinity, we obtain that

$$\widehat{\mathbb{E}}[l(t, Y_t - \widehat{\mathbb{E}}[Y_t] + \bar{z}'')] = z_t,$$

which implies that $z'' = \bar{z}_t$ by the definition of $\bar{z}$. The proof is complete. ■

In the following we give the proof of Theorem 3.6, which very different to that of Theorem 3.2.
Proof of Theorem 3.6. For any $t \in [0,T]$, set

$$s_t = \widehat{\mathbb{E}}[S_t] \quad \text{and} \quad \bar{l}_t = H^{-1}(t,0,S_t).$$

By Proposition 2.7 and Lemma 3.9, we have $s = \{s_t\}_{t\in[0,T]} \in C[0,T]$ and $\bar{l} = \{\bar{l}_t\}_{t\in[0,T]} \in C[0,T]$. The trivial equality

$$\widehat{\mathbb{E}}[l(0, S_0 - \widehat{\mathbb{E}}[S_0] + \widehat{\mathbb{E}}[S_0])] = \widehat{\mathbb{E}}[l(0,S_0)]$$

implies that

$$\widehat{\mathbb{E}}[S_0] = H^{-1}(0, \widehat{\mathbb{E}}[l(0,S_0)], S_0).$$

Noting that $H^{-1}(t,\cdot,Y)$ is strictly increasing for any $t \in [0,T]$ and $Y \in L^1_G(\Omega_T)$ and $\widehat{\mathbb{E}}[l(0,S_0)] \geq 0$, we have $\widehat{\mathbb{E}}[S_0] \geq \bar{l}_0$. Now, let (x,A) be the unique solution of the Skorokhod problem $\mathrm{SP}(\bar{l},s)$, which is defined to satisfy the following conditions:

(1') $x_t = s_t + A_t \geq \bar{l}_t$, $t \in [0,T]$,

(2') $A \in I[0,T]$ and $\int_0^T (x_t - \bar{l}_t)dA_t = 0$.

In fact, $A_t = \sup_{s\in[0,t]}(s_t - \bar{l}_t)^-$. Set $X_t = S_t + A_t$. We claim that (X,A) is the solution to the Skorokhod problem $\mathbb{SP}(l,S)$. First, simple calculation implies that

$$\widehat{\mathbb{E}}[l(t,X_t)] = \widehat{\mathbb{E}}[l(t,S_t + A_t)] = \widehat{\mathbb{E}}[l(t, S_t - \widehat{\mathbb{E}}[S_t] + x_t)] = H(t,x_t,S_t) \geq H(t,\bar{l}_t,S_t) = 0.$$

The above equation also indicates that $x_t = H^{-1}(t, \widehat{\mathbb{E}}[l(t,X_t)], S_t)$. Furthermore, note that

$$x_t > \bar{l}_t \Longleftrightarrow \widehat{\mathbb{E}}[l(t, S_t - \widehat{\mathbb{E}}[S_t] + x_t)] > 0 \Longleftrightarrow \widehat{\mathbb{E}}[l(t,X_t)] > 0,$$

which, together with the above condition (2'), implies that $\int_0^T \widehat{\mathbb{E}}[l(t,X_t)]dA_t = 0$.

It remains to prove the uniqueness. Suppose that (X',A') is another solution of the Skorokhod problem $\mathbb{SP}(l,S)$. Similar arguments as above could show that (x',A') where $x'_t = H^{-1}(t,\widehat{\mathbb{E}}[l(t,X'_t)],S_t)$, is the solution of Skorokhod problem $\mathrm{SP}(\bar{l},s)$. Due to the uniqueness of solutions to classical Skorokhod problem, we have $A = A'$. Consequently, $X = X'$. The proof is complete. ■

The preceding proof establishes the connection between the solution to a Skorokhod problem with mean reflection and the solution to a classical Skorokhod problem, thereby aiding in obtaining the following a priori estimates.

Proposition 3.10 *Suppose l^i and S^i satisfy Assumptions (H_l), (H'_l), and (H_S), for $i = 1,2$. Let (X^i,A^i) be the solution of the Skorokhod problem $\mathbb{SP}(l^i,S^i)$, $i = 1,2$. Then, there exists a constant C depending on c_l, C_l, such that*

$$\sup_{t\in[0,T]} |A^1_t - A^2_t| \leq C\left\{ \sup_{t\in[0,T]} \widehat{\mathbb{E}} \sup_{x\in\mathbb{R}} |l^1(t,x) - l^2(t,x)| + \sup_{t\in[0,T]} \widehat{\mathbb{E}}|S^1_t - S^2_t| \right\},$$
$$\textit{and} \quad \widehat{\mathbb{E}} \sup_{t\in[0,T]} |X^1_t - X^2_t| \leq C\left\{ \sup_{t\in[0,T]} \widehat{\mathbb{E}} \sup_{x\in\mathbb{R}} |l^1(t,x) - l^2(t,x)| + \widehat{\mathbb{E}} \sup_{t\in[0,T]} |S^1_t - S^2_t| \right\}.$$

Proof. It suffices to prove the first estimate, for the reason that the second estimate can be obtained by the representation of X and the triangle inequality. By the proof of Theorem 3.2, (x^i,A^i) are the solutions of the Skorokhod problem $\mathrm{SP}(\bar{l}^i, s^i)$, where $s^i_t = \widehat{\mathbb{E}}[S^i_t]$, $\bar{l}^i_t = H_i^{-1}(t, 0, S^i_t)$ and $H_i^{-1}(t, \cdot, S^i_t)$ is the inverse map of $H_i(t, \cdot, S^i_t) = \widehat{\mathbb{E}}[l^i(t, S^i_t - \widehat{\mathbb{E}}[S^i_t] + \cdot)]$, for $i = 1, 2$. Since

$$A^i_t = \sup_{s\in[0,t]} (s^i_s - \bar{l}^i_s)^-,$$

we have

$$\sup_{t\in[0,T]} |A^1_t - A^2_t| \leq \sup_{t\in[0,T]} |s^1_t - s^2_t| + \sup_{t\in[0,T]} |\bar{l}^1_t - \bar{l}^2_t|. \tag{3.5}$$

It is easy to check that

$$0 = \widehat{\mathbb{E}}[l^1(t, S^1_t - \widehat{\mathbb{E}}[S^1_t] + \bar{l}^1_t)] - \widehat{\mathbb{E}}[l^2(t, S^2_t - \widehat{\mathbb{E}}[S^2_t] + \bar{l}^2_t)] =: \mathcal{I}^1_t + \mathcal{I}^2_t + \mathcal{I}^3_t,$$

where

$$\begin{aligned}
\mathcal{I}^1_t =& \widehat{\mathbb{E}}[l^1(t, S^1_t - \widehat{\mathbb{E}}[S^1_t] + \bar{l}^1_t)] - \widehat{\mathbb{E}}[l^2(t, S^1_t - \widehat{\mathbb{E}}[S^1_t] + \bar{l}^1_t)],\\
\mathcal{I}^2_t =& \widehat{\mathbb{E}}[l^2(t, S^1_t - \widehat{\mathbb{E}}[S^1_t] + \bar{l}^1_t)] - \widehat{\mathbb{E}}[l^2(t, S^2_t - \widehat{\mathbb{E}}[S^2_t] + \bar{l}^1_t)],\\
\mathcal{I}^3_t =& \widehat{\mathbb{E}}[l^2(t, S^2_t - \widehat{\mathbb{E}}[S^2_t] + \bar{l}^1_t)] - \widehat{\mathbb{E}}[l^2(t, S^2_t - \widehat{\mathbb{E}}[S^2_t] + \bar{l}^2_t)].
\end{aligned}$$

Without loss of generality, assume that $\bar{l}^1_t > \bar{l}^2_t$. By the assumption of l^2, for any $(t, x) \in [0, T] \times \mathbb{R}$, we have

$$c_l(\bar{l}^1_t - \bar{l}^2_t) \leq l^2(t, x + \bar{l}^1_t) - l^2(t, x + \bar{l}^2_t) \leq C_l(\bar{l}^1_t - \bar{l}^2_t).$$

Consequently,

$$c_l(\bar{l}^1_t - \bar{l}^2_t) \leq \mathcal{I}^3_t \leq C_l(\bar{l}^1_t - \bar{l}^2_t).$$

Hence, there exists a constant $C \in [c_l, C_l]$, such that $\mathcal{I}^3_t = C(\bar{l}^1_t - \bar{l}^2_t)$. The above analysis indicates that

$$|\bar{l}^1_t - \bar{l}^2_t| \leq \frac{1}{c_l}|I^3_t| \leq \frac{1}{c_l}(|\mathcal{I}^1_t| + |\mathcal{I}^2_t|) \leq \frac{1}{c_l}\left\{\widehat{\mathbb{E}} \sup_{x\in\mathbb{R}} |l^1(t, x) - l^2(t, x)| + 2C_l \widehat{\mathbb{E}}|S^1_t - S^2_t|\right\}.$$

Plugging the above inequality into Equation (3.5), we obtain the desired result. ■

Corollary 3.11 *Suppose l satisfies (H_l) and (H'_l) and S satisfies (H_S). Let (X, A) be the solution of the Skorokhod problem $\mathbb{SP}(l, S)$. For any $0 \leq s \leq t \leq T$, we have*

$$|A_t - A_s| \leq C\left\{\sup_{r\in[s,t]} \widehat{\mathbb{E}}|S_r - S_s| + F(|t - s|)\right\},$$

$$\textit{and} \quad \widehat{\mathbb{E}} \sup_{r\in[s,t]} |X_r - X_s| \leq C\left\{\widehat{\mathbb{E}} \sup_{r\in[s,t]} |S_r - S_s| + F(|t - s|)\right\}.$$

Proof. We only prove the first estimate, while the second estimate can be obtained by the representation of X and the triangle inequality. For any fixed $s \in [0, T]$ and $r \in [0, T]$, set

$$S'_r = S_{r\wedge s} \quad \text{and} \quad l'(r, x) = l(r \wedge s, x).$$

Let (X', A') be the solution to the Skorokhod problem $\mathbb{SP}(l', S')$. It is easy to check that for any $r \in [0, T]$,

$$X'_r = X_{r\wedge s} \quad \text{and} \quad A'_r = A_{r\wedge s}.$$

By Proposition 3.10, we have

$$\begin{aligned}
|A_t - A_s| &\leq \sup_{r\in[s,t]} |A_r - A'_r| \\
&\leq C\left\{\sup_{r\in[s,t]} \widehat{\mathbb{E}} \sup_{x\in\mathbb{R}} |l(r,x) - l'(r,x)| + \sup_{r\in[s,t]} \widehat{\mathbb{E}}|S_r - S'_r|\right\} \\
&\leq C\left\{\sup_{r\in[s,t]} \widehat{\mathbb{E}}|S_r - S_s| + \sup_{r\in[s,t]} F(|r-s|)\right\} \\
&\leq C\left\{\sup_{r\in[s,t]} \widehat{\mathbb{E}}|S_r - S_s| + F(|t-s|)\right\},
\end{aligned}$$

as desired. ■

4 Mean reflected G-SDEs

In this section, we establish the well-posedness of the mean reflected G-SDE (1.2), recalled here as follows: for $t \in [0, T]$,

$$\begin{cases} X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t h(x, X_s)d\langle B\rangle_s + \int_0^t \sigma(s, X_s)dB_s + A_t, \\ \widehat{\mathbb{E}}[l(t, X_t)] \geq 0 \quad \text{and} \quad \int_0^T \widehat{\mathbb{E}}[l(t, X_t)]dA_t = 0. \end{cases}$$

We consider the coefficient functions $b, h, \sigma : \Omega_T \times [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(A1) For some $p \geq 1$ and each $x \in \mathbb{R}$, $b(\cdot, \cdot, x), h(\cdot, \cdot, x) \in M_G^p(0, T)$ and $\sigma(\cdot, \cdot, x) \in H_G^p(0, T)$;

(A2) For any $(\omega, t) \in \Omega_T \times [0, T]$ and any $x, x' \in \mathbb{R}$, there exists a constant $\kappa > 0$, such that

$$|b(\omega, t, x) - b(\omega, t, x')| + |h(\omega, t, x) - h(\omega, t, x')| + |\sigma(\omega, t, x) - \sigma(\omega, t, x')| \leq \kappa|x - x'|.$$

Now, we state the main result of this section.

Theorem 4.1 *Suppose Assumptions (A1), (A2), (H_l) and (H'_l) hold and $\widehat{\mathbb{E}}[l(0, x_0)] \geq 0$. The mean reflected G-SDE (1.2) admits a unique pair of solution $(X, A) \in S_G^p(0, T) \times I[0, T]$.*

Proof. First, we fix a positive constant δ, which will be determined later. Given a process $U \in S_G^p(0, \delta)$, for any $t \in [0, \delta]$, set

$$\widetilde{X}_t = x_0 + \int_0^t b(s, U_s)ds + \int_0^t h(s, U_s)d\langle B\rangle_s + \int_0^t \sigma(s, U_s)dB_s.$$

By Assumptions (A1) and (A2), Hölder's inequality and the BDG inequality under the G-expectation (Proposition 2.6), we may check that $\widetilde{X} \in S_G^p(0, \delta)$. Theorem 3.2 indicates that there exists a unique solution (X, A) to the Skorokhod problem $\mathbb{SP}(l, \widetilde{X})$ on the time interval $[0, \delta]$. We define the mapping $\Gamma : S_G^p(0, \delta) \to S_G^p(0, T)$ as

$$\Gamma(U) = X.$$

We then show that Γ is a contraction mapping when δ is small enough. Similarly, for given $U' \in S_G^p(0,\delta)$, define $\widetilde{X}'$ as above. Let (X,A) and (X',A') be the solutions to the Skorokhod problem associated with $(l,\widetilde{X})$ and $(l,\widetilde{X}')$, respectively. We define

$$\widehat{b}_t = b(t,U_t) - b(t,U_t'), \quad \widehat{h}_t = h(t,U_t) - h(t,U_t'), \quad \widehat{\sigma}_t = \sigma(t,U_t) - \sigma(t,U_t'),$$
$$\widehat{A}_t = A_t - A_t' \quad \text{and} \quad \widehat{X}_t = X_t - X_t'.$$

Simple calculation implies that

$$\begin{aligned}
\widehat{\mathbb{E}} \sup_{t\in[0,\delta]} |\widehat{X}_t|^p \leq & C\left\{ \widehat{\mathbb{E}} \sup_{t\in[0,\delta]} |\widetilde{X}_t - \widetilde{X}_t'|^p + \sup_{t\in[0,\delta]} |\widehat{A}_t|^p \right\} \\
\leq & C\left\{ \widehat{\mathbb{E}} \sup_{t\in[0,\delta]} \left| \int_0^t \widehat{b}_s ds \right|^p + \widehat{\mathbb{E}} \sup_{t\in[0,\delta]} \left| \int_0^t \widehat{h}_s d\langle B\rangle_s \right|^p \right. \\
& \left. + \widehat{\mathbb{E}} \sup_{t\in[0,\delta]} \left| \int_0^t \widehat{\sigma}_s dB_s \right|^p + \sup_{t\in[0,\delta]} |\widehat{A}_t|^p \right\} \\
\leq & C\left\{ \widehat{\mathbb{E}} \int_0^\delta |\widehat{U}_t|^p ds + \sup_{t\in[0,\delta]} |\widehat{A}_t|^p \right\},
\end{aligned} \tag{4.1}$$

where we have used Hölder's inequality and the BDG inequality under the G-expectation (Proposition 2.6). Here, C is a constant depending only on $p, \delta, \kappa, \overline{\sigma}, \underline{\sigma}$. Recalling the first proof of Theorem 3.2, we have

$$\sup_{t\in[0,\delta]} |\widehat{A}_t|^p \leq \sup_{t\in[0,\delta]} |L_t(\widetilde{X}_t) - L_t(\widetilde{X}_t')|^p \leq \frac{C_l^p}{c_l^p} \sup_{t\in[0,\delta]} \widehat{\mathbb{E}}|\widetilde{X}_t - \widetilde{X}_t'|^p \leq C\widehat{\mathbb{E}} \int_0^\delta |\widehat{U}_t|^p ds, \tag{4.2}$$

where we have used Remark 3.7 in the second inequality and C is a constant depending on p, δ, κ, $\overline{\sigma}, \underline{\sigma}$, C_l, c_l. Equations (4.1) and (4.2) yield that

$$\widehat{\mathbb{E}} \sup_{t\in[0,\delta]} |\widehat{X}_t|^p \leq C\delta \widehat{\mathbb{E}} \sup_{t\in[0,\delta]} |\widehat{U}_t|^p.$$

Therefore, for δ being sufficiently small, Γ is a contraction mapping. We then obtain the existence and uniqueness of the solution, denoted by $(X^{(1)}, A^{(1)})$, to the mean reflected G-SDE (1.2) on the interval $[0,\delta]$. Now, let N be such that $N = [\frac{T}{\delta}] + 1$. For any $2 \leq n \leq N$, by a similar analysis as above, the following reflected G-SDEs on the time interval $[(n-1)\delta, n\delta \wedge T]$ admits a unique solution $(X^{(n)}, A^{(n)})$:

$$\begin{cases}
X_t^{(n)} = X_{(n-1)\delta}^{(n-1)} + \int_{(n-1)\delta}^t b(s, X_s^{(n)})ds + \int_{(n-1)\delta}^t h(s, X_s^{(n)})d\langle B\rangle_s + \int_{(n-1)\delta}^t \sigma(s, X_s^{(n)})dB_s + A_t^{(n)}, \\
\widehat{\mathbb{E}}[l(t, X_t^{(n)})] \geq 0 \quad \text{and} \quad \int_{(n-1)\delta}^{n\delta\wedge T} \widehat{\mathbb{E}}[l(t, X_t^{(n)})]dA_t^n = 0.
\end{cases}$$

We define, for $t \in ((n-1)\delta, n\delta \wedge T]$ and $1 \leq n \leq N$,

$$X_t = X_t^{(1)} \mathbb{1}_{[0,\delta]}(t) + \sum_{n=2}^N X_t^{(n)} \mathbb{1}_{((n-1)\delta, n\delta\wedge T]}(t) \quad \text{and} \quad A_t = A_t^{(n)} + \sum_{j=1}^{n-1} A_{jh}^{(j)},$$

with the convention $\sum_{j=1}^0 A_{jh}^j = 0$. It is easy to check that (X,A) is the solution to the mean reflected G-SDE (1.2). The uniqueness follows from the uniqueness for each small interval. The proof is complete. ■

Let (X, A) be the solution to the mean reflected G-SDE (1.2) and we provide its some moment estimates in the proposition below. Set

$$U_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t h(s, X_s)d\langle B\rangle_s + \int_0^t \sigma(s, X_s)dB_s. \tag{4.3}$$

Proposition 4.2 *Suppose Assumptions (A1), (A2), (H_l), (H_l') hold and $\widehat{\mathbb{E}}[l(0, x_0)] \geq 0$. Then, there exists a constant C depending on $p, \kappa, C_l, c_l, \overline{\sigma}, \underline{\sigma}, T$, such that*

$$\begin{aligned} &\widehat{\mathbb{E}} \sup_{t\in[0,T]} |X_t|^p \qquad (4.4)\\ &\leq C\left(1 + |x_0|^p + \widehat{\mathbb{E}}\left[\int_0^T |b(s, 0)|^p ds\right] + \widehat{\mathbb{E}}\left[\int_0^T |h(s, 0)|^p ds\right] + \widehat{\mathbb{E}}\left(\int_0^T |\sigma(s, 0)|^2 ds\right)^{p/2}\right). \end{aligned}$$

Furthermore, suppose for each $x \in \mathbb{R}$, $b(\cdot, \cdot, x), h(\cdot, \cdot, x), \sigma(\cdot, \cdot, x) \in S_G^p(0, T)$. Then, for any $0 \leq s \leq t \leq T$,

$$|A_t - A_s| \leq C\Big\{|t-s|^{\frac{1}{2}} + F(|t-s|)\Big\} \quad \textit{and} \quad \widehat{\mathbb{E}}|X_t - X_s|^p \leq C\Big\{|t-s|^{\frac{p}{2}} + F^p(|t-s|)\Big\}. \tag{4.5}$$

Proof. Note that A can be viewed as the second component of the solution to the Skorokhod problem $\mathbb{SP}(l, U)$. By Corollary 3.11, for any $0 \leq s \leq t \leq T$, we have

$$|A_t| \leq C\left\{\sup_{r\in[0,t]} \widehat{\mathbb{E}}|U_r - U_0| + F(|t|)\right\} \leq C\left\{\sup_{r\in[0,t]} \widehat{\mathbb{E}}|U_r - x_0| + F(|T|)\right\}, \tag{4.6}$$

$$|A_t - A_s| \leq C\left\{\sup_{r\in[s,t]} \widehat{\mathbb{E}}|U_r - U_s| + F(|t-s|)\right\}. \tag{4.7}$$

Simple calculation gives that

$$\begin{aligned} \widehat{\mathbb{E}} \sup_{s\in[0,t]} |X_s|^p &\leq C\left\{\widehat{\mathbb{E}} \sup_{s\in[0,t]} |U_s|^p + |A_t|^p\right\} \\ &\leq C\left\{1 + |x_0|^p + \widehat{\mathbb{E}}\left[\int_0^t |b(s, 0)|^p ds\right] + \widehat{\mathbb{E}}\left[\int_0^t |h(s, 0)|^p ds\right]\right. \\ &\qquad \left. + \widehat{\mathbb{E}}\left(\int_0^t |\sigma(s, 0)|^2 ds\right)^{p/2} + \widehat{\mathbb{E}}\left[\int_0^t |X_s|^p ds\right]\right\} \\ &\leq C\left\{1 + |x_0|^p + \widehat{\mathbb{E}}\left[\int_0^t |b(s, 0)|^p ds\right] + \widehat{\mathbb{E}}\left[\int_0^t |h(s, 0)|^p ds\right]\right. \\ &\qquad \left. + \widehat{\mathbb{E}}\left(\int_0^t |\sigma(s, 0)|^2 ds\right)^{p/2} + \int_0^t \widehat{\mathbb{E}}\left[\sup_{r\in[0,s]} |X_r|^p\right] dr\right\}. \end{aligned}$$

Applying Grönwall's inequality, we obtain the moment estimate (4.4).

Next, by Hölder's inequality and Proposition 2.6, we have

$$
\begin{aligned}
\widehat{\mathbb{E}}|U_t-U_s|^p \leq & C\left\{\widehat{\mathbb{E}}\left|\int_s^t b(r,X_r)dr\right|^p+\widehat{\mathbb{E}}\left|\int_s^t h(r,X_r)d\langle B\rangle_r\right|^p+\widehat{\mathbb{E}}\left|\int_s^t \sigma(r,X_r)dB_r\right|^p\right\} \\
\leq & C\left\{\widehat{\mathbb{E}}\left|\int_s^t |b(r,X_r)|dr\right|^p+\widehat{\mathbb{E}}\left|\int_s^t |h(r,X_r)|dr\right|^p+\widehat{\mathbb{E}}\left(\int_s^t |\sigma(r,X_r)|^2dr\right)^{p/2}\right\} \\
\leq & C\left\{\widehat{\mathbb{E}}\left|\int_s^t \Big(\sup_{r\in[s,t]}|b(r,0)|+\sup_{r\in[s,t]}|h(r,0)|+\sup_{r\in[s,t]}|X_r|\Big)dr\right|^p\right. \\
& \left.\qquad\qquad +\widehat{\mathbb{E}}\left|\int_s^t \Big(\sup_{r\in[s,t]}|\sigma(r,0)|^2+\sup_{r\in[s,t]}|X_r|^2\Big)dr\right|^{p/2}\right\} \\
\leq & |t-s|^{p/2},
\end{aligned}
$$

where we have used the estimate (4.4) and the fact that $b,h,\sigma\in S_G^p(0,T)$ in the last step. Plugging the above inequality into equation (4.7), we obtain the first result in equation (4.5). Noting that

$$X_t-X_s=U_t-U_s+A_t-A_s,$$

we obtain the estimate for $\widehat{\mathbb{E}}|X_t-X_s|^p$. The proof is complete. ■

Proposition 4.2 establishes the continuity of the second component A in the solution to the mean reflected G-SDE. In the following, we show that subject to certain regularity conditions on the loss function l, the function A exhibits Lipschitz continuity.

Proposition 4.3 *Let Assumptions (A1) and (A2) hold with $p\geq 2$. Assume that $l\in C_b^{1,2}([0,T]\times\mathbb{R})$ is bi-Lipschtiz (i.e., satisfies (3.2)) and strictly increasing in its second component with $l(0,x_0)\geq 0$ and*

$$\sup_{t\in[0,T]}\Big(\widehat{\mathbb{E}}|b(t,0)|^2+\widehat{\mathbb{E}}|h(t,0)|^2+\widehat{\mathbb{E}}|\sigma(t,0)|^2\Big)<\infty,$$

where $C_b^{1,2}([0,T]\times\mathbb{R})$ is the space of all functions of class $C^{1,2}([0,T]\times\mathbb{R})$ whose partial derivatives are bounded. Let (X,A) be the solution to the mean reflected G-SDE (1.2). Then the function A is Lipschitz continuous.

Proof. We define the operator $\widetilde{L}_t : L_G^1(\Omega_T)\to\mathbb{R}$ as

$$\widetilde{L}_t(X)=\inf\Big\{x\in\mathbb{R}:\widehat{\mathbb{E}}[l(t,x+X)]\geq 0\Big\}. \tag{4.8}$$

Clearly, we have $L_t(X)=(\widetilde{L}_t(X))^+$. Let (X,A) be the solution to the Skorokhod problem $\mathbb{SP}(l,S)$. By the proof of Theorem 3.2, we have $A_t=\sup_{s\in[0,T]}(\widetilde{L}_s(S_s))^+$. We first prove that $t\to\widetilde{L}_t(U_t)$ is Lipschitz continuous on $[0,T]$. In fact, by the definition of $\widetilde{L}_t(U_t)$, we have

$$\widehat{\mathbb{E}}[l(t,\widetilde{L}_t(U_t)+U_t)]=0.$$

If $x\geq y$, since l is bi-Lipschitz and $\widehat{\mathbb{E}}[\cdot]$ is sub-additive, we obtain that

$$c_l(x-y)\leq-\widehat{\mathbb{E}}[l(t,y+U_t)-l(t,x+U_t)]\leq\widehat{\mathbb{E}}[l(t,x+U_t)]-\widehat{\mathbb{E}}[l(t,y+U_t)].$$

The above analysis implies that, for any $0 \le s < t \le T$,

$$
\begin{aligned}
|\widetilde{L}_s(U_s) - \widetilde{L}_t(U_t)| &\le \frac{1}{c_l}\Big|\widehat{\mathbb{E}}[l(t, \widetilde{L}_s(U_s) + U_t)] - \widehat{\mathbb{E}}[l(t, \widetilde{L}_t(U_t) + U_t)]\Big| \\
&= \frac{1}{c_l}\Big|\widehat{\mathbb{E}}[l(t, \widetilde{L}_s(U_s) + U_t)] - \widehat{\mathbb{E}}[l(s, \widetilde{L}_s(U_s) + U_s)]\Big|.
\end{aligned}
\tag{4.9}
$$

Applying G-Itô's formula (Li and Peng, 2011) to $l(t, \widetilde{L}_s(U_s) + U_t)$, we obtain that

$$
\begin{aligned}
l(t, \widetilde{L}_s(U_s) + U_t) =& l(s, \widetilde{L}_s(U_s) + U_s) + \int_s^t \Big[\partial_t l(r, \widetilde{L}_s(U_s) + U_r) + \partial_x l(r, \widetilde{L}_s(U_s) + U_r) b(r, X_r)\Big] dr \\
&+ \int_s^t \Big[\partial_x l(r, \widetilde{L}_s(U_s) + U_r) h(r, X_r) + \frac{1}{2}\partial_x^2 l(r, \widetilde{L}_s(U_s) + U_r)\sigma^2(r, X_r)\Big] d\langle B\rangle_r \\
&+ \int_s^t \partial_x l(r, \widetilde{L}_s(U_s) + U_r)\sigma(r, X_r) dB_r.
\end{aligned}
\tag{4.10}
$$

By Equations (4.9) and (4.10), together with the estimate (4.4) in Proposition 4.2, we have

$$
\begin{aligned}
&|\widetilde{L}_s(U_s) - \widetilde{L}_t(U_t)| \\
&\le \frac{1}{c_l}\widehat{\mathbb{E}}\Bigg[\Bigg|\int_s^t \Big[\partial_t l(r, \widetilde{L}_s(U_s) + U_r) + \partial_x l(r, \widetilde{L}_s(U_s) + U_r) b(r, X_r)\Big] dr \\
&\qquad + \int_s^t \Big[\partial_x l(r, \widetilde{L}_s(U_s) + U_r) h(r, X_r) + \frac{1}{2}\partial_x^2 l(r, \widetilde{L}_s(U_s) + U_r)\sigma^2(r, X_r)\Big] d\langle B\rangle_r\Bigg|\Bigg] \\
&\le \frac{C}{c_l}\widehat{\mathbb{E}}\Bigg[\int_s^t \Big(1 + |b(r,0)| + |h(r,0)| + |\sigma(r,0)|^2 + |X_r| + |X_r|^2\Big) dr\Bigg] \\
&\le C|t-s|.
\end{aligned}
$$

Now we are ready to prove that A is Lipschitz continuous. Since $L_t(U_t) = (\widetilde{L}_t(U_t))^+$, $L_\cdot(U_\cdot)$ is Lipschitz continuous with Lipschitz constant C. Note that A can be viewed as the second component of the solution to the Skorokhod problem $\mathbb{SP}(l, U)$. By the first proof of Theorem 3.2, for any $0 \le s \le t \le T$, we have

$$
A_t = \sup_{r\in[0,t]} L_r(U_r) = \max\left(\sup_{r\in[0,s]} L_r(U_r),\ \sup_{r\in[s,t]} L_r(U_r)\right) = \max\left(A_s,\ \sup_{r\in[s,t]} L_r(U_r)\right).
$$

For any $r \in [s,t]$, it is easy to check that

$$
L_r(U_r) \le L_s(U_s) + C(r-s) \le A_s + C(t-s),
$$

which implies that $\sup_{r\in[s,t]} L_r(U_r) \le A_s + C(t-s)$. Therefore,

$$
0 \le A_s \le A_t \le A_s + C(t-s).
$$

The proof is complete. ■

5 Approximation by an interacting particle system

The aim of this section is to demonstrate that the solution of the mean reflected G-SDE can be viewed as the limit of an interacting reflected particle system. To achieve this, we make the following assumptions: the loss function is defined as $l(t,x) = ax - r_t$, the coefficient function $h \equiv 0$, and the coefficient functions b and σ are deterministic and independent of t, where a represents a positive constant and $\{r_t\}_{t\in[0,T]}$ is a continuous function. The rationale behind considering this particular form of mean reflected G-SDEs is discussed in Remark 5.4. Without loss of generality, we assume $a = 1$. The solution of the mean reflected G-SDE (1.2) can be written as

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s + \sup_{s\in[0,t]} L_s(U_s), \tag{5.1}$$

where the operator L_t is defined in Equation (3.1) and

$$U_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s. \tag{5.2}$$

It is easy to check that

$$L_t(U_t) = (r_t - \widehat{\mathbb{E}}[U_t])^+ = \left(r_t - x_0 - \widehat{\mathbb{E}}\left[\int_0^t b(X_s)ds\right]\right)^+.$$

Now, fix a positive integer N. Let B^1 be a G-Brownian motion independent of B and let B^i be a G-Brownian motions independent of $(B, B^1, \cdots, B^{i-1})$ for any $i = 2, \cdots, N$. The dynamics of the candidate particle system is of the following form: for any $t \in [0,T]$ and $1 \leq i \leq N$,

$$X_t^{i,N} = x_0 + \int_0^t b(X_s^{i,N})ds + \int_0^t \sigma(X_s^{i,N})dB_s^i + \sup_{s\in[0,t]} \widetilde{G}_s(U_s^{(N)}), \tag{5.3}$$

where $U_t^{(N)} = \frac{1}{N}\sum_{i=1}^N U_t^{i,N}$ and

$$\widetilde{G}_t(U_t^{(N)}) = \inf\left\{x \geq 0 : \frac{1}{N}\sum_{i=1}^N l(t, x + U_t^{i,N}) = 0\right\},$$

with $U_t^{i,N}$ evolving according to

$$U_t^{i,N} = x_0 + \int_0^t b(X_s^{i,N})ds + \int_0^t \sigma(X_s^{i,N})dB_s^i.$$

It is easy to check that

$$\widetilde{G}_t(U_t^{(N)}) = (r_t - U_t^{(N)})^+.$$

Then we introduce the independent copies of X, for $1 \leq i \leq N$,

$$X_t^i = x_0 + \int_0^t b(X_s^i)ds + \int_0^t \sigma(X_s^i)dB_s^i + \sup_{s\in[0,t]} L_s(U_s),$$

and the particles

$$U_t^i = x_0 + \int_0^t b(X_s^i)ds + \int_0^t \sigma(X_s^i)dB_s^i.$$

By Proposition 4.3, the function $\{\sup_{s\in[0,t]} L_s(U_s)\}_{t\in[0,T]}$ is Lipschitz continuous. Denote its Radon-Nikodym derivative by ψ. Then X and X^i can be viewed as the solution to the SDE with coefficients $\tilde{b}, \sigma$ where $\tilde{b}(t,x) = b(x) + \psi_t$, driven by G-Brownian motion B and B^i, respectively. That is,

$$X_t = x_0 + \int_0^t (b(X_s) + \psi_s)ds + \int_0^t \sigma(X_s)dB_s,$$
$$X_t^i = x_0 + \int_0^t (b(X_s^i) + \psi_s)ds + \int_0^t \sigma(X_s^i)dB_s^i.$$

We now enforce b, σ and ψ enough regularities to suffice the technical assumption below.

(H_0) For any fixed $t_0 \in [0,T]$, let $u(t_0;t,x)$ be the solution to the following fully nonlinear PDE:

$$\begin{cases} \partial_t u(t_0;t,x) + G(\sigma^2(x)\partial_x^2 u(t_0;t,x)) + (b(x) + \psi_t)\partial_x u(t_0;t,x) = 0, \\ u(t_0;t_0,x) = b(x), \end{cases} \tag{5.4}$$

where $u(t_0;t,x) \in C^{1,2}([0,t_0] \times \mathbb{R})$. Additionally, for any $t \in [0,t_1]$ with $0 \le t_1 < t_2 \le T$,

$$\partial_x^2 u(t_1;t,x)\partial_x^2 u(t_2;t,x) \ge 0. \tag{5.5}$$

We validate the above assumption in the following remark.

Remark 5.1 *Let b be a convex and nondecreasing function and $\sigma(x) = a_1 x + a_2$, where a_1, a_2 are constants. Then, for any $t \in [0,t_0]$ where $t_0 \in [0,T]$, the function $u(t_0;t,x)$ is convex in x. In fact, for any $(t,x) \in [0,t_0] \times \mathbb{R}$, consider the following FBSDE driven by G-Brownian motion*

$$\begin{cases} X_s^{t,x} = x + \int_t^s (b(X_r^{t,x}) + \psi_r)dr + \int_t^s \sigma(X_r^{t,x})dB_r, \\ Y_s^{t_0;t,x} = b(X_{t_0}^{t,x}) - \int_s^{t_0} Z_r^{t_0;t,x}dB_r - (K_{t_0}^{t_0;t,x} - K_s^{t_0;t,x}). \end{cases}$$

By Theorem 4.5 in Hu et al. (2014), $u(t_0;t,x) = Y_t^{t_0;t,x}$ is the solution to PDE (5.4). Hence, it suffices to show that $Y_t^{t_0;t,x}$ is convex in x. Noting that b is convex and nondecreasing, by the comparison theorem for G-BSDE (Theorem 3.7 in Hu et al. (2014)), it then suffices to show that $X^{t,x}$ is convex in x. Define, for any $x, x' \in \mathbb{R}$ and $\lambda \in (0,1)$,

$$X_s^{\lambda,t,x,x'} = \lambda X_s^{t,x} + (1-\lambda)X_s^{t,x'}.$$

It is easy to check that

$$\begin{aligned} X_s^{\lambda,t,x,x'} &= \lambda x + (1-\lambda)y + \int_t^s \Big(\lambda b(X_r^{t,x}) + (1-\lambda)b(X_r^{t,x'}) + \psi_r\Big)dr + \int_t^s (a_1 X_r^{\lambda,t,x,x'} + a_2)dB_r \\ &= \lambda x + (1-\lambda)y + \int_t^s \Big(b(X_r^{\lambda,t,x,x'}) + \tilde{b}_r + \psi_r\Big)dr + \int_t^s (a_1 X_r^{\lambda,t,x,x'} + a_2)dB_r, \end{aligned}$$

where

$$\tilde{b}_r = \lambda b(X_r^{t,x}) + (1-\lambda)b(X_r^{t,x'}) - b(X_r^{\lambda,t,x,x'}) \ge 0.$$

Set $\widehat{X}_s = X_s^{\lambda,t,x,x'} - X_s^{t,\lambda x + (1-\lambda)x'}$. Simple calculation yields that

$$\widehat{X}_s = \int_t^s (b(X_r^{\lambda,t,x,x'}) - b(X_r^{t,\lambda x+(1-\lambda)x'}) + \tilde{b}_r)dr + \int_t^s a_1 \widehat{X}_r dB_r. \tag{5.6}$$

Similar construction as in the proof of Theorem 3.6 in Hu et al. (2014), for any $\varepsilon > 0$, there exists a pair of processes $(b^\varepsilon, m^\varepsilon)$ such that $b^\varepsilon \in M_G^2(0,T)$ with $|b_r^\varepsilon| \le \kappa$ and $|m_r^\varepsilon| \le 2\kappa\varepsilon$ satisfying

$$b(X_r^{\lambda,t,x,x'}) - b(X_r^{t,\lambda x+(1-\lambda)x'}) = b_r^\varepsilon \widehat{X}_r + m_r^\varepsilon.$$

Then, we may solve the G-SDE (5.6) explicitly, i.e.,

$$\widehat{X}_s = (\Gamma_s^\varepsilon)^{-1} \int_t^s (\tilde{b}_r + m_r^\varepsilon)\Gamma_r^\varepsilon dr \ge -2\kappa\varepsilon(\Gamma_s^\varepsilon)^{-1} \int_t^s \Gamma_r^\varepsilon dr, \tag{5.7}$$

where

$$\Gamma_s^\varepsilon = \exp\left(-\int_t^s b_r^\varepsilon dr + \frac{1}{2}a_1^2(\langle B\rangle_s - \langle B\rangle_t) - a_1(B_s - B_t) \right).$$

Letting $\varepsilon \to 0$ in Equation (5.7), we obtain the desired result. Furthermore, if for any $t_0 \in [0,T]$, the function $u(t_0;t,x)$ belongs to $C^{1,2}([0,t_0]\times\mathbb{R})$, then for any $t \in [0,t_0]$, we have $\partial_x^2 u(t_0;t,x) \ge 0$. That is, Equation (5.5) holds.

Condition (H_0) ensures that we may change the order of integral and the nonlinear expectation $\widehat{\mathbb{E}}[\cdot]$ in certain cases, which will be needed in the following lemma.

Lemma 5.2 *Suppose that the functions b, σ are deterministic satisfying (A2) and the loss function $l(t,x)$ satisfies $l(0,x_0) \ge 0$. Under Assumption (H_0), let X^{x_0} be the solution to the G-SDE (5.1). Then we have*

$$\widehat{\mathbb{E}} \int_0^T b(X_t^{x_0})dt = \int_0^T \widehat{\mathbb{E}}[b(X_t^{x_0})]dt.$$

Proof. For any fixed $n \in \mathbb{N}$, let $\{t_i\}_{i=1}^n$ be a partition of $[0,T]$. Let $\{a_i\}_{i=1}^n$ be positive constants. We first show that

$$\widehat{\mathbb{E}} \sum_{i=1}^n a_i \varphi(X_{t_i}^{x_0}) = \sum_{i=1}^n a_i \widehat{\mathbb{E}}[\varphi(X_{t_i}^{x_0})].$$

Without loss of generality, we assume that $a_i = 1$ for $i = 1, \cdots, n$. Set

$$Y_t^{t_i,x_0} = \begin{cases} u(t_i;t,X_t^{x_0}), & t \in [0,t_i],, \\ b(X_{t_i}^{x_0}), & t \in (t_i,T], \end{cases}$$

where $u(t_i;t,x)$ is the solution to the PDE (5.4). Applying G-Itô's formula (Li and Peng, 2011) to $Y_t^{t_i,x_0}$, for any $t \in [0,t_i]$, we obtain that

$$\begin{aligned} Y_t^{t_i,x_0} =& u(t_i;t_i,X_{t_i}^{x_0}) + \int_t^{t_i} \Big[\partial_t u(t_i;s,X_s^{x_0}) + (b(X_s^{x_0}) + \psi_s)\partial_x u(t_i;s,X_s^{x_0})\Big] ds \\ &+ \int_t^{t_i} \frac{1}{2}\sigma^2(X_s^{x_0})\partial_x^2 u(t_i;s,X_s^{x_0})d\langle B\rangle_s + \int_t^{t_i} \sigma(X_s^x)\partial_x u(t_i;s,X_s^x)dB_s \\ =& b(X_{t_i}^{x_0}) + \int_t^T \zeta_s^{t_i,x_0} dB_s + \int_t^T \eta_s^{t_i,x_0} d\langle B\rangle_s - \int_t^T 2G(\eta_s^{t_i,x_0})ds, \end{aligned}$$

where for any $s \in [0,t_i]$,

$$\zeta_s^{t_i,x_0} = \sigma(X_s^{x_0})\partial_x u(t_i;s,X_s^{x_0}) \quad \text{and} \quad \eta_s^{t_i,x_0} = \frac{1}{2}\sigma^2(X_s^{x_0})\partial_x^2 u(t_i;s,X_s^{x_0}),$$

and for any $s \in (t_i, T]$,
$$\zeta_s^{t_i,x_0} = \eta_s^{t_i,x_0} = 0.$$
By Equation (5.5), for any $0 \le i < j \le n$ and $s \in [0,T]$, we have
$$\eta_s^{t_i,x_0}\eta_s^{t_j,x_0} \ge 0.$$
It follows that
$$\begin{aligned}\sum_{i=1}^n Y_t^{t_i,x_0} &= \sum_{i=1}^n b(X_{t_i}^{x_0}) + \int_t^T \sum_{i=1}^n \zeta_s^{t_i,x_0} dB_s + \int_t^T \sum_{i=1}^n \eta_s^{t_i,x_0} d\langle B\rangle_s - \int_t^T 2\sum_{i=1}^n G(\eta_s^{t_i,x_0})ds \\ &= \sum_{i=1}^n b(X_{t_i}^{x_0}) + \int_t^T \sum_{i=1}^n \zeta_s^{t_i,x_0} dB_s + \int_t^T \sum_{i=1}^n \eta_s^{t_i,x_0} d\langle B\rangle_s - \int_t^T 2G\left(\sum_{i=1}^n \eta_s^{t_i,x_0}\right) ds.\end{aligned}$$
Therefore, we have
$$\widehat{\mathbb{E}}\sum_{i=1}^n b(X_{t_i}^{x_0}) = \sum_{i=1}^n Y_0^{t_i,x_0} = \sum_{i=1}^n \widehat{\mathbb{E}}[b(X_{t_i}^{x_0})].$$

At last, for any $n \in \mathbb{N}$, let $t_i = \frac{iT}{n}$ where $i = 0,1,\dots,n$. Recalling Equation (4.5), it is easy to check that
$$\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \widehat{\mathbb{E}}|b(X_t^{x_0}) - b(X_{t_i}^{x_0})|dt \le C\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |t-t_i|^{1/2}dt \to 0, \quad \text{as } n \to \infty.$$
Then we have
$$\widehat{\mathbb{E}}\int_0^T b(X_t^{x_0})dt = \lim_{n\to\infty}\widehat{\mathbb{E}}\sum_{i=0}^{n-1} b(X_{t_i}^{x_0})(t_{i+1}-t_i) = \lim_{n\to\infty}\sum_{i=0}^{n-1}\widehat{\mathbb{E}}[b(X_{t_i}^{x_0})](t_{i+1}-t_i) = \int_0^T \widehat{\mathbb{E}}[b(X_t^{x_0})]dt,$$
as desired. ■

The following theorem is the main result of this section.

Theorem 5.3 *Suppose that the functions b, σ are deterministic satisfying (A2) and the loss function $l(t,x)$ satisfies $l(0,x_0) \ge 0$. Under Assumption (H_0), there exists a constant C independent of N such that, for $j = 1,\cdots,N$,*
$$\widehat{\mathbb{E}}\sup_{r\in[0,T]}|X_r^{j,N} - X_r^j|^2 \le \frac{C}{N}\left(1 + \sup_{s\in[0,T]}\widehat{\mathbb{E}}|X_u|^2\right).$$

Proof. The proof proceeds in two steps. In the first step, we will show that, there exists a constant C independent of N, such that for any $t \in [0,T]$,
$$\widehat{\mathbb{E}}\sup_{s\in[0,t]}|\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)|^2 \le \frac{C}{N}\left(1 + \sup_{s\in[0,T]}\widehat{\mathbb{E}}|X_u|^2\right),$$
where $\overline{U}_t^{(N)} = \frac{1}{N}\sum_{i=1}^N U_t^i$ and $\widetilde{G}_t(\overline{U}_t^{(N)}) = (r_t - \overline{U}_t^{(N)})^+$. Then we plug that into the result obtained in the second step to complete the proof.

Step 1. By the representations of $\widetilde{G}_s(\overline{U}_s^{(N)})$ and $L_s(U_s)$, noting that $\widehat{\mathbb{E}}[U_s] = \widehat{\mathbb{E}}[U_s^i]$ for $i = 1, 2, \cdots, N$, we obtain that

$$\begin{aligned}
|\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)| &= \left|(r_t - U_s^{(N)})^+ - (r_s - \widehat{\mathbb{E}}[U_s])^+\right| \\
&\le \left|\frac{1}{N}\sum_{i=1}^N U_s^i - \frac{1}{N}\sum_{i=1}^N \widehat{\mathbb{E}}[U_s^i]\right| \\
&= \left|\frac{1}{N}\sum_{i=1}^N \left\{\int_0^s b(X_u^i)du - \widehat{\mathbb{E}}\left[\int_0^s b(X_u^i)du\right]\right\} + \frac{1}{N}\sum_{i=1}^N \int_0^s \sigma(X_u^i)dB_u^i\right| \\
&\le \left|\int_0^s \frac{1}{N}\sum_{i=1}^N (b(X_u^i) - \widehat{\mathbb{E}}[b(X_u^i)])du\right| + \left|\frac{1}{N}\sum_{i=1}^N \int_0^s \sigma(X_u^i)dB_u^i\right|,
\end{aligned}$$

where we have used Lemma 5.2 in the last inequality. Set

$$\mathcal{X}_u^i = b(X_u^i) - \widehat{\mathbb{E}}[b(X_u^i)].$$

As a byproduct, we have

$$\sup_{s\in[0,t]} |\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)| \le \int_0^t \left|\frac{1}{N}\sum_{i=1}^N \mathcal{X}_u^i\right| du + \sup_{s\in[0,t]}\left|\frac{1}{N}\sum_{i=1}^N \int_0^s \sigma(X_u^i)dB_u^i\right|.$$

For any $u \in [0, T]$, simple calculation yields that

$$\begin{aligned}
\widehat{\mathbb{E}}\left|\frac{1}{N}\sum_{i=1}^N \mathcal{X}_u^i\right|^2 &\le \frac{1}{N^2}\left\{\sum_{i=1}^N \widehat{\mathbb{E}}|\mathcal{X}_u^i|^2 + 2\sum_{1\le i<j\le N} \widehat{\mathbb{E}}[\mathcal{X}_u^i \mathcal{X}_u^j]\right\} \\
&= \frac{1}{N^2}\sum_{i=1}^N \widehat{\mathbb{E}}\left[(b(X_u^i) - \widehat{\mathbb{E}}[b(X_u^i)])^2\right] \\
&= \frac{1}{N}\widehat{\mathbb{E}}\left[(b(X_u) - \widehat{\mathbb{E}}[b(X_u)])^2\right] \\
&\le \frac{C}{N}\left(1 + \sup_{s\in[0,T]} \widehat{\mathbb{E}}[|X_u|^2]\right).
\end{aligned}$$

Applying Proposition 2.6, we obtain that for any $t \in [0, T]$,

$$\widehat{\mathbb{E}}\sup_{s\in[0,t]}\left|\frac{1}{N}\sum_{i=1}^N \int_0^s \sigma(X_u^i)dB_u^i\right|^2 \le \frac{C}{N^2}\sum_{i=1}^N \int_0^t \widehat{\mathbb{E}}[\sigma^2(X_u^i)]du \le \frac{C}{N}\left(1 + \sup_{s\in[0,T]} \widehat{\mathbb{E}}[|X_u|^2]\right).$$

All the above analysis gives

$$\widehat{\mathbb{E}}\sup_{s\in[0,t]} |\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)|^2 \le \frac{C}{N}\left(1 + \sup_{s\in[0,T]} \widehat{\mathbb{E}}|X_u|^2\right),$$

as desired

Step 2. Let $t \in (0, T]$. For $j = 1, \cdots, N$, set

$$\widehat{X}^j = X^{j,N} - X^j.$$

For any $r \leq t$, we have

$$|\widehat{X}_r^j| \leq \int_0^r |b(X_s^{j,N}) - b(X_s^j)|ds + \left|\int_0^r (\sigma(X_s^{j,N}) - \sigma(X_s^j))dB_s^j\right| + \left|\sup_{s\in[0,r]} \widetilde{G}_s(U_s^{(N)}) - \sup_{s\in[0,r]} L_s(U_s)\right|.$$

By the representations of $\widetilde{G}_s(U_s^{(N)})$ and $\widetilde{G}_s(U_s^{(N)})$, it is easy to check that

$$\begin{aligned}
&\left|\sup_{s\in[0,r]} \widetilde{G}_s(U_s^{(N)}) - \sup_{s\in[0,r]} L_s(U_s)\right| \\
&\leq \left|\sup_{s\in[0,r]} \widetilde{G}_s(U_s^{(N)}) - \sup_{s\in[0,r]} \widetilde{G}_s(\overline{U}_s^{(N)})\right| + \left|\sup_{s\in[0,r]} \widetilde{G}_s(\overline{U}_s^{(N)}) - \sup_{s\in[0,r]} L_s(U_s)\right| \\
&\leq \sup_{s\in[0,r]} |\widetilde{G}_s(U_s^{(N)}) - \widetilde{G}_s(\overline{U}_s^{(N)})| + \sup_{s\in[0,r]} |\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)| \\
&\leq \sup_{s\in[0,r]} \left|\frac{1}{N}\sum_{i=1}^N (U_s^{i,N} - U_s^i)\right| + \sup_{s\in[0,r]} |\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)|.
\end{aligned}$$

Combining the above two inequalities yields that

$$\sup_{r\in[0,t]} |\widehat{X}_r^j| \leq I_t^j + \sup_{s\in[0,t]} \left|\frac{1}{N}\sum_{i=1}^N (U_s^{i,N} - U_s^i)\right| + \sup_{s\in[0,t]} |\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)|, \tag{5.8}$$

where

$$I_t^j = \int_0^t |b(X_s^{j,N}) - b(X_s^j)|ds + \sup_{r\in[0,t]} \left|\int_0^r (\sigma(X_s^{j,N}) - \sigma(X_s^j))dB_s^j\right|.$$

Applying Hölder's inequality and Proposition 2.6, we obtain that

$$\widehat{\mathbb{E}}|I_t^j|^2 \leq C\int_0^t \widehat{\mathbb{E}}|\widehat{X}_s^j|^2 ds. \tag{5.9}$$

Note that

$$\sup_{s\in[0,t]} \left|\frac{1}{N}\sum_{i=1}^N (U_s^{i,N} - U_s^i)\right| \leq \frac{1}{N}\sum_{i=1}^N \sup_{s\in[0,t]} |U_s^{i,N} - U_s^i| \leq \frac{1}{N}\sum_{i=1}^N I_t^i.$$

Since the variables are exchangeable, it follows that

$$\widehat{\mathbb{E}} \sup_{s\in[0,t]} \left|\frac{1}{N}\sum_{i=1}^N (U_s^{i,N} - U_s^i)\right|^2 \leq \frac{C}{N}\sum_{i=1}^N \widehat{\mathbb{E}}|I_t^i|^2 = C\widehat{\mathbb{E}}|I_t^j|^2 \leq C\int_0^t \widehat{\mathbb{E}}|\widehat{X}_s^j|^2 ds. \tag{5.10}$$

Combining Equations (5.8)-(5.10), we obtain that

$$\begin{aligned}
\widehat{\mathbb{E}} \sup_{r\in[0,t]} |\widehat{X}_r^j|^2 &\leq C\int_0^t \widehat{\mathbb{E}}|\widehat{X}_s^j|^2 ds + C\widehat{\mathbb{E}} \sup_{s\in[0,t]} |\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)|^2 \\
&\leq C\int_0^t \widehat{\mathbb{E}} \sup_{r\in[0,s]} |\widehat{X}_r^j|^2 ds + C\widehat{\mathbb{E}} \sup_{s\in[0,T]} |\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)|^2
\end{aligned}$$

It follows from Grönwall's lemma that

$$\widehat{\mathbb{E}}\sup_{r\in[0,T]}|\widehat{X}_r^j|^2 \le C\widehat{\mathbb{E}}\sup_{s\in[0,T]}|\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)|^2.$$

Using the result obtained in Step 1, we complete the proof. ■

Finally, we address the reason for omitting the $d\langle B\rangle$ term in the following remark, aiming to provide insights that may inspire future research.

Remark 5.4 *If there exists the $d\langle B\rangle$-term, we cannot get the convergence result. For example, suppose that $h = 1$, $b = \sigma = 0$. Recall that the convergence result requires the estimate for the following term*

$$\widehat{\mathbb{E}}\sup_{s\in[0,t]}|\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)|^2.$$

In this case, we have

$$\widetilde{G}_s(\overline{U}_s^{(N)}) = (r_s - x_0 - \frac{1}{N}\sum_{i=1}^{N}\langle B^i\rangle_s)^+ \quad \textit{and} \quad L_s(U_s) = (r_s - \widehat{\mathbb{E}}[x_0 + \langle B\rangle_s])^+ = (r_s - x_0 - \bar{\sigma}^2 s)^+.$$

It follows that

$$|\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)| \le \left|\frac{1}{N}\sum_{i=1}^{N}(\langle B^i\rangle_s - \bar{\sigma}^2 s)\right|$$

However, simple calculation yields that

$$\widehat{\mathbb{E}}\sup_{s\in[0,t]}\left|\frac{1}{N}\sum_{i=1}^{N}(\langle B^i\rangle_s - \bar{\sigma}^2 s)\right|^2 = \widehat{\mathbb{E}}\left(\frac{1}{N}\sum_{i=1}^{N}(\bar{\sigma}^2 t - \langle B^i\rangle_t)\right)^2 = (\bar{\sigma}^2 - \underline{\sigma}^2)^2 t^2,$$

which clearly is independent of N. Therefore, we cannot get the desired estimate

$$\widehat{\mathbb{E}}\sup_{s\in[0,t]}|\widetilde{G}_s(\overline{U}_s^{(N)}) - L_s(U_s)|^2 \le \frac{C}{N}.$$

That is also the rationale behind our focus solely on the linear loss function l. Alternatively, if l were a nonlinear smooth function and $\sigma \neq 0$, estimating the term $|\widetilde{G}_s(U^{(N)}s) - L_s(U_s)|$ would necessitate the application of G-Itô's formula (Li and Peng, 2011) to $l(s, U_s^i + A_s)$, where $A_s = \sup_{r\in[0,s]} L_r(U)_r$ (see the proof of Theorem 3.3 (ii) in Briand et al. (2020a)). It is evident that this approach would introduce a $d\langle B\rangle$ term, rendering it untenable to obtain the convergence result outlined earlier.

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