

Convergence Rate of LQG Mean Field Games with Common Noise

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Abstract

This paper focuses on exploring the convergence properties of a generic player's trajectory and empirical measures in an N -player Linear-Quadratic-Gaussian Nash game, where Brownian motion serves as the common noise. The study establishes three distinct convergence rates concerning the representative player and empirical measure. To investigate the convergence, the methodology relies on a specific decomposition of the equilibrium path in the N -player game and utilizes the associated Mean Field Game framework.

1 Introduction

Mean Field Game (MFG) theory was introduced by Lasry and Lions in their seminal paper ([19]), and by Huang, Caines, and Malhame ([15, 13, 14, 12]). It aims to provide a framework for studying the asymptotic behavior of N -player differential games being invariant under the reshuffling of the players' indices. For a comprehensive overview of recent advancements and relevant applications of MFG theory, it is recommended to refer to the two-volume book by Carmona and Delarue ([4, 5]) published in 2018 and the references provided therein.

Mean Field Games (MFG) have become widely accepted as an approximation for N -player games, particularly when the number of players, N , is large enough. A fundamental question that arises in this context concerns the convergence rate of this approximation. Convergence can be analyzed from different perspectives, such as convergence in value, the trajectory followed by the representative player, or the behavior of the mean field term. Each of these perspectives offers valuable insights into the behavior and characteristics of the MFG approximation. Furthermore, they raise a variety of intriguing questions within this context.

To be more concrete, we examine the behavior of the triangular array $\hat{X}_t^{(N)} = (\hat{X}_{it}^{(N)} : 1 \leq i \leq N)$ as $N \rightarrow \infty$, where $\hat{X}_{it}^{(N)}$ represents the equilibrium state of the i -th player at time t in the N -player game, defined within the probability space $(\Omega^{(N)}, \mathcal{F}^{(N)}, \mathbb{F}^{(N)}, \mathbb{P}^{(N)})$. Additionally, we denote \hat{X}_t as the equilibrium path at time t derived from the associated MFG, defined in the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Considering the identical but not independent distribution $\mathcal{L}(\hat{X}_{it}^{(N)})$, the first question pertains to the convergence of $\hat{X}_{1t}^{(N)}$, which represents the generic path. It can be framed as follows:

(Q1) The \mathbb{W}_p -convergence rate of the representative equilibrium path,

$$\mathbb{W}_p \left(\mathcal{L} \left(\hat{X}_{1t}^{(N)} \right), \mathcal{L} \left(\hat{X}_t \right) \right) = O \left(N^{-?} \right).$$

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Here, \mathbb{W}_p denotes the p -Wasserstein metric.

The existing literature extensively explores the convergence rate in this context. For (Q1), Theorem 2.4.9 of the monograph [3] establishes a convergence rate of $O(N^{-1/2})$ using the \mathbb{W}_1 metric. More recently, [17] addresses (Q1) by introducing displacement monotonicity and controlled common noise, and Theorem 2.23 applies the maximum principle of forward-backward propagation of chaos to achieve the same convergence rate. Within the LQG framework, [18] also provides a convergence rate of $1/2$ for the representative player.

The second question pertains to the convergence of the mean-field term, which is equivalent to the convergence of the empirical measure $\rho(\hat{X}_t^{(N)}) = \frac{1}{N} \sum_{i=1}^N \delta_{\hat{X}_{it}^{(N)}}$ of N players. Given the Brownian motion, denoted as \tilde{W}_t , to be the common noise, the problem lies in determining the rate of convergence of the empirical measures to the MFG equilibrium measure

$$\hat{m}_t = \mathcal{L} \left(\hat{X}_t \middle| \mathcal{F}_t^{\tilde{W}} \right), \quad \forall t \in (0, T].$$

Thus, the second question can be stated as follows:

(Q2) The \mathbb{W}_p -convergence rate of empirical measures in L^p sense,

$$\left(\mathbb{E} \left[\mathbb{W}_p^p \left(\rho \left(\hat{X}_t^{(N)} \right), \mathcal{L} \left(\hat{X}_t \middle| \mathcal{F}_t^{\tilde{W}} \right) \right) \right] \right)^{\frac{1}{p}} = O \left(N^{-?} \right).$$

As for (Q2), Theorem 3.1 of [8] provides an answer, stating that the empirical measures exhibit a convergence rate of $O(N^{-1/(2p)})$ in the \mathbb{W}_p distance for $p \in [1, 2]$. In [8], they also explore a related question that is both similar and more intriguing, which concerns the uniform \mathbb{W}_p -convergence rate:

(Q3) The t -uniform \mathbb{W}_p -convergence rate of empirical measures in L^p sense,

$$\left(\mathbb{E} \left[\sup_{t \in [0, T]} \mathbb{W}_p^p \left(\rho \left(\hat{X}_t^{(N)} \right), \mathcal{L} \left(\hat{X}_t \middle| \mathcal{F}_t^{\tilde{W}} \right) \right) \right] \right)^{\frac{1}{p}} = O \left(N^{-?} \right).$$

The answer provided by Theorem 3.1 in [8] reveals that the uniform convergence rate, as formulated in (Q3), is considerably slower compared to the convergence rate mentioned in (Q2). Specifically, the convergence rate for (Q3) is $O(N^{-1/(d+8)})$ when $p = 2$, where d represents the dimension of the state space.

In our paper, we specifically focus on a class of one-dimensional Linear-Quadratic-Gaussian (LQG) Mean Field Nash Games with Brownian motion as the common noise. It is important to note that the assumptions made in the aforementioned papers except [18] only account for linear growth in the state and control elements for the running cost, thus excluding the consideration of LQG. It is also noted that differences between [18] and the current paper lie in various aspects: (1) The problem setting in our paper considers Brownian motion as the common noise, whereas [18] employs a Markov chain. This discrepancy leads to significant differences in the subsequent analysis; (2) The work in [18] does not address the questions posed in (Q2) and (Q3).

Our main contribution is the establishment of the convergence rate of all three questions in the above in LQG framework. Firstly, the paper establishes that the convergence rate of the p -Wasserstein metric for the distribution of the representative player is $O(N^{-1/2})$ for $p \in [1, 2]$. Secondly, it demonstrates that the convergence rate of the p -Wasserstein metric for the empirical measure in the L^p sense is $O(N^{-1/(2p)})$ for $p \in [1, 2]$. Lastly, the paper shows that the convergence rate of the uniform p -Wasserstein metric for the empirical measure in the L^p sense is $O(N^{-1/(2p)})$ for $p \in (1, 2]$, and $O(N^{-1/2} \ln(N))$ for $p = 1$.

It is worth noting that the convergence rates obtained for (Q1) and (Q2) in the LQG framework align with the results found in existing literature, albeit under different conditions. Additionally, it is revealed that the uniform convergence rate of (Q3) may be slower than that of (Q2), which is consistent with the observations made by [8] from a similar perspective. Interestingly, when considering the specific case where $p = 2$ and $d = 1$, the uniform convergence rate of (Q3) is established as $O(N^{-1/9})$ according to [8], while it is determined to be $O(N^{-1/4})$ within our framework that incorporates the LQG structure.

Regarding (Q2), if the states $(\hat{X}_{it}^{(N)} : 1 \leq i \leq N)$ were independent, the convergence rate could be determined as $1/(2p)$ based on Theorem 1 of [10] and Theorem 5.8 of [4], which provide convergence rates for empirical measures of independent and identically distributed sequences. However, in the mean-field game, the states $\hat{X}_{it}^{(N)}$ are not independent of each other, despite having identical distributions. The correlation is introduced mainly by two factors: One is the system coupling arising from the mean-field term and the other is the common noise. Consequently, determining the convergence rate requires understanding the contributions of these two factors to the correlation among players.

In our proof, we rely on a specific decomposition (refer to Lemma 6 and the proof of the main theorem) of the underlying states. This decomposition reveals that the states can be expressed as a sum of a weakly correlated triangular array and a common noise. By analyzing the behavior of these components, we can address the correlation and establish the convergence rate.

Additionally, it is worth mentioning that a similar technique of dimension reduction in N -player LQG games have been previously utilized in [16] and related papers to establish decentralized Nash equilibria and the convergence rate in terms of value functions.

The remainder of the paper is organized as follows: Section 2 outlines the problem setup and presents the main result. The proof of the main result, which relies on two propositions, is provided in Section 3. We establish the proof for these two propositions in Section 4 and Section 5. Some lemmas are given in the Appendix.

2 Problem setup and main results

2.1 The formulation of equilibrium in Mean Field Game

In this section, we present the formulation of the Mean Field Game in the sample space Ω .

Let $T > 0$ be a given time horizon. We assume that $W = \{W_t\}_{t \geq 0}$ is a standard Brownian motion constructed on the probability space $(\bar{\Omega}, \bar{\mathcal{F}} = \bar{\mathcal{F}}_T, \bar{\mathbb{P}}, \bar{\mathbb{F}} = \{\bar{\mathcal{F}}_t\}_{t \geq 0})$. Similarly, the process $\tilde{W} = \{\tilde{W}_t\}_{t \geq 0}$ is a standard Brownian motion constructed on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}} = \tilde{\mathcal{F}}_T, \tilde{\mathbb{P}}, \tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{t \geq 0})$. We define the product structure as follows:

$$\Omega = \bar{\Omega} \times \tilde{\Omega}, \quad \mathcal{F} = \bar{\mathcal{F}} \otimes \tilde{\mathcal{F}}, \quad \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \quad \mathbb{P},$$

where $(\mathcal{F}, \mathbb{P})$ is the completion of $(\bar{\mathcal{F}} \otimes \tilde{\mathcal{F}}, \bar{\mathbb{P}} \otimes \tilde{\mathbb{P}})$ and \mathbb{F} is the complete and right continuous augmentation of $\{\mathcal{F}_t \otimes \tilde{\mathcal{F}}_t\}_{t \geq 0}$.

Note that, W and \tilde{W} are two Brownian motions from separate sample spaces $\bar{\Omega}$ and $\tilde{\Omega}$, they are independent of each other in their product space Ω . In our manuscript, W is called individual or idiosyncratic noise, and \tilde{W} is called common noise, see their different roles in the problem formulation later defined via fixed point condition (4). To proceed, we denote by $L^p := L^p(\Omega, \mathbb{P})$ the set of random variables X on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite p -th moment with norm $\|X\|_p = (\mathbb{E}[|X|^p])^{1/p}$ and by $L_{\mathbb{F}}^p := L_{\mathbb{F}}^p(\Omega \times [0, T])$ the space of all \mathbb{R} valued \mathbb{F} -progressively measurable random processes

α such that

$$\mathbb{E} \left[\int_0^T |\alpha_t|^p dt \right] < \infty.$$

Let $\mathcal{P}_p(\mathbb{R})$ denote the Wasserstein space of probability measures μ on \mathbb{R} satisfying $\int_{\mathbb{R}} x^p d\mu(x) < \infty$ endowed with p -Wasserstein metric $\mathbb{W}_p(\cdot, \cdot)$ defined by

$$\mathbb{W}_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left(\int_{\mathbb{R} \times \mathbb{R}} |x - y|^p d\pi(x, y) \right)^{\frac{1}{p}},$$

where $\Pi(\mu, \nu)$ is the collection of all probability measures on $\mathbb{R} \times \mathbb{R}$ with its marginals agreeing with μ and ν .

Let $X_0 \in L^2$ be a random variable that is independent with W and \tilde{W} . For any control $\alpha \in L^2_{\mathbb{F}}$, consider the state $X = \{X_t\}_{t \geq 0}$ of the generic player is governed by a stochastic differential equation (SDE)

$$dX_t = \alpha_t dt + dW_t + d\tilde{W}_t \quad (1)$$

with the initial value X_0 , where the underlying process $X : [0, T] \times \Omega \mapsto \mathbb{R}$. Given a random measure flow $m : (0, T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$, the generic player wants to minimize the expected accumulated cost on $[0, T]$:

$$J(x, \alpha) = \mathbb{E} \left[\int_0^T \left(\frac{1}{2} \alpha_s^2 + F(X_s, m_s) \right) ds \middle| X_0 = x \right] \quad (2)$$

with some given cost function $F : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{R}$.

The objective of the control problem for the generic player is to find its optimal control $\hat{\alpha} \in \mathcal{A} := L^4_{\mathbb{F}}$ to minimize the total cost, i.e.,

$$V[m](x) = J[m](x, \hat{\alpha}) \leq J[m](x, \alpha), \quad \forall \alpha \in \mathcal{A}. \quad (3)$$

Associated to the optimal control $\hat{\alpha}$, we denote the optimal path by $\hat{X} = \{\hat{X}_t\}_{t \geq 0}$.

Next, to introduce the MFG Nash equilibrium, it is useful to emphasize the dependence of the optimal path and optimal control of the generic player, as well as its associated value, on the underlying measure flow m . These quantities are denoted as $\hat{X}_t[m]$, $\hat{\alpha}_t[m]$, $J[m]$, and $V[m]$, respectively.

We now present the definitions of the equilibrium measure, equilibrium path, and equilibrium control. Please also refer to page 127 of [5] for a general setup with a common noise.

Definition 1. *Given an initial distribution $\mathcal{L}(X_0) = m_0 \in \mathcal{P}_2(\mathbb{R})$, a random measure flow $\hat{m} = \hat{m}(m_0)$ is said to be an MFG equilibrium measure if it satisfies the fixed point condition*

$$\hat{m}_t = \mathcal{L} \left(\hat{X}_t[\hat{m}] \middle| \tilde{\mathcal{F}}_t \right), \quad \forall 0 < t \leq T, \quad \text{almost surely in } \mathbb{P}. \quad (4)$$

The path \hat{X} and the control $\hat{\alpha}$ associated with \hat{m} are called the MFG equilibrium path and equilibrium control, respectively.

The flowchart of the MFG diagram is given in Figure 1. It is noted from the optimality condition (3) and the fixed point condition (4) that

$$J[\hat{m}](x, \hat{\alpha}) \leq J[\hat{m}](x, \alpha), \quad \forall \alpha$$

holds for the equilibrium measure \hat{m} and its associated equilibrium control $\hat{\alpha}$, while it is not

$$J[\hat{m}](x, \hat{\alpha}) \leq J[m](x, \alpha), \quad \forall \alpha, m.$$

Otherwise, this problem turns into a McKean-Vlasov control problem, which is essentially different from the current Mean Field Games setup. Readers refer to [7, 6] to see the analysis of this different model as well as some discussion of the differences between these two problems.

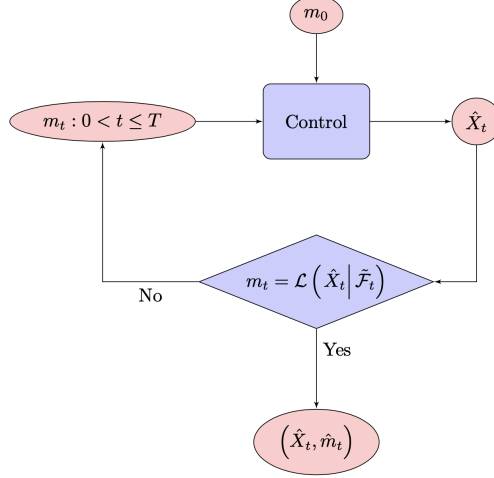


Figure 1: The MFG diagram.

2.2 The formulation of Nash equilibrium in N -player game

In this subsection, we set up N -player game and define the Nash equilibrium of N -player game in the sample space $\Omega^{(N)}$. Firstly, let $W^{(N)} = (W_i^{(N)} : i = 1, 2, \dots, N)$ be an N -dimensional standard Brownian motion constructed on the space $(\bar{\Omega}^{(N)}, \bar{\mathcal{F}}^{(N)}, \bar{\mathbb{P}}^{(N)}, \bar{\mathbb{F}}^{(N)} = \{\bar{\mathcal{F}}_t^{(N)}\}_{t \geq 0})$ and $\tilde{W} = \{\tilde{W}_t\}_{t \geq 0}$ be the common noise in MFG defined in Section 2.1 on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. The probability space for the N -player game is $(\Omega^{(N)}, \mathcal{F}^{(N)}, \mathbb{F}^{(N)}, \mathbb{P}^{(N)})$, which is constructed via the product structure with

$$\Omega^{(N)} = \bar{\Omega}^{(N)} \times \tilde{\Omega}, \quad \mathcal{F}^{(N)}, \quad \mathbb{F}^{(N)} = \left\{ \mathcal{F}_t^{(N)} \right\}_{t \geq 0}, \quad \mathbb{P}^{(N)}.$$

where $(\mathcal{F}^{(N)}, \mathbb{P}^{(N)})$ is the completion of $(\bar{\mathcal{F}}^{(N)} \otimes \tilde{\mathcal{F}}, \bar{\mathbb{P}}^{(N)} \otimes \tilde{\mathbb{P}})$ and $\mathbb{F}^{(N)}$ is the complete and right continuous augmentation of $\{\bar{\mathcal{F}}_t^{(N)} \otimes \tilde{\mathcal{F}}_t\}_{t \geq 0}$.

Consider a stochastic dynamic game with N players, where each player $i \in \{1, 2, \dots, N\}$ controls a state process $X_i^{(N)} = \{X_{it}^{(N)}\}_{t \geq 0}$ in \mathbb{R} given by

$$dX_{it}^{(N)} = \alpha_{it}^{(N)} dt + dW_{it}^{(N)} + d\tilde{W}_t, \quad X_{i0}^{(N)} = x_i^{(N)} \quad (5)$$

with a control $\alpha_i^{(N)}$ in an admissible set $\mathcal{A}^{(N)} := L^4_{\mathbb{F}^{(N)}}$ and random initial state $x_i^{(N)}$.

Given the strategies $\alpha_{-i}^{(N)} = (\alpha_1^{(N)}, \dots, \alpha_{i-1}^{(N)}, \alpha_{i+1}^{(N)}, \dots, \alpha_N^{(N)})$ from other players, the objective of player i is to select a control $\alpha_i^{(N)} \in \mathcal{A}^{(N)}$ to minimize her expected total cost given by

$$J_i^N(x^{(N)}, \alpha_i^{(N)}; \alpha_{-i}^{(N)}) = \mathbb{E} \left[\int_0^T \left(\frac{1}{2} (\alpha_{it}^{(N)})^2 + F(X_{it}^{(N)}, \rho(X_t^{(N)})) \right) dt \middle| X_0^{(N)} = x^{(N)} \right], \quad (6)$$

where $x^{(N)} = (x_1^{(N)}, x_2^{(N)}, \dots, x_N^{(N)})$ is a \mathbb{R}^N -valued random vector in $\Omega^{(N)}$ to denote the initial state for N players, and

$$\rho(x^{(N)}) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^{(N)}}$$

is the empirical measure of the vector $x^{(N)}$ with Dirac measure δ . We use the notation $\alpha^{(N)} := (\alpha_i^{(N)}, \alpha_{-i}^{(N)}) = (\alpha_1^{(N)}, \alpha_2^{(N)}, \dots, \alpha_N^{(N)})$ to denote the control from N players as a whole. Next, we give the equilibrium value function and equilibrium path in the sense of the Nash game.

Definition 2. 1. The value function of player i for $i = 1, 2, \dots, N$ of the Nash game is defined by $V^N = (V_i^N : i = 1, 2, \dots, N)$ satisfying the equilibrium condition

$$V_i^N \left(x^{(N)} \right) := J_i^N \left(x^{(N)}, \hat{\alpha}_i^{(N)}; \hat{\alpha}_{-i}^{(N)} \right) \leq J_i^N \left(x^{(N)}, \alpha_i^{(N)}; \hat{\alpha}_{-i}^{(N)} \right), \quad \forall \alpha_i^{(N)} \in \mathcal{A}^{(N)}. \quad (7)$$

2. The equilibrium path of the N -player game is the N -dimensional random path $\hat{X}_t^{(N)} = (\hat{X}_{1t}^{(N)}, \hat{X}_{2t}^{(N)}, \dots, \hat{X}_{Nt}^{(N)})$ driven by (5) associated to the control $\hat{\alpha}_t^{(N)}$ satisfying the equilibrium condition of (7).

2.3 Main result

We consider three convergence questions on N -player game defined in $\Omega^{(N)}$: The first one is the convergence of the representative path $\hat{X}_{it}^{(N)}$, the second one is the convergence of the empirical measure $\rho(\hat{X}_t^{(N)})$, while the last one is the t -uniform convergence of the empirical measure $\rho(\hat{X}_t^{(N)})$. To be precise, we shall assume the following throughout the paper:

Assumption 1. • $\mathbb{E}[|X_0|^q] < \infty$ for some $q > 4$.

- The initials $X_{i0}^{(N)}$ of the N -player game is i.i.d. random variables in $\Omega^{(N)}$ with the same distribution as $\mathcal{L}(X_0)$ in the MFG.

Note that the equilibrium path $\hat{X}_t^{(N)} = (\hat{X}_{it}^{(N)} : i = 1, 2, \dots, N)$ is a vector-valued stochastic process. Due to the Assumption 1, the game is invariant to index reshuffling of N players and the elements in $(\hat{X}_{it}^{(N)} : i = 1, 2, \dots, N)$ have identical distributions, but they are not independent of each other.

So, the first question on the representative path is indeed about $\hat{X}_{1t}^{(N)}$ in $\Omega^{(N)}$ and we are interested in how fast it converges to \hat{X}_t in Ω in distribution:

(Q1) The \mathbb{W}_p -convergence rate of the representative equilibrium path,

$$\mathbb{W}_p \left(\mathcal{L} \left(\hat{X}_{1t}^{(N)} \right), \mathcal{L} \left(\hat{X}_t \right) \right) = O \left(N^{-?} \right).$$

The second question is about the convergence of the empirical measure $\rho(\hat{X}_t^{(N)})$ of the N -player game defined by

$$\rho \left(\hat{X}_t^{(N)} \right) = \frac{1}{N} \sum_{i=1}^N \delta_{\hat{X}_{it}^{(N)}}.$$

We are interested in how fast this converges to the MFG equilibrium measure given by

$$\hat{m}_t = \mathcal{L} \left(\hat{X}_t \middle| \tilde{\mathcal{F}}_t \right), \quad \forall t \in (0, T].$$

(Q2') The \mathbb{W}_p -convergence rate of empirical measures,

$$\mathbb{W}_p \left(\rho \left(\hat{X}_t^{(N)} \right), \mathcal{L} \left(\hat{X}_t \middle| \tilde{\mathcal{F}}_t \right) \right) = O \left(N^{-?} \right).$$

Note that the left-hand side of the above equality is a random quantity and one shall be more precise about what the Big O notation means in this context. Indeed, by the definition of the empirical measure, $\rho(\hat{X}_t^{(N)})$ is a random distribution measurable by σ -algebra generated

by the random vector $\hat{X}_t^{(N)}$. On the other hand, $\mathcal{L}(\hat{X}_t|\tilde{\mathcal{F}}_t)$ is a random distribution measurable by the σ -algebra $\tilde{\mathcal{F}}_t$. Therefore, from the construction of the product probability space $\Omega^{(N)}$ in Section 2.2, both random distributions $\rho(\hat{X}_t^{(N)})$ and $\mathcal{L}(\hat{X}_t|\tilde{\mathcal{F}}_t)$ are measurable with respect to $\mathcal{F}_t^{(N)} = \tilde{\mathcal{F}}_t^{(N)} \otimes \tilde{\mathcal{F}}_t$. Consequently, $\mathbb{W}_p(\rho(\hat{X}_t^{(N)}), \mathcal{L}(\hat{X}_t|\tilde{\mathcal{F}}_t))$ is a random variable in the probability space $(\Omega^{(N)}, \mathcal{F}^{(N)}, \mathbb{P}^{(N)})$ and we will focus on a version of (Q2') in the L^p sense:

(Q2) The \mathbb{W}_p -convergence rate of empirical measures in L^p sense for each $t \in [0, T]$,

$$\left(\mathbb{E} \left[\mathbb{W}_p^p \left(\rho \left(\hat{X}_t^{(N)} \right), \mathcal{L} \left(\hat{X}_t \middle| \tilde{\mathcal{F}}_t \right) \right) \right] \right)^{\frac{1}{p}} = O \left(N^{-?} \right).$$

In addition, we also study the following related question:

(Q3) The t -uniform \mathbb{W}_p -convergence rate of empirical measures in L^p sense,

$$\left(\mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{W}_p^p \left(\rho \left(\hat{X}_t^{(N)} \right), \mathcal{L} \left(\hat{X}_t \middle| \tilde{\mathcal{F}}_t \right) \right) \right] \right)^{\frac{1}{p}} = O \left(N^{-?} \right).$$

In this paper, we will study the above three questions (Q1), (Q2), and (Q3) in the framework of LQG structure with Brownian motion as a common noise with the following function F in the cost functional (2).

Assumption 2. Let the function $F : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{R}$ be given in the form of

$$F(x, m) = k \int_{\mathbb{R}} (x - z)^2 m(dz) = k(x^2 - 2x[m]_1 + [m]_2) \quad (8)$$

for some $k > 0$, where $[m]_1, [m]_2$ are the first and second moment of the measure m .

The main result of this paper is presented below. Let us recall that q denotes the parameter defined in Assumption 1.

Theorem 1. Under Assumptions 1-2, for any $p \in [1, 2]$, we have

1. The \mathbb{W}_p -convergence rate of the representative equilibrium path is $1/2$, i.e.,

$$\mathbb{W}_p \left(\mathcal{L} \left(\hat{X}_{1t}^{(N)} \right), \mathcal{L} \left(\hat{X}_t \right) \right) = O \left(N^{-\frac{1}{2}} \right).$$

2. The \mathbb{W}_p -convergence rate of empirical measures in L^p sense is

$$\mathbb{E} \left[\mathbb{W}_p^p \left(\rho \left(\hat{X}_t^{(N)} \right), \mathcal{L} \left(\hat{X}_t \middle| \tilde{\mathcal{F}}_t \right) \right) \right] = O \left(N^{-\frac{1}{2}} \right).$$

3. The uniform \mathbb{W}_p -convergence rate of empirical measures in L^p sense is

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \mathbb{W}_p^p \left(\rho \left(\hat{X}_t^{(N)} \right), \mathcal{L} \left(\hat{X}_t \middle| \tilde{\mathcal{F}}_t \right) \right) \right] = \begin{cases} O \left(N^{-\frac{1}{2}} \ln(N) \right), & \text{if } p = 1, \\ O \left(N^{-\frac{1}{2}} \right), & \text{if } 1 < p \leq 2. \end{cases}$$

We would like to provide some additional remarks on our main result. Firstly, the cost function F defined in (6) applies to the running cost for the i -th player in the N -player game, and it takes the form:

$$F\left(X_{it}^{(N)}, \rho\left(X_t^{(N)}\right)\right) = \frac{k}{N} \sum_{j=1}^N \left(X_{it}^{(N)} - X_{jt}^{(N)}\right)^2. \quad (9)$$

Interestingly, if $k < 0$, although F does satisfy the Lasry-Lions monotonicity ([2]) as demonstrated in Appendix 6.1 of [18], there is no global solution for MFG due to the concavity in x . On the contrary, when $k > 0$, F satisfies the displacement monotonicity proposed in [11] as shown by the following derivation:

$$\mathbb{E}[(F_x(X_1, \mathcal{L}(X_1)) - F_x(X_2, \mathcal{L}(X_2)))(X_1 - X_2)] = 2k \left(\mathbb{E}[(X_1 - X_2)^2] - (\mathbb{E}[X_1 - X_2])^2 \right) \geq 0.$$

3 Proof of the main result with two propositions

Our objective is to investigate the relations between $(\hat{X}_{1t}^{(N)}, \hat{X}_{2t}^{(N)}, \dots, \hat{X}_{Nt}^{(N)})$ and \hat{X}_t described in (Q1), (Q2), and (Q3). In this part, we will give the proof of Theorem 1 based on two propositions whose proof will be given later.

Proposition 1. *Under Assumptions 1-2, the MFG equilibrium path $\hat{X} = \hat{X}[\hat{m}]$ is given by*

$$d\hat{X}_t = -2a(t) \left(\hat{X}_t - \hat{\mu}_t \right) dt + dW_t + d\tilde{W}_t, \quad \hat{X}_0 = X_0, \quad (10)$$

where a is the solution of

$$a'(t) - 2a^2(t) + k = 0, \quad a(T) = 0, \quad (11)$$

and $\hat{\mu}$ is

$$\hat{\mu}_t := \mathbb{E} \left[\hat{X}_t \middle| \tilde{\mathcal{F}}_t \right] = \mathbb{E}[X_0] + \tilde{W}_t.$$

Moreover, the equilibrium control follows

$$\hat{\alpha}_t = -2a(t) \left(\hat{X}_t - \hat{\mu}_t \right). \quad (12)$$

Proposition 2. *Suppose Assumptions 1-2 hold. For the N -player game, the path and the control of player i under the equilibrium are given by*

$$d\hat{X}_{it}^{(N)} = -2a^N(t) \left(\hat{X}_{it}^{(N)} - \frac{1}{N-1} \sum_{j \neq i}^N \hat{X}_{jt}^{(N)} \right) dt + dW_{it}^{(N)} + d\tilde{W}_t, \quad (13)$$

and

$$\hat{\alpha}_{it}^{(N)} = -2a^N(t) \left(\hat{X}_{it}^{(N)} - \frac{1}{N-1} \sum_{j \neq i}^N \hat{X}_{jt}^{(N)} \right)$$

respectively for $i = 1, 2, \dots, N$, where a^N is the solution of

$$a' - \frac{2(N+1)}{N-1}a^2 + \frac{N-1}{N}k = 0, \quad a(T) = 0. \quad (14)$$

3.1 Preliminaries

We first recall the convergence rate of empirical measures of i.i.d. sequence provided in Theorem 1 of [10] and Theorem 5.8 of [4].

Lemma 1. *Let $d = 1$ or 2 . Suppose $\{X_i : i \in \mathbb{N}\}$ is a sequence of d dimensional i.i.d. random variables with $\mathbb{E}[|X_1|^q] < \infty$ for some $q > 4$. Then, the empirical measure*

$$\rho^N(X) = \frac{1}{N} \sum_{i=1}^N \delta_{X_i}$$

satisfies

$$\mathbb{E} [\mathbb{W}_p^p(\rho^N(X), \mathcal{L}(X_1))] = \begin{cases} O(N^{-1/2}), & \text{if } p \in (1, 2], \\ O(N^{-1/2}), & \text{if } p = 1, d = 1, \\ O(N^{-1/2} \ln N), & \text{if } p = 1, d = 2. \end{cases}$$

Next, we give the definition of some notations that will be used in the following part. Denote $C_b(\mathbb{R}^d)$ to be the collection of bounded and continuous functions on \mathbb{R}^d , and let $C_b^1(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ be the space of functions on \mathbb{R}^d whose first order derivative is also bounded and continuous.

Lemma 2. *Suppose m_1, m_2 are two probability measures on $\mathcal{B}(\mathbb{R}^d)$ and $f \in C_b^1(\mathbb{R}^d, \mathbb{R})$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel set on \mathbb{R}^d . Then,*

$$\mathbb{W}_p(f_*m_1, f_*m_2) \leq |Df|_0 \mathbb{W}_p(m_1, m_2),$$

where f_*m_j is the pushforward measure for $j = 1, 2$, and $|Df|_0 = \sup_{x \in \mathbb{R}^d} \max\{|\partial_{x_i} f(x)| : i = 1, 2, \dots, d\}$.

Proof. We define a function $F(x, y) = (f(x), f(y)) : \mathbb{R}^{2d} \mapsto \mathbb{R}^2$. Note that, for any $\pi \in \Pi(m_1, m_2)$, $F_*\pi \in \Pi(f_*m_1, f_*m_2)$, i.e.,

$$F_*\Pi(m_1, m_2) \subset \Pi(f_*m_1, f_*m_2).$$

Therefore, we have the following inequalities:

$$\begin{aligned} \mathbb{W}_p^p(f_*m_1, f_*m_2) &= \inf_{\pi' \in \Pi(f_*m_1, f_*m_2)} \int_{\mathbb{R}^2} |x - y|^p \pi'(dx, dy) \\ &\leq \inf_{\pi' \in F_*\Pi(m_1, m_2)} \int_{\mathbb{R}^2} |x - y|^p \pi'(dx, dy) \\ &= \inf_{\pi \in \Pi(m_1, m_2)} \int_{\mathbb{R}^{2d}} |f(x) - f(y)|^p \pi(dx, dy) \\ &\leq |Df|_0^p \inf_{\pi \in \Pi(m_1, m_2)} \int_{\mathbb{R}^{2d}} |x - y|^p \pi(dx, dy) \\ &= |Df|_0^p \mathbb{W}_p^p(m_1, m_2). \end{aligned}$$

□

Lemma 3. *Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of d dimensional random variables in $(\Omega, \mathcal{F}, \mathbb{P})$. Let $f \in C_b^1(\mathbb{R}^d)$. We also denote by $f(X)$ the sequence $\{f(X_i) : i \in \mathbb{N}\}$. Then*

$$\mathbb{W}_p(\rho^N(f(X)), \mathcal{L}(f(X_1))) \leq |Df|_0 \mathbb{W}_p(\rho^N(X), \mathcal{L}(X_1)), \quad \text{almost surely}$$

where $|Df|_0 = \sup_{x \in \mathbb{R}^d} \max\{|\partial_{x_i} f(x)| : i = 1, 2, \dots, d\}$.

Proof. For any sequence $\{c_i : i \in \mathbb{N}\}$ in \mathbb{R}^d , the empirical measure $\rho^N(c) := \frac{1}{N} \sum_{i=1}^N \delta_{c_i}$ satisfies

$$\rho^N(f(c)) = f_* \rho^N(c),$$

since

$$\langle \phi, \rho^N(f(c)) \rangle = \frac{1}{N} \sum_{i=1}^N \phi(f(c_i)) = \langle \phi \circ f, \rho^N(c) \rangle, \quad \forall \phi \in C_b(\mathbb{R}^d).$$

This implies that

$$\rho^N(f(X)) = f_* \rho^N(X), \quad \text{almost surely.}$$

On the other hand, we also have

$$\mathcal{L}(f(X_1))(A) = \mathbb{P}(f(X_1) \in A) = \mathbb{P}(X_1 \in f^{-1}(A)) = f_* \mathcal{L}(X_1)(A), \quad \forall A \in \mathcal{B}(\mathbb{R}^d).$$

Therefore, the conclusion follows by applying Lemma 2. \square

3.2 Empirical measures of a sequence with a common noise

We are going to apply lemmas from the previous subsection to study the convergence of empirical measures of a sequence with a common noise in the following sense.

Definition 3. We say a sequence of random variables $X = \{X_i : i \in \mathbb{N}\}$ is a sequence with a common noise, if there exists a random variable β such that

- $X - \beta = \{X_i - \beta : i \in \mathbb{N}\}$ is a sequence of i.i.d. random variables,
- β is independent to $X - \beta$.

By this definition, a sequence with a common noise is i.i.d. if and only if β is a deterministic constant.

Example 1. Let $q > 4$ be a given constant and $X = \{X_i : i \in \mathbb{N}\}$ be a 1-dimensional sequence of L^q random variables with a common noise term β , where

$$X_i - \beta = \gamma_i + \sigma \alpha_i.$$

In above, $\{(\alpha_i, \gamma_i) : i \in \mathbb{N}\}$ is a sequence of 2-dimensional i.i.d. random variables independent to β , and σ is a given non-negative constant. Let $\rho^N(X)$ be the empirical measure defined by

$$\rho^N(X) = \frac{1}{N} \sum_{i=1}^N \delta_{X_i}.$$

The first question is

(Qa) In Example 1, where does $\rho^N(X)$ converge to?

For any test function $\phi \in C_b(\mathbb{R})$,

$$\langle \phi, \rho^N(X) \rangle = \frac{1}{N} \sum_{i=1}^N \phi(X_i) = \frac{1}{N} \sum_{i=1}^N \phi(\gamma_i + \sigma \alpha_i + \beta).$$

Since β is independent to (α_i, γ_i) , by Example 4.1.5 of [9] together with the Law of Large Numbers, we have

$$\frac{1}{N} \sum_{i=1}^N \phi(\gamma_i + \sigma \alpha_i + c) \rightarrow \mathbb{E}[\phi(\gamma_1 + \sigma \alpha_1 + c)] = \mathbb{E}[\phi(\gamma_1 + \sigma \alpha_1 + \beta) | \beta = c], \quad \forall c \in \mathbb{R}.$$

Therefore, we conclude that

$$\begin{aligned} \langle \phi, \rho^N(X) \rangle &\rightarrow \mathbb{E}[\phi(\gamma_1 + \sigma \alpha_1 + \beta) | \beta], \quad \beta - a.s. \\ &= \langle \phi, \mathcal{L}(\gamma_1 + \sigma \alpha_1 + \beta | \beta) \rangle, \quad \beta - a.s. \end{aligned}$$

Hence, the answer for the (Qa) is

- $\rho^N(X) \Rightarrow \mathcal{L}(X_1 | \beta)$, β -a.s. More precisely, since all random variables are square-integrable, the weak convergence implies, for all $p \in [1, 2]$,

$$\mathbb{W}_p(\rho^N(X), \mathcal{L}(X_1 | \beta)) \rightarrow 0, \quad \beta - a.s.$$

The next question is

(Qb) In Example 1, what's the convergence rate in the sense $\mathbb{E}[\mathbb{W}_p^p(\rho^N(X), \mathcal{L}(X_1 | \beta))]$?

Since β is independent to $\gamma_1 + \sigma \alpha_1$, by Example 4.1.5 of [9], we have

$$\mathbb{E}[\phi(\gamma_1 + \sigma \alpha_1 + \beta) | \beta = c] = \mathbb{E}[\phi(\gamma_1 + \sigma \alpha_1 + c)], \quad \forall \phi \in C_b(\mathbb{R}), c \in \mathbb{R},$$

or equivalently, if one takes $c = \beta(\omega)$,

$$\mathcal{L}(X_1 | \beta)(\omega) = \mathcal{L}(\gamma_1 + \sigma \alpha_1 + \beta | \beta)(\omega) = \mathcal{L}(\gamma_1 + \sigma \alpha_1 + c).$$

On the other hand, with $c = \beta(\omega)$,

$$\rho^N(X)(\omega) = \rho^N(X(\omega)) = \frac{1}{N} \sum_{i=1}^N \delta_{\gamma_i(\omega) + \sigma \alpha_i(\omega) + c}.$$

From the above two identities, with $c = \beta(\omega)$, we can write

$$\mathbb{W}_p(\rho^N(X)(\omega), \mathcal{L}(X_1 | \beta = c)(\omega)) = \mathbb{W}_p\left(\frac{1}{N} \sum_{i=1}^N \delta_{\gamma_i(\omega) + \sigma \alpha_i(\omega) + c}, \mathcal{L}(\gamma_1 + \sigma \alpha_1 + c)\right). \quad (15)$$

Now we can conclude (Qb) in the next lemma.

Lemma 4. *Let $p \in [1, 2]$ be a given constant. For a sequence $X = \{X_i : i \in \mathbb{N}\}$ with a common noise β as of Example 1, we have*

$$\mathbb{E}[\mathbb{W}_p^p(\rho^N(X), \mathcal{L}(X_1 | \beta))] = O\left(N^{-\frac{1}{2}}\right).$$

Proof. Originally, $X_i = \gamma_i + \sigma \alpha_i + \beta$ of Example 1 are dependent due to the common term β . We apply (49) in Lemma 11 in Appendix to (15) and obtain

$$\begin{aligned} \mathbb{W}_p(\rho^N(X)(\omega), \mathcal{L}(X_1 | \beta)(\omega)) &= \mathbb{W}_p\left(\frac{1}{N} \sum_{i=1}^N \delta_{\gamma_i(\omega) + \sigma \alpha_i(\omega) + \beta(\omega)}, \mathcal{L}(\gamma_1 + \sigma \alpha_1 + \beta(\omega))\right) \\ &= \mathbb{W}_p(\rho^N(\gamma(\omega) + \sigma \alpha(\omega)), \mathcal{L}(\gamma_1 + \sigma \alpha_1)). \end{aligned}$$

Now, the convergence of empirical measures is equivalent to the ones of i.i.d. sequence $\{\gamma_i + \sigma \alpha_i : i \in \mathbb{N}\}$. The conclusion follows from Lemma 1. \square

Next, we present the uniform convergence rate by combining Lemma 3.

Lemma 5. *In Example 1, we use $X(\sigma)$ to denote X to emphasize its dependence on σ . Then,*

$$\mathbb{E} \left[\sup_{\sigma \in [0,1]} \mathbb{W}_p^p(\rho^N(X(\sigma)), \mathcal{L}(X_1(\sigma)|\beta)) \right] = \begin{cases} O\left(N^{-\frac{1}{2}} \ln(N)\right), & \text{if } p = 1, \\ O\left(N^{-\frac{1}{2}}\right), & \text{if } 1 < p \leq 2. \end{cases}$$

Proof. Note that, by (49) in Lemma 11 in Appendix,

$$\mathbb{W}_p^p(\rho^N(X(\sigma)), \mathcal{L}(X_1(\sigma)|\beta)) = \mathbb{W}_p^p(\rho^N(\gamma_i + \sigma\alpha_i), \mathcal{L}(\gamma_1 + \sigma\alpha_1)).$$

Next, applying Lemma 3 with $f(x, y) = x + \sigma y$, we obtain

$$\begin{aligned} \sup_{\sigma \in [0,1]} \mathbb{W}_p^p(\rho^N(\gamma_i + \sigma\alpha_i), \mathcal{L}(\gamma_1 + \sigma\alpha_1)) &\leq \sup_{\sigma \in [0,1]} \max\{1, \sigma^p\} \mathbb{W}_p^p(\rho^N((\gamma, \alpha)), \mathcal{L}((\gamma_1, \alpha_1))) \\ &= \mathbb{W}_p^p(\rho^N((\gamma, \alpha)), \mathcal{L}((\gamma_1, \alpha_1))). \end{aligned}$$

At last, using Lemma 1 for the 2-dimensional i.i.d. sequence $\{(\gamma_i, \alpha_i) : i \in \mathbb{N}\}$, we obtain the desired conclusion. \square

3.3 Generalization of the convergence to triangular arrays

Unfortunately, $(\hat{X}_{1t}^{(N)}, \hat{X}_{2t}^{(N)}, \dots, \hat{X}_{Nt}^{(N)})$ of the N -player's game does not have a clean structure with a common noise term β given in Example 1. Therefore, we need a generalization of the convergence result in Example 1 to a triangular array. To proceed, we provide the following lemma.

Lemma 6. *Let $\sigma > 0$, $q > 4$, and*

$$X_i^N(\sigma) = \gamma_i^N + \sigma\alpha_i^N + \Delta_i^N(\sigma) + \beta, \text{ and } \hat{X}(\sigma) = \hat{\gamma} + \sigma\hat{\alpha} + \beta,$$

where

- $(\gamma^N, \alpha^N) = \{(\gamma_i^N, \alpha_i^N) : i \in \mathbb{N}\}$ is a sequence of 2-dimensional i.i.d. random variables with distribution identical to $\mathcal{L}((\hat{\gamma}, \hat{\alpha}))$ with $(\hat{\gamma}, \hat{\alpha}) \in L^q$ for some $q > 4$,
- $\beta \in L^q$ is independent to the random variables $(\gamma_i^N, \alpha_i^N, \hat{\gamma}, \hat{\alpha})$,
- $\max_{i=1,2,\dots,N} \mathbb{E} \left[\sup_{\sigma \in [0,1]} |\Delta_i^N(\sigma)|^2 \right] = O(N^{-1})$.

Let $\rho^N(X^N)$ be the empirical measure given by

$$\rho^N(X^N) = \frac{1}{N} \sum_{i=1}^N \delta_{X_i^N}.$$

Then, we have the following three results: For $p \in [1, 2]$,

$$\mathbb{W}_p \left(\mathcal{L}(X_1^N(\sigma)), \mathcal{L}(\hat{X}(\sigma)) \right) = O\left(N^{-\frac{1}{2}}\right), \quad (16)$$

$$\sup_{\sigma \in [0,1]} \mathbb{E} \left[\mathbb{W}_p^p \left(\rho^N(X^N(\sigma)), \mathcal{L}(\hat{X}(\sigma)|\beta) \right) \right] = O\left(N^{-\frac{1}{2}}\right), \quad (17)$$

and

$$\mathbb{E} \left[\sup_{\sigma \in [0,1]} \mathbb{W}_p^p \left(\rho^N(X^N(\sigma)), \mathcal{L}(\hat{X}(\sigma)|\beta) \right) \right] = \begin{cases} O\left(N^{-\frac{1}{2}} \ln(N)\right), & \text{if } p = 1, \\ O\left(N^{-\frac{1}{2}}\right), & \text{if } p > 1. \end{cases} \quad (18)$$

Proof. We will omit the dependence of σ if there is no confusion, for instance, we use X in lieu of $X(\sigma)$. Since $\mathcal{L}(\hat{X}) = \mathcal{L}(X_1^N - \Delta_1^N)$, the first result (16) directly follows from

$$\mathbb{W}_p^p(\mathcal{L}(X_1^N), \mathcal{L}(\hat{X})) \leq \mathbb{E}[|\Delta_1^N|^p] \leq \left(\mathbb{E}[|\Delta_1^N|^2]\right)^{\frac{p}{2}} = O\left(N^{-\frac{p}{2}}\right).$$

Next, we set $Y_i^N(\sigma) = \gamma_i^N + \sigma\alpha_i^N + \beta$. By the definition of empirical measures, we have

$$\mathbb{W}_p^p(\rho^N(X^N), \rho^N(Y^N)) \leq \frac{1}{N} \sum_{i=1}^N |X_i^N - Y_i^N|^p = \frac{1}{N} \sum_{i=1}^N |\Delta_i^N|^p. \quad (19)$$

From the third condition on Δ_i^N , we obtain

$$\mathbb{E}[\mathbb{W}_p^p(\rho^N(X^N), \rho^N(Y^N))] = O\left(N^{-\frac{p}{2}}\right).$$

By Lemma 4, we also have

$$\mathbb{E}[\mathbb{W}_p^p(\rho^N(Y^N), \mathcal{L}(\hat{X}|\beta))] = O\left(N^{-\frac{1}{2}}\right).$$

In the end, (17) follows from the triangle inequality together with the fact that $p \geq 1$. Finally, for the proof of (18), we first use (19) to write

$$\mathbb{W}_p^p(\rho^N(X^N(\sigma)), \mathcal{L}(\hat{X}(\sigma)|\beta)) \leq 2^{p-1} \left(\mathbb{W}_p^p(\rho^N(Y^N(\sigma)), \mathcal{L}(\hat{X}(\sigma)|\beta)) + \frac{1}{N} \sum_{i=1}^N |\Delta_i^N(\sigma)|^p \right).$$

Applying Lemma 5 and the third condition on $\Delta_i^N(\sigma)$, we can conclude (18). \square

3.4 Proof of Theorem 1

For simplicity, let us introduce the following notations:

$$\mathcal{E}_t(a) = \exp\left\{\int_0^t a(s)ds\right\}, \quad \mathcal{E}_t(a, M) = \int_0^t \mathcal{E}_s(a) dM_s$$

for a deterministic function $a(\cdot)$ and a martingale $M = \{M_t\}_{t \geq 0}$. With these notations, one can write the solution to the Ornstein–Uhlenbeck process

$$dX_t = -a_t X_t dt + dM_t$$

for a determinant function a in the form of

$$\mathcal{E}_t(a) X_t = X_0 + \mathcal{E}_t(a, M). \quad (20)$$

For MFG equilibrium, we define

$$\tilde{X}_t = \hat{X}_t - \hat{\mu}_t.$$

According to (10) in Proposition 1, \tilde{X} satisfies the following equation:

$$\tilde{X}_t = \tilde{X}_0 - \int_0^t 2a_s \tilde{X}_s ds + W_t.$$

Next, we express the solution of the above SDE in the form of

$$\tilde{Y}_t := \mathcal{E}_t(2a)\tilde{X}_t = \tilde{X}_0 + \mathcal{E}_t(2a, W).$$

Note that \tilde{Y} and $\hat{\mu}$ are independent. Therefore, \hat{X} admits a decomposition of two independent processes as

$$\hat{X}_t = \tilde{X}_t + \hat{\mu}_t.$$

Furthermore, we have

$$\hat{Y}_t := \mathcal{E}_t(2a)\hat{X}_t = \tilde{X}_0 + \mathcal{E}_t(2a, W) + \mathcal{E}_t(2a) \left(\hat{\mu}_0 + \tilde{W}_t \right).$$

In the N -player game, we define the following quantities:

$$\bar{X}_t^{(N)} = \frac{1}{N} \sum_{i=1}^N \hat{X}_{it}^{(N)}, \quad \bar{W}_t^{(N)} = \frac{1}{N} \sum_{i=1}^N W_{it}^{(N)},$$

and

$$\tilde{X}_{it}^{(N)} = \hat{X}_{it}^{(N)} - \bar{X}_t^{(N)}.$$

It is worth noting that, by Proposition 2, we have

$$\hat{X}_{it}^{(N)} = \hat{X}_{i0}^{(N)} - \int_0^t 2 \frac{N}{N-1} a^N(s) \left(\hat{X}_{is}^{(N)} - \frac{1}{N} \sum_{j=1}^N \hat{X}_{js}^{(N)} \right) ds + W_{it}^{(N)} + \tilde{W}_t$$

for all $i = 1, 2, \dots, N$, then the mean-field term satisfies

$$\bar{X}_t^{(N)} = \bar{X}_0^{(N)} + \bar{W}_t^{(N)} + \tilde{W}_t$$

and the i -th player's path deviated from the mean-field path can be rewritten by

$$\tilde{X}_{it}^{(N)} = \tilde{X}_{i0}^{(N)} - \int_0^t 2\hat{a}^N(s) \tilde{X}_{is}^{(N)} ds + W_{it}^{(N)} - \bar{W}_t^{(N)},$$

where

$$\hat{a}^N = \frac{N}{N-1} a^N.$$

Next, we introduce

$$\hat{Y}_{it}^{(N)} = \mathcal{E}_t(2\hat{a}^N) \hat{X}_{it}^{(N)}, \quad \tilde{Y}_{it}^{(N)} = \mathcal{E}_t(2\hat{a}^N) \tilde{X}_{it}^{(N)}, \quad \bar{Y}_t^{(N)} = \mathcal{E}_t(2\hat{a}^N) \bar{X}_t^{(N)}.$$

Consequently, we obtain the following relationships:

$$\tilde{Y}_{it}^{(N)} = \tilde{X}_{i0}^{(N)} + \mathcal{E}_t \left(2\hat{a}^N, W_i^{(N)} - \bar{W}^{(N)} \right),$$

$$\bar{Y}_t^{(N)} = \mathcal{E}_t(2\hat{a}^N) \left(\bar{W}_t^{(N)} + \tilde{W}_t + \bar{X}_0^{(N)} \right),$$

and

$$\hat{Y}_{it}^{(N)} = \bar{Y}_{it}^{(N)} + \tilde{Y}_{it}^{(N)}.$$

To compare the process $\hat{Y}_{it}^{(N)}$ with the target process

$$\begin{aligned}\hat{Y}_t &= \tilde{X}_0 + \mathcal{E}_t(2a, W) + \mathcal{E}_t(2a) (\hat{\mu}_0 + \tilde{W}_t) \\ &= \tilde{X}_0 + \sigma_t Z_t + \mathcal{E}_t(2a) (\hat{\mu}_0 + \tilde{W}_t),\end{aligned}\tag{21}$$

where

$$\sigma_t = \left(\int_0^t \mathcal{E}_s(4a) ds \right)^{1/2},$$

and

$$Z_t = \sigma_t^{-1} \mathcal{E}_t(2a, W) \sim \mathcal{N}(0, 1),$$

we write $\hat{Y}_{it}^{(N)}$ by

$$\begin{aligned}\hat{Y}_{it}^{(N)} &= \tilde{X}_{i0}^{(N)} + \mathcal{E}_t(2a, W_i^{(N)}) + \Delta_{it}^{(N)} + \mathcal{E}_t(2a) (\hat{\mu}_0 + \tilde{W}_t) \\ &= \tilde{X}_{i0}^{(N)} + \sigma_t Z_{it}^{(N)} + \Delta_{it}^{(N)} + \mathcal{E}_t(2a) (\hat{\mu}_0 + \tilde{W}_t),\end{aligned}\tag{22}$$

where

$$Z_{it}^{(N)} = \sigma_t^{-1} \mathcal{E}_t(2a, W_i^{(N)}) \sim \mathcal{N}(0, 1),$$

and

$$\begin{aligned}\Delta_{it}^{(N)} &= \left(\mathcal{E}_t(2\hat{a}^N, W_i^{(N)}) - \mathcal{E}_t(2a, W_i^{(N)}) \right) \\ &\quad - \mathcal{E}_t(2\hat{a}^N, \bar{W}^{(N)}) \\ &\quad + (\mathcal{E}_t(2\hat{a}^N) - \mathcal{E}_t(2a)) (\hat{\mu}_0 + \tilde{W}_t) \\ &\quad + \mathcal{E}_t(2\hat{a}^N) (\bar{X}_0^{(N)} - \hat{\mu}_0 + \bar{W}_t^{(N)}) \\ &:= I_{it}^{(N)} + II_{it}^{(N)} + III_{it}^{(N)} + IV_{it}^{(N)}.\end{aligned}\tag{23}$$

To apply Lemma 6 to the processes of (22) and (21), we only need to show the second moment on $\sup_{t \in [0, T]} \Delta_{it}^{(N)}$ is $O(N^{-1})$ for each $i = 1, 2, \dots, N$. In the following analysis, we will utilize the explicit solution of the ODE:

- Let $c, d > 0$ be two constants. The solution of

$$v'(t) - c^2 v^2(t) + d^2 = 0, \quad v(T) = 0$$

is

$$v(t) = \frac{d}{c} \cdot \frac{1 - e^{2dc(t-T)}}{1 + e^{2dc(t-T)}}.\tag{24}$$

We will employ this solution to derive the second-moment estimations of $\sup_{t \in [0, T]} \Delta_{it}^{(N)}$.

1. From (24), we have an estimation of

$$|a^N(t) - a(t)| = \frac{k|T-t|}{N} + o(N^{-1}).\tag{25}$$

Therefore, we have

$$|\mathcal{E}_t(2\hat{a}^N) - \mathcal{E}_t(2a)| = \frac{2t(T-t)}{N} + o(N^{-1})\tag{26}$$

and thus by Burkholder-Davis-Gundy (BDG) inequality

$$\begin{aligned}
\mathbb{E} \left[\sup_{t \in [0, T]} \left(I_{it}^{(N)} \right)^2 \right] &= \mathbb{E} \left[\sup_{t \in [0, T]} \left(\int_0^t (\mathcal{E}_s(2\hat{a}^N) - \mathcal{E}_s(2a)) dW_{is}^{(N)} \right)^2 \right] \\
&\leq C \mathbb{E} \left[\left(\int_0^T (\mathcal{E}_s(2\hat{a}^N) - \mathcal{E}_s(2a)) dW_{is}^{(N)} \right)^2 \right] \quad \text{for some constant } C > 0 \\
&= C \int_0^T (\mathcal{E}_s(2\hat{a}^N) - \mathcal{E}_s(2a))^2 ds \\
&= O(N^{-2}).
\end{aligned}$$

2. Since \hat{a}^N is uniformly bounded by $\sqrt{k/2}$, $II_t^{(N)}$ is a martingale with its quadratic variance

$$[II^{(N)}]_T = \frac{1}{N} \int_0^T \mathcal{E}_s(4\hat{a}^N) ds = O(N^{-1}).$$

So, we have

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left(II_t^{(N)} \right)^2 \right] = O(N^{-1}).$$

3. From the estimation (26), we also have

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left(III_t^{(N)} \right)^2 \right] = O(N^{-2}).$$

4. By the assumption of i.i.d. initial states, we have

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left(IV_t^{(N)} \right)^2 \right] = \mathcal{E}_T(4\hat{a}^N) \left(\text{Var}(\bar{X}_0^{(N)}) + \mathbb{E} \left[\sup_{t \in [0, T]} \left(\bar{W}_t^{(N)} \right)^2 \right] \right) = O(N^{-1}).$$

As a result, we have the following expression:

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left(\Delta_{it}^{(N)} \right)^2 \right] = O(N^{-1}), \quad \forall i = 1, 2, \dots, N. \quad (27)$$

By combining equations (21), (22), and (27), we can conclude Theorem 1 by applying Lemma 6.

4 Proposition 1: Derivation of the MFG path

This section is dedicated to proving Proposition 1, which provides insights into the MFG solution. To proceed, in Subsection 4.1, we begin by reformulating the MFG problem, assuming a Markovian structure for the equilibrium. Then, in Subsection 4.2, we solve the underlying control problem and derive the corresponding Riccati system. Finally, in Subsection 4.3, we examine the fixed-point condition of the MFG problem, leading to the conclusion.

4.1 Reformulation

To determine the equilibrium measure, as defined in Definition 2, one needs to explore the infinite-dimensional space of random measure flows $m : (0, T] \times \Omega \rightarrow \mathcal{P}_2(\mathbb{R})$ until a measure flow satisfies the fixed-point condition $m_t = \mathcal{L}(\hat{X}_t | \tilde{\mathcal{F}}_t)$ for all $t \in (0, T]$, as illustrated in Figure 1.

The first observation is that the cost function F in (8) is only dependent on the measure m through the first two moments with the quadratic cost structure, which is given by

$$F(x, m) = k(x^2 - 2x[m]_1 + [m]_2).$$

Consequently, the underlying stochastic control problem for MFG can be entirely determined by the input given by the \mathbb{R}^2 valued random processes $\mu_t = [m_t]_1$ and $\nu_t = [m_t]_2$, which implies that the fixed point condition can be effectively reduced to merely checking two conditions:

$$\mu_t = \mathbb{E} \left[\hat{X}_t \middle| \tilde{\mathcal{F}}_t \right], \quad \nu_t = \mathbb{E} \left[\hat{X}_t^2 \middle| \tilde{\mathcal{F}}_t \right].$$

This observation effectively reduces our search from the space of random measure-valued processes $m : (0, T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$ to the space of \mathbb{R}^2 -valued random processes $(\mu, \nu) : (0, T] \times \Omega \mapsto \mathbb{R}^2$.

It is important to note that if the underlying MFG does not involve common noise, the aforementioned observation is adequate to transform the original infinite-dimensional MFG into a finite-dimensional system. In this case, the moment processes (μ, ν) become deterministic mappings $[0, T] \rightarrow \mathbb{R}^2$. However, the following example demonstrates that this is not applicable to MFG with common noise, which presents a significant drawback in characterizing LQG-MFG using a finite-dimensional system.

Example 2. *To illustrate this point, let's consider the following uncontrolled mean field dynamics: Let the mean field term $\mu_t := \mathbb{E}[\hat{X}_t | \tilde{\mathcal{F}}_t]$, where the underlying dynamic is given by*

$$d\hat{X}_t = -\mu_t \tilde{W}_t dt + dW_t + d\tilde{W}_t, \quad \hat{X}_0 = X_0.$$

Here are two key observations:

- μ_t is path dependent on entire path of \tilde{W} , i.e.,

$$\mu_t = \mu_0 e^{-\int_0^t \tilde{W}_s ds} + e^{-\int_0^t \tilde{W}_s ds} \int_0^t e^{\int_0^s \tilde{W}_r dr} d\tilde{W}_s.$$

This implies that the $(t, \tilde{W}) \mapsto \mu_t$ is a function on an infinite dimensional domain.

- μ_t is Markovian, i.e.,

$$d\mu_t = -\mu_t \tilde{W}_t dt + d\tilde{W}_t.$$

It is possible to express the μ_t via a SDE with finite-dimensional coefficient functions of (t, μ_t) .

To make the previous idea more concrete, we propose the assumption of a Markovian structure for the first and second moments of the MFG equilibrium. In other words, we restrict our search for equilibrium to a smaller space \mathcal{M} of measure flows that capture the Markovian structure of the first and second moments.

Definition 4. *The space \mathcal{M} is the collection of all $\tilde{\mathcal{F}}_t$ -adapted measure flows $m : [0, T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$, whose first moment $[m_t]_1 := \mu_t$ and second moment $[m_t]_2 := \nu_t$ satisfy a system of SDE*

$$\begin{aligned} \mu_t &= \mu_0 + \int_0^t (w_1(s)\mu_s + w_2(s)) ds + \tilde{W}_t, \\ \nu_t &= \nu_0 + \int_0^t (w_3(s)\mu_s + w_4(s)\nu_s + w_5(s)\mu_s^2 + w_6(s)) ds + 2 \int_0^t \mu_s d\tilde{W}_s, \end{aligned} \tag{28}$$

for some smooth deterministic functions $(w_i : i = 1, 2, \dots, 6)$ for all $t \in [0, T]$.

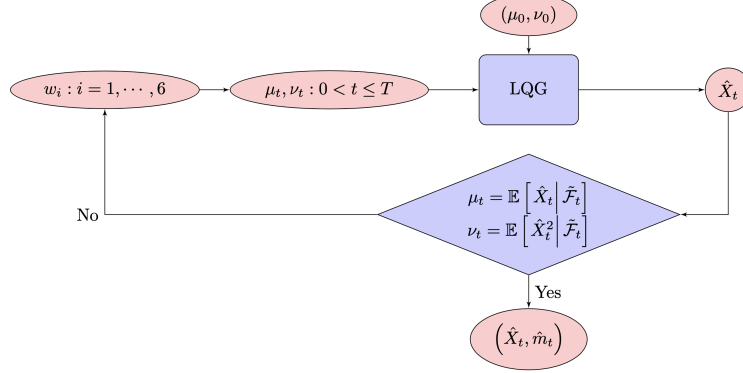


Figure 2: Equivalent MFG diagram with $\mu_0 = [m_0]_1$ and $\nu_0 = [m_0]_2$.

The MFG problem originally given by Definition 1 can be recast as the following combination of stochastic control problem and fixed point condition:

- RLQG(Revised LQG):

Given smooth functions $w = (w_i : i = 1, 2, \dots, 6)$, we want to find the value function $\bar{V} = \bar{V}[w] : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ and optimal path $(\hat{X}, \hat{\mu}, \hat{\nu})[w]$ from the following control problem:

$$\bar{V}(t, x, \bar{\mu}, \bar{\nu}) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_t^T \left(\frac{1}{2} \alpha_s^2 + \bar{F}(X_s, \mu_s, \nu_s) \right) ds \middle| X_t = x, \mu_t = \bar{\mu}, \nu_t = \bar{\nu} \right]$$

with the underlying process X of (1) and (μ, ν) of (28) and with the cost functions: $\bar{F} : \mathbb{R}^3 \mapsto \mathbb{R}$ given by

$$\bar{F}(x, \bar{\mu}, \bar{\nu}) = k(x^2 - 2x\bar{\mu} + \bar{\nu}),$$

where $\bar{\mu}, \bar{\nu}$ are scalars, while μ, ν are used as processes.

- RFP(Revised fixed point condition):

Determine w satisfying the following fixed point condition:

$$\hat{\mu}_s = \mathbb{E} \left[\hat{X}_s \middle| \tilde{\mathcal{F}}_s \right] \text{ and } \hat{\nu}_s = \mathbb{E} \left[\hat{X}_s^2 \middle| \tilde{\mathcal{F}}_s \right], \quad \forall s \in [0, T]. \quad (29)$$

The equilibrium measure is then $\mathcal{N}(\hat{\mu}_t, \hat{\nu}_t - \hat{\mu}_t^2)$.

Remark 1. It is important to highlight that the Markovian structure for the first and second moments of the MFG equilibrium in this manuscript differs significantly from that presented in [18]. In [18], the processes μ_t and ν_t are pairs of processes with finite variation, while in our case, they are quadratic variation processes.

Specifically, in [18], the coefficient functions depend on the common noise Y , whereas in (28), the coefficient functions $(w_i : i = 1, 2, \dots, 6)$ are independent of the common noise \tilde{W} . Instead, the first and second moments of the MFG equilibrium are only influenced by the common noise through an additive term.

4.2 The generic player's control with a given population measure

This section is devoted to the control problem RLQG parameterized by w .

4.2.1 HJB equation

To simplify the notation, let's denote each function $w_i(t)$ as w_i for $i \in \{1, 2, \dots, 6\}$. Assuming sufficient regularity conditions, and according to the dynamic programming principle (refer to [20] for more details), the value function \bar{V} defined in the RLQG problem can be obtained as a solution v of the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} \partial_t v + \inf_{a \in \mathbb{R}} \left(a \partial_x v + \frac{1}{2} a^2 \right) + (w_1 \bar{\mu} + w_2) \partial_{\bar{\mu}} v + (w_3 \bar{\mu} + w_4 \bar{\nu} + w_5 \bar{\mu}^2 + w_6) \partial_{\bar{\nu}} v + \partial_{xx} v + \frac{1}{2} \partial_{\bar{\mu}\bar{\mu}} v \\ \quad + \partial_{x\bar{\mu}} v + 2\bar{\mu}^2 \partial_{\bar{\nu}\bar{\nu}} v + 2\bar{\mu} \partial_{\bar{\mu}\bar{\nu}} v + 2\bar{\mu} \partial_{x\bar{\nu}} v + k(x^2 - 2\bar{\mu}x + \bar{\nu}) = 0, \\ v(T, x, \mu_T, \nu_T) = 0. \end{cases}$$

Therefore, the optimal control has to admit the feedback form of

$$\hat{\alpha}(t) = -\partial_x v(t, \hat{X}_t, \mu_t, \nu_t), \quad (30)$$

and then the HJB equation can be reduced to

$$\begin{cases} \partial_t v - \frac{1}{2} (\partial_x v)^2 + (w_1 \bar{\mu} + w_2) \partial_{\bar{\mu}} v + (w_3 \bar{\mu} + w_4 \bar{\nu} + w_5 \bar{\mu}^2 + w_6) \partial_{\bar{\nu}} v + \partial_{xx} v + \frac{1}{2} \partial_{\bar{\mu}\bar{\mu}} v \\ \quad + \partial_{x\bar{\mu}} v + 2\bar{\mu}^2 \partial_{\bar{\nu}\bar{\nu}} v + 2\bar{\mu} \partial_{\bar{\mu}\bar{\nu}} v + 2\bar{\mu} \partial_{x\bar{\nu}} v + k(x^2 - 2\bar{\mu}x + \bar{\nu}) = 0, \\ v(T, x, \mu_T, \nu_T) = 0. \end{cases} \quad (31)$$

Next, we identify what conditions are needed for equating the control problem RLQG and the above HJB equation. Denote \mathcal{S} to be the set of v such that $v \in C^\infty$ satisfies

$$(1 + |x|^2)^{-1} (|v| + |\partial_t v|) + (1 + |x| + |\mu|)^{-1} (|\partial_x v| + |\partial_\mu v|) + (|\partial_{xx} v| + |\partial_{x\mu} v| + |\partial_{\mu\mu} v| + |\partial_\nu v|) < K$$

for all (t, x, μ, ν) for some positive constant K .

Lemma 7. *Consider the control problem RLQG with some given smooth functions $w = (w_i : i = 1, 2, \dots, 6)$.*

1. *(Verification theorem) Suppose there exists a solution $v \in \mathcal{S}$ of (31). Then $v(t, x, \bar{\mu}, \bar{\nu}) = \bar{V}(t, x, \bar{\mu}, \bar{\nu})$, and an optimal control is provided by (30).*
2. *Suppose that the value function \bar{V} belongs to \mathcal{S} , and then $\bar{V}(t, x, \bar{\mu}, \bar{\nu})$ solves HJB equation (31). Moreover, $\hat{\alpha}$ of (30) is an optimal control.*

Proof. 1. First, we prove the verification theorem. Since $v \in \mathcal{S}$, for any admissible $\alpha \in \mathcal{H}_{\mathbb{F}}^4$, the process X^α is well defined and one can apply Itô's formula to obtain

$$\mathbb{E}[v(T, X_T, \mu_T, \nu_T)] = v(t, x, \bar{\mu}, \bar{\nu}) + \mathbb{E} \left[\int_t^T \mathcal{G}^{\alpha(s)} v(s, X_s, \mu_s, \nu_s) ds \right],$$

where

$$\begin{aligned} \mathcal{G}^a f(s, x, \bar{\mu}, \bar{\nu}) = & \left(\partial_t + a \partial_x + \partial_{xx} + (w_1 \bar{\mu} + w_2) \partial_{\bar{\mu}} + (w_3 \bar{\mu} + w_4 \bar{\nu} + w_5 \bar{\mu}^2 + w_6) \partial_{\bar{\nu}} \right. \\ & \left. + \frac{1}{2} \partial_{\bar{\mu}\bar{\mu}} + 2\bar{\mu}^2 \partial_{\bar{\nu}\bar{\nu}} + \partial_{x\bar{\mu}} + 2\bar{\mu} \partial_{\bar{\mu}\bar{\nu}} + 2\bar{\mu} \partial_{x\bar{\nu}} \right) f(s, x, \bar{\mu}, \bar{\nu}). \end{aligned}$$

Note that the HJB equation actually implies that

$$\inf_a \left\{ \mathcal{G}^a v + \frac{1}{2} a^2 \right\} = -\bar{F},$$

which again yields

$$-\mathcal{G}^a v \leq \frac{1}{2} a^2 + \bar{F}.$$

Hence, we obtain that for all $\alpha \in \mathcal{H}_{\mathbb{F}}^4$,

$$\begin{aligned} & v(t, x, \bar{\mu}, \bar{\nu}) \\ &= \mathbb{E} \left[\int_t^T -\mathcal{G}^{\alpha(s)} v(s, X_s, \mu_s, \nu_s) ds \right] + \mathbb{E} [v(T, X_T, \mu_T, \nu_T)] \\ &\leq \mathbb{E} \left[\int_t^T \left(\frac{1}{2} \alpha^2(s) + \bar{F}(X_s, \mu_s, \nu_s) \right) ds \right] \\ &= J(t, x, \alpha, \bar{\mu}, \bar{\nu}). \end{aligned}$$

In the above, if α is replaced by $\hat{\alpha}$ given by the feedback form (30), then since $\partial_x v$ is Lipschitz continuous in x , there exists corresponding optimal path $\hat{X} \in \mathcal{H}_{\mathbb{F}}^4$. Thus, $\hat{\alpha}$ is also in $\mathcal{H}_{\mathbb{F}}^4$. One can repeat all the above steps by replacing X and α by \hat{X} and $\hat{\alpha}$, and \leq sign by $=$ sign to conclude that v is indeed the optimal value.

2. The opposite direction of the verification theorem follows by taking $\theta \rightarrow t$ for the dynamic programming principle, for all stopping time $\theta \in [t, T]$,

$$\begin{aligned} & \bar{V}(t, x, \bar{\mu}, \bar{\nu}) \\ &= \mathbb{E} \left[\int_t^\theta \left(\frac{1}{2} \alpha_s^2 + \bar{F}(X_s, \mu_s, \nu_s) \right) ds + \bar{V}(\theta, X_\theta, \mu_\theta, \nu_\theta) \middle| X_t = x, \mu_t = \bar{\mu}, \nu_t = \bar{\nu} \right], \end{aligned}$$

which is valid under our regularity assumptions on all the partial derivatives. □

4.2.2 LQG solution

It is worth noting that the costs \bar{F} of RLQG are quadratic functions in $(x, \bar{\mu}, \bar{\nu})$, while the drift function of the process ν of (28) is not linear in $(x, \bar{\mu}, \bar{\nu})$. Therefore, the stochastic control problem RLQG does not fit into the typical LQG control structure. Nevertheless, similarly to the LQG solution, we guess the value function to be a quadratic function in the form of

$$v(t, x, \bar{\mu}, \bar{\nu}) = a(t)x^2 + b(t)\bar{\mu}^2 + c(t)\bar{\nu} + d(t) + e(t)x + f(t)\bar{\mu} + g(t)x\bar{\mu}. \quad (32)$$

Under the above setup for the value function v , for $t \in [0, T]$, the optimal control is given by

$$\hat{\alpha}_t = -\partial_x v(t, \hat{X}_t, \mu_t, \nu_t) = -2a(t)\hat{X}_t - e(t) - g(t)\mu_t, \quad (33)$$

and the optimal path \hat{X} is

$$\begin{cases} d\hat{X}_t = \left(-2a(t)\hat{X}_t - e(t) - g(t)\mu_t \right) dt + dW_t + d\tilde{W}_t, \\ \hat{X}_0 = X_0. \end{cases} \quad (34)$$

To proceed, we introduce the following Riccati system of ODEs for $t \in [0, T]$,

$$\begin{cases} a' - 2a^2 + k = 0, \\ b' - \frac{1}{2}g^2 + 2bw_1 + cw_5 = 0, \\ c' + cw_4 + k = 0, \\ d' - \frac{1}{2}e^2 + fw_2 + cw_6 + 2a + b + g = 0, \\ e' - 2ae + w_2g = 0, \\ f' - eg + w_1f + 2bw_2 + cw_3 = 0, \\ g' - 2ag + w_1g - 2k = 0, \end{cases} \quad (35)$$

with terminal conditions

$$a(T) = b(T) = c(T) = d(T) = e(T) = f(T) = g(T) = 0. \quad (36)$$

Lemma 8. *Suppose there exists a unique solution (a, b, c, d, e, f, g) to the Riccati system of ODEs (35)-(36) on $[0, T]$. Then the value function of (RMFG) is given by*

$$\begin{aligned} \bar{V}(t, x, \bar{\mu}, \bar{\nu}) &= v(t, x, \bar{\mu}, \bar{\nu}) \\ &= a(t)x^2 + b(t)\bar{\mu}^2 + c(t)\bar{\nu} + d(t) + e(t)x + f(t)\bar{\mu} + g(t)x\bar{\mu} \end{aligned} \quad (37)$$

for $t \in [0, T]$ and the optimal control and optimal path are given by (33) and (34), respectively.

Proof. With the form of value function v given in (32) and the conditional first and second moment of \hat{X}_t under the σ -algebra $\tilde{\mathcal{F}}_t$ given in (28), we have

$$\begin{aligned} \partial_t v &= a'(t)x^2 + e'(t)x + b'(t)\bar{\mu}^2 + f'(t)\bar{\mu} + g'(t)x\bar{\mu} + c'(t)\bar{\nu} + d'(t), \\ \partial_x v &= 2xa(t) + e(t) + g(t)\bar{\mu}, \\ \partial_{xx} v &= 2a(t), \\ \partial_{\bar{\mu}} v &= 2b(t)\bar{\mu} + f(t) + g(t)x, \\ \partial_{\bar{\nu}} v &= c(t), \\ \partial_{\bar{\mu}\bar{\mu}} v &= 2b(t), \\ \partial_{x\bar{\mu}} v &= g(t), \\ \partial_{\bar{\mu}\bar{\nu}} v &= \partial_{\bar{\nu}\bar{\nu}} v = \partial_{x\bar{\nu}} v = 0. \end{aligned}$$

Plugging them back to the HJB equation in (31), we get a system of ODEs in (35) by equating x , $\bar{\mu}$, $\bar{\nu}$ -like terms in each equation with the terminal conditions given in (36).

Therefore, any solution (a, b, c, d, e, f, g) of a system of ODEs (35) leads to the solution of HJB (31) in the form of the quadratic function given by (37). Since the (a, b, c, d, e, f, g) are differentiable functions on the closed set $[0, T]$, they are also bounded, and thus the regularity conditions needed for $v \in \mathcal{S}$ is valid. Finally, we invoke the verification theorem given by Lemma 7 to conclude the desired result. \square

4.3 Fixed point condition and the proof of Proposition 1

Returning to the ODE system (35), there are 7 equations, whereas we need to determine a total of 13 deterministic functions of $[0, T] \times \mathbb{R}$ to characterize MFG. These are

$$(a, b, c, d, e, f, g) \quad \text{and} \quad (w_i : i = 1, 2, \dots, 6).$$

In this below, we identify the missing 6 equations by checking the fixed point condition of RFP. This leads to a complete characterization of the equilibrium for MFG in Definition 1.

Lemma 9. *With the dynamic of the optimal path \hat{X} defined in (34), the fixed point condition (29) implies that the first moment $\hat{\mu}_s := \mathbb{E}[\hat{X}_s | \tilde{\mathcal{F}}_s]$ and the second moment $\hat{\nu}_s := \mathbb{E}[\hat{X}_s^2 | \tilde{\mathcal{F}}_s]$ of the optimal path conditioned on $\tilde{\mathcal{F}}_t$ satisfy*

$$\begin{cases} \hat{\mu}_s = \bar{\mu} + \int_t^s ((-2a(r) - g(r)) \hat{\mu}_r - e(r)) dr + \tilde{W}_s, \\ \hat{\nu}_s = \bar{\nu} + \int_t^s (2 - 4a(r) \hat{\nu}_r - 2e(r) \hat{\mu}_r - 2g(r) \hat{\mu}_r^2) dr + \int_t^s 2\hat{\mu}_r d\tilde{W}_r, \end{cases} \quad (38)$$

for $s \geq t$, and thus the coefficient functions $w = (w_i : i = 1, 2, \dots, 6)$ in (28) satisfy the following equations:

$$w_1 = -2a - g, \quad w_2 = -e, \quad w_3 = -2e, \quad w_4 = -4a, \quad w_5 = -2g, \quad w_6 = 2, \quad \forall t \in [0, T]. \quad (39)$$

Proof. With the dynamic of the optimal path \hat{X} given by (34), we have

$$\hat{X}_t = X_0 + \int_0^t (-2a(s) \hat{X}_s - e(s) - g(s) \hat{\mu}_s) ds + W_t + \tilde{W}_t,$$

and since the functions a, e, g are continuous on $[0, T]$, then we can change of order of integration and expectation and it yields

$$\begin{aligned} \hat{\mu}_t &= \mathbb{E} \left[\hat{X}_t \middle| \tilde{\mathcal{F}}_t \right] \\ &= \mathbb{E} \left[X_0 \middle| \tilde{\mathcal{F}}_t \right] + \int_0^t (-2a(s) \hat{\mu}_s - e(s) - g(s) \hat{\mu}_s) ds + \mathbb{E} \left[W_t \middle| \tilde{\mathcal{F}}_t \right] + \mathbb{E} \left[\tilde{W}_t \middle| \tilde{\mathcal{F}}_t \right] \\ &= \mathbb{E} \left[X_0 \middle| \tilde{\mathcal{F}}_t \right] + \int_0^t (-2a(s) \hat{\mu}_s - e(s) - g(s) \hat{\mu}_s) ds + \tilde{W}_t. \end{aligned}$$

Similarly, applying Itô's formula, we obtain

$$\hat{X}_t^2 = X_0^2 + \int_0^t (2 - 4a(s) \hat{X}_s^2 - 2e(s) \hat{X}_s - 2g(s) \hat{\mu}_s \hat{X}_s) ds + \int_0^t 2\hat{X}_s dW_s + \int_0^t 2\hat{X}_s d\tilde{W}_s,$$

and it follows that

$$\begin{aligned} \hat{\nu}_t &= \mathbb{E} \left[\hat{X}_t^2 \middle| \tilde{\mathcal{F}}_t \right] \\ &= \mathbb{E} \left[X_0^2 \middle| \tilde{\mathcal{F}}_t \right] + \int_0^t (2 - 4a(s) \hat{\nu}_s - 2e(s) \hat{\mu}_s - 2g(s) \hat{\mu}_s^2) ds + \mathbb{E} \left[\int_0^t 2\hat{X}_s dW_s \middle| \tilde{\mathcal{F}}_t \right] + \mathbb{E} \left[\int_0^t 2\hat{X}_s d\tilde{W}_s \middle| \tilde{\mathcal{F}}_t \right] \\ &= \mathbb{E} \left[X_0^2 \middle| \tilde{\mathcal{F}}_t \right] + \int_0^t (2 - 4a(s) \hat{\nu}_s - 2e(s) \hat{\mu}_s - 2g(s) \hat{\mu}_s^2) ds + \int_0^t 2\hat{\mu}_s d\tilde{W}_s. \end{aligned}$$

Thus the desired result in (38) is obtained. Next, comparing the terms in (28) and (38), to satisfy the fixed point condition in MFG, we require another 6 equations in (39) for the coefficient functions $w = (w_i : i = 1, 2, \dots, 6)$. \square

Using further algebraic structures, one can reduce the ODE system of 13 equations composed by (35) and (39) into a system of 4 equations.

Proof of Proposition 1. Let the smooth and bounded functions $\{w_i : i = 1, 2, \dots, 6\}$ be given, the functions (a, b, c, d, e, f, g) in (35) is a coupled linear system, and thus their existence, uniqueness and boundedness is shown by Theorem 12.1 in [1].

Plugging the 6 equations in (39) to the ODE system (35), we obtain

$$\begin{cases} a' - 2a^2 + k = 0, \\ b' - \frac{1}{2}g^2 - 4ab - 2bg - 2cg = 0, \\ c' - 4ac + k = 0, \\ d' - \frac{1}{2}e^2 - ef + 2c + 2a + b + g = 0, \\ e' - 2ae - eg = 0, \\ f' - eg - 2af - gf - 2be - 2ce = 0, \\ g' - 4ag - g^2 - 2k = 0, \end{cases}$$

with the terminal conditions

$$a(T) = b(T) = c(T) = d(T) = e(T) = f(T) = g(T) = 0.$$

Let $l = 2a + g$, and it easily to obtain

$$l'(t) - l^2(t) = 0, \quad l(T) = 0,$$

which implies that $l(t) = 2a(t) + g(t) = 0$ for all $t \in [0, T]$. This gives the result that $g = -2a$ and it yields $e' = 0$. Then with $e(T) = 0$, we have $e(t) = 0$ for all $t \in [0, T]$ and thus one can obtain $f' = 0$, which indicates that $f(t) = 0$ for all $t \in [0, T]$ as $f(T) = 0$. Therefore the ODE system (35) can be simplified to the following form about $(a(t), b(t), c(t), d(t) : t \in [0, T])$:

$$\begin{cases} a'(t) - 2a^2(t) + k = 0, \\ b'(t) - 2a^2(t) + 4a(t)c(t) = 0, \\ c'(t) - 4a(t)c(t) + k = 0, \\ d'(t) + b(t) + 2c(t) = 0, \end{cases} \quad (40)$$

with the terminal conditions

$$a(T) = b(T) = c(T) = d(T) = 0. \quad (41)$$

The unique solvability of the Riccati system (40)-(41) is proven in Lemma 12 in the Appendix. Note that the solution a of (11) is consistent with the solution of the Riccati system given by equations (40)-(41).

In this case, since $2a + g = 0$ and $e = 0$ for all $t \in [0, T]$, it follows that $\hat{\mu}_s = \bar{\mu} + \tilde{W}_s$ for all $s \in [t, T]$ from the fixed point result (38). Similarly,

$$\hat{\nu}_s = \bar{\nu} + \int_t^s (2 + 4a(r)\hat{\mu}_r^2 - 4a(r)\hat{\nu}_r) dr + \int_t^s 2\hat{\mu}_r d\tilde{W}_r, \quad \forall s \in [t, T].$$

Plugging $e = 0$ and $\hat{\mu}_s = \bar{\mu} + \tilde{W}_r$ back to (33), we obtain the optimal control by

$$\hat{\alpha}_s = 2a(s)(\bar{\mu} + \tilde{W}_s - \hat{X}_s).$$

Moreover, since $e = f = 0$ and $g = -2a$ for $s \in [t, T]$, the value function can be simplified from (32) to

$$v(t, x, \bar{\mu}, \bar{\nu}) = a(t)x^2 - 2a(t)x\bar{\mu} + b(t)\bar{\mu}^2 + c(t)\bar{\nu} + d(t).$$

This concludes Proposition 1. □

5 The N -Player Game

This section focuses on proving Proposition 2 regarding the corresponding N -player game. For simplicity, we can omit the superscript (N) when referring to the processes in the sample space $\Omega^{(N)}$.

To begin, we address the N -player game in Subsection 5.1, where we solve it and obtain a Riccati system containing $O(N^3)$ equations. Subsequently, we reduce the relevant Riccati system to an ODE system in Subsection 5.2, which has a dimension independent of N . This simplified system forms the fundamental component of the convergence result.

5.1 Characterization of the N -player game by Riccati system

It is important to emphasize that based on the problem setting in Subsection 2.2 and the running cost for each player specified in (9), the N -player game can be classified as an N -coupled stochastic LQG problem. As a result, the value function and optimal control for each player can be determined by means of the following Riccati system:

For $i = 1, 2, \dots, N$, consider

$$\begin{cases} A'_i - 2A_i^\top e_i e_i^\top A_i - 4 \sum_{j \neq i}^N A_j^\top e_j e_j^\top A_i + \frac{k}{N} \sum_{j \neq i}^N (e_i - e_j)(e_i - e_j)^\top = 0, \\ B'_i - 2A_i^\top e_i e_i^\top B_i - 2 \sum_{j \neq i}^N \left(A_i^\top e_j e_j^\top B_j + A_j^\top e_j e_j^\top B_i \right) = 0, \\ C'_i - \frac{1}{2} B_i^\top e_i e_i^\top B_i - \sum_{j \neq i}^N B_j^\top e_j e_j^\top B_i + 2\text{tr}(A_i) = 0, \\ A_i(T) = B_i(T) = C_i(T) = 0, \end{cases} \quad (42)$$

where A_i is $N \times N$ symmetric matrix, B_i is N -dimensional vector, $C_i \in \mathbb{R}$ is a real constant, and e_i is the i -th natural basis in \mathbb{R}^N for each $i = 1, 2, \dots, N$.

Lemma 10. *Suppose $(A_i, B_i, C_i : i = 1, 2, \dots, N)$ is the solution of the Riccati system (42). Then, the value functions of N -player game defined by (7) is*

$$V_i(x^{(N)}) = \left(x^{(N)}\right)^\top A_i(0)x^{(N)} + \left(x^{(N)}\right)^\top B_i(0) + C_i(0), \quad i = 1, 2, \dots, N.$$

Moreover, the path and the control under the equilibrium are given by

$$d\hat{X}_{it}^{(N)} = \left(-2(A_i(t))_i^\top \hat{X}_t^{(N)} - (B_i(t))_i\right) dt + dW_{it}^{(N)} + d\tilde{W}_t, \quad (43)$$

and

$$\hat{\alpha}_{it}^{(N)} = -2(A_i(t))_i^\top \hat{X}_t^{(N)} - (B_i(t))_i$$

for each $i = 1, 2, \dots, N$, where $(A)_i$ denotes the i -th column of matrix A , $(B)_i$ denotes the i -th entry of vector B and $\hat{X}_t^{(N)} = [\hat{X}_{1t}^{(N)}, \hat{X}_{2t}^{(N)}, \dots, \hat{X}_{Nt}^{(N)}]^\top$.

Proof. From the dynamic programming principle, it is standard that, under enough regularities, the players' value function $V(x^{(N)}) = (V_1, V_2, \dots, V_N)(x^{(N)})$ can be lifted to the solution $v_i(t, x^{(N)})$ of the following system of HJB equations, for $i = 1, 2, \dots, N$,

$$\begin{cases} \partial_t v_i + \inf_{a_{it} \in \mathbb{R}} \left(a_{it} \partial_{x_i} v_i + \frac{1}{2} a_{it}^2 \right) + \sum_{j \neq i}^N a_{jt} \partial_{x_j} v_i + \Delta v_i + \frac{k}{N} \sum_{j \neq i}^N \left((e_i - e_j)^\top x^{(N)} \right)^2 = 0, \\ v_i(T, x^{(N)}) = 0. \end{cases}$$

Note that with $a_{it} = -\partial_{x_i} v_i(t, x^{(N)})$ for each $i = 1, 2, \dots, N$, the term in the infimum attains the optimal value and thus the HJB equation can be reduced to

$$\begin{cases} \partial_t v_i - \frac{1}{2} (\partial_{x_i} v_i)^2 - \sum_{j \neq i}^N \partial_{x_j} v_j \partial_{x_j} v_i + \Delta v_i + \frac{k}{N} \sum_{j \neq i}^N \left((e_i - e_j)^\top x^{(N)} \right)^2 = 0, \\ v_i(T, x^{(N)}) = 0. \end{cases} \quad (44)$$

Then, the value functions V of N -player game defined by (7) is $V_i(x^{(N)}) = v_i(0, x^{(N)})$ for all $i = 1, 2, \dots, N$. Moreover, the path and the control under the equilibrium are given by

$$d\hat{X}_{it}^{(N)} = -\partial_{x_i} v_i(t, \hat{X}_t^{(N)}) dt + dW_{it}^{(N)} + d\tilde{W}_t,$$

and

$$\hat{\alpha}_{it}^{(N)} = -\partial_{x_i} v_i(t, \hat{X}_t^{(N)})$$

for $i = 1, 2, \dots, N$. The proof is the application of Itô's formula and the details are omitted here. Due to its LQG structure, the value function leads to a quadratic function of the form

$$v_i(t, x^{(N)}) = (x^{(N)})^\top A_i(t) x^{(N)} + (x^{(N)})^\top B_i(t) + C_i(t).$$

Plugging V_i into (44), and matching the coefficient of variables, we get the Riccati system of ODEs in (42) and the desired results are obtained. \square

5.2 Proof of Proposition 2: Reduced Riccati form for the equilibrium

At present, the MFG and the corresponding N -player game can be characterized by Proposition 1 and Lemma 10, respectively. One of our primary objectives is to examine the convergence of the representative optimal path $\hat{X}_{1t}^{(N)}$ generated by the N -player game defined in (42)-(43) to the optimal path \hat{X}_t of the MFG described in Proposition 1.

It should be noted that \hat{X}_t is solely dependent on the function $a(t)$, as indicated in the ODE (11). In contrast, $\hat{X}_{1t}^{(N)}$ depends on $O(N^3)$ many functions derived from the solutions of a substantial Riccati system (42) involving matrices $(A_{it}, B_{it} : i = 1, 2, \dots, N)$. Consequently, comparing these two processes meaningfully becomes an exceedingly challenging task without gaining further insight into the intricate structure of the Riccati system (42).

Proof of Proposition 2. Inspired from the setup in [18] and [16], we may seek a pattern for the matrix A_i in the following form:

$$(A_i)_{pq} = \begin{cases} a_1(t), & \text{if } p = q = i, \\ a_2(t), & \text{if } p = q \neq i, \\ a_3(t), & \text{if } p \neq q, p = i \text{ or } q = i, \\ a_4(t), & \text{otherwise.} \end{cases} \quad (45)$$

The next result justifies the above pattern: the N^2 entries of the matrix A_i can be embedded to a 2-dimensional vector space no matter how big N is.

For the Riccati system (42), with the given of A_i and suppose each function in A_i is continuous on $[0, T]$, it is obvious to see that $B_i = 0$ for all $t \in [0, T]$ and for all $i = 1, 2, \dots, N$. Note that in this case, for $i = 1, 2, \dots, N$, the optimal control is given by

$$\hat{\alpha}_i = -2 \sum_{j=1}^N (A_i)_{ij} \hat{X}_{jt}^{(N)} = -2 (A_i)_i^\top \hat{X}_t^{(N)},$$

where $(A)_i$ is the i -th column of matrix A .

Plugging the pattern (45) into the differential equation of A_i , we obtain the following system of ODEs:

$$\begin{cases} a_1' - 2a_1^2 - 4(N-1)a_3^2 + \frac{N-1}{N}k = 0, \\ a_2' - 2a_3^2 - 4a_1a_2 - 4(N-2)a_3a_4 + \frac{k}{N} = 0, \\ a_3' - 2a_1a_3 - 4a_1a_3 - 4(N-2)a_3^2 - \frac{k}{N} = 0, \\ a_3' - 2a_1a_3 - 4a_2a_3 - 4(N-2)a_3a_4 - \frac{k}{N} = 0, \\ a_4' - 2a_3^2 - 4a_2a_3 - 4a_1a_4 - 4(N-3)a_3a_4 = 0 \end{cases}$$

with the terminal conditions

$$a_1(T) = a_2(T) = a_3(T) = a_4(T) = 0.$$

It is worth noting that there are two ODEs for a_3 , and the two expressions should be equal, thus

$$a_1a_3 + (N-2)a_3^2 = a_2a_3 + (N-2)a_3a_4,$$

which implies that $(a_1 + (N-2)a_3)' = (a_2 + (N-2)a_4)'$ or

$$\begin{aligned} & 2a_1^2 + 2(N-2)a_1a_3 + 4(N-1)a_3^2 + 4(N-2)a_2a_3 + 4(N-2)^2a_3a_4 - \frac{k}{N} \\ &= 2(N-1)a_3^2 + 4a_1a_2 + 4(N-2)(a_2a_3 + a_3a_4 + a_1a_4) + 4(N-2)(N-3)a_3a_4 - \frac{k}{N}. \end{aligned}$$

After combining terms and substituting $a_2 + (N-2)a_4$ with $a_1 + (N-2)a_3$, we get

$$a_1^2 + (N-2)a_1a_3 - (N-1)a_3^2 = 0,$$

which yields $a_3 = a_1$ or $a_3 = -\frac{1}{N-1}a_1$. Note that, since a_1 and a_3 satisfies different differential equations, it follows that $a_3 \neq a_1$. Hence, we can conclude that $a_3 = -\frac{1}{N-1}a_1$. Next, from the equation $a_1 + (N-2)a_3 = a_2 + (N-2)a_4$, we have

$$a_4 = \frac{1}{N-2}a_1 + a_3 - \frac{1}{N-2}a_2.$$

In conclusion, for $i = 1, 2, \dots, N$, A_i has the following expressions:

$$(A_i)_{pq} = \begin{cases} a_1(t), & \text{if } p = q = i, \\ a_2(t), & \text{if } p = q \neq i, \\ -\frac{1}{N-1}a_1(t), & \text{if } p \neq q, p = i \text{ or } q = i, \\ \frac{1}{(N-1)(N-2)}a_1(t) - \frac{1}{N-2}a_2(t), & \text{otherwise,} \end{cases}$$

where a_1 and a_2 satisfies the system of ODEs (46)

$$\begin{cases} a_1' - \frac{2(N+1)}{N-1}a_1^2 + \frac{N-1}{N}k = 0, \\ a_2' + \frac{2}{(N-1)^2}a_1^2 - \frac{4N}{N-1}a_1a_2 + \frac{k}{N} = 0, \\ a_1(T) = a_2(T) = 0. \end{cases} \quad (46)$$

The existence and uniqueness of A_i in (42) are equivalent to the existence and uniqueness of (46). Firstly, the existence, uniqueness, and boundness of a_1 in (46) is from the same argument for a in (40), which is shown as the proof of Lemma 12 in Appendix. The explicit solution of a_1 is given by

$$a_1(t) = \sqrt{\frac{k}{2} \frac{(N-1)^2}{N(N+1)}} \frac{1 - e^{-2\sqrt{2}\sqrt{\frac{N+1}{N}}k(T-t)}}{1 + e^{-2\sqrt{2}\sqrt{\frac{N+1}{N}}k(T-t)}}$$

for all $t \in [0, T]$. Next, with the given of a_1 , the existence, uniqueness, and boundness of a_2 in (46) is guaranteed by Theorem 12.1 in [1]. Therefore, we can express the equilibrium paths and associated controls as the following:

$$d\hat{X}_{it}^{(N)} = -2a_1^N(t) \left(\hat{X}_{it}^{(N)} - \frac{1}{N-1} \sum_{j \neq i}^N \hat{X}_{jt}^{(N)} \right) dt + dW_{it}^{(N)} + d\tilde{W}_t, \quad (47)$$

and

$$\hat{\alpha}_{it}^{(N)} = -2a_1^N(t) \left(\hat{X}_{it}^{(N)} - \frac{1}{N-1} \sum_{j \neq i}^N \hat{X}_{jt}^{(N)} \right)$$

respectively for $i = 1, 2, \dots, N$, where a_1^N is the solution to the ODE for a_1 in (46). This concludes Proposition 2. \square

6 Further remark

We have now established Proposition 1 concerning the MFG in Section 4 and Proposition 2 regarding the N -player game in Section 5. With these propositions proven, we are now able to conclude the proof of Theorem 1, which was presented in Section 3.4.

7 Appendix

Lemma 11. *Let \mathbb{W}_p be the p -Wasserstein metric. If X and Y are two real-valued random variables and c is a constant, then*

$$\mathbb{W}_p(\mathcal{L}(X), \mathcal{L}(Y)) = \mathbb{W}_p(\mathcal{L}(X + c), \mathcal{L}(Y + c)). \quad (48)$$

Moreover, if $\alpha = \{\alpha_i : i \in \mathbb{N}\}$ is a sequence of random variables, then

$$\mathbb{W}_p\left(\frac{1}{N} \sum_{i=1}^N \delta_{\alpha_i+c}, \mathcal{L}(Y + c)\right) = \mathbb{W}_p\left(\frac{1}{N} \sum_{i=1}^N \delta_{\alpha_i}, \mathcal{L}(Y)\right). \quad (49)$$

Proof. By definition of the p -Wasserstein metric, we have:

$$\mathbb{W}_p(\mathcal{L}(X), \mathcal{L}(Y)) = \left(\inf_{\pi \in \Pi(\mathcal{L}(X), \mathcal{L}(Y))} \int_{\mathbb{R}^2} |x - y|^p d\pi(x, y) \right)^{\frac{1}{p}},$$

where $\Pi(\mathcal{L}(X), \mathcal{L}(Y))$ is the set of all joint probability measures with marginals $\mathcal{L}(X)$ and $\mathcal{L}(Y)$. Similarly,

$$\mathbb{W}_p(\mathcal{L}(X + c), \mathcal{L}(Y + c)) = \left(\inf_{\pi \in \Pi(\mathcal{L}(X+c), \mathcal{L}(Y+c))} \int_{\mathbb{R}^2} |x - y|^p d\pi(x, y) \right)^{\frac{1}{p}},$$

where $\Pi(\mathcal{L}(X + c), \mathcal{L}(Y + c))$ is the set of all joint probability measures with marginals $\mathcal{L}(X + c)$ and $\mathcal{L}(Y + c)$.

Now, consider the mapping $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $\Phi(x, y) = (x + c, y + c)$. For any $\pi \in \Pi(\mathcal{L}(X), \mathcal{L}(Y))$, the pushforward measure of π under Φ belongs to $\Pi(\mathcal{L}(X + c), \mathcal{L}(Y + c))$, i.e., $\pi' = \Phi_*\pi \in \Pi(\mathcal{L}(X + c), \mathcal{L}(Y + c))$. Thus, we have

$$\Phi_*\Pi(\mathcal{L}(X), \mathcal{L}(Y)) \subset \Pi(\mathcal{L}(X + c), \mathcal{L}(Y + c)).$$

Moreover, Φ is bijective and measure preserving, then

$$\int_{\mathbb{R}^2} |x - y|^p d\pi'(x, y) = \int_{\mathbb{R}^2} |(x + c) - (y + c)|^p d\pi(x, y) = \int_{\mathbb{R}^2} |x - y|^p d\pi(x, y).$$

Therefore, we know that

$$\begin{aligned} \mathbb{W}_p^p(\mathcal{L}(X), \mathcal{L}(Y)) &= \inf_{\pi \in \Pi(\mathcal{L}(X), \mathcal{L}(Y))} \int_{\mathbb{R}^2} |x - y|^p d\pi(x, y) \\ &= \inf_{\pi \in \Pi(\mathcal{L}(X), \mathcal{L}(Y))} \int_{\mathbb{R}^2} |x - y|^p d\Phi_*\pi(x, y) \\ &= \inf_{\pi' \in \Phi_*\Pi(\mathcal{L}(X), \mathcal{L}(Y))} \int_{\mathbb{R}^2} |x - y|^p d\pi'(x, y) \\ &\geq \mathbb{W}_p^p(\mathcal{L}(X + c), \mathcal{L}(Y + c)). \end{aligned}$$

by the definition of the p -Wasserstein metric. If we apply the above inequality to $X' = X + c$, $Y' = Y + c$, and $c' = -c$, the opposite inequality is provided. Thus, it completes the proof of (48).

Next, we note that

$$\frac{1}{N} \sum_{i=1}^N \delta_{\alpha_i+c} = \mathcal{L}(\alpha_u + c | \alpha),$$

where u be a uniform random variable on $\{1, 2, \dots, N\}$ independent to α . Using (48), we conclude (49) from

$$\begin{aligned}\mathbb{W}_p\left(\frac{1}{N}\sum_{i=1}^N\delta_{\alpha_i+c}, \mathcal{L}(Y+c)\right) &= \mathbb{W}_p(\mathcal{L}(\alpha_u+c|\alpha), \mathcal{L}(Y+c)) \\ &= \mathbb{W}_p(\mathcal{L}(\alpha_u|\alpha), \mathcal{L}(Y)) \\ &= \mathbb{W}_p\left(\frac{1}{N}\sum_{i=1}^N\delta_{\alpha_i}, \mathcal{L}(Y)\right).\end{aligned}$$

□

Lemma 12. *Under the Assumption 2, there exists a unique solution $(a(t), b(t), c(t), d(t)) : t \in [0, T]$ for the Riccati system of ODEs (40)-(41) and the solution can given explicitly by*

$$\begin{cases} a(t) = \sqrt{\frac{k}{2}} \frac{1 - e^{-2\sqrt{2k}(T-t)}}{1 + e^{-2\sqrt{2k}(T-t)}}, \\ b(t) = \int_t^T (4a(s)c(s) - 2a^2(s)) ds, \\ c(t) = k \int_t^T e^{\int_t^s -4a(r)dr} ds, \\ d(t) = \int_t^T (b(s) + 2c(s)) ds. \end{cases}$$

Proof. Firstly, with the given of $k > 0$, we can solve the ODE

$$a'(t) - 2a^2(t) + k = 0, \quad a(T) = 0$$

explicitly by the method of separating variables. Note that with the differential form, we have

$$\frac{da}{(\sqrt{2}a - \sqrt{k})(\sqrt{2}a + \sqrt{k})} = \frac{1}{2\sqrt{k}} \left(\frac{1}{\sqrt{2}a - \sqrt{k}} - \frac{1}{\sqrt{2}a + \sqrt{k}} \right) da = dt.$$

It follows that

$$\ln \left(\left| \frac{\sqrt{2}a - \sqrt{k}}{\sqrt{2}a + \sqrt{k}} \right| \right) = 2\sqrt{2k}t + C_1$$

for some constant C_1 by taking integration on both sides. Thus by calculation, we obtain

$$a(t) = \sqrt{\frac{k}{2}} \frac{1 - C_2 e^{2\sqrt{2k}t}}{1 + C_2 e^{2\sqrt{2k}t}}$$

for some constant C_2 to be determined. Since $a(T) = 0$, it yields that $C_2 = e^{-2\sqrt{2k}T}$ and thus

$$a(t) = \sqrt{\frac{k}{2}} \frac{1 - e^{-2\sqrt{2k}(T-t)}}{1 + e^{-2\sqrt{2k}(T-t)}}.$$

It is easily to verify that $a(\cdot)$ is in $C^\infty([0, T])$ and is bounded. With the given of a , the functions (b, c, d) in the Riccati system (40)-(41) is a coupled linear system, and thus their existence, uniqueness, and boundedness are given by Theorem 12.1 in [1]. □

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