PROPERTIES AND CONJECTURES REGARDING DISCRETE RENEWAL SEQUENCES

NIKOLAI NIKOLOV[1,2] AND MLADEN SAVOV[3,4]

ABSTRACT. In this work we review and derive some elementary properties of the discrete renewal sequences based on a positive, finite and integer-valued random variable. Our results consider these sequences as dependent on the probability masses of the underlying random variable. In particular we study the minima and the maxima of these sequences and prove that they are attained for indices of the sequences smaller or equal than the support of the underlying random variable. Noting that the minimum itself is a minimum of multi-variate polynomials we conjecture that one universal polynomial envelopes the minimum from below and that it is *maximal* in some sense and *largest* in another. We prove this conjecture in a special case.

1. INTRODUCTION

Renewal theory is at the heart of classical probability theory. For this reason new results keep on appearing thereby adding further knowledge about these classical objects, see [2; 3] for more information. In this note we consider discrete renewal sequences, whose basic theoretical results can be found in [1]. We review and obtain some elementary properties of the sequence of renewal masses. First, using the basic renewal equation we establish that the minimum and the maximum of the whole renewal sequence is attained within the support of the underlying probability masses. Second, we show that under natural restrictions the members of the aforementioned sequence are sandwiched between two monotone sequences which jointly converge to the limit of the well-known Blackwell's theorem. Third, given that each renewal mass is a polynomial of the probability masses we consider the minimum of the renewal sequences as a minimum of a finite number of polynomials on the respective simplex. We then introduce a universal polynomial which envelops from below this minimum and we conjecture that it is *maximal* within a natural subset of polynomials. We also show that it is not *largest* amongst the members of this subset. We confirm the conjecture when the renewal sequence is based on probability masses with at most three non-zero terms (the random variable takes values only in $\{1, 2, 3\}$). We show that within a subclass of polynomials

the aforementioned universal polynomial is *largest*. These conjectures are interesting from the standpoint of approximation theory as they involve multivariate polynomial approximation from below of a minima of a finite set of multivariate polynomials.

2. Notation and preliminary facts

In this note we work with a sequence of independent, positive, integer-valued random variables $(X_i)_{i \geq 1}$ defined on a common probability space. Furthermore, we assume that they are mutually independent and identically distributed with finite support, that is $\mathbb{P}(X_1 = l) = p_l \geq 0, 1 \leq l \leq k$, and $\sum_{l=1}^{k} p_l = 1$. Extend $p_n = 0$ for $n \geq k+1$. We consider the increasing random walk $S := (S_n)_{n \geq 0}$ defined as $S_0 = 0$ and $S_n = \sum_{k=1}^{n} X_k, n \geq 1$, and the related to it renewal masses

$$(2.1) \qquad u_n = \mathbb{P}(n \in \{S_0, S_1, \cdots\}) = \sum_{l=0}^{\infty} \mathbb{P}(S_l = n), \ n \geq 0.$$

Obviously $u_0 = 1$ and the well-known recurrent relation holds

$$(2.2) \qquad u_n = \sum_{l=1}^{n} p_l u_{n-l} = \sum_{l=1}^{\min\{n,k\}} p_l u_{n-l},$$

where we have used that $p_l = 0, l \geq k+1$. Blackwell's theorem states that for aperiodic random walks, that is their support is $\mathbb{N}$ or equivalently for all large n, S_n does not live on a sub-lattice of $\mathbb{N}$,

$$(2.3) \qquad \lim_{n \to \infty} u_n = \frac{1}{\mathbb{E}[X_1]} = \frac{1}{\sum_{l=1}^{k} l p_l}$$

see [1, XIII Theorem 3]. For example, the random walk is clearly aperiodic if $p_1 > 0$. For the purpose of our investigation we introduce

$$(2.4) \qquad M_k = \max_{l \geq 1}\{u_l\} \ \text{and} \ m_k = \min_{l \geq 1}\{u_l\}.$$

From Proposition 3.1 we see that $m_k = m_k(p_1, \cdots, p_{k-1})$ and it is the minimum of $k-1$ multi-variate polynomials. Next, with each random walk as defined above, **we introduce for** $1 \leq n \leq k$, **with the convention** $\prod_{j=1}^{0} = 1$,

$$(2.5) \qquad \mathbf{Q_n(p_1, p_2, \cdots, p_{n-1})} = \prod_{j=1}^{n-1} \sum_{l=1}^{j} \mathbf{p_l},$$

Note that if $\sum_{l=1}^{j} p_l = 1, 1 \leq j < k$, and $\sum_{l=1}^{j-1} p_l < 1$ then nominally Q_k depends at most on $p_1, \cdots, p_{j-1}$. We note that in this context we

are interested in the behaviour of Q_k on

(2.6)
$$A_k = \left\{ (p_1, \cdots, p_{k-1}) : p_l \geq 0, 1 \leq l \leq k-1; \sum_{l=1}^{k-1} p_l \leq 1 \right\} \subseteq \mathbb{R}^{k-1}.$$

We set $\mathcal{P}_k$ for the set of polynomials of $k-1$ variables. We introduce partial ordering in $\mathcal{P}_k$ in the following manner: we say that $P_1 \prec P_2, P_1, P_2 \in \mathcal{P}_k$, if and only if $P_1 \leq P_2$ on A_k. This is inherited from the partial ordering of functions of $k-1$ variables on A_k defined as $f \prec g \iff f \leq g$ on A_k. Furthermore, we consider

(2.7)
$$\mathtt{A}_k := \{P \in \mathcal{P}_k : deg(P) \leq k-1, P \prec m_k\},$$

where $deg(P)$ is the power of P, i.e. the highest combined power of every monomial constituting P. We say that $P \in \mathtt{A}_k$ is *maximal* if and only if

(2.8)
$$\tilde{P} \in \mathtt{A}_k \text{ and } \tilde{P} \succ P \Rightarrow \tilde{P} = P.$$

Largest element is $P \in \mathtt{A}_k$ such that $P \succ Q$ for all $Q \in \mathtt{A}_k$. Furthermore, for any $P \in \mathtt{A}_k$ we set $\hat{P}(p_1, \cdots, p_{k-1}) = P(p_1, p_2 - p_1, \ldots, p_k - p_{k-1})$ and $\hat{P}_j(p_1, \cdots, p_j) = \hat{P}(p_1, \cdots, p_j, 1, \cdots, 1), j \leq k-1$. We set

(2.9)
$$\hat{\mathtt{A}}_k = \left\{ P \in \mathtt{A}_k : \deg(\hat{P}_j) \leq j \, \forall 1 \leq j \leq k-1 \right\}.$$

Let us comment briefly on the classes $\mathtt{A}_k$ and $\hat{\mathtt{A}}_k$. According to Corollary 3.4 the running minimum m_k as defined in (2.4) is a minimum of polynomials of $p_1, \ldots, p_{k-1}$. Then $\mathtt{A}_k$ is a natural choice for class polynomials that envelop from below m_k uniformly in k. Theorem 3.6, however, reveals that $\mathtt{A}_k$ does not possess a maximal element, and therefore an unique element that envelops m_k from below. The structure of u_l suggests our choice of $\hat{A}_k$ and for $k = 3$ it possesses a largest element thereby confirming our Conjecture 3.7.

Here and hereafter we shall consider solely random walks S and their renewal masses as defined above. Extensive account of renewal theory can be found in [1; 2].

3. Main results and Conjectures

The first result shows that M_k (respectively m_k) are attained within the first k (respectively $k-1$) of the renewal masses.

Proposition 3.1. *Let $(u_n)_{n \geq 0}$ be the renewal masses of the random walk S defined above. Then*

(3.1)
$$1 \geq M_k = \max_{1 \leq l \leq k} u_l \text{ and } m_k = \max_{1 \leq l \leq k-1} u_l \geq 0.$$

$M_k = 1 \iff p_j = 1$, *for some $k \geq j \geq 1$, and $m_k = 0 \iff p_1 = 0$. Moreover, if $p_j \neq 1$, for all $k \geq j \geq 1$, then for $n > k$, $u_n < M_k$, and,*

if the walk is aperiodic $u_n > m_k$, for $n > k - 1$. Next, if $p_1 \in (0,1)$ the sequence $\{u_n\}_{n\geq 1}$ is not ultimately monotone and

(3.2)
$$u_n < \max_{1\leq l\leq k} \{u_{n-l}\} =: b_n \text{ for } n > k;$$
$$u_n > \min_{1\leq l\leq k} \{u_{n-l}\} =: c_n \text{ for } n > k - 1.$$

Finally, the sequences $\{b_n\}_{n\geq k}$ and $\{c_n\}_{n\geq k-1}$ are respectively monotone non-increasing and monotone non-decreasing.

Remark 3.2. *The final claim establishes that, provided $p_1 \in (0,1)$, u_n is sandwiched between two monotone sequences.*

Next we formulate a simple result that is used below.

Proposition 3.3. *Let $(u_n)_{n\geq 0}$ be the renewal masses of the random walk S defined above. Then, for $l \geq 1$,*

(3.3)
$$u_l = P_l\left(p_1, \cdots, p_{\min\{k-1,l\}}\right),$$

where $P_l, l \geq 1$ satisfy $\deg(P_l) = \min\{k-1, l\}$. For each P_l each variable $p_j, j \leq l$, has a highest possible power of precisely $l|j$ or the integer division of l by j. Finally, for any $t > 0$, and any $l \geq 1$

(3.4)
$$P_l\left(t^{\frac{1}{l}}p_1, t^{\frac{2}{l}}p_2, \cdots, t^{\frac{\min\{k-1,l\}}{l}}p_{\min\{k-1,l\}}\right) = tP_l\left(p_1, \cdots, p_{\min\{k-1,l\}}\right).$$

We have the immediate corollary.

Corollary 3.4. *Let $(u_n)_{n\geq 0}$ be the renewal masses of the random walk S defined above. Then*

(3.5)
$$M_k = M_k\left(p_1, p_2, \cdots, p_{k-1}\right) = \max_{1\leq l\leq k}\left\{P_l\left(p_1, \cdots, p_{\min\{k-1,l\}}\right)\right\}$$
$$m_k = m_k\left(p_1, p_2, \cdots, p_{k-1}\right) = \min_{1\leq l\leq k-1}\left\{P_l\left(p_1, \cdots, p_l\right)\right\}.$$

Next, we state a result which estimates m_k from below with Q_k.

Proposition 3.5. *Let $(u_n)_{n\geq 0}$ be the renewal masses of the random walk S defined above. Then*

(3.6)
$$m_k \geq Q_k.$$

Furthermore, for any $1 \leq n \leq k - 1$,

(3.7)
$$u_n \geq Q_{n+1}.$$

We proceed to prove our main result concerning the number of maximal elements in $\mathtt{A}_k$.

Theorem 3.6. *Let $(u_n)_{n\geq 0}$ be the renewal masses of the random walk S defined above. For $k \geq 3$ there is no largest element in $\mathtt{A}_k$.*

Given that $\mathtt{A}_k, k \geq 3$, has no *largest* element we formulate the following conjecture.

Conjecture 3.7. *For any $k \geq 3$, Q_k is a maximal element in $\mathtt{A}_k$ and it is largest in $\hat{\mathtt{A}}_k$.*

In the direction of Conjecture 3.7 we prove a weaker version of it.

Proposition 3.8. *Conjecture 3.7 is valid for $k = 3$.*

4. Proofs

Proof of Proposition 3.1. Let $n > k$ and assume that $M := M_k = u_n$. We consider three separate cases. First, we assume that $1 > p_1 > 0$. Applying (2.2) we get that

$$M = \sum_{l=1}^{k} p_l u_{n-l}$$

which is only possible provided $u_{n-l} = M$ for all $1 \leq l \leq k$. Using this recursively we get that u_l must be equal to M, for all $l \geq 1$. Since the random walk is aperiodic and from (2.3) $M = \lim_{n\to\infty} u_n = \frac{1}{\mathbb{E}[X_1]} < 1$. However, then from (2.2)

$$M = u_k = p_k + \sum_{l=1}^{k-1} p_l u_{k-l} = p_k + M \sum_{l=1}^{k-1} p_l > M$$

and we arrive at a contradiction. Second, we assume $p_j = 1$ for some $1 \leq j \leq k$ and then clearly $M = 1$ and $n = jm$ for some $m \geq 1$. Third, if $p_1 = 0$ and no $p_j = 1$ then we set $A = \{1 \leq l \leq k : p_l > 0\}$ and from $M := M_k = u_n$ for some $n > k$ we again deduce from (2.2) that $u_l = M$ for all $l \in A$. This easily yields a contradiction from (2.2) when applied for u_j with $j \in A$ and j not the maximal element of A. Clearly, $m_k = 0 \iff p_1 = 0$ and then the second relation of (3.1) holds true. Assume that $m := m_k > 0$. Hence, $p_1 > 0$ and the random walk is aperiodic. Let $m := m_k = u_n > 0$, for some $n > k$. If $p_1 = 1$ then $M = m = 1$ and the second relation of (3.1) holds true. If $p_1 < 1$ then from (2.2) we have that

$$m = u_k = p_k + \sum_{l=1}^{k-1} p_l u_{k-l} > \min_{1 \leq l \leq k-1} u_l.$$

This yields a contradiction and validates (3.1). Since $0 < p_1 < 1$ and thus $M < 1$ then $u_n = \sum_{l=1}^{k} p_l u_{n-l}, n \geq k$, easily gives that $\{u_n\}_{n\geq 1}$ can be neither monotone increasing nor monotone decreasing. Finally, it is elementary to repeat the first argument of this proof to check that it is impossible that $u_n = \max_{1 \leq l \leq k} \{u_{n-l}\} = c$ for some for $n > k$ as otherwise we would again obtain that $u_n = c = p_1$, for all $n \geq 1$ and we would arrive at contradiction as above. The same can be done for the minimum and (3.2) is established. Let us next prove the monotonicity

of $\{b_n\}_{n \geq k}$ with $\{c_n\}_{n \geq k-1}$ being the same. Assume that $b_{n+1} > b_n$. Then from (2.2) and the definition of b_n we get a contradiction from

$$b_n < b_{n+1} = u_n = \sum_{l=1}^{k} p_l u_{n-l} \leq b_n.$$

$\square$

Proof of Proposition 3.3. The relation (3.3) is immediate from (2.2). The final claim is elementary from probabilistic standpoint: given $l \geq$ and $1 \leq j \leq l$ we cannot have a piece of path that reaches l with more than $l|j$ occurrences of value j among the random variables construing it; this means that the power of p_j does not exceed $l|j$; however, as $p_1 > 0$ then clearly we can reach l by $l|j$ steps of size j and then adding steps of value 1. To show (3.4) we observe that every summand in P_l consists of probabilities of possible path to l with the weighted sum of their powers by their index equalling l. Therefore, to total power contributed by the powers of t is simply 1. $\square$

Proof of Proposition 3.5. Clearly, $1 > Q_n$ is non-increasing in n and it suffices to prove (3.7). We do so by induction with obvious basis $u_1 = p_1 \geq p_1$. Assume that (3.7) holds for $n < k-1$. We have from (2.2), the monotonicity of v_n and the induction hypothesis that

$$u_{n+1} = \sum_{j=1}^{n+1} p_j u_{n+1-j} \geq Q_n \sum_{j=1}^{n+1} p_j = Q_{n+1}.$$

Thus the claim is furnished. $\square$

Proof of Theorem 3.6. Assume that $P \in \mathtt{A}_k, k \geq 3$ is *largest* element. We split the domain A_k, see (2.6), as subregions whereby each P_l defined in (3.3) is the smallest, that is $A_k \supseteq \bigcup_{l=1}^{k-1} A_k(l)$, $A_k(l) = A_k \cap \{P_l < P_j, \forall j \neq l\}$. Since P_l are polynomials then $A_k(l)$ is open set for each $1 \leq l \leq k-1$ and the Lebesgue measure of $A_k \backslash \bigcup_{l=1}^{k-1} A_k(l)$ is zero. Choose $p_0 \in A_k(l)$ and consider $\tilde{P}_l(p) = P_l(p) - a\langle p - p_0, p - p_0 \rangle, p, p_0 \in A_k$ with $\langle \cdot, \cdot \rangle$ standing for the scalar product in $\mathbb{R}^{k-1}$ and $a > 0$ a scalar. Since A_k is a compact then there exists $a_0 > 0$ such that $m_k \geq \tilde{P}_l$ on A_k and therefore $\tilde{P}_l \in \mathtt{A}_k$. Also, $\tilde{P}_l(p_0) = P_l(p_0) = m_k(p_0)$ and since P is the largest then $P(p_0) = \tilde{P}_l(p_0) = m_k(p_0)$. The latter can be made valid for any p_0 in any $A_k(l), 1 \leq l \leq k-1$ and hence $P = m_k$ which is impossible. $\square$

Proof of Proposition 3.8. When $k = 3$ we have, see (3.1),

$$m_3 = \min\left\{p_1, p_1^2 + p_2\right\} = p_1 \min\left\{1, \frac{p_2}{p_1} + p_1\right\}.$$

If we assume that $Q \succ Q_3, Q \in \mathtt{A}_3$ on $\mathtt{A}_3$ then clearly we obtain from the definition of Q_3, see (2.5), that

$$p_2 \leq \lim_{p_1 \to 0} \frac{Q(p_1, p_2)}{p_1} \leq \lim_{p_1 \to 0} \min \left\{ 1, \frac{p_2}{p_1} + p_1 \right\} = 1.$$

Since p_2 is arbitrary, choosing it to tend to 1 we get since $\deg Q \leq 2$, that $Q(p_1, p_2) = p_1(a + bp_1 + cp_2)$. Hence on $\mathtt{A}_3$

$$p_1 + p_2 \leq (a + bp_1 + cp_2) \leq \min \left\{ 1, \frac{p_2}{p_1} + p_1 \right\}.$$

Setting $p_2 = 0$ we first get that $a = 0$ and then $b = 1$. Substituting we get on $\mathtt{A}_3$ the inequalities

$$p_2 \leq cp_2 \leq \min \left\{ 1 - p_1, \frac{p_2}{p_1} \right\} \leq \frac{p_2}{p_1}.$$

Letting $p_1 \to 1$ we conclude that $c = 1$ and hence $Q = Q_3$. Next, let $P(p_1, p_2) = ap_1^2 + bp_1p_2 + cp_2^2 + dp_1 + ep_2 + f \in \hat{\mathtt{A}}_3$. Since $P(p_1, 1 - p_1)$ has by assumption degree 1, see (2.9), we conclude that $2b = a + c$. We consider $H := P - Q_3$ and note that on ∂A_3, $Q_3 = m_3$. Therefore, on ∂A_3 we have that $H \leq 0$. Note that $4\mathrm{Det}(H) = 4(a-1)c - (b-1)^2 = -(a - c - 1)^2$ and as it is well-known if $\mathrm{Det}(H) \leq 0$, then $\max_{\overline{G}} H = \max_{\partial G} H$ (and $\min_{\overline{G}} H = \min_{\partial G} H$) for each bounded region G in the plane. Therefore, $P \leq Q_3$ on A_3 and the claim that Q_3 is largest in $\hat{\mathtt{A}}_3$ is established. $\qquad\square$

ACKNOWLEDGEMENTS

REFERENCES

[1] William Feller *An Introduction to Probability Theory and Its Applications*, John Wiley and Sons, New York, Vol. 1, 2nd edition, 1967

[2] William Feller *An Introduction to Probability Theory and Its Applications*, John Wiley and Sons, New York, Vol. 2, 2nd edition, 1970

[3] Kosto Mitov and Edward Omey *Renewal Processes*, SpringerBriefs in Statistics, Springer, ISBN: 978-3-319-05854-2, 2014

1: INSTITUTE OF MATHEMATICS AND INFORMATICS, BULGARIAN ACADEMY OF SCIENCES, STR. AKAD. GEORGI BONCHEV, BLOCK 8, SOFIA - 1113, BULGARIA
Email address: nik@math.bas.bg

2: Faculty of Information Sciences, State University of Library Studies and Information Technologies, Shipchenski prohod 69A, 1574 Sofia, Bulgaria

3: Faculty of Mathematics and Informatics, Sofia University "St Kliment Ohridski", boul. James Boucher 5, Sofia - 1164, Bulgaria
Email address: msavov@fmi.uni-sofia.bg

4: Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, str. Akad. Georgi Bonchev, block 8, Sofia - 1113, Bulgaria
Email address: mladensavov@math.bas.bg