The two-sided exit problem for an additive functional of a time-inhomogeneous Markov chain

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Abstract

We consider an additive functional driven by a time-inhomogeneous Markov chain with a finite state space. Our study focuses on the joint distribution of the two-sided exit time and the state of the driving Markov chain at the time of exit, given in terms of expectation operators. These operators can be expressed as compositions of other operators related to some relevant one-sided exit (or first passage) problems. In addition, we study the law of the driving Markov chain at times prior to the exit time.

1 Introduction

In this paper we consider a time-inhomogeneous Markov process $X = (X_t)_{t\geq 0}$ with finite state space $\mathbf{E}$, and an additive functional

$$\phi_t(s) := \int_s^t v(X_u)\, du, \quad t \in [s, \infty), \tag{1.1}$$

where $s \geq 0$ and $v \neq 0$ is a real valued function defined on $\mathbf{E}$. For $\ell \geq 0$, we define the following first passage times

$$\tau_\ell^+(s) := \inf\{t \in [s,\infty] : \phi_t(s) > \ell\} \quad \text{and} \quad \tau_\ell^-(s) := \inf\{t \in [s,\infty] : \phi_t(s) < -\ell\}. \tag{1.2}$$

Our goal is to provide a analytical/algebraic expression for the following expectation

$$\mathbb{E}_{s,i}\left(g\left(\tau^-_{\ell^-}(s) \wedge \tau^+_{\ell^+}(s), X_{\tau^-_{\ell^-}(s)\wedge\tau^+_{\ell^+}(s)}\right)\right) \tag{1.3}$$

for any $s \geq 0$, $i \in \mathbf{E}$, $\ell^+, \ell^- \geq 0$ and bounded $g : \mathbb{R}_+ \times \mathbf{E} \to \mathbb{R}$. This is done via (3.5) and Theorem 3.2.

For the case of $\ell^- = -\infty$ (i.e., one-sided exit, or, first passage), under suitable conditions, one can derive equations that compute $\mathbb{E}(g(\tau^+_{\ell^+}(s), X_{\tau^+_{\ell^+}(s)}))$ for various g's. Most of the literature

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considers time-homogeneous setting. [BRW80] and [KW90] respectively study the one-sided exit problems without and with additive Brownian noise. That is, respectively, the case of

$$\varphi_t := \int_0^t v(X_u)\,du, \quad t \in [0,\infty), \tag{1.4}$$

and the case of

$$\widehat{\varphi}_t := \int_0^t v(X_u)\,du + W_t, \quad t \in [0,\infty), \tag{1.5}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion independent of X. In [Rog94], alternative proofs of the main results in [BRW80] and [KW90] are provided. The proofs are facilitated by introducing certain novel martingales (cf. [Rog94, (2.10) and (7.2)]). Using similar martingales, [JP08] generalizes the previous results to allow coexistence of unnoisy and noisy regions (we refer to [JP08, Sections 3 and 4] for details). Recently, [BCCG20] studied the one-sided exit problem and computation of $\mathbb{E}_{s,i}\left(g\left(\tau^+_{\ell^+}(s), X_{\tau^+_{\ell^+}(s)}\right)\right)$ in a time-inhomogeneous setting. Although martingales analogous to the aforementioned martingales are not explicitly used there, the proof in [BCCG20] is built upon a similar martingale idea.

However, the martingale method described above for the one-sided exit problems is not readily applicable to the two-sided exit problem here. Therefore, in this paper, we will express (1.3) in terms of some operators related to $\tau^{\pm}_{\ell^{\pm}}$, which are motivated by studies done in [BCCG20]. The analogous idea can also be found in [JP08] under time-homogeneous setting. However, some of the techniques used in [JP08] are not quite compatible with time-inhomogeneity, thus we adopt a different approach. Our approach is mainly based on the probabilistic decomposition such as (4.22) and its analytic counterpart (4.28), which originate from (3.5) and a reformulation of the problem in Section 2.2.

It is important to acknowledge the fact that the two-sided exit problem is used in various applications. For example, in [JP08] the solution of a two-sided exit problem was used as a tool for dealing with pricing a perpetual American option subject to a stock dynamics that is modulated by a Markov driver. In [ES2015] a two-sided exit problem was used a tool in studying issues arising in so called prospect theory. In [SN2014] a two-sided exit problem was considered in relation to precipitation statistics. Numerous applications in physics of two-sided exit problems are discussed in [Red2001].

The rest of the paper is organized as below. We first introduce our setup in Section 2. In Section 3 we present the main result of this paper, Theorem 3.2. Then, we give Proposition 3.5 which demonstrates that our main result can also be used to compute expectation of a function the driving state at times prior to the two-sided exit. The proofs are gathered in Section 4. Finally, in Section 5, we close with some concluding remarks and suggestions for a follow-up research.

2 Setup

Our setup is similar to that of [BCGH20], except that some assumptions on the regularity of the generator of the underlying Markov chain are relaxed here. The rest of the section details our setup.

2.1 Preliminaries

Throughout this paper we let $\mathbf{E}$ be a finite set, with $|\mathbf{E}| = m > 1$. We define $\overline{\mathbf{E}} := \mathbf{E} \cup \{\partial\}$, where ∂ denotes the coffin state isolated from $\mathbf{E}$. Let $(\Lambda_s)_{s\in\mathbb{R}_+}$, where $\mathbb{R}_+ := [0,\infty)$, be a family of $m \times m$ sub-Markovian generator matrices, i.e., their off-diagonal elements are non-negative, and the entries in their rows sum to a non-positive number. We additionally define $\Lambda_\infty := \mathsf{0}$, the $m \times m$ matrix with all entries equal to zero.

We make the following standing assumption:

Assumption 2.1. *There exists an absolute constant $K \in (0,\infty)$, such that $|\Lambda_s(i,j)| \le K$, for all $i, j \in \mathbf{E}$ and $s \in \mathbb{R}_+$.*

Let $v : \overline{\mathbf{E}} \to \mathbb{R}$ with $v(i) \neq 0$ for any $i \in \mathbf{E}$ and $v(\partial) = 0$. We will use the following partition of the set $\mathbf{E}$

$$\mathbf{E}_+ := \{i \in \mathbf{E} : \, v(i) > 0\} \quad \text{and} \quad \mathbf{E}_- := \{i \in \mathbf{E} : \, v(i) < 0\}\,.$$

We assume that both $\mathbf{E}_+$ and $\mathbf{E}_-$ are non-empty. Without loss of generality, we also assume that the indices of the first $m_+ = |\mathbf{E}_+|$ (respectively, last $m_- = |\mathbf{E}_-|$) rows and columns of any $m \times m$ matrix correspond to the elements in $\mathbf{E}_+$ (respectively, $\mathbf{E}_-$).

In what follows we let $\mathscr{X} := \mathbb{R}_+ \times \mathbf{E}$, and $\mathscr{X}_\pm := \mathbb{R}_+ \times \mathbf{E}_\pm$. The Borel σ-field on $\mathscr{X}$ (respectively, $\mathscr{X}_\pm$) is denoted by $\mathcal{B}(\mathscr{X}) := \mathcal{B}(\mathbb{R}_+) \otimes 2^{\mathbf{E}}$ (respectively, $\mathcal{B}(\mathscr{X}_\pm) := \mathcal{B}(\mathbb{R}_+) \otimes 2^{\mathbf{E}_\pm}$). Accordingly, we let $\overline{\mathscr{X}} := \mathscr{X} \cup (\infty, \partial)$ (respectively, $\overline{\mathscr{X}_\pm} := \mathscr{X}_\pm \cup (\infty,\partial)$) be the one-point completion of $\mathscr{X}$ (respectively, $\mathscr{X}_\pm$), and let $\mathcal{B}(\overline{\mathscr{X}}) := \sigma(\mathcal{B}(\mathscr{X}) \cup \{(\infty,\partial)\})$ (respectively, $\mathcal{B}(\overline{\mathscr{X}_\pm}) := \sigma(\mathcal{B}(\mathscr{X}_\pm) \cup \{(\infty,\partial)\})$). A pair $(s,i) \in \mathscr{X}$ consists of the time variable s and the space variable i.

We will also use the following notations for various spaces of real-valued functions:

- $\mathcal{B}_b(\overline{\mathscr{X}})$ is the space of $\mathcal{B}(\overline{\mathscr{X}})$-measurable, and bounded functions f on $\overline{\mathscr{X}}$, with $g(\infty,\partial) = 0$.
- $C_0(\overline{\mathscr{X}})$ is the space of functions $f \in \mathcal{B}_b(\overline{\mathscr{X}})$ such that $f(\cdot, i) \in C_0(\mathbb{R}_+)$ for all $i \in \mathbf{E}$, where $C_0(\mathbb{R}_+)$ is the space of functions vanishing at infinity.
- $C_c(\overline{\mathscr{X}})$ is the space of functions $f \in \mathcal{B}_b(\overline{\mathscr{X}})$ such that $f(\cdot, i) \in C_c(\mathbb{R}_+)$ for all $i \in \mathbf{E}$, where $C_c(\mathbb{R}_+)$ is the space of functions with compact support.

Sometimes $\overline{\mathscr{X}}$ will be replaced by $\overline{\mathscr{X}_+}$ or $\overline{\mathscr{X}_-}$ when the functions are defined on these spaces, in which case the set $\mathbf{E}$ will be replaced by $\mathbf{E}_+$ or $\mathbf{E}_-$, respectively, in the above definitions. Note that each function on $\overline{\mathscr{X}}$ can be viewed as a time-dependent vector of size m, which can be split into a time-dependent vector of size m_+ (a function on $\mathscr{X}_+$) and a time-dependent vector of size m_- (a function on $\mathscr{X}_-$).

2.2 A time-inhomogeneous Markov family corresponding to $(\Lambda_s)_{s\in\mathbb{R}_+}$ and related passage times

We start by introducing a time-inhomogeneous Markov Family corresponding to sub-Markovian matrix intensity function $(\Lambda_s)_{s\in\mathbb{R}_+}$. After the introduction of the time-inhomogeneous Markov Family, we proceed with a study of some passage times related to this family.

2.2.1 A time-inhomogeneous Markov family $\mathcal{M}$ corresponding to $(\Lambda_s)_{s\in\mathbb{R}_+}$

We take Ω as the collection of $\overline{\mathbf{E}}$-valued functions ω on $\mathbb{R}_+$, and $\mathscr{F} := \sigma\{X_t,\, t\in\mathbb{R}_+\}$, where X is the coordinate mapping $X_\cdot(\omega) := \omega(\cdot)$. Sometimes we may need the value of $\omega\in\Omega$ at infinity, and in such case we set $X_\infty(\omega)=\omega(\infty)=\partial$, for any $\omega\in\Omega$. We endow the space $(\Omega,\mathscr{F})$ with a family of filtrations $\mathbb{F}_s := \{\mathscr{F}_t^s,\, t\in[s,\infty]\}$, $s\in\overline{\mathbb{R}}_+$, where, for $s\in\mathbb{R}_+$,

$$\mathscr{F}_t^s := \bigcap_{r>t}\sigma\left(X_u,\ u\in[s,r]\right),\ \ t\in[s,\infty);\quad \mathscr{F}_\infty^s := \sigma\Bigg(\bigcup_{t\ge s}\mathscr{F}_t^s\Bigg),$$

and $\mathscr{F}_\infty^\infty := \{\emptyset,\Omega\}$. We denote by

$$\mathcal{M} := \left\{\left(\Omega,\mathscr{F},\mathbb{F}_s,(X_t)_{t\in[s,\infty]},\mathbb{P}_{s,i}\right),\,(s,i)\in\overline{\mathscr{X}}\right\}$$

a canonical *time-inhomogeneous* Markov family. That is,

- $\mathbb{P}_{s,i}$ is a probability measure on $(\Omega,\mathscr{F}_\infty^s)$ for $(s,i)\in\overline{\mathscr{X}}$;
- the function $P:\overline{\mathscr{X}}\times\overline{\mathbb{R}}_+\times 2^{\overline{\mathbf{E}}}\to[0,1]$ defined for $0\le s\le t\le\infty$ as

  $$P(s,i,t,B) := \mathbb{P}_{s,i}(X_t\in B)$$

  is measurable with respect to i for any fixed $s\le t$ and $B\in 2^{\overline{\mathbf{E}}}$;
- $\mathbb{P}_{s,i}(X_s=i)=1$ for any $(s,i)\in\overline{\mathscr{X}}$;
- for any $(s,i)\in\overline{\mathscr{X}}$, $s\le t\le r\le\infty$, and $B\in 2^{\overline{\mathbf{E}}}$, it holds that

  $$\mathbb{P}_{s,i}(X_r\in B\,|\,\mathscr{F}_t^s) = \mathbb{P}_{t,X_t}(X_r\in B)\,,\quad \mathbb{P}_{s,i}-\text{a.s.}\,,$$

Let $\mathsf{U} := (\mathsf{U}_{s,t})_{0\le s\le t<\infty}$ be the evolution system (cf. [Bot14]) corresponding to $\mathcal{M}$, defined by

$$\mathsf{U}_{s,t}f(i) := \mathbb{E}_{s,i}\left(f(X_t)\right),\quad 0\le s\le t<\infty,\quad i\in\mathbf{E}, \tag{2.1}$$

for all functions (column vectors) $f:\mathbf{E}\to\mathbb{R}$.[1] We assume that

$$\lim_{h\downarrow 0}\frac{1}{h}\left(\mathsf{U}_{s,s+h}f(i)-f(i)\right)=\Lambda_s f(i),\quad \text{for any }\ (s,i)\in\mathscr{X}, \tag{2.2}$$

for all $f:\mathbf{E}\to\mathbb{R}$. Without loss of generality,[2] we assume $\mathcal{M}$ is a standard Markov family (cf. [GS04, Definition I.6.6]).[3] In particular, $\mathcal{M}$ has the strong Markov property. Consequently, each process $(X_t)_{t\in[s,\infty]}$, for $s\in\mathbb{R}_+$, is a strong Markov process.

[1] Note that for $t\in\mathbb{R}_+$, X_t takes values in $\mathbf{E}$.
[2] We refer to the discussion in [BCCG20, Section 2.2.1].
[3] In terminology of [GS04, Definition I.6.6], $\mathcal{M}$ is called a Markov process instead of a Markov family.

2.2.2 Passage times related to $\mathcal{M}$

For all $s \in \overline{\mathbb{R}}_+$ and $\omega \in \Omega$, we define an additive functional $\phi_{\cdot}(s)$ as

$$\phi_t(s) := \int_s^t v(X_u)\,du, \quad t \in [s,\infty). \tag{2.3}$$

Additionally, we define $\phi_\infty(s,\omega) := 0$. Moreover, for any $s \in \overline{\mathbb{R}}_+$ and $\ell \in \mathbb{R}_+$, we define associated passage times

$$\tau_\ell^+(s) := \inf\{t \in [s,\infty] : \phi_t(s) > \ell\} \quad \text{and} \quad \tau_\ell^-(s) := \inf\{t \in [s,\infty] : \phi_t(s) < -\ell\}. \tag{2.4}$$

Both $\tau_\ell^+(s)$ and $\tau_\ell^-(s)$ are $\mathbb{F}_s$-stopping times since, $\phi_{\cdot}(s)$ is $\mathbb{F}_s$-adapted, has continuous sample paths, and $\mathbb{F}_s$ is right-continuous (cf. [JS03, Proposition 1.28]). For notational convenience, if no confusion arises, we will omit the parameter s in $\phi_t(s)$ and $\tau_\ell^\pm(s)$.

The following lemma is an immediate consequence of the definitions above. We refer to [BCCG20, Lemma 2.2] for the proof.

Lemma 2.2. *For any $s \in \overline{\mathbb{R}}_+$, $\ell \in \mathbb{R}_+$ and $\omega \in \Omega$ the following inclusion holds $X_{\tau_\ell^\pm(s)}(\omega) \in \mathbf{E}_\pm \cup \{\partial\}$. In particular, if $\tau_\ell^\pm(s,\omega) < \infty$, then $X_{\tau_\ell^\pm(s)}(\omega) \in \mathbf{E}_\pm$.*

In order to proceed, we introduce the following operators:

- $J^+ : \mathcal{B}_b(\overline{\mathscr{X}_+}) \to \mathcal{B}_b(\overline{\mathscr{X}_-})$ is defined as

$$\left(J^+ g^+\right)(s,i) := \mathbb{E}_{s,i}\left(g^+\left(\tau_0^+, X_{\tau_0^+}\right)\right), \quad (s,i) \in \overline{\mathscr{X}_-}. \tag{2.5}$$

  Clearly, for any $g^+ \in \mathcal{B}_b(\overline{\mathscr{X}_+})$ it holds that $|(J^+g^+)(s,i)| \le \|g^+\|_{\mathcal{B}_b(\overline{\mathscr{X}_+})} < \infty$ for any $(s,i) \in \mathscr{X}_-$, and $(J^+g^+)(\infty,\partial) = 0$, so that $J^+g^+ \in \mathcal{B}_b(\overline{\mathscr{X}_-})$.

- $J^- : \mathcal{B}_b(\overline{\mathscr{X}_-}) \to \mathcal{B}_b(\overline{\mathscr{X}_+})$ is defined as,

$$\left(J^- g^-\right)(s,i) := \mathbb{E}_{s,i}\left(g^-\left(\tau_0^-, X_{\tau_0^-}\right)\right), \quad (s,i) \in \overline{\mathscr{X}_+}. \tag{2.6}$$

- For any $\ell \in \mathbb{R}_+$, $\mathcal{P}_\ell^+ : \mathcal{B}_b(\overline{\mathscr{X}_+}) \to \mathcal{B}_b(\overline{\mathscr{X}_+})$ is defined as

$$\left(\mathcal{P}_\ell^+ g^+\right)(s,i) := \mathbb{E}_{s,i}\left(g^+\left(\tau_\ell^+, X_{\tau_\ell^+}\right)\right), \quad (s,i) \in \overline{\mathscr{X}_+}. \tag{2.7}$$

- For any $\ell \in \mathbb{R}_+$, $\mathcal{P}_\ell^- : \mathcal{B}_b(\overline{\mathscr{X}_-}) \to \mathcal{B}_b(\overline{\mathscr{X}_-})$ is defined as,

$$\left(\mathcal{P}_\ell^- g^-\right)(s,i) := \mathbb{E}_{s,i}\left(g^-\left(\tau_\ell^-, X_{\tau_\ell^-}\right)\right), \quad (s,i) \in \overline{\mathscr{X}_-}. \tag{2.8}$$

The proposition below follows from the strong Markov property of $(X_t)_{t\in[s,\infty]}$. We refer to [BCCG20, Section 2.3] for the proof.

Proposition 2.3. *For $g^+ \in \mathcal{B}_b(\overline{\mathscr{X}_+})$, $\ell \in (0,\infty)$, and $(s,i) \in \mathscr{X}_-$, we have*

$$\mathbb{E}_{s,i}\left(g^+\left(\tau_\ell^+, X_{\tau_\ell^+}\right)\right) = \left(J^+\mathcal{P}_\ell^+ g^+\right)(s,i). \tag{2.9}$$

Analogously, for $g^- \in \mathcal{B}_b(\overline{\mathscr{X}_-})$, $\ell \in (0,\infty)$, and $(s,i) \in \mathscr{X}_+$, we have

$$\mathbb{E}_{s,i}\left(g^-\left(\tau_\ell^-, X_{\tau_\ell^-}\right)\right) = \left(J^-\mathcal{P}_\ell^- g^-\right)(s,i).$$

Remark 2.4. Let $C_0^1(\overline{\mathscr{X}})$ be the space of functions $f \in C_0(\overline{\mathscr{X}})$ such that, for any $i \in \mathbf{E}$, $\partial f(\cdot, i)/\partial s$ exists and belongs to $C_0(\mathbb{R}_+)$. [BCCG20] shows that, if we assume additionally that $s \mapsto \Lambda_s$ is continuous, then $J^\pm$ and $\mathcal{P}^\pm$ can be uniquely characterized by certain operator equation. More precisely: Define $\mathsf{V} := \operatorname{diag}\{v(i) : i \in \mathbf{E}\}$, $\widetilde{\Lambda} g(s,i) := [\Lambda_s g(s,\cdot)]_i$ and consider the following equation in unknown (S^+, H^+, S^-, H^-)

$$\mathsf{V}^{-1}\left(\frac{\partial}{\partial s} + \widetilde{\Lambda}\right)\begin{pmatrix} I^+ & S^- \\ S^+ & I^- \end{pmatrix}\begin{pmatrix} g^+ \\ g^- \end{pmatrix} = \begin{pmatrix} I^+ & S^- \\ S^+ & I^- \end{pmatrix}\begin{pmatrix} H^+ & 0 \\ 0 & -H^- \end{pmatrix}\begin{pmatrix} g^+ \\ g^- \end{pmatrix}, \quad g^\pm \in C_0^1(\overline{\mathscr{X}_\pm}), \tag{2.10}$$

subject to the conditions below:

(a$^\pm$) $S^\pm : C_0(\overline{\mathscr{X}_\pm}) \to C_0(\overline{\mathscr{X}_\mp})$ is a bounded operator such that

(i) for any $g^\pm \in C_c(\overline{\mathscr{X}_\pm})$ with $\operatorname{supp} g^\pm \subset [0, \eta_{g^\pm}] \times \mathbf{E}_\pm$ for some constant $\eta_{g^\pm} \in (0,\infty)$, we have $\operatorname{supp} S^\pm g^\pm \subset [0, \eta_{g^\pm}] \times \mathbf{E}_\mp$;

(ii) for any $g^\pm \in C_0^1(\overline{\mathscr{X}_\pm})$, we have $S^\pm g^\pm \in C_0^1(\overline{\mathscr{X}_\mp})$.

(b$^\pm$) $H^\pm$ is the strong generator of a strongly continuous positive contraction semigroup $(\mathcal{Q}_\ell^\pm)_{\ell\in\mathbb{R}_+}$ on $C_0(\overline{\mathscr{X}_\pm})$ with domain $\mathscr{D}(H^\pm) = C_0^1(\overline{\mathscr{X}_\pm})$.

Then, (2.10) has a unique solution. Moreover, restricted to $C_0(\overline{\mathscr{X}_\pm})$, we have $J^\pm = S^\pm$. In addition, $(\mathcal{Q}_\ell^\pm)_{\ell\in\mathbb{R}^+}$ are strongly continuous positive contraction semigroups with generators $H^\pm$ and $\mathcal{P}_\ell^\pm = Q_\ell^\pm$.

3 Main Result

Note that $\phi_0(s) = 0$. To facilitate the investigation of the exit time of $(\phi_t(s))_{t\ge s}$ from interval $[-\ell^-, \ell^+]$, where $\ell^\pm \ge 0$, we define

$$\xi^+_{\ell^-,\ell^+}(s) := \inf\{t \in [s, \tau^-_{\ell^-}(s)) : \phi_t(s) > \ell^+\} = \mathbb{1}_{\left\{\tau^+_{\ell^+}(s) < \tau^-_{\ell^-}(s)\right\}}\tau^+_{\ell^+}(s) + \mathbb{1}_{\left\{\tau^+_{\ell^+}(s) \ge \tau^-_{\ell^-}(s)\right\}} \cdot \infty,$$

and

$$\xi^-_{\ell^-,\ell^+}(s) := \inf\{t \in [s, \tau^+_{\ell^+}(s)) : \phi_t(s) < -\ell^-\} = \mathbb{1}_{\left\{\tau^-_{\ell^-}(s) < \tau^+_{\ell^+}(s)\right\}}\tau^-_{\ell^-}(s) + \mathbb{1}_{\left\{\tau^-_{\ell^-}(s) \ge \tau^+_{\ell^+}(s)\right\}} \cdot \infty,$$

where we adopted the usual convention that $\inf\emptyset = \infty$. Note that for a fixed ω at least one of $\xi^+_{\ell^-,\ell^+}(s,\omega)$ and $\xi^-_{\ell^-,\ell^+}(s,\omega)$ equals to ∞. Clearly,

$$X_{\xi^+_{\ell^-,\ell^+}(s)} \in \mathbf{E}_+ \cup \{\partial\} \quad \text{and} \quad X_{\xi^-_{\ell^-,\ell^+}(s)} \in \mathbf{E}_- \cup \{\partial\}. \tag{3.1}$$

We define $\Xi^+_{\ell^-,\ell^+} : \mathcal{B}_b(\overline{\mathscr{X}_+}) \to \mathcal{B}_b(\overline{\mathscr{X}})$ as

$$\left(\Xi^+_{\ell^-,\ell^+} g^+\right)(s,i) := \mathbb{E}_{s,i}\left(g^+\left(\xi^+_{\ell^-,\ell^+}(s), X_{\xi^+_{\ell^-,\ell^+}(s)}\right)\right), \quad (s,i) \in \overline{\mathscr{X}}, \tag{3.2}$$

where $g^+ \in \mathcal{B}_b(\overline{\mathscr{X}_+})$. Note that we consider $g^+ \in \mathcal{B}_b(\overline{\mathscr{X}_+})$ instead of $g \in \mathcal{B}_b(\overline{\mathscr{X}})$ due to (3.1). Similarly, we define $\Xi^-_{\ell^-,\ell^+} : \mathcal{B}_b(\overline{\mathscr{X}_-}) \to \mathcal{B}_b(\overline{\mathscr{X}})$ as

$$\left(\Xi^-_{\ell^-,\ell^+} g^-\right)(s,i) := \mathbb{E}_{s,i}\left(g^-\left(\xi^-_{\ell^-,\ell^+}(s), X_{\xi^-_{\ell^-,\ell^+}(s)}\right)\right), \quad (s,i) \in \overline{\mathscr{X}}, \tag{3.3}$$

where $g^- \in \mathcal{B}_b(\overline{\mathscr{X}_-})$.

Note that events $\{\xi^+_{\ell^-,\ell^+}(s) < \infty\}$, $\{\xi^-_{\ell^-,\ell^+}(s) < \infty\}$, and $\{\xi^+_{\ell^-,\ell^+}(s) = \xi^-_{\ell^-,\ell^+}(s) = \infty\}$ forms a partition for Ω. It follows that

$$\tau^+_{\ell^+}(s)\wedge\tau^-_{\ell^-}(s)=\mathbb{1}_{\{\xi^+_{\ell^-,\ell^+}(s)<\infty\}}\xi^+_{\ell^-,\ell^+}(s)+\mathbb{1}_{\{\xi^-_{\ell^-,\ell^+}(s)<\infty\}}\xi^-_{\ell^-,\ell^+}(s)+\mathbb{1}_{\{\xi^+_{\ell^-,\ell^+}(s)=\xi^-_{\ell^-,\ell^+}(s)=\infty\}}\cdot\infty. \tag{3.4}$$

For any $g \in \mathcal{B}_b(\overline{\mathscr{X}})$, if we let $g^\pm \in \mathcal{B}_b(\overline{\mathscr{X}_\pm})$ satisfy $g^\pm(s,i) = g(s,i)$ for $(s,i) \in \mathscr{X}_\pm$, then by (3.1) and (3.4), we have

$$\mathbb{E}_{s,i}\left(g\left(\tau^-_{\ell^-}\wedge\tau^+_{\ell^+}, X_{\tau^-_{\ell^-}\wedge\tau^+_{\ell^+}}\right)\right) = \left(\Xi^+_{\ell^-,\ell^+}g^+\right)(s,i) + \left(\Xi^-_{\ell^-,\ell^+}g^-\right)(s,i). \tag{3.5}$$

Recall that the left hand side of (3.5) is the expectation showing in (1.3). We therefore focus on computing $\Xi^+_{\ell^-,\ell^+}$ and $\Xi^-_{\ell^-,\ell^+}$.

In what follows we say that $g^\pm \in \mathcal{B}_b(\overline{\mathscr{X}_\pm})$ decays exponentially fast to zero if there are constants $C, c > 0$ such that $\max_{i\in\mathbf{E}_\pm} |g^\pm(s,i)| \le Ce^{-cs}$ for $s \in \mathbb{R}_+$. The proofs of results below are presented in Section 4. We begin with the following lemma.

Lemma 3.1. *Suppose that $g^+ \in \mathcal{B}_b(\overline{\mathscr{X}_+})$ decays exponentially fast to zero. Then for any $\ell^+, \ell^- \in \mathbb{R}_+$,*

$$\sum_{n=1}^{\infty}\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)^n g^+ \tag{3.6}$$

converges in $\|\cdot\|_\infty$. Analogously, suppose that $g^- \in \mathcal{B}_b(\overline{\mathscr{X}_-})$ decays exponentially fast to zero. Then for any $\ell^+, \ell^- \in \mathbb{R}_+$,

$$\sum_{n=1}^{\infty}\left(J^+\mathcal{P}^+_{\ell^-+\ell^+}J^-\mathcal{P}^-_{\ell^-+\ell^+}\right)^n g^- \tag{3.7}$$

converges in $\|\cdot\|_\infty$.

The ensuing theorem is our main result.

Theorem 3.2. *For $g^\pm \in \mathcal{B}_b(\overline{\mathscr{X}_\pm})$ decaying exponentially fast to zero, for any $\ell^+, \ell^- \in \mathbb{R}_+$ and $(s,i) \in \mathscr{X}$, we have*

$$\left(\Xi^+_{\ell^-,\ell^+}g^+\right)(s,i)=\left[\left(\begin{pmatrix}I^+\\J^+\end{pmatrix}\mathcal{P}^+_{\ell^+}-\begin{pmatrix}J^-\\I^-\end{pmatrix}\mathcal{P}^-_{\ell^-}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)\sum_{n=0}^{\infty}(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+})^n g^+\right](s,i), \tag{3.8}$$

$$\left(\Xi^-_{\ell^-,\ell^+}g^-\right)(s,i)=\left[\left(\begin{pmatrix}J^-\\I^-\end{pmatrix}\mathcal{P}^-_{\ell^-}-\begin{pmatrix}J^+\\I^+\end{pmatrix}\mathcal{P}^+_{\ell^+}J^-\mathcal{P}^-_{\ell^++\ell^-}\right)\sum_{n=0}^{\infty}(J^+\mathcal{P}^+_{\ell^-+\ell^+}J^-\mathcal{P}^-_{\ell^-+\ell^+})^n g^-\right](s,i). \tag{3.9}$$

Remark 3.3. If one of J^+, J^-, $\mathcal{P}^+_{\ell^-+\ell^+}$ or $\mathcal{P}^-_{\ell^-+\ell^+}$ has an operator norm $\|\cdot\|_\infty$ that is strictly less than 1, then $\sum_{n=0}^\infty (J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+})^n$ converges in operator norm $\|\cdot\|_\infty$ to $(I^+ - J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+})^{-1}$, and further allows removing the condition that $g^\pm$ decays exponentially fast to zero from the statement of Theorem 3.2. The proof of this modification of Theorem 3.2 is similar to the proof in Section 4.3, and will be omitted. A sufficient condition for $\|\mathcal{P}^\pm_{\ell^-+\ell^+}\|_\infty < 1$ is that there is a $c > 0$ such that each row of Λ_s sums up to a number less than $-c$ for all $s \in \mathbb{R}_+$. Indeed, this introduces an exponential killing at the rate of at least c to $(X_t)_{t\in[s,\infty]}$, for all $s \in \mathbb{R}_+$. Consequently, $\|\mathcal{P}^\pm_{\ell^-+\ell^+}\|_\infty \le 1 - \exp(-c(\ell^- + \ell^+)\|v\|^{-1}_\infty)$, since for $\ell > 0$ and $s \in \mathbb{R}_+$ we have $\tau^\pm_\ell(s) - s \ge \ell\|v\|^{-1}_\infty$.

Remark 3.4. In this remark we relate (3.8) to [JP08, Proposition 1] without additive Brownian noise by letting the Markov family $\mathcal{M}$ be time-homogeneous, namely, for all $s \in \mathbb{R}_+$, $\mathsf{\Lambda}_s = \mathsf{\Lambda}_0 =: \mathsf{\Lambda}$. For simplicity, we assume that there is a $c > 0$ such that each row of $\mathsf{\Lambda}$ sums up to a number smaller than $-c$. We define $\zeta(s) := \inf\{t \geq s : X_t = \partial\}$; we will omit s in $\zeta(s)$ when no confusion arises. In this case, observe that

$$\mathbb{P}_{s,i}\Big(X_{\tau_\ell^\pm(s)} = k\Big) = \mathbb{P}_{s,i}\Big(\tau_\ell^\pm(s) < \zeta(s), X_{\tau_\ell^\pm(s)} = k\Big) = \mathbb{P}_{0,i}\Big(\tau_\ell^\pm(0) < \zeta(0), X_{\tau_\ell^\pm(0)} = k\Big)$$

for any $(s,i) \in \mathscr{X}$, $k \in \mathbf{E}_\pm$, and $\ell \in \mathbb{R}_+$. Moreover, it is shown in [JP08, Theorem 2, (24)] (see also [BRW80], [BCCG20, Remark 3.5]) that there are sub-Markovian matrices Q^+ and Q^- on $\mathbf{E}_+$ and $\mathbf{E}_-$, respectively, such that

$$\mathbb{P}_{0,i}\Big(\tau_\ell^+ < \zeta, X_{\tau_\ell^+} = k\Big) = \big[e^{\ell \mathsf{Q}^+}\big]_{ik}, \quad (i,k) \in \mathbf{E}_+^2, \quad \ell \in \mathbb{R}_+,$$
$$\mathbb{P}_{0,i}\Big(\tau_\ell^- < \zeta, X_{\tau_\ell^-} = k\Big) = \big[e^{\ell \mathsf{Q}^-}\big]_{ik}, \quad (i,k) \in \mathbf{E}_-^2, \quad \ell \in \mathbb{R}_+.$$

We define $\mathsf{J}^+ \in \mathbb{R}^{|\mathbf{E}_-| \times |\mathbf{E}_+|}$ and $\mathsf{J}^- \in \mathbb{R}^{|\mathbf{E}_+| \times |\mathbf{E}_-|}$ as

$$\mathsf{J}^+_{ik} := \mathbb{P}_{0,i}\Big(\tau_0^+ < \zeta, X_{\tau_0^+} = k\Big), \quad (i,k) \in \mathbf{E}_- \times \mathbf{E}_+,$$
$$\mathsf{J}^-_{ik} := \mathbb{P}_{0,i}\Big(\tau_0^- < \zeta, X_{\tau_0^-} = k\Big), \quad (i,k) \in \mathbf{E}_+ \times \mathbf{E}_-.$$

Then, for any g^+ satisfying $g^+(s,i) = g^+(0,i)$ for all $(s,i) \in \mathscr{X}_+$, setting $\mathsf{g}^+(\cdot) := g^+(0,\cdot)$, we have

$$(J^+ g^+)(s,i) = \sum_{k \in \mathbf{E}_+} \mathbb{E}_{s,i}\Big(\mathbb{1}_{\{\tau_0^+ < \zeta, X_{\tau_0^+} = k\}} \mathsf{g}^+(k)\Big) = [\mathsf{J}^+ \mathsf{g}^+]_i, \quad (s,i) \in \mathscr{X}^-,$$
$$\big(\mathcal{P}_\ell^+ g^+\big)(s,i) = \sum_{k \in \mathbf{E}_+} \mathbb{E}_{s,i}\Big(\mathbb{1}_{\{\tau_\ell^+ < \zeta, X_{\tau_\ell^+} = k\}} \mathsf{g}^+(k)\Big) = [e^{\ell \mathsf{Q}^+} \mathsf{g}^+]_i, \quad (s,i) \in \mathscr{X}^+, \quad \ell \geq 0.$$

In view of the exponential killing, with similar reasoning as in Remark 3.3, we have $\|e^{(\ell^+ + \ell^-)\mathsf{Q}^\pm}\|_\infty \leq e^{-c(\ell^+ + \ell^-)\|v\|_\infty^{-1}}$. It follows that

$$\begin{aligned}\sum_{n=0}^\infty \Big(\Big(J^- \mathcal{P}^-_{\ell^- + \ell^+} J^+ \mathcal{P}^+_{\ell^- + \ell^+}\Big)^n g^+\Big)(0,\cdot) &= \sum_{n=0}^\infty \Big(\mathsf{J}^- e^{(\ell^- + \ell^+)\mathsf{Q}^-} \mathsf{J}^+ e^{(\ell^- + \ell^+)\mathsf{Q}^+}\Big)^n \mathsf{g}^+ \\ &= \Big(\mathsf{I} - \mathsf{J}^- e^{(\ell^- + \ell^+)\mathsf{Q}^-} \mathsf{J}^+ e^{(\ell^- + \ell^+)\mathsf{Q}^+}\Big)^{-1} \mathsf{g}^+.\end{aligned}$$

Consequently, for any $(s,i) \in \mathscr{X}$, the right hand side of (3.8) becomes

$$\left[\left(\begin{pmatrix}\mathsf{I}^+ \\ \mathsf{J}^+\end{pmatrix} e^{\ell^+ \mathsf{Q}^+} - \begin{pmatrix}\mathsf{J}^- \\ \mathsf{I}^-\end{pmatrix} e^{\ell^- \mathsf{Q}^-} \mathsf{J}^+ e^{(\ell^- + \ell^+)\mathsf{Q}^+}\right)\Big(\mathsf{I} - \mathsf{J}^- e^{(\ell^- + \ell^+)\mathsf{Q}^-} \mathsf{J}^+ e^{(\ell^- + \ell^+)\mathsf{Q}^+}\Big)^{-1} \mathsf{g}^+\right]_i,$$

which shows that (3.8) as a special case of [JP08, Proposition 1, equation (31)] without additive Brownian noise. For (3.9), the sanity check can be carried out analogously.

The knowledge of $\Xi^\pm_{\ell^-,\ell^+}$ can also be used to study the law of X_T before the exit time, that is the law of X_T restricted to the set $\tau^-_{\ell^-} \wedge \tau^+_{\ell^+} > T$. Observe that

$$\mathbb{E}_{s,i}\left(h(X_T)\mathbb{1}_{\left\{\tau^-_{\ell^-} \wedge \tau^+_{\ell^+} > T\right\}}\right) = \mathbb{E}_{s,i}(h(X_T)) - \mathbb{E}_{s,i}\left(h(X_T)\mathbb{1}_{\left\{\tau^-_{\ell^-} \wedge \tau^+_{\ell^+} \leq T\right\}}\right), \tag{3.10}$$

it is sufficient to investigate the second term of the right hand side. The proposition below represents the desired quantity in terms of $\Xi^\pm$.

Proposition 3.5. *For any $h:\mathbf{E}\to\mathbb{R}$ and $\ell^{\pm}\in\mathbb{R}_+$, we have*

$$\mathbb{E}_{s,i}\left(h(X_T)\mathbb{1}_{\left\{\tau^-_{\ell^-}\wedge\tau^+_{\ell^+}\le T\right\}}\right)=\left(\Xi^+_{\ell^-,\ell^+}k^+\right)(s,i)+\left(\Xi^-_{\ell^-,\ell^+}k^-\right)(s,i),$$

where

$$k^{\pm}(t,j):=\mathbb{1}_{[0,T]}(t)\,\mathbb{E}_{t,j}\big(h(X_T)\big),\quad (t,j)\in\mathscr{X}_{\pm}. \tag{3.11}$$

4 Proofs

4.1 Proof of Lemma 3.1

The proof of Lemma 3.1 relies on the following inequality.

Lemma 4.1. *Suppose that $g^+(t,j)=\mathbb{1}_{[0,T]}(t)$ for some $T\in\mathbb{R}_+$. Then, for any $(s,i)\in\mathscr{X}_+$ and $\ell^{\pm}\in\mathbb{R}_+$, we have*

$$\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}g^+\right)(s,i)\le\mathbb{1}_{[0,T]}(s)\,\left(1-e^{-K(T-s)}\right)^2.$$

Proof. To start with, we let $\gamma(s):=\inf\{t\ge s: X_t\ne X_s\}$. Clearly, $\mathbb{P}_{s,i}(\gamma(s)\le T)=0$ for $s>T$. It is known that (cf. [RSST99, Section 8.4.2, (8.4.21)])

$$\mathbb{P}_{s,i}\big(\gamma(s)\le T\big)=1-\exp\left(-\int_s^T\Lambda_u(i,i)du\right)\le 1-e^{-K(T-s)},\quad s\le T, \tag{4.1}$$

where we used Assumption 2.1-(i) for the inequality. It follows from Lemma 2.2 and (4.1) that

$$\mathbb{P}_{s,i}\big(\tau^{\pm}_0(s)\le T\big)\le\mathbb{P}_{s,i}\big(\gamma(s)\le T\big)\le 1-e^{-K(T-s)},\quad s\le T, i\in\mathbf{E}_{\mp}. \tag{4.2}$$

Next, observe that $\tau^{\pm}_{\ell}(s)\ge s$. By (2.7), we have

$$\left(\mathcal{P}^+_{\ell^-+\ell^+}g^+\right)(s,i)=\mathbb{P}_{s,i}\big(\tau^+_{\ell^-+\ell^+}\le T\big)\le g^+(s,i)$$

Then, for $(s,i)\in\mathscr{X}_-$, by (2.5) and (4.2), we have

$$0\le\left(J^+\mathcal{P}^+_{\ell^-+\ell^+}g^+\right)(s,i)\le\big(J^+g^+\big)(s,i)=\mathbb{P}_{s,i}\big(\tau^{\pm}_0(s)\le T\big)\le\mathbb{1}_{[0,T]}(s)\Big(1-e^{-K(T-s)}\Big). \tag{4.3}$$

It follows from (2.8) and (4.3) that, for $(s,i)\in\mathscr{X}_-$,

$$0\le\left(\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}g^+\right)(s,i)\le\sup_{r\ge s}\mathbb{1}_{[0,T]}(r)\Big(1-e^{-K(T-r)}\Big)\le\mathbb{1}_{[0,T]}(s)\,\Big(1-e^{-K(T-s)}\Big). \tag{4.4}$$

Finally, by (2.6), (4.4) and (4.2),

$$\begin{aligned}\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}g^+\right)(s,i)&\le\mathbb{E}_{s,i}\left(\mathbb{1}_{[0,T]}(\tau^-_0)\sup_{r\ge s,\,i\in\mathbf{E}_-}\left(\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}g^+\right)(r,i)\right)\\&\le\mathbb{1}_{[0,T]}(s)\,\Big(1-e^{-K(T-s)}\Big)^2,\end{aligned}$$

which completes the proof. ∎

Proof of Lemma 3.1. We will only present the proof for the convergence of (3.6) as the proof for the convergence of (3.7) follows analogously. Without loss of generality, we assume that $g^+(t,i) = e^{-ct}$ for some $c > 0$ and define $g_k^+(s,i) := \sum_{j=1}^{2^k-1} \mathbb{1}_{[0,T_j^k]}(s)/2^k$, where $T_j^k := \inf\{t \ge 0 : e^{-ct} \le 1 - j2^{-k}\} = c^{-1}\ln(2^k/(2^k-j))$ for $j = 1, \ldots, 2^k - 1$. Therefore, by Lemma 4.1,

$$\left(\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)g_k^+\right)(0,i) \le \frac{1}{2^k}\sum_{j=1}^{2^k-1}\left(1-\left(1-\frac{j}{2^k}\right)^{K/c}\right)^2 \le \int_0^1 \left(1-(1-x)^{K/c}\right)^2 \mathrm{d}x =: C_{K,c} < 1.$$

Since g_k^+ is non-decreasing in $k \in \mathbb{N}$ and $\lim_{k\to\infty} g_k^+(s,i) = g^+(s,i)$ for $(s,i) \in \mathscr{X}_+$, by (2.7) and monotone convergence, we have that $\mathcal{P}^+_{\ell^-+\ell^+}g_k^+$ is non-decreasing in $k \in \mathbb{N}$ and that

$$\lim_{k\to\infty}\left(\mathcal{P}^+_{\ell^-+\ell^+}g_k^+\right)(s,i) = \left(\mathcal{P}^+_{\ell^-+\ell^+}g^+\right)(s,i), \quad (s,i) \in \mathscr{X}_+.$$

It follows from (2.5) and monotone convergence that

$$\lim_{k\to\infty}\left(J^+\mathcal{P}^+_{\ell^-+\ell^+}g_k^+\right)(s,i) = \left(J^+\mathcal{P}^+_{\ell^-+\ell^+}g^+\right)(s,i), \quad (s,i) \in \mathscr{X}_-.$$

By a similar reasoning we deduce that

$$\left(\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)g^+\right)(0,i) = \lim_{k\to\infty}\left(\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)g_k^+\right)(0,i) \le C_{K,c}.$$

Note that

$$\left(\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)g^+\right)(s,i) = e^{-cs}\left(\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)g^{+,s}\right)(s,i),$$

where $g^{+,s}(t,j) := e^{-c(t-s)}$ for $t \ge s$ and $j \in \mathbb{E}$. A similar argument as before shows that

$$\left(\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)g^+\right)(s,i) \le C_{K,c}\, e^{-cs} = C_{K,c}\, g^+(s,i).$$

Invoking the linearity and nonnegativity of $J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}$, by iteration, we have

$$\left(\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)^n g^+\right)(s,i) \le C^n_{K,c}\, g^+(s,i), \quad \text{for any } n \in \mathbb{N}.$$

Finally, we obtain that

$$\left\|\sum_{n=1}^{\infty}\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)^n g^+\right\|_\infty \le \sum_{n=1}^{\infty}\left\|\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)^n g^+\right\|_\infty \le \sum_{n=1}^{\infty} C^n_{K,c}\,\|g^+\|_\infty < \infty,$$

from which the strong convergence follows immediately. □

4.2 Auxiliary Markov families

In this subsection, we introduce an auxiliary time-inhomogenous Markov family $\widehat{\mathcal{M}}$ and an auxiliary time-homogenous Markov family $\widetilde{\mathcal{M}}$. Most of the technical details presented in this section, except for those involving two-sided exit times, are similar to [BCCG20, Section 4.1].

We start by introducing some more notations of spaces and σ-fields. Let $\mathscr{Y} := \overline{\mathbf{E}} \times \mathbb{R}$, and the Borel σ-field on $\mathscr{Y}$ is denoted by $\mathcal{B}(\mathscr{Y}) := 2^{\mathbf{E}} \otimes \mathcal{B}(\mathbb{R})$. Accordingly, let $\overline{\mathscr{Y}} := \mathscr{Y} \cup \{(\partial, \infty)\}$ be the one-point completion of $\mathscr{Y}$, and $\mathcal{B}(\overline{\mathscr{Y}}) := \sigma(\mathcal{B}(\mathscr{Y}) \cup \{(\partial, \infty)\})$. Moreover, we set $\mathscr{Z} := \mathbb{R}_+ \times \mathscr{Y} = \mathscr{X} \times \mathbb{R}$ and $\overline{\mathscr{Z}} := \mathscr{Z} \cup \{(\infty, \partial, \infty)\}$.

Let $\widehat{\Omega}$ be the set of càdlàg functions $\widehat{\omega}$ on $\mathbb{R}_+$ taking values in $\mathscr{Y}$. We define $\widehat{\omega}(\infty) := (\partial, \infty)$ for every $\widehat{\omega} \in \widehat{\Omega}$. As shown in [BCCG20, Appendix A], one can construct a *standard* canonical time-inhomogeneous Markov family (cf. [GS04, Definition I.6.6])

$$\widehat{\mathcal{M}} := \big\{ \big(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{\mathbb{F}}_s, (\widehat{X}_t, \widehat{\varphi}_t)_{t \in [s,\infty]}, \widehat{\mathbb{P}}_{s,(i,a)} \big), \, (s,i,a) \in \overline{\mathscr{Z}} \big\}$$

with transition function $\widehat{P}$ given by

$$\widehat{P}(s,(i,a),t,A) := \mathbb{P}_{s,i} \bigg(\bigg(X_t, \, a + \int_s^t v\big(X_u\big)\, du \bigg) \in A \bigg), \tag{4.5}$$

where $(s,i,a) \in \overline{\mathscr{Z}}$, $t \in [s,\infty]$, and $A \in \mathcal{B}(\overline{\mathscr{Y}})$. Note above $\widehat{X}_t$ takes values in $\overline{\mathbf{E}}$ and $\widehat{\varphi}_t$ takes values in $\mathbb{R}$.

The lemma below reveals the probabilistic relationship between $\widehat{\mathcal{M}}$ and $\mathcal{M}$. The proof can be found in [BCCG20, Lemma A3].

Lemma 4.2. *For any $0 \le s \le t < \infty$, let $\widehat{\Omega}_{s,t}$ be the collection of all càdlàg functions on $[s,t]$ taking values in $\mathscr{Y}$. Let $\widehat{\mathscr{G}}_{s,t}$ be the cylindrical σ-field on $\widehat{\Omega}_{s,t}$ generated by $(\widehat{X}_u, \widehat{\varphi}_u)_{u \in [s,t]}$. Then, for any $a \in \mathbb{R}$ and $C \in \widehat{\mathscr{G}}_{s,t}$,*

$$\widehat{\mathbb{P}}_{s,(i,a)} \big(\big(\widehat{X}_r, \widehat{\varphi}_r \big)_{r \in [s,t]} \in C \big) = \mathbb{P}_{s,i} \bigg(\bigg(X_r, \, a + \int_s^r v\big(X_u\big)\, du \bigg)_{r \in [s,t]} \in C \bigg). \tag{4.6}$$

As an immediate consequence of Lemma 4.2, $\widehat{\mathcal{M}}$ has the following properties:

(i) for any $(s,i,a) \in \overline{\mathscr{Z}}$,

$$\text{the law of } \widehat{X} \text{ under } \widehat{\mathbb{P}}_{s,(i,a)} \;=\; \text{the law of } X \text{ under } \mathbb{P}_{s,i}; \tag{4.7}$$

(ii) for any $(s,i,a) \in \mathscr{Z}$,

$$\widehat{\mathbb{P}}_{s,(i,a)} \bigg(\widehat{\varphi}_t = a + \int_s^t v(\widehat{X}_u)\, du, \quad \text{for all } t \in [s,\infty) \bigg) = 1. \tag{4.8}$$

Considering the standard Markov family $\widehat{\mathcal{M}}$, for any $s, \ell \in \mathbb{R}$ with $-\ell^- \le \ell^+$, we define the following auxiliary first passage time

$$\widehat{\tau}^+_\ell(s) := \inf \big\{ t \in [s,\infty) : \, \widehat{\varphi}_t > \ell \big\}, \quad \widehat{\tau}^-_\ell(s) := \inf \big\{ t \in [s,\infty) : \, \widehat{\varphi}_t < -\ell \big\}.$$

For any $s, \ell^\pm \in \mathbb{R}$ with $-\ell^- \le \ell^+$, we also define the two-sided exit time

$$\widehat{\xi}^+_{\ell^-,\ell^+}(s) := \inf \big\{ t \in \big[s, \widehat{\tau}^-_{\ell^-}(s)\big) : \widehat{\varphi}_t > \ell^+ \big\}, \quad \widehat{\xi}^-_{\ell^-,\ell^+}(s) := \inf \big\{ t \in \big[s, \widehat{\tau}^+_{\ell^+}(s)\big) : \widehat{\varphi}_t < -\ell^- \big\},$$

The above are $\widehat{\mathbb{F}}_s$-stopping times in light of the continuity of $\widehat{\varphi}$ and the right-continuity of the filtration $\widehat{\mathbb{F}}_s$. If no confusion arises, we will omit the s in $\widehat{\tau}^\pm_\ell(s)$ and $\widehat{\xi}^\pm_{\ell^-,\ell^+}(s)$.

By (4.8) and (4.7), for $a \le \ell$, we have

$$
\begin{aligned}
\widehat{\mathbb{E}}_{s,(i,a)}\Big(g^+\Big(\widehat{\tau}^+_\ell, \widehat{X}_{\widehat{\tau}^+_\ell}\Big)\Big) &= \widehat{\mathbb{E}}_{s,(i,a)}\bigg(g^+\bigg(\inf\Big\{u\ge s: a+\int_s^u v(\widehat{X}_r)dr>\ell\Big\}, \widehat{X}_{\inf\big\{u\ge s:\, a+\int_s^u v(\widehat{X}_r)dr>\ell\big\}}\bigg)\bigg) \\
&= \mathbb{E}_{s,i}\bigg(g^+\bigg(\inf\Big\{u\ge s: \int_s^u v\big(X_r\big)dr > \ell - a\Big\}, X_{\inf\big\{u\ge s:\int_s^u v(X_r)dr>\ell-a\big\}}\bigg)\bigg) \\
&= \mathbb{E}_{s,i}\Big(g^+\Big(\tau^+_{\ell-a},\, X_{\tau^+_{\ell-a}}\Big)\Big).
\end{aligned}
\tag{4.9}
$$

Similarly, for $a, \ell^\pm \in \mathbb{R}$ satisfying $-\ell^- \le a \le \ell^+$,

$$
\widehat{\mathbb{E}}_{s,(i,a)}\Big(g^+\Big(\widehat{\xi}^+_{\ell^-,\ell^+}, \widehat{X}_{\widehat{\xi}^+_{\ell^-,\ell^+}}\Big)\Big) = \mathbb{E}_{s,i}\Big(g^+\Big(\xi^+_{\ell^-+a,\ell^+-a}, X_{\xi^+_{\ell^-+a,\ell^+-a}}\Big)\Big). \tag{4.10}
$$

Equalities (4.9) and (4.10) provide useful representations of the expectation under $\mathbb{P}$. We will need still another representation of this expectation. Towards this end, we will first transform the time-inhomogeneous Markov family $\mathcal{M}$ into a *time-homogeneous* Markov family

$$
\widetilde{\mathcal{M}} = \big\{\big(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{F}}, (Z_t)_{t\in\overline{\mathbb{R}}_+}, (\theta_r)_{r\in\mathbb{R}_+}, \widetilde{\mathbb{P}}_z\big), z \in \overline{\mathscr{Z}}\big\}
$$

following the setup in [Bot14]. The construction of $\widetilde{\mathcal{M}}$ proceeds as follows.

- We let $\widetilde{\Omega} := \overline{\mathbb{R}}_+ \times \Omega$ to be the new sample space, with elements $\widetilde{\omega} = (s, \widehat{\omega})$, where $s \in \overline{\mathbb{R}}_+$ and $\widehat{\omega} \in \widehat{\Omega}$. On $\widetilde{\Omega}$ we consider the σ-field

$$
\widetilde{\mathscr{F}} := \Big\{\widetilde{A} \subset \widetilde{\Omega}:\ \widetilde{A}_s \in \widehat{\mathscr{F}}^s_\infty \text{ for any } s \in \overline{\mathbb{R}}_+\Big\},
$$

  where $\widetilde{A}_s := \{\widehat{\omega} \in \widehat{\Omega}:\ (s,\widehat{\omega}) \in \widetilde{A}\}$ and $\widehat{\mathscr{F}}^s_\infty$ is the last element in $\widehat{\mathbb{F}}_s$ (the filtration in $\mathcal{M}$).

- We let $\overline{\mathscr{Z}} = \mathscr{Z} \cup \{(\infty, \partial, \infty)\}$ to be the new state space, where $\mathscr{Z} = \mathbb{R}_+ \times \mathscr{Y} = \mathscr{X} \times \mathbb{R}$, with elements $z = (s, i, a)$. On $\mathscr{Z}$ we consider the σ-field

$$
\widetilde{\mathcal{B}}(\mathscr{Z}) := \Big\{\widetilde{B} \subset \mathscr{Z}:\ \widetilde{B}_s \in \mathcal{B}(\mathscr{Y}) \text{ for any } s \in \mathbb{R}_+\Big\},
$$

  where $\widetilde{B}_s := \big\{(i,a) \in \mathscr{Y}:\ (s,i,a) \in \widetilde{B}\big\}$. Let $\widetilde{\mathcal{B}}(\overline{\mathscr{Z}}) := \sigma(\widetilde{\mathcal{B}}(\mathscr{Z}) \cup \{(\infty,\partial,\infty)\})$.

- We consider a family of probability measures $(\widetilde{\mathbb{P}}_z)_{z\in\overline{\mathscr{Z}}}$, where, for $z = (s,i,a) \in \overline{\mathscr{Z}}$,

$$
\widetilde{\mathbb{P}}_z\big(\widetilde{A}\big) = \widetilde{\mathbb{P}}_{s,i,a}\big(\widetilde{A}\big) := \widehat{\mathbb{P}}_{s,(i,a)}\big(\widetilde{A}_s\big), \quad \widetilde{A} \in \widetilde{\mathscr{F}}. \tag{4.11}
$$

- We consider the process $Z := (Z_t)_{t\in\overline{\mathbb{R}}_+}$ on $(\widetilde{\Omega}, \widetilde{\mathscr{F}})$, where, for $t \in \overline{\mathbb{R}}_+$,

$$
Z_t(\widetilde{\omega}) := \big(s+t, \widehat{X}_{s+t}(\widehat{\omega}), \widehat{\varphi}_{s+t}(\widehat{\omega})\big), \quad \widetilde{\omega} = (s,\widehat{\omega}) \in \widetilde{\Omega}. \tag{4.12}
$$

  Hereafter, we denote the three components of Z by Z^1, Z^2, and Z^3, respectively.

- On $(\widetilde{\Omega}, \widetilde{\mathscr{F}})$, we define $\widetilde{\mathbb{F}} := (\widetilde{\mathscr{F}}_t)_{t\in\overline{\mathbb{R}}_+}$, where $\widetilde{\mathscr{F}}_t := \widetilde{\mathscr{G}}_{t+}$ (with the convention $\widetilde{\mathscr{G}}_{\infty+} = \widetilde{\mathscr{G}}_\infty$), and $(\widetilde{\mathscr{G}}_t)_{t\in\overline{\mathbb{R}}_+}$ is the completion of the natural filtration generated by $(Z_t)_{t\in\overline{\mathbb{R}}_+}$ with respect to the set of probability measures $\{\widetilde{\mathbb{P}}_z, z \in \overline{\mathscr{Z}}\}$ (cf. [GS04, Chapter I]).

- Finally, for any $r \in \mathbb{R}_+$, we consider the shift operator $\theta_r : \widetilde{\Omega} \to \widetilde{\Omega}$ defined by

$$\theta_r \widetilde{\omega} = (u + r, \omega_{\cdot + r}), \quad \widetilde{\omega} = (u, \omega) \in \widetilde{\Omega}.$$

It follows that $Z_t \circ \theta_r = Z_{t+r}$, for any $t, r \in \mathbb{R}_+$.

For $z = (s, i, a) \in \overline{\mathscr{Z}}$, $t \in \overline{\mathbb{R}}_+$, and $\widetilde{B} \in \mathcal{B}(\overline{\mathscr{Z}})$, we define the transition function $\widetilde{P}$ by

$$\widetilde{P}\big(z, t, \widetilde{B}\big) := \widetilde{\mathbb{P}}_z\big(Z_t \in \widetilde{B}\big).$$

In view of (4.11), we have

$$\widetilde{P}\big(z, t, \widetilde{B}\big) = \mathbb{P}_{s,(i,a)}\Big((\widehat{X}_{t+s}, \widehat{\varphi}_{t+s}) \in \widetilde{B}_{s+t}\Big) = P\big(s, (i, a), s + t, \widetilde{B}_{s+t}\big). \tag{4.13}$$

It can be shown that the transition function $\widehat{P}$, defined in (4.5), is associated with a Feller semigroup, so that $\widehat{P}$ is a Feller transition function. This and [Bot14, Theorem 3.2] imply that $\widetilde{P}$ is also a Feller transition function. In light of the right continuity of the sample paths, and invoking [GS04, Theorem I.4.7], we conclude that $\widetilde{\mathcal{M}}$ is a *time-homogeneous strong* Markov family.

In light of (4.8), (4.11), and (4.12), for any $(s, i, a) \in \mathscr{Z}$, we have

$$\widetilde{\mathbb{P}}_{s,i,a}\Big(Z_t^3 = a + \int_0^t v\big(Z_u^2\big)\, du, \text{ for all } t \in \mathbb{R}_+\Big) = 1. \tag{4.14}$$

For any $\ell \in \mathbb{R}$, we define the auxiliary first passage time

$$\widetilde{\tau}_\ell^+ := \inf\big\{t \in \overline{\mathbb{R}}_+ : Z_t^3 > \ell\big\}, \quad \widetilde{\tau}_\ell^- := \inf\big\{t \in \overline{\mathbb{R}}_+ : Z_t^3 < -\ell\big\}.$$

Let $-\ell^- \le \ell^+$. We define two auxiliary constrained passage times for $\widetilde{\mathcal{M}}$ as

$$\widetilde{\xi}^+_{\ell^-,\ell^+} := \inf\big\{t \in [0, \widetilde{\tau}^-_{-\ell^-}) : Z_t^3 > \ell^+\big\}, \quad \widetilde{\xi}^-_{\ell^-,\ell^+} := \inf\big\{t \in [0, \widetilde{\tau}^+_{\ell^+}) : Z_t^3 < -\ell^-\big\}, \tag{4.15}$$

which are $\widetilde{\mathbb{F}}$-stopping times since Z^3 has continuous sample paths and $\widetilde{\mathbb{F}}$ is right-continuous. By (4.9), (4.10), (4.11), (4.12) and (4.14), for any $g^+ \in \mathcal{B}_b(\overline{\mathscr{X}_+})$, $(s, i, a) \in \mathscr{Z}$, and $\ell \in [a, \infty)$,

$$\mathbb{E}_{s,i}\bigg(g^+\Big(\tau^+_{\ell-a}, X_{\tau^+_{\ell-a}}\Big)\bigg) = \widetilde{\mathbb{E}}_{s,i,a}\Big(g^+\Big(Z^1_{\widetilde{\tau}^+_\ell}, Z^2_{\widetilde{\tau}^+_\ell}\Big)\Big), \tag{4.16}$$

which, in particular, implies that

$$\widetilde{\mathbb{E}}_{s,i,a}\Big(g^+\Big(Z^1_{\widetilde{\tau}^+_\ell}, Z^2_{\widetilde{\tau}^+_\ell}\Big)\Big) = \widetilde{\mathbb{E}}_{s,i,0}\Big(g^+\Big(Z^1_{\widetilde{\tau}^+_{\ell-a}}, Z^2_{\widetilde{\tau}^+_{\ell-a}}\Big)\Big). \tag{4.17}$$

Similarly, for any $g^\pm \in \mathcal{B}_b(\overline{\mathscr{X}_\pm})$, $(s, i, a) \in \mathscr{Z}$ and $-\ell^- \le a \le \ell^+$,

$$\mathbb{E}_{s,i}\Big(g^+\Big(\xi^+_{\ell^-+a,\ell^+-a}, X_{\xi^+_{\ell^-+a,\ell^+-a}}\Big)\Big) = \widetilde{\mathbb{E}}_{s,i,a}\Big(g^+\Big(Z^1_{\widetilde{\xi}^+_{\ell^-,\ell^+}}, Z^2_{\widetilde{\xi}^+_{\ell^-,\ell^+}}\Big)\Big). \tag{4.18}$$

and

$$\mathbb{E}_{s,i}\Big(g^-\Big(\xi^-_{\ell^-+a,\ell^+-a}, X_{\xi^-_{\ell^-+a,\ell^+-a}}\Big)\Big) = \widetilde{\mathbb{E}}_{s,i,a}\Big(g^-\Big(Z^1_{\widetilde{\xi}^-_{\ell^-,\ell^+}}, Z^2_{\widetilde{\xi}^-_{\ell^-,\ell^+}}\Big)\Big). \tag{4.19}$$

We conclude this section with the following lemma. It is an exact adaption of [BCCG20, Lemma 4.2], and the proof is therefore omitted here.

Lemma 4.3. *Let $\widetilde{\tau}$ be any $\widetilde{\mathbb{F}}$-stopping time, and $g^+ \in \mathcal{B}_b(\overline{\mathscr{X}_+})$. Then, for any $(s, i, a) \in \overline{\mathscr{Z}}$ and $\ell \in [a, \infty)$, we have*

$$\mathbb{1}_{\{\widetilde{\tau} \le \widetilde{\tau}^+_\ell\}} \widetilde{\mathbb{E}}_{s,i,a}\Big(g^+\Big(Z^1_{\widetilde{\tau}^+_\ell}, Z^2_{\widetilde{\tau}^+_\ell}\Big) \,\Big|\, \widetilde{\mathscr{F}}_{\widetilde{\tau}}\Big) = \mathbb{1}_{\{\widetilde{\tau} \le \widetilde{\tau}^+_\ell\}} \widetilde{\mathbb{E}}_{Z^1_{\widetilde{\tau}}, Z^2_{\widetilde{\tau}}, Z^3_{\widetilde{\tau}}}\Big(g^+\Big(Z^1_{\widetilde{\tau}^+_\ell}, Z^2_{\widetilde{\tau}^+_\ell}\Big)\Big), \quad \widetilde{\mathbb{P}}_{s,i,a} - a.\,s.. \tag{4.20}$$

4.3 Proof of Theorem 3.2

Let

$$\widetilde{\eta}^{+}_{\ell^-,\ell^+} := \inf\left\{t \ge \widetilde{\xi}^{-}_{\ell^-,\ell^+} : Z^3_t > \ell^+\right\}. \tag{4.21}$$

The auxillary time $\widetilde{\eta}^{-}_{\ell^-,\ell^+}$ will be used to prove (3.8). Since the proof of (3.9) can be done in an analogous way, we do not introduce the 'minus' counterpart of $\widetilde{\eta}^{+}_{\ell^-,\ell^+}$. We first establish an important observation.

Lemma 4.4. *For any $g^+ \in \mathcal{B}_b(\overline{\mathscr{X}}_+)$ and $-\ell^- \le 0 \le \ell^+$, we have*

$$\widetilde{\mathbb{E}}_{s,i,0}\left(\mathbb{1}_{\{\widetilde{\xi}^{-}_{\ell^-,\ell^+}<\infty\}}\, g^+\left(Z^1_{\widetilde{\eta}^{+}_{\ell^-,\ell^+}}, Z^2_{\widetilde{\eta}^{+}_{\ell^-,\ell^+}}\right)\,\middle|\, \widetilde{\mathscr{F}}_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}\right) = \left(J^+\mathcal{P}^+_{\ell^++\ell^-}g^+\right)\left(Z^1_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}, Z^2_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}\right).$$

Proof. Observe that

$$\mathbb{1}_{\{\widetilde{\xi}^{-}_{\ell^-,\ell^+}<\infty\}}\, g^+\left(Z^1_{\widetilde{\eta}^{+}_{\ell^-,\ell^+}}, Z^2_{\widetilde{\eta}^{+}_{\ell^-,\ell^+}}\right) = \mathbb{1}_{\{\widetilde{\xi}^{-}_{\ell^-,\ell^+}<\infty\}}\, g^+\left(Z^1_{\widetilde{\tau}^{+}_{\ell^+}}, Z^2_{\widetilde{\tau}^{+}_{\ell^+}}\right).$$

In addition,

$$\left\{\widetilde{\xi}^{-}_{\ell^-,\ell^+} < \infty\right\} = \bigcup_{n\in\mathbb{N}}\left\{\widetilde{\xi}^{-}_{\ell^-,\ell^+} < n\right\} \in \widetilde{\mathscr{F}}_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}} \quad \text{and} \quad \left\{\widetilde{\xi}^{-}_{\ell^-,\ell^+} < \infty\right\} \subseteq \left\{\widetilde{\xi}^{-}_{\ell^-,\ell^+} \le \widetilde{\tau}^{+}_{\ell^+}\right\}.$$

The above together with Lemma 4.3 implies that

$$\begin{aligned}
&\widetilde{\mathbb{E}}_{s,i,0}\left(\mathbb{1}_{\left\{\widetilde{\xi}^{-}_{\ell^-,\ell^+}<\infty\right\}}\, g^+\left(Z^1_{\widetilde{\eta}^{+}_{\ell^-,\ell^+}}, Z^2_{\widetilde{\eta}^{+}_{\ell^-,\ell^+}}\right)\,\middle|\, \widetilde{\mathscr{F}}_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}\right)\\
&\quad = \mathbb{1}_{\left\{\widetilde{\xi}^{-}_{\ell^-,\ell^+}<\infty\right\}}\mathbb{1}_{\left\{\widetilde{\xi}^{-}_{\ell^-,\ell^+}\le\widetilde{\tau}^{+}_{\ell^+}\right\}}\widetilde{\mathbb{E}}_{s,i,0}\left(g^+\left(Z^1_{\widetilde{\tau}^{+}_{\ell^+}}, Z^2_{\widetilde{\tau}^{+}_{\ell^+}}\right)\,\middle|\, \widetilde{\mathscr{F}}_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}\right)\\
&\quad = \mathbb{1}_{\left\{\widetilde{\xi}^{-}_{\ell^-,\ell^+}<\infty\right\}}\mathbb{1}_{\left\{\widetilde{\xi}^{-}_{\ell^-,\ell^+}\le\widetilde{\tau}^{+}_{\ell^+}\right\}}\widetilde{\mathbb{E}}_{Z^1_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}, Z^2_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}, Z^3_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}}\left(g^+\left(Z^1_{\widetilde{\tau}^{+}_{\ell^+}}, Z^2_{\widetilde{\tau}^{+}_{\ell^+}}\right)\right)\\
&\quad = \mathbb{1}_{\left\{\widetilde{\xi}^{-}_{\ell^-,\ell^+}<\infty\right\}}\widetilde{\mathbb{E}}_{Z^1_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}, Z^2_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}, Z^3_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}}\left(g^+\left(Z^1_{\widetilde{\tau}^{+}_{\ell^+}}, Z^2_{\widetilde{\tau}^{+}_{\ell^+}}\right)\right).
\end{aligned}$$

Notice that under $\{\widetilde{\xi}^{-}_{\ell^-,\ell^+} < \infty\}$, $Z^3_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}} = -\ell^-$ thanks to the continuous sample path of Z^3. By (4.17), (4.16), and Proposition 2.3, we have

$$\begin{aligned}
&\mathbb{1}_{\{\widetilde{\xi}^{-}_{\ell^-,\ell^+}<\infty\}}\widetilde{\mathbb{E}}_{Z^1_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}, Z^2_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}, Z^3_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}}\left(g^+\left(Z^1_{\widetilde{\tau}^{+}_{\ell^+}}, Z^2_{\widetilde{\tau}^{+}_{\ell^+}}\right)\right)\\
&\quad = \mathbb{1}_{\{\widetilde{\xi}^{-}_{\ell^-,\ell^+}<\infty\}}\widetilde{\mathbb{E}}_{Z^1_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}, Z^2_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}, 0}\left(g^+\left(Z^1_{\widetilde{\tau}^{+}_{\ell^-+\ell^+}}, Z^2_{\widetilde{\tau}^{+}_{\ell^-+\ell^+}}\right)\right) = \left(J^+\mathcal{P}^+_{\ell^-+\ell^+}g^+\right)\left(Z^1_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}, Z^2_{\widetilde{\xi}^{-}_{\ell^-,\ell^+}}\right).
\end{aligned}$$

The proof is complete. $\square$

We are ready to prove Theorem 3.2.

Proof of Theorem 3.2. We will only present the proof for Ξ^+ as the proof for Ξ^- follows analogously. To start with, in view of (4.15), $\{\widetilde{\xi}^+_{\ell^-,\ell^+}<\infty\}, \{\widetilde{\xi}^-_{\ell^-,\ell^+}<\infty\}$, and $\{\widetilde{\tau}^+_{\ell^+}\wedge\widetilde{\tau}^-_{\ell^-}=\infty\}$ form a disjoint partition of $\widetilde{\Omega}$. It follows that

$$\widetilde{\tau}^+_{\ell^+}=\widetilde{\xi}^+_{\ell^-,\ell^+}\mathbb{1}_{\{\widetilde{\xi}^+_{\ell^-,\ell^+}<\infty\}}+\widetilde{\eta}^+_{\ell^-,\ell^+}\mathbb{1}_{\{\widetilde{\xi}^-_{\ell^-,\ell^+}<\infty\}}+\infty\mathbb{1}_{\{\widetilde{\tau}^+_{\ell^+}\wedge\widetilde{\tau}^-_{\ell^-}=\infty\}}, \tag{4.22}$$

where we recall the definition of $\widetilde{\eta}^+_{\ell^-,\ell^+}$ from (4.21). Therefore, for $g^+\in\mathcal{B}(\overline{\mathscr{X}_+})$,

$$\begin{aligned}\widetilde{\mathbb{E}}_{s,i,0}\left(g^+\left(Z^1_{\widetilde{\tau}^+_{\ell^+}},Z^2_{\widetilde{\tau}^+_{\ell^+}}\right)\right)&=\widetilde{\mathbb{E}}_{s,i,0}\left(\mathbb{1}_{\{\widetilde{\xi}^+_{\ell^-,\ell^+}<\infty\}}\,g^+\left(Z^1_{\widetilde{\xi}^+_{\ell^-,\ell^+}},Z^2_{\widetilde{\xi}^+_{\ell^-,\ell^+}}\right)\right)\\&\quad+\widetilde{\mathbb{E}}_{s,i,0}\left(\mathbb{1}_{\{\widetilde{\xi}^-_{\ell^-,\ell^+}<\infty\}}\,g^+\left(Z^1_{\widetilde{\eta}^+_{\ell^-,\ell^+}},Z^2_{\widetilde{\eta}^+_{\ell^-,\ell^+}}\right)\right).\end{aligned}\tag{4.23}$$

For the left hand side of (4.23), due to (4.16) and Proposition 2.3, we have

$$\widetilde{\mathbb{E}}_{s,i,0}\left(g^+\left(Z^1_{\widetilde{\tau}^+_{\ell^+}},Z^2_{\widetilde{\tau}^+_{\ell^+}}\right)\right)=\left(\begin{pmatrix}I^+\\J^+\end{pmatrix}\mathcal{P}^+_{\ell^+}g^+\right)(s,i),\quad (s,i)\in\mathscr{X}. \tag{4.24}$$

For the first term in the right hand of (4.23), due to (4.18) and (3.2), we have

$$\widetilde{\mathbb{E}}_{s,i,0}\left(\mathbb{1}_{\{\widetilde{\xi}^+_{\ell^-,\ell^+}<\infty\}}\,g^+\left(Z^1_{\widetilde{\xi}^+_{\ell^-,\ell^+}},Z^2_{\widetilde{\xi}^+_{\ell^-,\ell^+}}\right)\right)=\left(\Xi^+_{\ell^-,\ell^+}g^+\right)(s,i),\quad (s,i)\in\mathscr{X}. \tag{4.25}$$

For the second term in the right hand side of (4.23), due to Lemma 4.4, (4.19) and (3.3), we have

$$\widetilde{\mathbb{E}}_{s,i,0}\left(\mathbb{1}_{\{\widetilde{\xi}^-_{\ell^-,\ell^+}<\infty\}}\,g^+\left(Z^1_{\widetilde{\eta}^-_{\ell^-,\ell^+}},Z^2_{\widetilde{\eta}^-_{\ell^-,\ell^+}}\right)\right)=\left(\Xi^-_{\ell^-,\ell^+}J^+\mathcal{P}^+_{\ell^++\ell^-}g^+\right)(s,i),\quad (s,i)\in\mathscr{X}. \tag{4.26}$$

Note also that (4.24), (4.24), and (4.24) remains true for $(s,i)=(\infty,\partial)$ as both hand sides vanish at coffin state. By combining (4.23)-(4.26), we yield

$$\begin{pmatrix}I^+\\J^+\end{pmatrix}\mathcal{P}^+_{\ell^+}g^+=\Xi^+_{\ell^-,\ell^+}g^++\Xi^-_{\ell^-,\ell^+}J^+\mathcal{P}^+_{\ell^++\ell^-}g^+. \tag{4.27}$$

By analogue, for $g^-\in\mathcal{B}_b(\overline{\mathscr{X}_-})$ we also have

$$\begin{pmatrix}J^-\\I^-\end{pmatrix}\mathcal{P}^-_{\ell^-}g^-=\Xi^-_{\ell^-,\ell^+}g^-+\Xi^+_{\ell^-,\ell^+}J^-\mathcal{P}^-_{\ell^-+\ell^+}g^-. \tag{4.28}$$

Substituting $J^+\mathcal{P}^+_{\ell^-+\ell^+}g^+$ for g^- in (4.28) then subtracting (4.28) from (4.27), we yield

$$\left(\begin{pmatrix}I^+\\J^+\end{pmatrix}\mathcal{P}^+_{\ell^+}-\begin{pmatrix}J^-\\I^-\end{pmatrix}\mathcal{P}^-_{\ell^-}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)g^+=\Xi^+_{\ell^-,\ell^+}\left(I^+-J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)g^+. \tag{4.29}$$

Below we point out a useful observation, that is, for $N\in\mathbb{N}$ we have

$$\left(I^+-J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)\sum_{n=0}^{N}\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)^ng^+=g^+-\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)^{N+1}g^+.$$

By Lemma 3.1, letting $N\to\infty$, we yield

$$\left(I^+-J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)\sum_{k=0}^{\infty}\left(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+}\right)^ng^+=g^+,\quad \ell^\pm\in\mathbb{R}_+. \tag{4.30}$$

Finally, in view of (4.29) and (4.30), substituting $\sum_{k=0}^{\infty}(J^-\mathcal{P}^-_{\ell^-+\ell^+}J^+\mathcal{P}^+_{\ell^-+\ell^+})^kg^+$ for g^+ in(4.29), the proof is complete. □

4.4 Proof of Proposition 3.5

We present below the proof of Proposition 3.5.

Proof of Proposition 3.5. Notice that

$$\mathbb{E}_{s,i}\left(h(X_T)\mathbb{1}_{\left\{\tau^-_{\ell-}\wedge\tau^+_{\ell+}\le T\right\}}\right)=\mathbb{E}_{s,i}\left(\mathbb{E}_{s,i}\left(h(X_T)\mathbb{1}_{\left\{\tau^-_{\ell-}\wedge\tau^+_{\ell+}\le T\right\}}\,\middle|\,\mathscr{F}_{\tau^-_{\ell-}\wedge\tau^+_{\ell+}\wedge T}\right)\right).$$

Because

$$\left\{\tau^-_{\ell-}(s)\wedge\tau^+_{\ell+}(s)\le T\right\}\cap\left\{\tau^-_{\ell-}(s)\wedge\tau^+_{\ell+}(s)\wedge T\le t\right\}=\begin{cases}\left\{\tau^-_{\ell-}(s)\wedge\tau^+_{\ell+}(s)\le t\right\}\in\mathscr{F}_t, & t<T\\ \left\{\tau^-_{\ell-}(s)\wedge\tau^+_{\ell+}(s)\le T\right\}\in\mathscr{F}_t, & t\ge T\end{cases},$$

we have $\mathbb{1}_{\{\tau^-_{\ell-}(s)\wedge\tau^+_{\ell+}(s)\le T\}}$ is $\mathscr{F}_{\tau^-_{\ell-}(s)\wedge\tau^+_{\ell+}(s)\wedge T}$-measurable. Therefore,

$$\mathbb{E}_{s,i}\left(h(X_T)\mathbb{1}_{\left\{\tau^-_{\ell-}\wedge\tau^+_{\ell+}\le T\right\}}\right)=\mathbb{E}_{s,i}\left(\mathbb{1}_{\left\{\tau^-_{\ell-}\wedge\tau^+_{\ell+}\le T\right\}}\mathbb{E}_{s,i}\left(h(X_T)\,\middle|\,\mathscr{F}_{\tau^-_{\ell-}\wedge\tau^+_{\ell+}\wedge T}\right)\right).$$

By [GS04, Theorem I.4.6],

$$\begin{aligned}\mathbb{E}_{s,i}\left(h(X_T)\mathbb{1}_{\left\{\tau^-_{\ell-}\wedge\tau^+_{\ell+}\le T\right\}}\right)&=\mathbb{E}_{s,i}\left(\mathbb{1}_{\left\{\tau^-_{\ell-}\wedge\tau^+_{\ell+}\le T\right\}}\mathbb{E}_{\tau^-_{\ell-}\wedge\tau^+_{\ell+}\wedge T,X_{\tau^-_{\ell-}\wedge\tau^+_{\ell+}\wedge T}}(h(X_T))\right)\\&=\mathbb{E}_{s,i}\left(\mathbb{1}_{[0,T]}\left(\tau^-_{\ell-}\wedge\tau^+_{\ell+}\right)\mathbb{E}_{\tau^-_{\ell-}\wedge\tau^+_{\ell+},X_{\tau^-_{\ell-}\wedge\tau^+_{\ell+}}}(h(X_T))\right).\end{aligned}$$

In view of (3.11) and (3.5), the proof is complete. □

5 Concluding remarks and future work

In this paper, we have shown that certain expectation associated with two-sided exit time can be expressed in terms of expectation operator of one-sided exit time. Our proof is based on the probabilistic decomposition such as (4.22) and its analytic counterpart (4.28). This decomposition emerges from the Markov property of an appropriately extended time-space process, introduced in Section 4.2. We would like to highlight that Lemma 3.1 provides a crucial regularity that aids our proof. We conjecture that an analogous regularity holds true in the presence of additive Brownian noise (potentially also driven by the Markov chain), i.e.,

$$\phi_t(s)=\int_s^t v(X_u)du+\int_s^t\sigma(X_u)dW_u,$$

where X and W are independent. Consequently, a similar methodology can be employed to study the expectation associated with the two-sided exit time in this scenario.

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