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On a characterization theorem in the space $\mathbb{R}^n$

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By Heyde's theorem, the class of Gaussian distributions on the real line is characterized by the symmetry of the conditional distribution of one linear form of independent random variables given another. We prove an analogue of this theorem for two independent random vectors taking values in the space $\mathbb{R}^n$. The obtained class of distributions consists of convolutions of Gaussian distributions and a distribution supported in a subspace, which is determined by coefficients of the linear forms.

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1 Introduction

By the well-known Skitovich–Darmois theorem, Gaussian distributions on the real are characterized by the independence of two linear forms with nonzero coefficients of independent random variables. This result was generalized by S.G. Ghurye and I. Olkin to independent random vectors taking values in the space $\mathbb{R}^n$. Coefficients of the linear forms in this case are invertible linear operators in $\mathbb{R}^n$ ([14], see also [16, §3.2]). A theorem similar to the Skitovich–Darmois theorem was proved by C.C. Heyde. By Heyde's theorem, Gaussian distributions on the real line are characterized by the symmetry of the conditional distribution of one linear form given another. Coefficients of the linear forms are nonzero real numbers ([15], see also [16, §13.4]). For two independent random variables Heyde's theorem states the following. Let ξ_1 and ξ_2 be independent random variables with distributions μ_1 and μ_2. Let a_j and b_j be nonzero numbers such that $b_1a_1^{-1}+b_2a_2^{-1}\neq 0$. If the conditional distribution of linear form $L_2=b_1\xi_1+b_2\xi_2$ given $L_1=a_1\xi_1+a_2\xi_2$ is symmetric, then μ_j are Gaussian distributions. It is easy to see that in studying the possible distributions μ_j we can assume without loss of generality that $L_1=\xi_1+\xi_2$, $L_2=\xi_1+a\xi_2$, where $a\neq 0$. Hence the following statement holds.

Theorem A *Let ξ_1 and ξ_2 be independent random variables with distributions μ_1 and μ_2. Let $a\neq 0$ and $a\neq -1$. If the conditional distribution of the linear form $L_2=\xi_1+a\xi_2$ given $L_1=\xi_1+\xi_2$ is symmetric, then μ_j are Gaussian distributions.*

A number of works have been devoted to analogues of Heyde's theorem for various locally compact Abelian groups (see e.g. [5,6,8–13,17–20]). In so doing coefficients of the linear forms are topological automorphisms of the group. At the same time Heyde's theorem for the space $\mathbb{R}^n$ was not specially studied. The space $\mathbb{R}^n$ was considered only as a special case of a locally compact Abelian group. In particular, the following theorem was proved in [11] for arbitrary locally compact Abelian groups containing no nonzero elements of finite order. For the space $\mathbb{R}^n$ this theorem can be formulated as follows.

Theorem B *Let α be an invertible linear operator in the space $\mathbb{R}^n$ such that $I+\alpha$ is also an invertible operator. Let ξ_1 and ξ_2 be independent random vectors with values in $\mathbb{R}^n$ and distributions μ_1 and μ_2. If the conditional distribution of the linear form $L_1=\xi_1+\xi_2$ given $L_2=\xi_1+\alpha\xi_2$ is symmetric, then μ_j are Gaussian distributions.*

Obviously, Theorem A is a particular case of Theorem B for $n = 1$. The purpose of this note is to prove Heyde's theorem for the space $\mathbb{R}^n$ without any restrictions on an invertible linear operator α. In other words, we want to get a full description of the possible distributions μ_j in Theorem B in the case when the operator $I+\alpha$ need not be invertible. It turns out that if the operator $I+\alpha$ is not invertible, then μ_j are convolutions of Gaussian distributions and a distribution supported in a subspace which is determined by the operator α.

Denote by $x = (x_1, x_2, \dots, x_n)$, $x_j \in \mathbb{R}$, elements of the space $\mathbb{R}^n$. If $x, y \in \mathbb{R}^n$, then put

$$\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j.$$

Denote by $\mathbb{C}$ the complex plane. We will also use the notation $\langle x, y\rangle$ in the case when $x, y \in \mathbb{C}^n$. Let H be a subspace of $\mathbb{R}^n$. Denote by

$$A(\mathbb{R}^n, H) = \{x \in \mathbb{R}^n : \langle x, y\rangle = 0 \text{ for all } y \in H\}$$

the annihilator of H.

Let α be a linear operator in $\mathbb{R}^n$. Denote by $\widetilde{\alpha}$ its adjoint operator. If a subspace G of $\mathbb{R}^n$ is invariant with respect to α, then denote by α_G the restriction of α to G. Denote by I the identity operator. Denote by $\|x\|$ a norm of a vector $x \in \mathbb{R}^n$ and by $\|\alpha\|$ the norm of the operator α.

Let $P(y)$ be an arbitrary function on $\mathbb{R}^n$ and let $h \in \mathbb{R}^n$. Denote by Δ_h the finite difference operator

$$\Delta_h P(y) = P(y+h) - P(y), \quad y \in \mathbb{R}^n.$$

Recall that a continuous function $P(y)$ is a polynomial in some neighbourhood of zero in $\mathbb{R}^n$ if and only if for a nonnegative integer m the function $P(y)$ satisfies the equation

$$\Delta_h^{m+1} P(y) = 0, \tag{1}$$

for all y and h in a neighborhood of zero in $\mathbb{R}^n$. Moreover, the minimum m at which (1) is satisfied coincides with the degree of the poynomial $P(y)$.

Let μ be a probability distribution on $\mathbb{R}^n$. Denote by $\hat{\mu}(y)$, $y \in \mathbb{R}^n$, the characteristic function of μ. Define the distribution $\bar{\mu}$ by the formula $\bar{\mu}(B) = \mu(-B)$ for each Borel subset B in $\mathbb{R}^n$. Then $\hat{\bar{\mu}}(y) = \overline{\hat{\mu}(y)}$. Denote by E_x the degenerate distribution concentrated at a vector $x \in \mathbb{R}^n$.

2 Main theorem

The main result of this paper is the proof of the following theorem.

Theorem 2.1 *Let α be an invertible linear operator in the space $\mathbb{R}^n$. Put $K = \mathrm{Ker}\ (I+\alpha)$. Let ξ_1 and ξ_2 be independent random vectors with values in $\mathbb{R}^n$ and distributions μ_1 and μ_2. Assume that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. Then there exists an α-invariant subspace G such that μ_j are shifts of convolutions of symmetric Gaussian distributions supported in G and a distribution supported in K. Moreover, $K \cap G = \{0\}$.*

It is clear that if in Theorem 2.1 $I+\alpha$ is an invertible operator, i.e. $K = \{0\}$, then μ_j are Gaussian distributions. Thus, Theorem B follows from Theorem 2.1.

To prove Theorem 2.1 we need a series of lemmas. The following lemma holds for independent random variables taking values in an arbitrary locally compact Abelian group. We formulate it for the space $\mathbb{R}^n$.

Lemma 2.2 ([7, Lemma 16.1]) *Let α be an invertible linear operator in the space $\mathbb{R}^n$. Let ξ_1 and ξ_2 be independent random vectors with values in $\mathbb{R}^n$ and distributions μ_1 and μ_2. The conditional distribution of the linear form $L_2 = \xi_1 + \alpha\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric if and only if the characteristic functions $\hat\mu_j(y)$ satisfy the equation*

$$\hat\mu_1(u+v)\hat\mu_2(u+\widetilde{\alpha}v) = \hat\mu_1(u-v)\hat\mu_2(u-\widetilde{\alpha}v), \quad u,v\in\mathbb{R}^n. \tag{2}$$

Equation (2) is called the Heyde functional equation. Due to Lemma 2.2, the proof of Theorem 2.1 reduces to the description of solutions of equation (2) in the class of continuous normalized positive definite functions. Note that the proof of the Ghurye–Olkin theorem, mentioned in the introduction, for two independent random vectors reduces to solving the Skitovich–Darmois functional equation

$$\hat\mu_1(u+v)\hat\mu_2(u+\widetilde{\alpha}v) = \hat\mu_1(u)\hat\mu_2(u)\hat\mu_1(v)\hat\mu_2(\widetilde{\alpha}v), \quad u,v\in\mathbb{R}^n, \tag{3}$$

in the class of continuous normalized positive definite functions. Unlike equation (2), all solutions of equation (3) in the class of continuous normalized positive definite functions are characteristic functions of Gaussian distributions in the space $\mathbb{R}^n$.

Note that based on characterization of polynomials as the solutions set of some functional equations, J. M. Almira in [2] proposed a new approach to solving the Skitovich–Darmois functional equation in the space $\mathbb{R}^n$ for $m\geq 2$ functions. Then, using the fact that Aichinger's equation characterizes polynomial functions (see [1] by E. Aichinger and J. Moosbauer), J. M. Almira in [3] studied the solutions of the Skitovich–Darmois functional equation on an arbitrary Abelian group.

We will need the following easily verified statement, which we formulate as a lemma (see e.g. [4, Lemma 6.9]).

Lemma 2.3 *Let $\mathbb{R}^n = \mathbb{R}^p\times\mathbb{R}^q$ and let μ be a distribution on the space $\mathbb{R}^n$ with the characteristic function $\hat\mu(s_1,s_2)$, $s_1\in\mathbb{R}^p$, $s_2\in\mathbb{R}^q$. Assume that the function $\hat\mu(0,s_2)$, $s_2\in\mathbb{R}^q$, is extended to $\mathbb{C}^q$ as an entire function in s_2. Put $B_r=\{s_2=(s_{21},s_{22},\dots,s_{2q})\in\mathbb{C}^q : |s_{2j}|\leq r,\ j=1,2,\dots,q\}$. Then for each fixed $s_1\in\mathbb{R}^p$ the function $\hat\mu(s_1,s_2)$, $s_2\in\mathbb{R}^q$, is also extended to $\mathbb{C}^q$ as an entire function in s_2, and for each $s_1\in\mathbb{R}^p$ the inequality*

$$\max_{s_2\in B_r}|\hat\mu(s_1,s_2)| \leq \max_{s_2\in B_r}|\hat\mu(0,s_2)| \tag{4}$$

holds.

The following lemma holds for independent random variables taking values in an arbitrary locally compact Abelian group containing no elements of order 2. It follows directly from Lemma 2.2. We formulate it for the space $\mathbb{R}^n$.

Lemma 2.4 *Let ξ_1 and ξ_2 be independent random vectors with values in the space $\mathbb{R}^n$ and distributions μ_1 and μ_2. The conditional distribution of the linear form $L_2=\xi_1-\xi_2$ given $L_1=\xi_1+\xi_2$ is symmetric if and only if $\mu_1=\mu_2$.*

The following lemma is crucial in the proof of Theorem 2.1. It describes the possible distributions μ_j in Theorem B in the case when in a suitable basis a Jordan cell with the eigenvalue $\lambda=-1$ corresponds to a linear operator α, i.e.

$$\alpha=\alpha_n=\begin{pmatrix} -1 & 1 & 0 & \dots & 0\\ 0 & -1 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1\\ 0 & 0 & 0 & \dots & -1\end{pmatrix}.$$

Lemma 2.5 *Let α_n be the invertible linear operator in the space $\mathbb{R}^n$, $n \geq 2$, of the form*

$$\alpha_n(x_1, x_2, \dots, x_n) = (-x_1 + x_2, -x_2 + x_3, \dots, -x_{n-1} + x_n, -x_n), \quad x_j \in \mathbb{R}. \tag{5}$$

Let ξ_1 and ξ_2 be independent random vectors with values in $\mathbb{R}^n$ and distributions μ_1 and μ_2. Assume that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha_n\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. Put $K = \mathrm{Ker}\,(I + \alpha_n)$. Then we can replace the distributions μ_j by their shifts τ_j in such a way that $\tau_1 = \tau_2$, the distribution τ_j is supported in the subspace K, and if η_j are independent identically distributed random vectors with values in the space $\mathbb{R}^n$ and distribution τ_j, then the conditional distribution of the linear form $M_2 = \eta_1 + \alpha_n\eta_2$ given $M_1 = \eta_1 + \eta_2$ is symmetric.

Proof. Note that $K = \{(x_1, 0, \dots, 0) \in \mathbb{R}^n : x_1 \in \mathbb{R}\}$. We divide the proof of the lemma into three steps.

1. By Lemma 2.2, the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (2). Put $\nu_j = \mu_j * \bar{\mu}_j$. Then $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0$. Since $\hat{\nu}_j(0) = 1$, we can choose $\varepsilon > 0$ in such a way that the inequalities $\hat{\nu}_j(y) > 0$, $j = 1, 2$, are valid for all $\|y\| < \varepsilon$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (2). Put $P_j(y) = \log \hat{\nu}_j(y)$, where $\|y\| < \varepsilon$, $j = 1, 2$. Let $\|\alpha_n\| = M$. It is obvious that $M \geq 1$. Put $\delta = \varepsilon/8M$ and

$$H = (I + \widetilde{\alpha}_n)(\mathbb{R}^n).$$

We prove that $P_j(y)$ are polynomials of degree at most 2 in some neighbourhood of zero in the subspace H.

It follows from equation (2) for the characteristic functions $\hat{\nu}_j(y)$ that the functions $P_j(y)$ satisfy the equation

$$P_1(u + v) + P_2(u + \widetilde{\alpha}_n v) - P_1(u - v) - P_2(u - \widetilde{\alpha}_n v) = 0, \quad u, v \in \mathbb{R}^n,\ \|u\| < \delta,\ \|v\| < \delta. \tag{6}$$

Equation (6) arises in the study of Heyde's theorem on various locally compact Abelian groups. To solve equation (6) we use the finite difference method. This is a standard reasoning (see e.g. [9], [11]).

Take an arbitrary vector k_1 in the space $\mathbb{R}^n$ such that $\|k_1\| < \delta$. Put $h_1 = \widetilde{\alpha}_n k_1$. Replacing in (6) u by $u + h_1$ and v by $v + k_1$ and subtracting equation (6) from the obtained equation, we get

$$\Delta_{l_{11}} P_1(u + v) + \Delta_{l_{12}} P_2(u + \widetilde{\alpha}_n v) - \Delta_{l_{13}} P_1(u - v) = 0, \quad u, v \in \mathbb{R}^n,\ \|u\| < \delta,\ \|v\| < \delta, \tag{7}$$

where $l_{11} = (I + \widetilde{\alpha}_n)k_1$, $l_{12} = 2\widetilde{\alpha}_n k_1$, $l_{13} = (\widetilde{\alpha}_n - I)k_1$. Take an arbitrary vector k_2 in the space $\mathbb{R}^n$ such that $\|k_2\| < \delta$. Put $h_2 = k_2$. Replacing in (7) u by $u + h_2$ and v by $v + k_2$ and subtracting equation (7) from the obtained equation, we find

$$\Delta_{l_{21}} \Delta_{l_{11}} P_1(u + v) + \Delta_{l_{22}} \Delta_{l_{12}} P_2(u + \widetilde{\alpha}_n v) = 0, \quad u, v \in \mathbb{R}^n,\ \|u\| < \delta,\ \|v\| < \delta, \tag{8}$$

where $l_{21} = 2k_2$, $l_{22} = (I + \widetilde{\alpha}_n)k_2$. Take an arbitrary vector k_3 in the space $\mathbb{R}^n$ such that $\|k_3\| < \delta$. Put $h_3 = -\widetilde{\alpha}_n k_3$. Replacing in (8) u by $u + h_3$ and v by $v + k_3$ and subtracting equation (8) from the obtained equation, we get

$$\Delta_{l_{31}} \Delta_{l_{21}} \Delta_{l_{11}} P_1(u + v) = 0, \quad u, v \in \mathbb{R}^n,\ \|u\| < \delta,\ \|v\| < \delta, \tag{9}$$

where $l_{31} = (I - \widetilde{\alpha}_n)k_3$. Substituting $v = 0$ into (9) we find that

$$\Delta_{l_{31}} \Delta_{l_{21}} \Delta_{l_{11}} P_1(u) = 0, \quad u \in \mathbb{R}^n,\ \|u\| < \delta. \tag{10}$$

Since multiplication by 2 and $I - \widetilde{\alpha}_n$ are invertible linear operators in the space $\mathbb{R}^n$ and k_j are arbitrary vectors in $\mathbb{R}^n$ such that $\|k_j\| < \delta$, it follows from (10) that the function $P_1(y)$ satisfies the equation

$$\Delta_h^3 P_1(y) = 0, \quad y, h \in H,\ \|y\| < \delta,\ \|h\| < \delta. \tag{11}$$

Arguing similarly, we exclude the function $P_1(y)$ from equation (8) and obtain that the function $P_2(y)$ also satisfies equation (11). It follows from this that $P_j(y)$ are polynomials of degree at most 2 in some neighbourhood of zero in the subspace H.

2. Assume $n = 2$ and prove that $\hat\nu_1(y) = \hat\nu_2(y) = 1$ for all $y \in H$. Since $\alpha_2(x_1, x_2) = (-x_1 + x_2, -x_2)$, $x_j \in \mathbb{R}$, we have $\widetilde{\alpha}_2(y_1, y_2) = (-y_1, y_1 - y_2)$, $y_j \in \mathbb{R}$, and then $H = \{(0, y_2) \in \mathbb{R}^2 : y_2 \in \mathbb{R}\}$. Since $P_j(y)$ are polynomials of the degree at most 2 in some neighbourhood of zero in the subspace H and $P_j(y) = \log \hat\nu_j(y)$ for all $\|y\| < \delta$, $y \in H$, this implies that the characteristic functions $\hat\nu_j(0, y_2)$ coincide with the characteristic functions of some symmetric Gaussian distributions in some neighbourhood of zero on the real line. It is well known that then $\hat\nu_j(0, y_2)$, $y_2 \in \mathbb{R}$, are the characteristic functions of some symmetric Gaussian distributions ([16, §1.2]). Thus, we have

$$P_j(0, y_2) = -\sigma_j y_2^2, \quad y_2 \in \mathbb{R}, \tag{12}$$

where $\sigma_j \geq 0$, $j = 1, 2$.

Since $\widetilde{\alpha}_2(y_1, y_2) = (-y_1, y_1 - y_2)$, $y_j \in \mathbb{R}$, equation (2) for the functions $\hat\nu_j(y)$ takes the form

$$\begin{aligned}\hat\nu_1(u_1 + v_1, u_2 + v_2)&\hat\nu_2(u_1 - v_1, u_2 + v_1 - v_2)\\ &= \hat\nu_1(u_1 - v_1, u_2 - v_2)\hat\nu_2(u_1 + v_1, u_2 - v_1 + v_2), \quad u_j, v_j \in \mathbb{R}. \end{aligned}\tag{13}$$

By Lemma 2.3, it follows from (12) that for each fixed $y_1 \in \mathbb{R}$ the functions $\hat\nu_j(y_1, y_2)$ can be extended to the complex plane $\mathbb{C}$ as entire functions in y_2. It is obvious that equation (13) remains valid for all $u_1, v_1 \in \mathbb{R}$ and $u_2, v_2 \in \mathbb{C}$.

We will verify that if $\hat\nu_j(y_1, 0) > 0$ for all $|y_1| < \delta$, $j = 1, 2$, then the functions $\hat\nu_j(y_1, y_2)$ do not vanish for all $|y_1| < \delta$ and $y_2 \in \mathbb{C}$. Assume the contrary, let $\hat\nu_1(\widetilde{y}_1, \widetilde{y}_2) = 0$, where $|\widetilde{y}_1| < \delta$, $\widetilde{y}_2 \in \mathbb{C}$. Take arbitrary numbers $u_1 \in \mathbb{R}$ and $u_2 \in \mathbb{C}$. Put $v_1 = \widetilde{y}_1 - u_1$, $v_2 = \widetilde{y}_2 - u_2$ and substitute u_j and v_j into equation (13). Then the left-hand side of equation (13) is equal to zero. Hence the equality

$$\hat\nu_1(2u_1 - \widetilde{y}_1, 2u_2 - \widetilde{y}_2)\hat\nu_2(\widetilde{y}_1, \widetilde{y}_2 - \widetilde{y}_1 + u_1) = 0 \tag{14}$$

is satisfied for all u_j. Since $\hat\nu_2(\widetilde{y}_1, y_2)$ is not identically zero an entire function in y_2, take $u_1 = \widetilde{u}_1$ in such a way that $\hat\nu_2(\widetilde{y}_1, \widetilde{y}_2 - \widetilde{y}_1 + \widetilde{u}_1) \neq 0$ and $|2\widetilde{u}_1 - \widetilde{y}_1| < \delta$. Obviously, we can do it. Then it follows from (14) that $\hat\nu_1(2\widetilde{u}_1 - \widetilde{y}_1, 2u_2 - \widetilde{y}_2) = 0$ for all $u_2 \in \mathbb{C}$, that contradicts the fact that $\hat\nu_1(2\widetilde{u}_1 - \widetilde{y}_1, y_2)$ is a not identically zero entire function in y_2. Arguing similarly it is easy to make sure that the function $\hat\nu_2(y_1, y_2)$ does not vanish for all $|y_1| < \delta$, $y_2 \in \mathbb{C}$.

Taking into account (12), by Lemma 2.3, it follows from inequality (4) that $\hat\nu_j(y_1, y_2)$ are entire functions in y_2 of order at most 2 for all $|y_1| < \delta$. Since the functions $\hat\nu_j(y_1, y_2)$ for all $|y_1| < \delta$, $y_2 \in \mathbb{C}$ do not vanish, Hadamard's theorem on the representation of entire functions of finite order implies that the representation

$$\hat\nu_j(y_1, y_2) = \exp\{a_j(y_1)y_2^2 + b_j(y_1)y_2 + c_j(y_1)\}, \quad |y_1| < \delta,\ y_2 \in \mathbb{C}, \tag{15}$$

is valid, where $a_j(y_1), b_j(y_1), c_j(y_1) \in \mathbb{C}$, $j = 1, 2$. Substituting $u_1 = v_1 = 0$, $u_2 = v_2 = y_2/2$ into (13), we get

$$\hat\nu_1(0, y_2) = \hat\nu_2(0, y_2), \quad y_2 \in \mathbb{R}. \tag{16}$$

It follows from (12) and (16) that $\sigma_1 = \sigma_2 = \sigma$. Thus, (12) and (15) imply that $a_1(0) = a_2(0) = -\sigma$, $b_1(0) = b_2(0) = 0$. Obviously, without loss of generality, we can assume that $c_1(0) = c_2(0) = 0$.

Substituting $u_1 = v_1 = y_1/2$ into (13) and taking into account (15), we obtain

$$\begin{aligned}a_1(y_1)(u_2 + v_2)^2 + b_1(y_1)(u_2 + v_2) + c_1(y_1) - \sigma(u_2 + y_1/2 - v_2)^2&\\ = -\sigma(u_2 - v_2)^2 + a_2(y_1)(u_2 - y_1/2 + v_2)^2 + b_2(y_1)(u_2 - y_1/2 + v_2)&\\ + c_2(y_1) + 2\pi i n(y_1), \quad |y_1| < \delta,\ u_2, v_2 \in \mathbb{C},&\end{aligned}\tag{17}$$

where the function $n(y_1)$ takes integer values. Considering the left-hand side and the right-hand side of (17) as polynomials in u_2 and v_2 and equating in (17) the coefficients of u_2 and v_2, we get

$$b_1(y_1) - \sigma y_1 = -a_2(y_1)y_1 + b_2(y_1), \quad b_1(y_1) + \sigma y_1 = -a_2(y_1)y_1 + b_2(y_1), \quad |y_1| < \delta.$$

This implies that $\sigma = 0$. Hence $\hat{\nu}_1(0, y_2) = \hat{\nu}_2(0, y_2) = 1$ for all $y_2 \in \mathbb{R}$.

3. We will prove the lemma by induction. Let $n = 2$. We have,

$$H = (I + \widetilde{\alpha}_2)(\mathbb{R}^2) = \{(0, y_2) \in \mathbb{R}^2 : y_2 \in \mathbb{R}\}.$$

Substituting $u_1 = v_1 = 0$, $u_2 = v_2 = y_2/2$ into equation (13) for the functions $\hat{\mu}_j(y_1, y_2)$, we get that $\hat{\mu}_1(0, y_2) = \hat{\mu}_2(0, y_2)$ for all $y_2 \in \mathbb{R}$. It follows from what was proved in step 2 that $|\hat{\mu}_1(0, y_2)| = |\hat{\mu}_2(0, y_2)| = 1$ for all $y_2 \in \mathbb{R}$. Hence there is a real number x such that $\hat{\mu}_j(0, y_2) = \exp\{ixy_2\}$ for all $y_2 \in \mathbb{R}$, $j = 1, 2$. Put $t_1 = (x, -x)$, $t_2 = (0, -x)$. Consider the distributions $\tau_j = \mu_j * E_{t_j}$. It is obvious that $\hat{\tau}_j(0, y_2) = 1$ for all $y_2 \in \mathbb{R}$, $j = 1, 2$. It follows from this that the distributions τ_j are supported in the annihilator $A(\mathbb{R}^2, H)$. It is obvious that $A(\mathbb{R}^2, H) = K$. Since $t_1 + \alpha_2 t_2 = 0$, the characteristic functions $\hat{\tau}_j(y)$ satisfy equation (2). By Lemma 2.2, this implies that if η_j are independent random vectors with values in the space $\mathbb{R}^2$ and distributions τ_j, then the conditional distribution of the linear form $M_2 = \eta_1 + \alpha_2\eta_2$ given $M_1 = \eta_1 + \eta_2$ is symmetric. Since $K = \text{Ker } (I + \alpha_2)$, the restriction of the operator α_2 to the subspace K coincides with $-I$. It means that if we consider η_j as independent random vectors with values in K, then the conditional distribution of the linear form $M_2 = \eta_1 - \eta_2$ given $M_1 = \eta_1 + \eta_2$ is symmetric. It follows from Lemma 2.4, applying to the subspace K, that $\tau_1 = \tau_2$. Thus, when $n = 2$ the lemma is proved.

Let $n > 2$. We note that $\widetilde{\alpha}_n(y_1, y_2, \dots, y_n) = (-y_1, y_1 - y_2, \dots, y_{n-1} - y_n)$. Hence equation (2) for the functions $\hat{\nu}_j(y)$ takes the form

$$\begin{aligned} &\hat{\nu}_1(u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)\hat{\nu}_2(u_1 - v_1, u_2 + v_1 - v_2, \dots, u_n + v_{n-1} - v_n) \\ &\quad = \hat{\nu}_1(u_1 - v_1, u_2 - v_2, \dots, u_n - v_n)\hat{\nu}_2(u_1 + v_1, u_2 - v_1 + v_2, \dots, u_n - v_{n-1} + v_n), \quad u_j, v_j \in \mathbb{R}. \end{aligned} \tag{18}$$

Substituting $u_1 = u_2 = \dots = u_{n-2} = 0$, $v_1 = v_2 = \dots = v_{n-2} = 0$ into (18) we obtain

$$\begin{aligned} &\hat{\nu}_1(0, \dots, 0, u_{n-1} + v_{n-1}, u_n + v_n)\hat{\nu}_2(0, \dots, 0, u_{n-1} - v_{n-1}, u_n + v_{n-1} - v_n) \\ &\quad = \hat{\nu}_1(0, \dots, 0, u_{n-1} - v_{n-1}, u_n - v_n)\hat{\nu}_2(0, \dots, 0, u_{n-1} + v_{n-1}, u_n - v_{n-1} + v_n), \quad u_j, v_j \in \mathbb{R}. \end{aligned} \tag{19}$$

We see that equation (19), up to the notation, coincides with equation (13). Therefore, as proven in step 2, we have $\hat{\nu}_1(0, \dots, 0, y_n) = \hat{\nu}_2(0, \dots, 0, y_n) = 1$ for all $y_n \in \mathbb{R}$. Hence $|\hat{\mu}_1(0, \dots, 0, y_n)| = |\hat{\mu}_2(0, \dots, 0, y_n)| = 1$ for all $y_n \in \mathbb{R}$. Put $L = \{(0, \dots, 0, y_n) \in \mathbb{R}^n : y_n \in \mathbb{R}\}$. Arguing as in the case when $n = 2$ we can replace the distributions μ_j by their shifts τ_j in such a way that the distributions τ_j are supported in the annihilator $A(\mathbb{R}^n, L) = \mathbb{R}^{n-1}$. Moreover, if η_j are independent random vectors with values in the space $\mathbb{R}^n$ and distributions τ_j, then the conditional distribution of the linear form $M_2 = \eta_1 + \alpha_n\eta_2$ given $M_1 = \eta_1 + \eta_2$ is symmetric. We note that $\alpha_n(\mathbb{R}^{n-1}) = \mathbb{R}^{n-1}$ and the restriction of the operator α_n to the subspace $\mathbb{R}^{n-1}$ coincides with the operator α_{n-1}. It means that if we consider η_j as independent random vectors with values in $\mathbb{R}^{n-1}$, then the conditional distribution of the linear form $M_2 = \eta_1 + \alpha_{n-1}\eta_2$ given $M_1 = \eta_1 + \eta_2$ is symmetric. The lemma is proved by induction.

The statement of the lemma can not be strengthened. Indeed, let ω be an arbitrary distribution supported in the subspace $K = \text{Ker } (I + \alpha_n)$. Let t_1 and t_2 be some vectors in $\mathbb{R}^n$ such that $t_1 + \alpha_n t_2 = 0$. Put $\mu_j = \omega * E_{t_j}$, $j = 1, 2$. Taking into account that the restriction of the operator α_n to the subspace K coincides with $-I$, Lemma 2.2 and Lemma 2.4 imply that if ξ_1 and ξ_2 are independent random vectors with values in the space $\mathbb{R}^n$ and distributions μ_j, then the conditional distribution of the linear form $L_2 = \xi_1 + \alpha_n\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. □

The following statement implies from Lemma 2.5.

Corollary 2.6 *Let α be an invertible linear operator in the space $\mathbb{R}^n$. Put $K = \mathrm{Ker}\ (I + \alpha)$ and suppose $K \neq \{0\}$. Let ξ_1 and ξ_2 be independent random vectors with values in $\mathbb{R}^n$ and distributions μ_1 and μ_2. Assume that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. Then there exists an α-invariant subspace G satisfying the condition $K \cap G = \{0\}$ and such that some shifts τ_j of the distributions μ_j are supported in the subspace $K \times G$. Moreover, if η_j are independent random vectors with values in the space $\mathbb{R}^n$ and distributions τ_j, then the conditional distribution of the linear form $M_2 = \eta_1 + \alpha\eta_2$ given $M_1 = \eta_1 + \eta_2$ is symmetric, and the restriction of the operator α to the subspace $K \times G$ is of the form $(-I, \alpha_G)$.*

Proof. Represent the space $\mathbb{R}^n$ as a direct sum of two α-invariant subspaces $\mathbb{R}^n = F \times G$, where F is the root subspace corresponding to the eigenvalue $\lambda = -1$ of the operator α, and the operator $I + \alpha$ is invertible in the subspace G. The operator α can be written in the form $\alpha = (\alpha_F, \alpha_G)$. In order not to complicate the notation we assume that $F = \mathbb{R}^p$, $G = \mathbb{R}^q$, a basis in the space $\mathbb{R}^p$ is chosen in such a way that $\mathbb{R}^p = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k}$, and the operator α_F is of the form $\alpha_F = (\alpha_{n_1}, \alpha_{n_2}, \dots, \alpha_{n_k})$, where $\alpha_{n_j} = -I$, if $n_j = 1$, and α_{n_j} is an invertible linear operator in the space $\mathbb{R}^{n_j}$ which is given by formula (5), if $n_j \geq 2$. By Lemma 2.2, the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (2) which takes the form

$$\begin{aligned}\hat{\mu}_1(u_1 + v_1, u_2 + v_2)&\hat{\mu}_2(u_1 + \widetilde{\alpha}_F v_1, u_2 + \widetilde{\alpha}_G v_2)\\ &= \hat{\mu}_1(u_1 - v_1, u_2 - v_2)\hat{\mu}_2(u_1 - \widetilde{\alpha}_F v_1, u_2 - \widetilde{\alpha}_G v_2), \quad u_1, v_1 \in \mathbb{R}^p,\ u_2, v_2 \in \mathbb{R}^q. \quad (20)\end{aligned}$$

Substitute $u_2 = v_2 = 0$ into (20). Taking into account Lemma 2.2 and applying successively Lemma 2.5 to each of the subspaces $\mathbb{R}^{n_j}$, where $n_j \geq 2$, we find as a result from the obtained equation that there exist vectors $t_j \in \mathbb{R}^p$ such that the distributions $\tau_j = \mu_j * E_{t_j}$ are supported in the subspace $K \times G$. Moreover, if η_j are independent random vectors with values in $K \times G$ and distributions τ_j, then the conditional distribution of the linear form $M_2 = \eta_1 + \alpha\eta_2$ given $M_1 = \eta_1 + \eta_2$ is symmetric. Obviously, the restriction of the operator α to the subspace $K \times G$ is of the form $(-I, \alpha_G)$. □

Remark 2.7 Assume that under the conditions of Theorem 2.1 $K \neq \{0\}$, i.e. $\lambda = -1$ is an eigenvalue of the operator α. Assume also that the root subspace corresponding to the eigenvalue $\lambda = -1$ does not coincide with the eigenspace. Then, as was proved in Corollary 2.6, some shifts of the distributions μ_j are supported in a proper subspace of the space $\mathbb{R}^n$. Hence μ_j are singular distributions.

Proof of Theorem 2.1. By Theorem B, if $K = \{0\}$, i.e. $\lambda = -1$ is not an eigenvalue of the operator α, then μ_j are Gaussian distributions. Therefore, we assume $K \neq \{0\}$, i.e. $\lambda = -1$ is an eigenvalue of the operator α. Let F be the root subspace corresponding to the eigenvalue $\lambda = -1$ of the operator α. Corollary 2.6 allows us to prove the theorem, assuming that $\alpha_F = -I$. In other words, the root subspace corresponding to the eigenvalue $\lambda = -1$ of the operator α is the eigenspace. Then equation (20) takes the form

$$\begin{aligned}\hat{\mu}_1(u_1 + v_1, u_2 + v_2)&\hat{\mu}_2(u_1 - v_1, u_2 + \widetilde{\alpha}_G v_2)\\ &= \hat{\mu}_1(u_1 - v_1, u_2 - v_2)\hat{\mu}_2(u_1 + v_1, u_2 - \widetilde{\alpha}_G v_2), \quad u_1, v_1 \in \mathbb{R}^p,\ u_2, v_2 \in \mathbb{R}^q. \quad (21)\end{aligned}$$

Substitute $u_1 = v_1 = 0$ into (21). Taking into account Lemma 2.2, it follows from Theorem B applying to the space $\mathbb{R}^q$ that $\hat{\mu}_j(0, y_2)$ are the characteristic functions of Gaussian distributions, i.e.

$$\hat{\mu}_j(0, y_2) = \exp\{-\langle A_j y_2, y_2\rangle + i\langle b_j, y_2\rangle\}, \quad y_2 \in \mathbb{R}^q, \quad (22)$$

where A_j is a symmetric positive semidefinite $q \times q$ matrix, $b_j \in \mathbb{R}^q$, $j = 1, 2$. Substituting $u_2 = v_2 = 0$ into (21) and taking into account (22) we obtain that $b_1 + \alpha_G b_2 = 0$. Hence we can replace the distributions μ_j by their shifts $\tau_j = \mu_j * E_{-b_j}$ and suppose that $b_1 = b_2 = 0$ in (22), i.e.

$$\hat{\mu}_j(0, y_2) = \exp\{-\langle A_j y_2, y_2 \rangle\}, \quad y_2 \in \mathbb{R}^q, \ j = 1, 2. \tag{23}$$

By Lemma 2.3, it follows from (23) that for each fixed $y_1 \in \mathbb{R}^p$ the function $\hat{\mu}_j(y_1, y_2)$ can be extended to $\mathbb{C}^q$ as an entire function in y_2. It is obvious that equation (21) remains valid for all $u_1, v_1 \in \mathbb{R}^p$, $u_2, v_2 \in \mathbb{C}^q$.

Substituting $u_1 = v_1 = y_1/2$, $u_2 = v_2 = 0$ into (21) we see that $\hat{\mu}_1(y_1, 0) = \hat{\mu}_2(y_1, 0)$ for all $y_1 \in \mathbb{R}^p$. Make sure that if for a fixed $\widetilde{y}_1 \in \mathbb{R}^p$ the inequalities

$$\hat{\mu}_j(\widetilde{y}_1, 0) \neq 0, \quad j = 1, 2, \tag{24}$$

hold, then $\hat{\mu}_j(\widetilde{y}_1, y_2) \neq 0$ for all $y_2 \in \mathbb{C}^q$, $j = 1, 2$. Suppose the contrary, let $\hat{\mu}_1(\widetilde{y}_1, \widetilde{y}_2) = 0$ for some $\widetilde{y}_2 \in \mathbb{C}^q$. Substitute $u_1 = v_1 = \widetilde{y}_1/2$, $u_2 = (I + \widetilde{\alpha}_G)^{-1}\widetilde{\alpha}_G\widetilde{y}_2$, $v_2 = (I + \widetilde{\alpha}_G)^{-1}\widetilde{y}_2$ into (21). Then the left-hand side of equation (21) is equal to zero. In view of (23) and (24), the right-hand side of equation (21) is nonzero. If $\hat{\mu}_2(\widetilde{y}_1, \widetilde{y}_2) = 0$ for some $\widetilde{y}_2 \in \mathbb{C}^q$, then substituting $u_1 = -v_1 = \widetilde{y}_1/2$, $u_2 = v_2 = (I + \widetilde{\alpha}_G)^{-1}\widetilde{y}_2$ into (21), we get the contradiction, because the left-hand side of equation (21) is equal to zero and the right-hand side is not. Thus, if inequalities (24) hold, then the function $\hat{\mu}_2(\widetilde{y}_1, y_2)$ is also nonzero for all $y_2 \in \mathbb{C}^q$.

So, we proved that if $\hat{\mu}_j(y_1, 0) \neq 0$, $j = 1, 2$, for some $y_1 \in \mathbb{R}^p$, then the functions $\hat{\mu}_j(y_1, y_2)$ can be extended to $\mathbb{C}^q$ as entire functions in y_2 without zeros. Hence we have the representations

$$\hat{\mu}_j(y_1, y_2) = \exp\{Q_j(y_1, y_2)\}, \quad y_2 \in \mathbb{C}^q, \ j = 1, 2,$$

where $Q_j(y_1, y_2)$ are entire functions in y_2 in $\mathbb{C}^q$.

By Lemma 2.3, it follows from inequality (4) and (23) that by Hadamard's theorem on the representation of entire functions of finite order, the restriction of the functions $Q_j(y_1, y_2)$ to each complex plane in $\mathbb{C}^q$ passing through zero are polynomials of degree at most 2. Hence the functions $Q_j(y_1, y_2)$ are polynomials of degree at most 2 in y_2. Thus, we have a representation

$$\hat{\mu}_j(y_1, y_2) = \exp\{\langle A_j(y_1)y_2, y_2\rangle + \langle b_j(y_1), y_2\rangle + c_j(y_1)\}, \quad y_2 \in \mathbb{C}^q, \tag{25}$$

where $A_j(y_1)$ are symmetric complex $q \times q$ matrices, $b_j(y_1) \in \mathbb{C}^q$, $c_j(y_1) \in \mathbb{C}$.

Assume $\hat{\mu}_j(y_1, 0) \neq 0$, $j = 1, 2$. Substituting $u_1 = v_1 = y_1/2$ into (21) and taking into account (25), we find from the obtained equation

$$\begin{aligned} &\langle A_1(y_1)(u_2 + v_2), u_2 + v_2\rangle + \langle A_2(0)(u_2 + \widetilde{\alpha}_G v_2), u_2 + \widetilde{\alpha}_G v_2\rangle \\ &\qquad = \langle A_1(0)(u_2 - v_2), u_2 - v_2\rangle + \langle A_2(y_1)(u_2 - \widetilde{\alpha}_G v_2), u_2 - \widetilde{\alpha}_G v_2\rangle, \quad u_2, v_2 \in \mathbb{C}^q. \end{aligned} \tag{26}$$

$$\langle b_1(y_1), u_2 + v_2\rangle + \langle b_2(0), u_2 + \widetilde{\alpha}_G v_2\rangle = \langle b_1(0), u_2 - v_2\rangle + \langle b_2(y_1), u_2 - \widetilde{\alpha}_G v_2\rangle, \quad u_2, v_2 \in \mathbb{C}^q. \tag{27}$$

The equalities

$$A_1(y_1) + A_2(0) = A_1(0) + A_2(y_1) \tag{28}$$

and

$$A_1(y_1) + \alpha_G A_2(0)\widetilde{\alpha}_G = A_1(0) + \alpha_G A_2(y_1)\widetilde{\alpha}_G \tag{29}$$

follow from (26). Substituting $u_1 = y_1$, $v_1 = 0$ into (21) and taking into account (25), we get from the obtained equation

$$\begin{aligned} &\langle A_1(y_1)(u_2 + v_2), u_2 + v_2\rangle + \langle A_2(y_1)(u_2 + \widetilde{\alpha}_G v_2), u_2 + \widetilde{\alpha}_G v_2\rangle \\ &\qquad = \langle A_1(y_1)(u_2 - v_2), u_2 - v_2\rangle + \langle A_2(y_1)(u_2 - \widetilde{\alpha}_G v_2), u_2 - \widetilde{\alpha}_G v_2\rangle, \quad u_2, v_2 \in \mathbb{C}^q. \end{aligned} \tag{30}$$

Equation (30) implies that

$$A_1(y_1) + \alpha_G A_2(y_1) = 0. \tag{31}$$

Taking into account that $I + \alpha_G$ is an invertible operator, (23), (29) and (31) imply that $A_1(y_1) = A_1(0) = -A_1$. Then it follows from (23) and (28) that $A_2(y_1) = A_2(0) = -A_2$. Thus, we have proved that if $\hat{\mu}_j(y_1, 0) \neq 0$, then

$$A_1(y_1) = -A_1, \quad A_2(y_1) = -A_2. \tag{32}$$

We find from (27) that

$$b_1(y_1) + b_2(0) = b_1(0) + b_2(y_1), \quad b_1(y_1) + \alpha_G b_2(0) = -b_1(0) - \alpha_G b_2(y_1). \tag{33}$$

It follows from (23) and (25) that $b_1(0) = b_2(0) = 0$. Given this, and taking into account that $I + \alpha_G$ is an invertible operator, (33) implies that if $\hat{\mu}_j(y_1, 0) \neq 0$, then $b_1(y_1) = b_2(y_1) = 0$ and hence in view of (25) and (32), the representations

$$\hat{\mu}_j(y_1, y_2) = \exp\{-\langle A_j y_2, y_2\rangle + c_j(y_1)\}, \quad y_2 \in \mathbb{C}^q, \quad j = 1, 2, \tag{34}$$

are valid for the functions $\hat{\mu}_j(y_1, y_2)$. Moreover, in this case substituting $u_1 = v_1 = y_1/2$, $u_2 = v_2 = 0$ in equation (21) and taking into account (34), we obtain that

$$\hat{\mu}_1(y_1, 0) = \hat{\mu}_2(y_1, 0) = \exp\{c_1(y_1)\} = \exp\{c_2(y_1)\}, \quad y_1 \in \mathbb{R}^p. \tag{35}$$

Assume $\hat{\mu}_j(\widetilde{y}_1, 0) = 0$, $j = 1, 2$, for some $\widetilde{y}_1 \in \mathbb{R}^p$. We verify that then $\hat{\mu}_j(\widetilde{y}_1, y_2) = 0$ for all $y_2 \in \mathbb{C}^q$, $j = 1, 2$. Put $u_1 = v_1 = \widetilde{y}_1/2$, $u_2 = -v_2 = y_2$ in equation (21). Then the left-hand side of equation (21) is equal to zero. In view of (23), we have $\hat{\mu}_2(\widetilde{y}_1, (I + \widetilde{\alpha}_G)y_2) = 0$ for all $y_2 \in \mathbb{C}^q$. Since $(I + \widetilde{\alpha}_G)$ is an invertible operator, we have $\hat{\mu}_2(\widetilde{y}_1, y_2) = 0$ for all $y_2 \in \mathbb{C}^q$. Substituting $u_1 = -v_1 = \widetilde{y}_1/2$, $u_2 = \widetilde{\alpha}_G y_2$, $v_2 = -y_2$ into equation (21), we make sure that $\hat{\mu}_1(\widetilde{y}_1, y_2) = 0$ for all $y_2 \in \mathbb{C}^q$.

Denote by γ_j the symmetric Gaussian distribution in the space $\mathbb{R}^q$ with the characteristic function

$$\hat{\gamma}_j(y_2) = \exp\{-\langle A_j y_2, y_2\rangle\}, \quad y_2 \in \mathbb{R}^q, \ j = 1, 2. \tag{36}$$

In view of $\hat{\mu}_1(y_1, 0) = \hat{\mu}_2(y_1, 0)$ for all $y_1 \in \mathbb{R}^p$, denote by ω a distribution in the space $\mathbb{R}^p$ with the characteristic function

$$\hat{\omega}(y_1) = \hat{\mu}_1(y_1, 0) = \hat{\mu}_2(y_1, 0), \quad y_1 \in \mathbb{R}^p. \tag{37}$$

Taking into account that $\hat{\omega}(y_1) = 0$ if and only if $\hat{\mu}_2(y_1, y_2) = 0$ for all $y_2 \in \mathbb{R}^q$, (34)–(37) imply the representations

$$\hat{\mu}_j(y_1, y_2) = \exp\{-\langle A_j y_2, y_2\rangle\}\hat{\omega}(y_1), \quad y_1 \in \mathbb{R}^p, \ y_2 \in \mathbb{R}^q, \ j = 1, 2.$$

This implies the statement of the theorem.

Note that we also proved that the intersection of each of the supports of the symmetric Gaussian distributions γ_j with the root subspace corresponding to the eigenvalue $\lambda = -1$ of the operator α is equal to zero.

The statement of the theorem can not be strengthened. Indeed, consider an invertible linear operator α in the space $\mathbb{R}^n$. Let G be an α-invariant subspace in $\mathbb{R}^n$ such that the operator $I + \alpha$ is invertible in G. Let G be isomorphic to $\mathbb{R}^q$. Denote by γ_j a Gaussian distribution in the space $\mathbb{R}^q$ with the characteristic function (36). Moreover, we will also assume that the equality

$$A_1 + A_2\widetilde{\alpha}_G = 0 \tag{38}$$

holds. Put $K = \text{Ker}\,(I+\alpha)$. Let ω be a distribution on K. Let x_1 and x_2 be some vectors in $\mathbb{R}^n$ such that

$$x_1 + \alpha x_2 = 0. \tag{39}$$

Put $\mu_j = \gamma_j * \omega * E_{x_j}$, $j = 1, 2$. Let ξ_1 and ξ_2 be independent random vectors with values in $\mathbb{R}^n$ and distributions μ_j. It is easy to see that (38) implies that the characteristic functions $\hat\gamma_j(y)$ satisfy equation (2). It follows from Lemma 2.2 and Lemma 2.4, applying to the subspace K, that the characteristic function $\hat\omega(y)$ satisfies equation (2). Moreover, (39) implies that the functions (x_j, y) satisfy equation (2). Hence the characteristic functions $\hat\mu_j(y)$ satisfy equation (2). By Lemma 2.2, the conditional distribution of the linear form $L_2 = \xi_1 + \alpha\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. $\square$

We complement Theorem 2.1 by a more detailed description of possible distributions μ_j in Theorem 2.1 for the space $\mathbb{R}^2$ depending on the spectrum of the linear operator α. If λ is an eigenvalue of the operator α, denote by L_λ the corresponding eigenspace. In particular, if $K = \text{Ker}\,(I+\alpha) \neq \{0\}$, then $K = L_{-1}$. We consider two cases: $\lambda = -1$ is an eigenvalue of the operator α or not.

1. $\lambda = -1$ is an eigenvalue of the operator α. Then the characteristic equation of the operator α has only real roots. Two cases are possible.

1A. The operator α has another eigenvalue $\lambda = \lambda_0$, where $\lambda_0 \neq -1$. In this case in a basis consisting of eigenvectors of α, the diagonal matrix $\alpha = \text{diag}\{-1, \lambda_0\}$ corresponds to the operator α. Theorem 2.1 easily implies the following alternative. If $\lambda_0 > 0$, then $\mu_j = \omega * E_{x_j}$, where ω is a distribution supported in K, $j = 1, 2$. In so doing K is a one-dimensional subspace of $\mathbb{R}^2$. If $\lambda_0 < 0$, then $\mu_j = \gamma_j * \omega$, where γ_j are Gaussian distributions supported in L_{λ_0}, and ω is a distribution supported in K.

1B. The operator α has the only eigenvalue $\lambda = -1$. If the root subspace corresponding to the eigenvalue $\lambda = -1$ does not coincide with K, then the Jordan cell

$$\alpha = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

in a suitable basis corresponds to the operator α. In this case by Lemma 2.5, $\mu_j = \omega * E_{x_j}$, where ω is a distribution supported in K, $j = 1, 2$. In so doing K is a one-dimensional subspace of $\mathbb{R}^2$. If the root subspace corresponding to the eigenvalue $\lambda = -1$ coincides with K, then $\alpha = -I$, and by Lemma 2.4, $\mu_1 = \mu_2 = \mu$, where μ is an arbitrary distribution on $\mathbb{R}^2$.

2. $\lambda = -1$ is not an eigenvalue of the operator α. By Theorem B, in this case μ_j are Gaussian distributions. Find out what can be said about the supports of the Gaussian distributions μ_j, in particular, when μ_j are degenerate distributions.

The characteristic functions $\hat\mu_j(y_1, y_2)$ are represented in the form

$$\hat\mu_j(y_1, y_2) = \exp\{-\langle A_j(y_1, y_2), (y_1, y_2)\rangle + i\langle b_j, (y_1, y_2)\rangle\}, \quad (y_1, y_2) \in \mathbb{R}^2, \tag{40}$$

where A_j is a symmetric positive semidefinite 2×2 matrix, $b_j \in \mathbb{R}^2$, $j = 1, 2$. By Lemma 2.2, the characteristic functions $\hat\mu_j(y_1, y_2)$ satisfy equation (2). Depending on the spectrum of the operator α, a matrix of a rather simple form in a suitable basis corresponds to the operator α. Substituting (40) into (2) we get that the matrices A_j satisfy the equation

$$A_1 + A_2\widetilde\alpha = 0. \tag{41}$$

The description of solutions of equation (41) in the class of symmetric positive semidefinite 2×2 matrices implies the description of the supports of the Gaussian distributions μ_j. Two cases are possible.

2A. The characteristic equation of the operator α has only real roots.

(i). The operator α has two different eigenvalues $\lambda = \lambda_1$ and $\lambda = \lambda_2$. In this case in a basis consisting of eigenvectors of α, the diagonal matrix $\alpha = \mathrm{diag}\{\lambda_1, \lambda_2\}$ corresponds to the operator α. Then the following statements are valid. If $\lambda_1 > 0$ and $\lambda_2 > 0$, then μ_j are degenerate distributions. If $\lambda_1\lambda_2 < 0$, then $\mu_j = \gamma_j * E_{x_j}$, where γ_j are symmetric Gaussian distributions supported in L_{λ_j}, where $\lambda_j < 0$. If $\lambda_1 < 0$ and $\lambda_2 < 0$, then $\mu_j = \gamma_j * E_{x_j}$, where γ_j are symmetric Gaussian distributions. In so doing either γ_j are degenerate distributions concentrated at zero, or the supports of γ_j are one of subspaces L_{λ_j}, or the supports of γ_j are the space $\mathbb{R}^2$.

(ii). The operator α has the only eigenvalue $\lambda = \lambda_0$. If the root space corresponding to the eigenvalue $\lambda = \lambda_0$ does not coincide with L_{λ_0}, then the Jordan cell

$$\alpha = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix}$$

in a suitable basis corresponds to the operator α. Then, if $\lambda_0 > 0$, then μ_j are degenerate distributions. If $\lambda_0 < 0$, then $\mu_j = \gamma_j * E_{x_j}$, where γ_j are symmetric Gaussian distributions supported in L_{λ_0}. If the root space corresponding to the eigenvalue $\lambda = \lambda_0$ coincides with L_{λ_0}, then $\alpha = \lambda_0 I$. In this case, if $\lambda_0 > 0$, then μ_j are degenerate distributions. If $\lambda_0 < 0$, then $\mu_j = \gamma_j * E_{x_j}$, where γ_j are symmetric Gaussian distributions. In so doing either γ_j are degenerate distributions concentrated at zero, or the supports of γ_j are the same one-dimensional subspace of $\mathbb{R}^2$, or the supports of γ_j are the space $\mathbb{R}^2$.

2B. The roots of the characteristic equation of the operator α are of the form $\lambda_1 = x_0 + iy_0$, $\lambda_1 = x_0 - iy_0$, where $y_0 \neq 0$. Then the matrix

$$\alpha = \begin{pmatrix} x_0 & y_0 \\ -y_0 & x_0 \end{pmatrix}$$

in a suitable basis corresponds to the operator α. In this case μ_j are degenerate distributions.

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