Finite time mixing and enhanced dissipation for 2D Navier-Stokes equations by Ornstein–Uhlenbeck flow

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Abstract

We consider the vorticity form of 2D Navier-Stokes equations perturbed by an Ornstein–Uhlenbeck flow of transport type. Contrary to previous works where the random perturbation was interpreted as Stratonovich transport noise, here we understand the equation in a pathwise manner and show the properties of mixing and enhanced dissipation for suitable choice of the flow.

Keywords: 2D Navier-Stokes equation, Ornstein–Uhlenbeck process, mixing, dissipation enhancement

1 Introduction

Let $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional (2D) torus; we consider the vorticity form of 2D Navier-Stokes equations on $\mathbb{T}^2$, perturbed by a smooth transport term:

$$\begin{cases} \partial_t \xi + u \cdot \nabla \xi + \boldsymbol{b} \cdot \nabla \xi = \kappa \Delta \xi, \\ u = K * \xi, \quad \xi|_{t=0} = \xi_0, \end{cases} \tag{1.1}$$

where ξ and u are the fluid vorticity and velocity field, K being the Biot-Savart kernel on $\mathbb{T}^2$:

$$K * \xi := -\nabla^\perp (-\Delta)^{-1} \xi,$$

with $\nabla^\perp = (\partial_2, -\partial_1)$, $\partial_i = \frac{\partial}{\partial x_i}$. $\kappa > 0$ is a fixed small number representing molecular diffusivity, and $\boldsymbol{b} : [0, \infty) \times \mathbb{T}^2 \to \mathbb{R}^2$ is a random time-dependent and divergence free vector field, continuous in time and smooth in space variables, standing for the small-scale parts of fluid velocity, thus the term $\boldsymbol{b} \cdot \nabla \xi$ models the effects of fluid small scales on the larger component ξ. It is well known that for L^2-initial condition ξ_0, $\mathbb{P}$-a.s., equation (1.1) admits a unique weak solution satisfying the usual energy estimate. Our purpose is to study, under suitable conditions on $\boldsymbol{b}$, the properties of mixing and dissipation enhancement for the system (1.1).

The above model can be heuristically derived from the deterministic 2D Navier-Stokes equation by separating the fluid into large-scale components and small-scale ones, and modeling the corresponding small-scale velocity by a random field $\boldsymbol{b}$, see for instance [17, Section 1.2] for derivations in the 3D case and also [31, Section 1.2] for similar discussions on 2D Boussinesq

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systems. In these papers, the small-scale perturbations are interpreted as Stratonovich multiplicative noises of transport type, delta-correlated in time and colored in space, and thus the first equation in (1.1) is understood as a stochastic partial differential equation (SPDE); we refer to the recent works [8, 22, 23, 28] for more rigorous derivations of such stochastic fluid dynamics models. Indeed, the investigations of regularizing effects of multiplicative transport noise on various models began much earlier, see e.g. [9, 14, 15]. More recently, inspired by the scaling limit method introduced by Galeati [25], stochastic fluid equations with transport noise have been studied intensively, and it is by now well understood that transport noise produces eddy dissipation/viscosity under certain rescaling of spatial covariance, see for instance [7, 10, 13, 19, 20, 21, 31, 34]. Moreover, the larger the noise intensity, the stronger the additional viscous term in the limit equations; the extra strong viscosity can be used to suppress possible blow-up of various deterministic equations, yielding long-term (even global) existence of strong solutions with large probability, cf. [1, 11, 16, 17, 32]. In a sense, the above results can be regarded as partial verifications of Boussinesq's eddy viscosity hypothesis [6], which is one of the basis for large eddy simulation (see [5]).

However, noises with delta-correlation in time are just idealized approximations of real objects, and it is worthy of considering more practical perturbations, see [35, Section 4] for related discussions. As an attempt in this respect, Flandoli and Russo [24] showed that the dissipation properties of a stochastic transport term of fractional Brownian motion with Hurst parameter $H > 1/2$ are weaker than standard Brownian motion. In a slightly earlier work [36], Pappalettera studied the mixing and dissipation enhancement properties of Ornstein-Uhlenbeck flows for passive scalar on d-dimensional torus $\mathbb{T}^d$:

$$\partial_t h + \boldsymbol{b} \cdot \nabla h = \kappa \Delta h, \quad h|_{t=0} = h_0 \in L^2(\mathbb{T}^d). \tag{1.2}$$

The time-dependent vector field $\boldsymbol{b}$ takes the form

$$\boldsymbol{b}(t,x) = \sum_{j\in J} \boldsymbol{b}_j(x)\, \eta^{\alpha,j}(t),$$

where J is a finite index set, $\{\boldsymbol{b}_j\}_{j\in J}$ are divergence free vector fields on $\mathbb{T}^d$ and $\{\eta^{\alpha,j}\}_{j\in J}$ are independent real Ornstein-Uhlenbeck processes with covariance $\mathrm{Cov}(\eta^{\alpha,j}(t), \eta^{\alpha,j}(s)) = \frac{\alpha}{2}\exp(-\alpha|t-s|)$, $\alpha > 0$ being a parameter. As $\alpha \to \infty$, the covariance $\frac{\alpha}{2}\exp(-\alpha|t-s|)$ converges in distribution to the Dirac delta function, and thus $\eta^{\alpha,j}$ can be seen as approximations of the white noise. Assuming suitable conditions on the spatial properties of the family $\{\boldsymbol{b}_j\}_{j\in J}$, it is shown in [36] that the solution h is close, in negative Sobolev norms, to the solution of the deterministic equation

$$\partial_t \bar{h} = (\kappa\Delta + \mathcal{L})\bar{h}, \quad \bar{h}_0 = h_0, \tag{1.3}$$

where the second order differential operator $\mathcal{L}\bar{h} = \sum_j \boldsymbol{b}_j \cdot \nabla(\boldsymbol{b}_j \cdot \nabla \bar{h})$ stands for the enhanced dissipation. As mentioned above, the small-scale perturbations are understood in previous works (e.g. [10, 13, 25]) as Stratonovich transport noise, and thus the additional operator $\mathcal{L}$ arises naturally as Stratonovich-Itô corrector. Here, however, one has to compute the iterated integral of Ornstein-Uhlenbeck processes and to borrow some ideas from the proof of Wong-Zakai type results; see [23, Section 3] for related computations. There are also many other advanced and very sophisticated works on mixing and dissipation enhancement properties, using different methods from ergodic theory, see e.g. [3, 4, 26].

Motivated by [36], we aim at studying the properties of mixing and enhanced dissipation of Ornstein-Uhlenbeck flow $\boldsymbol{b}$ for 2D Navier-Stokes equations (1.1) in vorticity form. To overcome difficulties arising from the nonlinearity, we follow [10, 13, 25] and assume that the time-dependent vector field $\boldsymbol{b}$ takes the more precise form

$$\boldsymbol{b}(t,x) = 2\sqrt{\nu} \sum_{k\in\mathbb{Z}_0^2} \theta_k\, \sigma_k(x)\, \eta^{\alpha,k}(t),$$

where $\nu > 0$ is the intensity of perturbation, $\mathbb{Z}_0^2 = \mathbb{Z}^2\backslash\{0\}$ is the set of nonzero lattice points, and $\theta = \{\theta_k\}_{k\in\mathbb{Z}_0^2} \in \ell^2(\mathbb{Z}_0^2)$, the latter being the space of square summable sequences. We always assume that θ is a radial function of $k \in \mathbb{Z}_0^2$ with only finitely many nonzero components, and $\|\theta\|_{\ell^2} = (\sum_{k\in\mathbb{Z}_0^2} \theta_k^2)^{1/2} = 1$. The vector fields $\sigma_k(x) = \frac{k^\perp}{|k|} e^{2\pi i k\cdot x}$, where $k^\perp = (k_2, -k_1)$ and $k\cdot x = k_1x_1 + k_2x_2$, constitute a CONS of the space $L^2(\mathbb{T}^2, \mathbb{R}^2)$ of divergence free vector fields on $\mathbb{T}^2$ with zero mean, while $\eta^{\alpha,k}$ are independent Ornstein-Uhlenbeck processes as above. Thanks to the exact choice of $\boldsymbol{b}$, the additional operator $\mathcal{L}$ takes the much simpler form $\nu\Delta$ (see (5.3) below for related computations), and thus our purpose is to show that the solution ξ of (1.1) is close to that of the deterministic 2D Navier-Stokes equation with extra viscosity:

$$\begin{cases} \partial_t\bar\xi + \bar u\cdot\nabla\bar\xi = (\kappa+\nu)\Delta\bar\xi, \\ \bar u = K * \bar\xi, \quad \bar\xi|_{t=0} = \xi_0. \end{cases} \tag{1.4}$$

Note that $\bar\xi$ has a fast exponential decay in L^2-norm (and also in negative Sobolev norms) for large ν which comes from the intensity of noise.

To state more exactly our main results, we need some notation. For $s \in \mathbb{R}$, let $H^s = H^s(\mathbb{T}^2)$ be the usual Sobolev space on $\mathbb{T}^2$ endowed with the norm $\|\cdot\|_{H^s}$; we will write H^0 as L^2 and $\|\cdot\|_{H^0}$ as $\|\cdot\|_{L^2}$. Unless mentioned explicitly, we will use the same notation for spaces of functions and vector fields on $\mathbb{T}^2$. Since the equations (1.1) and (1.4) preserve the spatial average of solutions, we shall assume that the spaces H^s consist of functions of zero average. We write $\|\theta\|_{\ell^\infty}$ for the supremum norm of $\theta \in \ell^2(\mathbb{Z}_0^2)$. In the sequel, the notation $a \lesssim b$ means that $a \le Cb$ for some constant $C > 0$; if we want to emphasize the dependence of C on some parameters γ, p, then we write $a \lesssim_{\gamma,p} b$.

Here is the first main result of our work; it gives us a quantitative estimate on the distance, in terms of negative Sobolev norms, between the solutions ξ and $\bar\xi$. Since $\bar\xi$ has a much faster decay in such norms, we can regard the result as a mixing property of the Ornstein-Uhlenbeck flow $\boldsymbol{b}$, valid on finite time intervals.

Theorem 1.1. *Let $\xi_0 \in L^2(\mathbb{T}^2)$ and $\xi, \bar\xi$ be the unique solutions of* (1.1) *and* (1.4) *respectively. Then for any $\gamma \in (0, \frac{1}{3})$, $\vartheta > 0$ and $T \ge 1$, there exist $\zeta \in (0,1)$ and $\epsilon > 0$ such that for α sufficiently large, it holds*

$$\mathbb{E}\big[\|\xi - \bar\xi\|_{C([0,T],H^{-\vartheta})}\big] \le C_1\|\xi_0\|_{L^2} \exp\big(C_2\|\xi_0\|_{L^2}^2\big)\big(\nu^{1+\frac{\gamma}{2}}\alpha^{-\epsilon} + \nu^{\frac{1}{2}}\|\theta\|_{\ell^\infty}\big)^\zeta, \tag{1.5}$$

where $C_1 > 0$ is a constant depending on $\kappa, \nu, \zeta, \gamma, T$ and $C_2 > 0$ only depends on κ, ν, T.

We can make the right-hand of inequality (1.5) small by first choosing $\theta \in \ell^2(\mathbb{Z}_0^2)$ with small norm $\|\theta\|_{\ell^\infty}$ (see Examples 2.7 and 2.8 in Section 2 below), and then taking α big enough. This result is an analogue of [36, Theorem 1.1], where the author measured the closedness of solutions in the stronger Hölder space $C^\delta([0,1], H^{-\vartheta})$, $\delta > 0$. The key idea in the proof of [36, Theorem 1.1] is to express the difference $h - \bar h$ of solutions to (1.2) and (1.3) in terms of a random distribution f, see the beginning of [36, Section 4] or (4.1) below for a similar quantity. If f were differentiable in time, then $h - \bar h$ could be estimated using the mild expression involving f and an analytic semigroup; in the absence of time regularity on f, one needs to apply [27, Theorem 1] which can be thought of as a generalization of such estimates. In our case, we have to deal with the extra nonlinear terms in equations (1.1) and (1.4), thus we shall combine the above idea with the quantitative arguments developed in [12], and then apply the Gronwall lemma to get the desired estimate. Compared to [12, Theorem 1.1], the coefficient C_1 in (1.5) might explode as $\kappa \to 0$, thus we cannot prove a similar estimate for the 2D Euler equation.

Our second main result shows the phenomenon of dissipation enhancement.

Theorem 1.2. *Given $\lambda > 0$, $p \geq 1$ and $R > 0$, we can find parameters $\nu > 0$, $\alpha > 0$ and $\theta \in \ell^2$, such that for every ξ_0 with $\|\xi_0\|_{L^2} \leq R$, there exists a random constant $C = C(\omega) > 0$ with finite p-th moment, such that the solution of* (1.1) *satisfies the following exponential decay: $\mathbb{P}$-a.s.,*

$$\|\xi_t\|_{L^2} \leq C e^{-\lambda t} \|\xi_0\|_{L^2} \quad \text{for all } t \geq 0.$$

This theorem improves [36, Theorem 1.2] in two aspects: first, we deal with the nonlinear equation (1.1) rather than the linear heat equation (1.2); second, the enhanced exponential decay of $\|\xi_t\|_{L^2}$ is shown for all sufficiently large $t > 0$, instead of on a finite interval. We briefly discuss the key ingredients for proving the latter result. Note that the solution ξ to (1.1) is time homogeneous, due to the stationarity of the Ornstein-Uhlenbeck flow $\boldsymbol{b}$; combined with the estimate (1.5) restricted to the unit interval $[0, 1]$, we conclude easily that similar result, up to taking conditional expectation, holds on any interval $[n, n + 1]$ if equation (1.4) is restarted at time $t = n$ with the initial value ξ_n. As a consequence, we can show that $\mathbb{E}\|\xi_{n+1}\|_{L^2} \leq c_0 \mathbb{E}\|\xi_n\|_{L^2}$ where $c_0 > 0$ can be very small by choosing parameters ν, α and $\|\theta\|_{\ell^\infty}$ in a suitable way. Once we have such estimate, it is relatively standard to show the enhanced exponential decay; see Section 4.2 for the detailed proofs. We mention that the initial condition ξ_0 is restricted in a ball of arbitrary (but fixed) radius R; this is due to the nonlinearity of (1.1), see the end of [33, Section 2.1] for similar discussions.

We make some further comments on the differences between our methods and those in [36]. First, the main results of [36] are stated in dimension $d \geq 3$, while the corresponding 2D assertions are derived by assuming translation invariance in one direction, see the discussions in [36, Remark 2.2]. The reason is due to a technical constraint on the Sobolev indices for product of functions: if $\phi \in H^a(\mathbb{T}^d)$ and $\psi \in H^b(\mathbb{T}^d)$ with $a, b < d/2$ and $a + b > 0$, then one has $\phi\,\psi \in H^{a+b-d/2}$ and $\|\phi\,\psi\|_{H^{a+b-d/2}} \lesssim_{a,b,d} \|\phi\|_{H^a}\|\psi\|_{H^b}$, cf. [36, Lemma 2.1] for the general case $d \geq 2$, or Lemma 2.2 below for the 2D case. If one wants to directly apply this result to estimate the H^1-norm of $\boldsymbol{b}(t) \cdot \nabla\phi$, where $\phi \in H^{2+\gamma}$ for some small $\gamma > 0$, then a possible choice of parameters would be $a = d/2 - \gamma < d/2$, $b = 1 + \gamma < d/2$, and one has

$$\|\boldsymbol{b}(t) \cdot \nabla\phi\|_{H^1} \lesssim_\gamma \|\boldsymbol{b}(t)\|_{H^{d/2-\gamma}} \|\nabla\phi\|_{H^{1+\gamma}} \leq \|\boldsymbol{b}(t)\|_{H^{d/2-\gamma}} \|\phi\|_{H^{2+\gamma}};$$

however, the above choice of parameters results in $d \geq 3$, $\gamma \in (0, (d-2)/2)$. In order to treat directly the 2D case, we make the following simple but key observation: since $\boldsymbol{b}(t)$ is divergence free in space, the function $\boldsymbol{b}(t) \cdot \nabla\phi$ has zero spatial average and thus one can apply Poincaré's inequality to get

$$\|\boldsymbol{b}(t) \cdot \nabla\phi\|_{H^1} \lesssim \|\nabla(\boldsymbol{b}(t) \cdot \nabla\phi)\|_{L^2} \leq \|\nabla\boldsymbol{b}(t) \cdot \nabla\phi\|_{L^2} + \|\boldsymbol{b}(t) \cdot \nabla^2\phi\|_{L^2};$$

note that now we only need to estimate L^2-norm of products, it is possible to choose suitable parameters such that the above product rule of Sobolev functions is applicable in the 2D case, see (3.4) below for details.

Next, we have tried to avoid using the supremum in time of Sobolev norms of $\boldsymbol{b}(t, \cdot)$, with one exception in Lemma 5.3; in this way, most of the estimates do not involve logarithmic terms, making them look simpler than those in [36].

We finish the short introduction with the structure of the paper. We present some preliminary results in Section 2 which will be frequently used below. Then we prove in Section 3 a few useful estimates on the solution ξ to equation (1.1); as in [36], the main technical estimate is Proposition 3.3 whose proof will be postponed to Section 5 in order not to interrupt the readability of the paper. The main results (Theorems 1.1 and 1.2) will be proved in Section 4, again following some ideas in [36] with suitable modifications to deal with the nonlinearities.

2 Preparations

Recall that $\mathbb{Z}_0^2$ consists of 2D nonzero integer points; let $\{W^k\}_{k\in\mathbb{Z}_0^2}$ be a family of independent two-sided Brownian motions defined on some filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$. For every $\alpha > 1$, the processes

$$\eta^{\alpha,k}(t) := \int_{-\infty}^t \alpha e^{-\alpha(t-s)}\, dW_s^k, \quad t \ge 0,\ k \in \mathbb{Z}_0^2$$

constitute a family of independent Ornstein-Uhlenbeck processes, which are solutions of the 1D SDE

$$d\eta^{\alpha,k} = -\alpha\, \eta^{\alpha,k}\, dt + \alpha\, dW_t^k.$$

It is clear that $\eta^{\alpha,k}$ is a stationary process, with the invariant Gaussian measure $N(0, \alpha/2)$. For the reader's convenience, we recall that the random vector field $\boldsymbol{b}$ is defined as

$$\boldsymbol{b}(t,x) = 2\sqrt{\nu} \sum_{k\in\mathbb{Z}_0^2} \theta_k\, \sigma_k(x)\, \eta^{\alpha,k}(t),$$

where $\nu > 0$, $\theta \in \ell^2(\mathbb{Z}_0^2)$ is radially symmetric and has compact support, $\|\theta\|_{\ell^2} = 1$, and $\sigma_k(x) = \frac{k^\perp}{|k|} e^{2\pi i k\cdot x}$, $k \in \mathbb{Z}_0^2$ constitute a CONS of the space of divergence free vector fields in $L^2(\mathbb{T}^2, \mathbb{R}^2)$.

We next introduce the definition of weak solutions for the equation (1.1), namely

$$\partial_t \xi + u \cdot \nabla\xi + \boldsymbol{b} \cdot \nabla\xi = \kappa\Delta\xi,$$

with $u = K * \xi$ and initial data $\xi_0 \in L^2(\mathbb{T}^2)$.

Definition 2.1. *Suppose $\xi_0 \in L^2(\mathbb{T}^2)$. A stochastic process $\xi : \Omega \times [0,\infty) \to L^2(\mathbb{T}^2)$ is called a weak solution of (1.1), if there exists a $\mathbb{P}$-negligible set $\mathcal{N} \subset \Omega$ such that for every $\omega \in \mathcal{N}^c$, it holds $\xi(\omega, \cdot) \in L^\infty\big([0,\infty), L^2(\mathbb{T}^2)\big)$ and*

$$\langle \phi, \xi_t\rangle = \langle \phi, \xi_s\rangle + \int_s^t \langle u_r \cdot \nabla\phi, \xi_r\rangle\, dr + \int_s^t \langle \boldsymbol{b}(r)\cdot\nabla\phi, \xi_r\rangle\, dr + \kappa \int_s^t \langle \Delta\phi, \xi_r\rangle\, dr,$$

for every test function $\phi \in C^\infty(\mathbb{T}^2)$ and every $0 \le s < t < \infty$.

It is easy to know that, given any L^2-initial condition ξ_0, (1.1) admits a unique weak solution satisfying the following $\mathbb{P}$-a.s. energy estimate:

$$\sup_{t\in[0,\infty)} \Big(\|\xi_t\|_{L^2}^2 + 2\kappa \int_0^t \|\nabla\xi_s\|_{L^2}^2\, ds \Big) \le \|\xi_0\|_{L^2}^2. \tag{2.1}$$

Similarly, for the solution of (1.4), it holds

$$\sup_{t\in[0,\infty)} \Big(\|\bar\xi_t\|_{L^2}^2 + 2(\kappa+\nu) \int_0^t \|\nabla\bar\xi_s\|_{L^2}^2\, ds \Big) \le \|\xi_0\|_{L^2}^2. \tag{2.2}$$

Now we state several technical lemmas for later use; as they are classical results in harmonic analysis, we omit their proofs. The first result is concerned with the product of Sobolev functions, see e.g. [2, Corollary 2.55].

Lemma 2.2. *For any $s_1, s_2 < 1$, if $s_1 + s_2 > 0$, then for any $u \in H^{s_1}(\mathbb{T}^2)$ and $v \in H^{s_2}(\mathbb{T}^2)$, we have $uv \in H^{s_1+s_2-1}(\mathbb{T}^2)$, and the following inequality holds:*

$$\|uv\|_{H^{s_1+s_2-1}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}.$$

The following result follows easily from Gagliardo-Nirenberg's characterization of H^α-norm for $\alpha \in (0,1)$, cf. [2, Proposition 1.37]

Lemma 2.3. *Let $\alpha \in (0,1)$ and $\epsilon > 0$ be such that $\alpha + \epsilon < 1$, then for any $u \in C^{\alpha+\epsilon}(\mathbb{T}^2)$ and $v \in H^\alpha(\mathbb{T}^2)$, it holds*

$$\|uv\|_{H^\alpha} \lesssim \|u\|_{C^{\alpha+\epsilon}} \|v\|_{H^\alpha}.$$

Lemma 2.4 (Interpolation inequality)**.** *For any $s_1 < s < s_2$, there exists $\alpha \in (0,1)$ satisfying $s = \alpha s_1 + (1-\alpha) s_2$, such that*

$$\|u\|_{H^s} \le \|u\|_{H^{s_1}}^\alpha \|u\|_{H^{s_2}}^{1-\alpha}.$$

The next lemma gives a useful estimate on the Sobolev norms of the vector field $\boldsymbol{b}$.

Lemma 2.5. *For every $p \ge 2$ and $\tau > 0$, we have the following estimate:*

$$\sup_{s\ge 0} \mathbb{E}\big[\|\boldsymbol{b}(s)\|_{H^\tau}^p\big] \lesssim \nu^{\frac{p}{2}} \alpha^{\frac{p}{2}} C_{\theta,\tau,p},$$

where $C_{\theta,\tau,p} := \sum_{k\in\mathbb{Z}_0^2} \theta_k^2 |k|^{p\tau} \in (0,\infty)$ is a constant depending on θ, τ, p.

Proof. Recall that $\sum_{k\in\mathbb{Z}_0^2} \theta_k^2 = 1$, then by the definition of $\boldsymbol{b}(s)$, Jensen's inequality yields

$$\|\boldsymbol{b}(s)\|_{H^\tau}^p = \Big(4\nu \sum_{k\in\mathbb{Z}_0^2} \theta_k^2 \, (\eta^{\alpha,k}(s))^2 \, |k|^{2\tau}\Big)^{\frac{p}{2}} \le (4\nu)^{\frac{p}{2}} \sum_{k\in\mathbb{Z}_0^2} \theta_k^2 \, \big|\eta^{\alpha,k}(s)\big|^p \, |k|^{p\tau}. \tag{2.3}$$

Taking expectation, we arrive at

$$\mathbb{E}\big[\|\boldsymbol{b}(s)\|_{H^\tau}^p\big] \le (4\nu)^{\frac{p}{2}} \sum_{k\in\mathbb{Z}_0^2} \theta_k^2 \, |k|^{p\tau} \, \mathbb{E}\Big[\big|\eta^{\alpha,k}(s)\big|^p\Big] \lesssim (\nu\alpha)^{\frac{p}{2}} \sum_{k\in\mathbb{Z}_0^2} \theta_k^2 \, |k|^{p\tau}$$

which gives us the desired estimate. □

Remark 2.6. *For $n > 1$, by Jensen's inequality we have $C_{\theta,\tau,p/n} \le C_{\theta,\tau,p}^{1/n}$.*

In the following two examples, we compute explicitly the values of $\|\theta\|_{\ell^\infty}$ and $C_{\theta,\tau,p}$ for special choices of coefficients θ.

Example 2.7. *For $N \ge 1$, we define $\theta \in \ell^2(\mathbb{Z}_0^2)$ as follows:*

$$\theta_k = \varepsilon_N \frac{1}{|k|^a} \mathbf{1}_{\{1\le |k|\le N\}}, \quad k \in \mathbb{Z}_0^2,$$

where $a \in (0,1)$, ε_N is a normalizing constant depending on N such that $\|\theta\|_{\ell^2} = 1$. Then for every $p \ge 1$, it holds

$$\|\theta\|_{\ell^\infty} \sim \Big(\frac{1-a}{\pi}\Big)^{1/2} \big(N^{2-2a} - 1\big)^{-1/2} \to 0 \quad \text{as } N \to \infty,$$

$$C_{\theta,\tau,p} \sim \frac{2-2a}{2-2a+p\tau} \frac{N^{2-2a+p\tau} - 1}{N^{2-2a} - 1} \sim N^{p\tau} \quad \text{as } N \to \infty.$$

Proof. By the definition of $C_{\theta,\tau,p}$, we can get

$$C_{\theta,\tau,p} = \varepsilon_N^2 \sum_{1\le|k|\le N} |k|^{-2a+p\tau} =: \varepsilon_N^2\, d_N\,,$$

where $d_N = \sum_{1\le|k|\le N} |k|^{-2a+p\tau}$. Notice that $\|\theta\|_{\ell^2}^2 = \sum_{1\le|k|\le N} \varepsilon_N^2 |k|^{-2a} = 1$, then we can estimate $\varepsilon_N^2 = \big(\sum_{1\le|k|\le N} |k|^{-2a}\big)^{-1}$ by integration as follows:

$$\varepsilon_N^2 \sim \Big(\int_{1\le|x|\le N} |x|^{-2a}\, dx\Big)^{-1} = \Big(\int_1^N \int_0^{2\pi} \frac{r}{r^{2a}}\, d\varphi\, dr\Big)^{-1} = \frac{1-a}{\pi}\big(N^{2-2a}-1\big)^{-1}.$$

Furthermore, we can easily get $\|\theta\|_{\ell^\infty} = \varepsilon_N$ for a fixed N. In the same way, we can estimate $d_N \sim \frac{2\pi}{2-2a+p\tau}\big(N^{2-2a+p\tau}-1\big)$, and therefore we obtain the value of $C_{\theta,\tau,p}$. □

Example 2.8. *If we change the support set of the above example and define*

$$\theta_k = \varepsilon_N \frac{1}{|k|^a} \mathbf{1}_{\{N\le|k|\le 2N\}}, \quad k\in\mathbb{Z}_0^2,$$

where $a>0$ and ε_N is still a normalizing constant, then for every $p\ge 1$, we can obtain

$$\|\theta\|_{\ell^\infty} \sim \begin{cases} \big(\frac{1-a}{\pi}\big)^{1/2}\big(2^{2-2a}-1\big)^{-1/2} N^{-1}, & 0<a<1,\\ \big(2\pi\log 2\big)^{-1/2} N^{-1}, & a=1,\\ \big(\frac{a-1}{\pi}\big)^{1/2}\big(1-2^{2-2a}\big)^{-1/2} N^{-1}, & a>1; \end{cases}$$

$$C_{\theta,\tau,p} \sim \begin{cases} \frac{(2-2a)(2^{2-2a+p\tau}-1)}{(2-2a+p\tau)(2^{2-2a}-1)} N^{p\tau}, & a\ne 1,\\ \frac{2^{p\tau}-1}{p\tau\log 2} N^{p\tau}, & a=1. \end{cases}$$

Finally we present a moment estimate of $\boldsymbol{b}$ in the space $C([0,T],H^\tau)$, $T\ge 1$, which will be used in Section 5.4.

Lemma 2.9. *Consider $\boldsymbol{b}$ as defined before, then for every $p\ge 2$ and $\tau>0$, it holds*

$$\mathbb{E}\Big[\sup_{s\in[0,T]} \|\boldsymbol{b}(s)\|_{H^\tau}^p\Big] \lesssim \nu^{\frac{p}{2}} \alpha^{\frac{p}{2}}\, C_{\theta,\tau,p} \log^{\frac{p}{2}}(1+\alpha T), \quad \forall\, T\ge 1.$$

Proof. Recall the useful estimate from [30]: for every fixed $p\ge 1$, it holds

$$\mathbb{E}\Big[\sup_{s\in[0,T]} |\eta^{\alpha,k}(s)|^p\Big] \lesssim \alpha^{\frac{p}{2}} \log^{\frac{p}{2}}(1+\alpha T) \quad \text{for all } k\in\mathbb{Z}_0^2;$$

by (2.3), for $p\ge 2$, we take supremum and then expectation on $\|\boldsymbol{b}(s)\|_{H^\tau}^p$ to obtain

$$\mathbb{E}\Big[\sup_{s\in[0,T]} \|\boldsymbol{b}(s)\|_{H^\tau}^p\Big] \le (4\nu)^{\frac{p}{2}} \sum_{k\in\mathbb{Z}_0^2} |k|^{p\tau}\theta_k^2\, \mathbb{E}\Big[\sup_{s\in[0,T]} |\eta^{\alpha,k}(s)|^p\Big] \lesssim \nu^{\frac{p}{2}}\alpha^{\frac{p}{2}}\, C_{\theta,\tau,p} \log^{\frac{p}{2}}(1+\alpha T).$$

□

3 Useful Estimates

We first prove an estimate on the time increment of ξ in H^{-1}-norm, which will be repeatedly used in the proofs of Lemma 3.2 and Proposition 3.3. Thanks to the estimate (2.1), we often control $\|\xi_s\|_{L^2}$ by $\|\xi_0\|_{L^2}$ in the following proofs.

Lemma 3.1. *Let $t \geq 0$, $\delta \in (0,1)$ satisfy $\delta\alpha \gtrsim 1$, then for every $p \geq 2$ and $\gamma \in (0,1)$, it holds*

$$\mathbb{E}\Big[\|\xi_{t+\delta} - \xi_t\|_{H^{-1}}^p\Big] \lesssim \delta^p \nu^{\frac{p}{2}} \alpha^{\frac{p}{2}} C_{\theta,1+\gamma,p} \|\xi_0\|_{L^2}^p \big(1 + \|\xi_0\|_{L^2}^p\big).$$

Proof. By Definition 2.1, for every test function $\phi \in C^\infty(\mathbb{T}^2)$, we have

$$\big|\langle \phi, \xi_{t+\delta} - \xi_t \rangle\big| \leq \int_t^{t+\delta} \big|\langle u_s \cdot \nabla\phi, \xi_s\rangle\big|\, ds + \int_t^{t+\delta} \big|\langle \boldsymbol{b}(s)\cdot\nabla\phi, \xi_s\rangle\big|\, ds + \kappa \int_t^{t+\delta} \big|\langle \Delta\phi, \xi_s\rangle\big|\, ds.$$

Now we will deal with each term separately. First, according to Sobolev embedding theorem and Lemma 2.4, for $\gamma \in (0,1)$, we have the following estimate:

$$\big|\langle u_s \cdot \nabla\phi, \xi_s\rangle\big| \;\leq\; \|u_s\|_{L^\infty} \|\nabla\phi\|_{L^2} \|\xi_s\|_{L^2} \;\lesssim\; \|\xi_s\|_{H^\gamma} \|\phi\|_{H^1} \|\xi_0\|_{L^2} \;\lesssim\; \|\xi_s\|_{H^1}^\gamma \|\phi\|_{H^1} \|\xi_0\|_{L^2}^{2-\gamma}.$$

Hence we can use Hölder's inequality and (2.1) to get

$$\begin{aligned}\int_t^{t+\delta} \big|\langle u_s \cdot \nabla\phi, \xi_s\rangle\big|\, ds &\lesssim \|\phi\|_{H^1} \|\xi_0\|_{L^2}^{2-\gamma} \int_t^{t+\delta} \|\xi_s\|_{H^1}^\gamma\, ds \\ &\leq \|\phi\|_{H^1} \|\xi_0\|_{L^2}^{2-\gamma} \Big(\int_t^{t+\delta} \|\xi_s\|_{H^1}^2\, ds\Big)^{\frac{\gamma}{2}} \Big(\int_t^{t+\delta} 1\, ds\Big)^{1-\frac{\gamma}{2}} \\ &\leq \kappa^{-\frac{\gamma}{2}} \delta^{1-\frac{\gamma}{2}} \|\phi\|_{H^1} \|\xi_0\|_{L^2}^2.\end{aligned} \tag{3.1}$$

In the same way, we can estimate the second term. For $\gamma \in (0,1)$, we have

$$\big|\langle \boldsymbol{b}(s)\cdot\nabla\phi, \xi_s\rangle\big| \;\leq\; \|\boldsymbol{b}(s)\|_{L^\infty} \|\nabla\phi\|_{L^2} \|\xi_s\|_{L^2} \;\lesssim\; \|\boldsymbol{b}(s)\|_{H^{1+\gamma}} \|\phi\|_{H^1} \|\xi_0\|_{L^2},$$

then the following inequality holds:

$$\int_t^{t+\delta} \big|\langle \boldsymbol{b}(s)\cdot\nabla\phi, \xi_s\rangle\big|\, ds \lesssim \|\phi\|_{H^1} \|\xi_0\|_{L^2} \int_t^{t+\delta} \|\boldsymbol{b}(s)\|_{H^{1+\gamma}}\, ds. \tag{3.2}$$

As for the last term, (2.1) and Hölder's inequality yield

$$\begin{aligned}\kappa \int_t^{t+\delta} \big|\langle \Delta\phi, \xi_s\rangle\big|\, ds &\leq \kappa \int_t^{t+\delta} \|\Delta\phi\|_{H^{-1}} \|\xi_s\|_{H^1}\, ds \\ &\leq \kappa \|\phi\|_{H^1} \Big(\int_t^{t+\delta} \|\xi_s\|_{H^1}^2\, ds\Big)^{\frac{1}{2}} \Big(\int_t^{t+\delta} 1\, ds\Big)^{\frac{1}{2}} \\ &\leq \kappa^{\frac{1}{2}} \delta^{\frac{1}{2}} \|\phi\|_{H^1} \|\xi_0\|_{L^2}.\end{aligned} \tag{3.3}$$

Having (3.1)-(3.3) at hand, we deduce

$$\big|\langle \phi, \xi_{t+\delta} - \xi_t\rangle\big| \lesssim \|\phi\|_{H^1} \bigg[\kappa^{-\frac{\gamma}{2}} \delta^{1-\frac{\gamma}{2}} \|\xi_0\|_{L^2}^2 + \|\xi_0\|_{L^2} \int_t^{t+\delta} \|\boldsymbol{b}(s)\|_{H^{1+\gamma}}\, ds + \kappa^{\frac{1}{2}} \delta^{\frac{1}{2}} \|\xi_0\|_{L^2}\bigg].$$

Since ϕ is arbitrary, the above formula yields

$$\|\xi_{t+\delta} - \xi_t\|_{H^{-1}} \lesssim \kappa^{-\frac{\gamma}{2}} \delta^{1-\frac{\gamma}{2}} \|\xi_0\|_{L^2}^2 + \|\xi_0\|_{L^2} \int_t^{t+\delta} \|\boldsymbol{b}(s)\|_{H^{1+\gamma}}\, ds + \kappa^{\frac{1}{2}} \delta^{\frac{1}{2}} \|\xi_0\|_{L^2}.$$

Taking the p-th moment, we finally get

$$
\begin{aligned}
&\mathbb{E}\Big[\|\xi_{t+\delta}-\xi_t\|_{H^{-1}}^p\Big] \\
&\lesssim \kappa^{-\frac{\gamma}{2}p}\,\delta^{(1-\frac{\gamma}{2})p}\|\xi_0\|_{L^2}^{2p}+\|\xi_0\|_{L^2}^p\,\mathbb{E}\bigg[\Big(\int_t^{t+\delta}\|\boldsymbol{b}(s)\|_{H^{1+\gamma}}\,ds\Big)^p\bigg]+\kappa^{\frac{p}{2}}\delta^{\frac{p}{2}}\|\xi_0\|_{L^2}^p \\
&\le \kappa^{-\frac{\gamma}{2}p}\,\delta^{(1-\frac{\gamma}{2})p}\|\xi_0\|_{L^2}^{2p}+\|\xi_0\|_{L^2}^p\delta^{p-1}\int_t^{t+\delta}\mathbb{E}\big[\|\boldsymbol{b}(s)\|_{H^{1+\gamma}}^p\big]ds+\kappa^{\frac{p}{2}}\delta^{\frac{p}{2}}\|\xi_0\|_{L^2}^p \\
&\lesssim \kappa^{-\frac{\gamma}{2}p}\,\delta^{(1-\frac{\gamma}{2})p}\|\xi_0\|_{L^2}^{2p}+\delta^p\nu^{\frac{p}{2}}\alpha^{\frac{p}{2}}\,C_{\theta,1+\gamma,p}\,\|\xi_0\|_{L^2}^p+\kappa^{\frac{p}{2}}\delta^{\frac{p}{2}}\|\xi_0\|_{L^2}^p.
\end{aligned}
$$

Noting that $\kappa>0$ is a fixed parameter, we finish the proof by taking into account our restrictions on δ and α. □

Based on the conclusion of Lemma 3.1, we can further deduce another useful estimate.

Lemma 3.2. *Let $t\ge 0$, $\delta\in(0,1)$ satisfy $\delta\alpha\gtrsim 1$ and $\delta^4\alpha^3\lesssim 1$, then for every $p\ge 2$ and $\gamma\in(0,1)$, it holds*

$$
\mathbb{E}\Big[\|\xi_{t+\delta}-\xi_t\|_{H^{-2-\gamma}}^p\Big]\lesssim \nu^p\big(\delta^{\frac{p}{2}}+\alpha^{-\frac{p}{2}}\big)C_{\theta,1+\gamma,2p}\|\xi_0\|_{L^2}^p\big(1+\|\xi_0\|_{L^2}^p\big)^2.
$$

Proof. The idea of proof is similar to that of Lemma 3.1, but we divide the right hand side into more terms: for every test function $\phi\in C^\infty(\mathbb{T}^2)$,

$$
\begin{aligned}
\big|\langle\phi,\xi_{t+\delta}-\xi_t\rangle\big| \le{}& \int_t^{t+\delta}\big|\langle u_s\cdot\nabla\phi,\xi_s-\xi_t\rangle\big|\,ds+\int_t^{t+\delta}\big|\langle u_s\cdot\nabla\phi,\xi_t\rangle\big|\,ds \\
&+\int_t^{t+\delta}\big|\langle\boldsymbol{b}(s)\cdot\nabla\phi,\xi_s-\xi_t\rangle\big|\,ds+\Big|\int_t^{t+\delta}\langle\boldsymbol{b}(s)\cdot\nabla\phi,\xi_t\rangle\,ds\Big|+\kappa\int_t^{t+\delta}\big|\langle\Delta\phi,\xi_s\rangle\big|\,ds.
\end{aligned}
$$

We estimate each term respectively. For the first one,

$$
\begin{aligned}
\int_t^{t+\delta}\big|\langle u_s\cdot\nabla\phi,\xi_s-\xi_t\rangle\big|\,ds &\le \int_t^{t+\delta}\|u_s\cdot\nabla\phi\|_{H^1}\,\|\xi_s-\xi_t\|_{H^{-1}}\,ds \\
&\lesssim \int_t^{t+\delta}\|\nabla(u_s\cdot\nabla\phi)\|_{L^2}\,\|\xi_s-\xi_t\|_{H^{-1}}\,ds,
\end{aligned}
$$

where in the second step we have used Poincaré's inequality. By Lemma 2.2 and Sobolev embedding theorem, for $\gamma\in(0,1)$, we have the following estimate:

$$
\begin{aligned}
\|\nabla(u_s\cdot\nabla\phi)\|_{L^2} &\le \|\nabla u_s\cdot\nabla\phi\|_{L^2}+\|u_s\cdot\nabla^2\phi\|_{L^2} \\
&\lesssim \|\nabla u_s\|_{L^2}\|\nabla\phi\|_{L^\infty}+\|u_s\|_{H^{1-\gamma}}\|\nabla^2\phi\|_{H^\gamma} \\
&\lesssim \|u_s\|_{H^1}\|\phi\|_{H^{2+\gamma}}+\|\xi_s\|_{H^{-\gamma}}\|\phi\|_{H^{2+\gamma}} \\
&\lesssim \|\xi_s\|_{L^2}\|\phi\|_{H^{2+\gamma}},
\end{aligned}
\tag{3.4}
$$

substituting this estimate into the above inequality, we arrive at

$$
\int_t^{t+\delta}\big|\langle u_s\cdot\nabla\phi,\xi_s-\xi_t\rangle\big|\,ds\lesssim\|\phi\|_{H^{2+\gamma}}\|\xi_0\|_{L^2}\int_t^{t+\delta}\|\xi_s-\xi_t\|_{H^{-1}}\,ds.
$$

For the second term, we use Sobolev embedding theorem to get

$$
\begin{aligned}
\int_t^{t+\delta}\big|\langle u_s\cdot\nabla\phi,\xi_t\rangle\big|\,ds &\le \int_t^{t+\delta}\|u_s\|_{L^2}\|\nabla\phi\|_{L^\infty}\|\xi_t\|_{L^2}\,ds \\
&\lesssim \|\phi\|_{H^{2+\gamma}}\|\xi_0\|_{L^2}\int_t^{t+\delta}\|\xi_s\|_{H^{-1}}\,ds \\
&\lesssim \delta\,\|\phi\|_{H^{2+\gamma}}\|\xi_0\|_{L^2}^2.
\end{aligned}
$$

Similarly to (3.4), we can estimate the third term:

$$\begin{aligned}\int_t^{t+\delta} \big|\langle \boldsymbol{b}(s)\cdot\nabla\phi, \xi_s-\xi_t\rangle\big|\,ds &\lesssim \int_t^{t+\delta} \|\nabla(\boldsymbol{b}(s)\cdot\nabla\phi)\|_{L^2}\|\xi_s-\xi_t\|_{H^{-1}}\,ds\\ &\lesssim \|\phi\|_{H^{2+\gamma}}\int_t^{t+\delta} \|\boldsymbol{b}(s)\|_{H^1}\|\xi_s-\xi_t\|_{H^{-1}}\,ds.\end{aligned}$$

As for the next term,

$$\Big|\int_t^{t+\delta}\langle \boldsymbol{b}(s)\cdot\nabla\phi,\xi_t\rangle\,ds\Big| = \Big|\Big\langle\Big(\int_t^{t+\delta}\boldsymbol{b}(s)\,ds\Big)\cdot\nabla\phi,\xi_t\Big\rangle\Big| \le \|\nabla\phi\|_{L^\infty}\|\xi_t\|_{L^2}\Big\|\int_t^{t+\delta}\boldsymbol{b}(s)\,ds\Big\|_{L^2};$$

recalling the definition of $\boldsymbol{b}$, we have

$$\int_t^{t+\delta}\boldsymbol{b}(s)\,ds = 2\sqrt{\nu}\sum_{k\in\mathbb{Z}_0^2}\theta_k\sigma_k\big(W_{t+\delta}^k-W_t^k\big)-\alpha^{-1}\big(\boldsymbol{b}(t+\delta)-\boldsymbol{b}(t)\big),$$

and therefore, for $\gamma\in(0,1)$, we can get

$$\begin{aligned}\Big|\int_t^{t+\delta}\langle \boldsymbol{b}(s)\cdot\nabla\phi,\xi_t\rangle\,ds\Big| &\lesssim \nu^{\frac12}\|\phi\|_{H^{2+\gamma}}\|\xi_0\|_{L^2}\Big[\sum_{k\in\mathbb{Z}_0^2}\theta_k^2\big(W_{t+\delta}^k-W_t^k\big)^2\Big]^{\frac12}\\ &\quad+\alpha^{-1}\,\|\phi\|_{H^{2+\gamma}}\|\xi_0\|_{L^2}\big(\|\boldsymbol{b}(t+\delta)\|_{L^2}+\|\boldsymbol{b}(t)\|_{L^2}\big).\end{aligned}$$

Finally, the fifth term can be estimated as follows:

$$\kappa\int_t^{t+\delta}\big|\langle\Delta\phi,\xi_s\rangle\big|\,ds \le \kappa\int_t^{t+\delta}\|\Delta\phi\|_{L^2}\|\xi_s\|_{L^2}\,ds \lesssim \kappa\delta\,\|\phi\|_{H^{2+\gamma}}\|\xi_0\|_{L^2}.$$

Combining these results together and noticing the arbitrariness of ϕ, we arrive at

$$\begin{aligned}\|\xi_{t+\delta}-\xi_t\|_{H^{-2-\gamma}} &\lesssim \|\xi_0\|_{L^2}\int_t^{t+\delta}\|\xi_s-\xi_t\|_{H^{-1}}\,ds+\delta\,\|\xi_0\|_{L^2}^2\\ &\quad+\int_t^{t+\delta}\|\boldsymbol{b}(s)\|_{H^1}\|\xi_s-\xi_t\|_{H^{-1}}\,ds+\nu^{\frac12}\|\xi_0\|_{L^2}\Big[\sum_{k\in\mathbb{Z}_0^2}\theta_k^2\big(W_{t+\delta}^k-W_t^k\big)^2\Big]^{\frac12}\\ &\quad+\alpha^{-1}\|\xi_0\|_{L^2}\big(\|\boldsymbol{b}(t+\delta)\|_{L^2}+\|\boldsymbol{b}(t)\|_{L^2}\big)+\kappa\delta\,\|\xi_0\|_{L^2}.\end{aligned}\tag{3.5}$$

To complete the proof, we also need the following several estimates. By Lemma 3.1,

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_t^{t+\delta}\|\xi_s-\xi_t\|_{H^{-1}}ds\Big)^p\Big] &\le \delta^{p-1}\int_t^{t+\delta}\mathbb{E}\big[\|\xi_s-\xi_t\|_{H^{-1}}^p\big]ds\\ &\lesssim \delta^{2p}\nu^{\frac p2}\alpha^{\frac p2}\,C_{\theta,1+\gamma,p}\,\|\xi_0\|_{L^2}^p\big(1+\|\xi_0\|_{L^2}^p\big).\end{aligned}\tag{3.6}$$

Hölder's inequality yields

$$\begin{aligned}&\mathbb{E}\Big[\Big(\int_t^{t+\delta}\|\boldsymbol{b}(s)\|_{H^1}\|\xi_s-\xi_t\|_{H^{-1}}\,ds\Big)^p\Big]\\ &\le \mathbb{E}\Big[\delta^{p-1}\int_t^{t+\delta}\|\boldsymbol{b}(s)\|_{H^1}^p\|\xi_s-\xi_t\|_{H^{-1}}^p\,ds\Big]\\ &\le \delta^{p-1}\int_t^{t+\delta}\Big(\mathbb{E}\big[\|\boldsymbol{b}(s)\|_{H^1}^{2p}\big]\Big)^{\frac12}\Big(\mathbb{E}\big[\|\xi_s-\xi_t\|_{H^{-1}}^{2p}\big]\Big)^{\frac12}\,ds\\ &\lesssim \delta^{2p}\nu^p\alpha^p\,C_{\theta,1+\gamma,2p}\,\|\xi_0\|_{L^2}^p\big(1+\|\xi_0\|_{L^2}^p\big).\end{aligned}\tag{3.7}$$

Besides, by Jensen's inequality, for $p \geq 2$, the following formula holds:

$$\mathbb{E}\bigg[\Big(\sum_{k\in\mathbb{Z}_0^2}\theta_k^2\,\big(W_{t+\delta}^k - W_t^k\big)^2\Big)^{\frac{p}{2}}\bigg] \leq \mathbb{E}\bigg[\sum_{k\in\mathbb{Z}_0^2}\theta_k^2\,\big|W_{t+\delta}^k - W_t^k\big|^p\bigg] \lesssim \sum_{k\in\mathbb{Z}_0^2}\theta_k^2\,\delta^{\frac{p}{2}} = \delta^{\frac{p}{2}}. \tag{3.8}$$

According to the definition of $\boldsymbol{b}(t)$, for $t \geq 0$, we have

$$\mathbb{E}\Big[\|\boldsymbol{b}(t+\delta)\|_{L^2}^p\Big] = \mathbb{E}\Big[\|\boldsymbol{b}(t)\|_{L^2}^p\Big] \leq (4\nu)^{\frac{p}{2}}\sum_{k\in\mathbb{Z}_0^2}\theta_k^2\,\mathbb{E}\Big[\big|\eta^{\alpha,k}(t)\big|^p\Big] \lesssim \nu^{\frac{p}{2}}\alpha^{\frac{p}{2}}. \tag{3.9}$$

Inserting (3.6)-(3.9) into (3.5), we can easily get

$$\begin{aligned}\mathbb{E}\Big[\|\xi_{t+\delta} - \xi_t\|_{H^{-2-\gamma}}^p\Big] &\lesssim \delta^{2p}\nu^{\frac{p}{2}}\alpha^{\frac{p}{2}}\,C_{\theta,1+\gamma,p}\,\|\xi_0\|_{L^2}^{2p}\big(1 + \|\xi_0\|_{L^2}^p\big) + \delta^p\,\|\xi_0\|_{L^2}^{2p}\\ &\quad + \delta^{2p}\nu^p\alpha^p\,C_{\theta,1+\gamma,2p}\,\|\xi_0\|_{L^2}^p\big(1 + \|\xi_0\|_{L^2}^p\big) + \delta^{\frac{p}{2}}\nu^{\frac{p}{2}}\,\|\xi_0\|_{L^2}^p\\ &\quad + \nu^{\frac{p}{2}}\alpha^{-\frac{p}{2}}\,\|\xi_0\|_{L^2}^p + \kappa^p\delta^p\,\|\xi_0\|_{L^2}^p.\end{aligned}$$

It is clear that the parts involving the L^2-norm of initial data are dominated by $\|\xi_0\|_{L^2}^p\big(1 + \|\xi_0\|_{L^2}^p\big)^2$; the conclusion follows by noticing our previous assumptions on the parameters. □

The following result is analogous to [36, Proposition 3.3] where a similar estimate, against test functions, was proved for the solution h of (1.2). We remark that the stronger estimate as below is needed at the end of the proof of Proposition 4.2. We first divide the interval $[0,T]$ into many subintervals of the form $[n\delta,(n+1)\delta]$, $n \in \mathbb{N}$, where $\delta \in (0,1)$ is a small parameter such that T/δ is an integer, then we estimate the quantity in each interval of length δ, and finally sum them up.

Proposition 3.3. *Fix $\beta > 3$, $\gamma \in (0,\frac{1}{3})$, then there exist $\epsilon > 0$, $\delta \in (0,1)$ and $\rho \in (0,\frac{1}{4})$ such that for α large enough, the following estimate holds:*

$$\begin{aligned}&\mathbb{E}\bigg[\sup_{1\leq m<n\leq T/\delta-1}\frac{1}{(|n-m|\delta)^\rho}\Big\|\xi_{n\delta} - \xi_{m\delta} - (\kappa+\nu)\int_{m\delta}^{n\delta}\Delta\xi_s\,ds + \int_{m\delta}^{n\delta}u_s\cdot\nabla\xi_s\,ds\Big\|_{H^{-\beta}}\bigg]\\ &\quad\lesssim T\|\xi_0\|_{L^2}\big(1 + \|\xi_0\|_{L^2}\big)^2\big(\nu^{1+\frac{\gamma}{2}}\alpha^{-\epsilon} + \nu^{\frac{1}{2}}\|\theta\|_{\ell^\infty}\big).\end{aligned}$$

As the proof of Proposition 3.3 is very long, we postpone it to Section 5. We mention that some restrictions on the parameters α and δ are necessary in order to obtain a sufficiently small estimate, and one can find the specific details in Section 5.7.

4 Proofs of main results

This section consists of two parts which are devoted to the proofs of Theorems 1.1 and 1.2, respectively.

4.1 Proof of Theorem 1.1

To prove Theorem 1.1, we first define a random distribution f as follows:

$$f_t = \xi_t - \xi_0 - (\kappa+\nu)\int_0^t\Delta\xi_s\,ds + \int_0^t u_s\cdot\nabla\xi_s\,ds. \tag{4.1}$$

If we replace ξ by $\bar{\xi}$ and u by $\bar{u}$, then the right-hand side vanishes; since we expect that ξ is close to $\bar{\xi}$, the distribution f would be small in suitable norms. We first prove a regularity estimate on f, which will be used in the proof of Proposition 4.2.

Lemma 4.1. *For every $0 \le s < t \le T$ and $\gamma > 0$, it holds*

$$\|f_t - f_s\|_{H^{-2}} \lesssim \|\xi_0\|_{L^2}^2 \, |t-s| + \|\xi_0\|_{L^2} \int_s^t \|\boldsymbol{b}(r)\|_{H^\gamma} \, dr + (\kappa+\nu) \, \|\xi_0\|_{L^2} \, |t-s|.$$

Proof. By (1.1), for every $s, t \in [0,T]$ and $s < t$, we have

$$\xi_t - \xi_s = -\int_s^t u_r \cdot \nabla \xi_r \, dr - \int_s^t \boldsymbol{b}(r) \cdot \nabla \xi_r \, dr + \kappa \int_s^t \Delta \xi_r \, dr.$$

Then we can further get

$$\begin{aligned} \|\xi_t - \xi_s\|_{H^{-2}} &\le \Big\| \int_s^t u_r \cdot \nabla \xi_r \, dr \Big\|_{H^{-2}} + \Big\| \int_s^t \boldsymbol{b}(r) \cdot \nabla \xi_r \, dr \Big\|_{H^{-2}} + \kappa \Big\| \int_s^t \Delta \xi_r \, dr \Big\|_{H^{-2}} \\ &\lesssim \int_s^t \|u_r \cdot \nabla \xi_r\|_{H^{-2}} \, dr + \int_s^t \|\boldsymbol{b}(r) \cdot \nabla \xi_r\|_{H^{-2}} \, dr + \kappa \int_s^t \|\xi_0\|_{L^2} \, dr. \end{aligned} \tag{4.2}$$

Now we will estimate the first and the second terms respectively. Using the divergence free property of u, we have

$$\|u_r \cdot \nabla \xi_r\|_{H^{-2}} = \|\nabla \cdot (\xi_r u_r)\|_{H^{-2}} \lesssim \|\xi_r u_r\|_{H^{-1}};$$

besides, by Hölder's inequality and Sobolev embedding theorem, for $\phi \in C^\infty(\mathbb{T}^2)$,

$$\big|\langle \xi_r u_r, \phi \rangle\big| \le \|\xi_r\|_{L^2} \|u_r\|_{L^4} \|\phi\|_{L^4} \lesssim \|\xi_0\|_{L^2} \|u_r\|_{H^{\frac12}} \|\phi\|_{H^{\frac12}} \lesssim \|\xi_0\|_{L^2}^2 \|\phi\|_{H^1}.$$

Then we get $\|\xi_r u_r\|_{H^{-1}} \lesssim \|\xi_0\|_{L^2}^2$ and thus $\|u_r \cdot \nabla \xi_r\|_{H^{-2}} \lesssim \|\xi_0\|_{L^2}^2$.

As for the next term, notice that $\boldsymbol{b}(r)$ is divergence free, we use the same method as above to estimate it: for $\gamma > 0$, we have

$$\|\boldsymbol{b}(r) \cdot \nabla \xi_r\|_{H^{-2}} \lesssim \|\xi_r \, \boldsymbol{b}(r)\|_{H^{-1}} \lesssim \|\xi_0\|_{L^2} \|\boldsymbol{b}(r)\|_{H^\gamma}.$$

Having the above results at hand, we combine (4.2) with (4.1) and arrive at

$$\begin{aligned} \|f_t - f_s\|_{H^{-2}} &\le \|\xi_t - \xi_s\|_{H^{-2}} + (\kappa+\nu) \int_s^t \|\Delta \xi_r\|_{H^{-2}} \, dr + \int_s^t \|u_r \cdot \nabla \xi_r\|_{H^{-2}} \, dr \\ &\le \int_s^t \|u_r \cdot \nabla \xi_r\|_{H^{-2}} \, dr + \int_s^t \|\boldsymbol{b}(r) \cdot \nabla \xi_r\|_{H^{-2}} \, dr + (\kappa+\nu) \int_s^t \|\xi_0\|_{L^2} \, dr \\ &\lesssim \|\xi_0\|_{L^2}^2 \, |t-s| + \|\xi_0\|_{L^2} \int_s^t \|\boldsymbol{b}(r)\|_{H^\gamma} \, dr + (\kappa+\nu) \|\xi_0\|_{L^2} \, |t-s|. \end{aligned}$$

□

Proposition 4.2. *Let $\beta > 3$, $\gamma \in (0, \frac13)$ and $T \ge 1$, then there are $\rho \in (0, \frac14)$ and $\epsilon > 0$ such that for every α sufficiently large, it holds*

$$\mathbb{E}\Big[\|f\|_{C^\rho([0,T], H^{-\beta})}\Big] \lesssim T \|\xi_0\|_{L^2} \big(1 + \|\xi_0\|_{L^2}\big)^2 \big(\nu^{1+\frac{\gamma}{2}} \alpha^{-\epsilon} + \nu^{\frac12} \|\theta\|_{\ell^\infty}\big).$$

Proof. First, as $f_0 = 0$, we give an equivalent norm as follows:

$$\|f\|_{C^\rho([0,T], H^{-\beta})} \sim \sup_{0 \le s < t \le T} \frac{\|f_t - f_s\|_{H^{-\beta}}}{|t-s|^\rho}. \tag{4.3}$$

Then we will prove Proposition 4.2 in the following two cases.

Case 1: $|t-s| \le \delta$. By Hölder's inequality, for $\rho \in (0, \frac{1}{4})$, we have

$$\int_s^t \|\boldsymbol{b}(r)\|_{H^\gamma}\, dr \le \Big(\int_s^t 1\, dr\Big)^{1-\rho} \Big(\int_s^t \|\boldsymbol{b}(r)\|_{H^\gamma}^{\frac{1}{\rho}}\, dr\Big)^{\rho} \le |t-s|^{1-\rho} \Big(\int_s^t \|\boldsymbol{b}(r)\|_{H^\gamma}^{\frac{1}{\rho}}\, dr\Big)^{\rho}.$$

Furthermore, we can get

$$\mathbb{E}\Bigg[\sup_{\substack{0\le s<t\le T\\ |t-s|\le\delta}} \frac{\int_s^t \|\boldsymbol{b}(r)\|_{H^\gamma}\, dr}{|t-s|^\rho}\Bigg] \le \delta^{1-2\rho}\, \mathbb{E}\Big(\int_0^T \|\boldsymbol{b}(r)\|_{H^\gamma}^{\frac{1}{\rho}}\, dr\Big)^{\rho} \lesssim \delta^{1-2\rho}\nu^{\frac{1}{2}}\alpha^{\frac{1}{2}}\, T^\rho C^\rho_{\theta,\gamma,1/\rho}\,,$$

where in the last step we have used Lemma 2.5. Then for $\beta > 3$, Lemma 4.1 yields

$$\begin{aligned}\mathbb{E}\Bigg[\sup_{\substack{0\le s<t\le T\\ |t-s|\le\delta}} \frac{\|f_t - f_s\|_{H^{-\beta}}}{|t-s|^\rho}\Bigg] &\lesssim \delta^{1-\rho}\, \|\xi_0\|_{L^2}^2 + \delta^{1-2\rho}\nu^{\frac{1}{2}}\alpha^{\frac{1}{2}}\, T^\rho C^\rho_{\theta,\gamma,1/\rho}\, \|\xi_0\|_{L^2} + (\kappa+\nu)\,\delta^{1-\rho}\, \|\xi_0\|_{L^2} \\ &\lesssim \delta^{1-2\rho}\nu^{\frac{1}{2}}\alpha^{\frac{1}{2}}\, T^\rho C^\rho_{\theta,\gamma,1/\rho}\, \|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big).\end{aligned} \tag{4.4}$$

Case 2: $|t-s| > \delta$. We suppose $s \in [(m-1)\delta, m\delta)$ and $t \in (n\delta, (n+1)\delta]$, where $n, m \in \mathbb{N}$ and $m \le n$. Hence, if $n > m$, we have

$$\frac{\|f_t - f_s\|_{H^{-\beta}}}{|t-s|^\rho} \le \frac{\|f_t - f_{n\delta}\|_{H^{-\beta}}}{|t-n\delta|^\rho} + \frac{\|f_{n\delta} - f_{m\delta}\|_{H^{-\beta}}}{|n\delta-m\delta|^\rho} + \frac{\|f_{m\delta} - f_s\|_{H^{-\beta}}}{|m\delta-s|^\rho},$$

while for $n = m$, the following formula holds:

$$\frac{\|f_t - f_s\|_{H^{-\beta}}}{|t-s|^\rho} \le \frac{\|f_t - f_{n\delta}\|_{H^{-\beta}}}{|t-n\delta|^\rho} + \frac{\|f_{n\delta} - f_s\|_{H^{-\beta}}}{|n\delta-s|^\rho}.$$

Then we can combine the above two cases and get

$$\sup_{0\le s<t\le T} \frac{\|f_t - f_s\|_{H^{-\beta}}}{|t-s|^\rho} \lesssim \sup_{\substack{0\le s<t\le T\\ |t-s|\le\delta}} \frac{\|f_t - f_s\|_{H^{-\beta}}}{|t-s|^\rho} + \sup_{1\le m<n\le T/\delta-1} \frac{\|f_{n\delta} - f_{m\delta}\|_{H^{-\beta}}}{|n\delta-m\delta|^\rho}. \tag{4.5}$$

By the definition of f and Proposition 3.3, for every $\beta > 3$, it holds

$$\mathbb{E}\Bigg[\sup_{1\le m<n\le T/\delta-1} \frac{\|f_{n\delta} - f_{m\delta}\|_{H^{-\beta}}}{|n\delta-m\delta|^\rho}\Bigg] \lesssim T\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)^2\big(\nu^{1+\frac{\gamma}{2}}\alpha^{-\epsilon} + \nu^{\frac{1}{2}}\|\theta\|_{\ell^\infty}\big). \tag{4.6}$$

Taking into account (4.4)–(4.6) and noticing the restrictions on the parameters in Proposition 3.3, we use (4.3) to complete the proof of Proposition 4.2. ☐

Now we give the following proposition which indicates that f is bounded in H^{-1}.

Proposition 4.3. *Suppose f is defined as in (4.1), then for $\gamma \in (0,1)$ and $T \ge 1$, it holds*

$$\sup_{t\in[0,T]} \|f_t\|_{H^{-1}} \lesssim T^{1-\frac{\gamma}{2}}\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)\big(\nu\kappa^{-\frac{1}{2}} + \kappa^{-\frac{\gamma}{2}}\big).$$

Proof. According to (4.1), the following formula holds:

$$\begin{aligned}\|f_t\|_{H^{-1}} &\le \|\xi_t-\xi_0\|_{H^{-1}}+(\kappa+\nu)\int_0^t\|\Delta\xi_s\|_{H^{-1}}\,ds+\int_0^t\|u_s\cdot\nabla\xi_s\|_{H^{-1}}\,ds\\ &\lesssim 2\|\xi_0\|_{L^2}+(\kappa+\nu)\int_0^t\|\xi_s\|_{H^1}\,ds+\int_0^t\|u_s\cdot\nabla\xi_s\|_{H^{-1}}\,ds.\end{aligned}\tag{4.7}$$

Applying Hölder's inequality and (2.1), we have

$$\int_0^t\|\xi_s\|_{H^1}\,ds\le\Big(\int_0^t 1\,ds\Big)^{\frac12}\Big(\int_0^t\|\xi_s\|_{H^1}^2\,ds\Big)^{\frac12}\lesssim\kappa^{-\frac12}\,t^{\frac12}\,\|\xi_0\|_{L^2}.\tag{4.8}$$

As for the last term, notice that for $\gamma\in(0,1)$, the following inequality holds:

$$\|u_s\cdot\nabla\xi_s\|_{H^{-1}}\lesssim\|\xi_s\,u_s\|_{L^2}\le\|\xi_s\|_{L^2}\,\|u_s\|_{L^\infty}\lesssim\|\xi_0\|_{L^2}\,\|\xi_s\|_{H^\gamma},$$

then we can use Lemma 2.4 and (2.1) to further get

$$\begin{aligned}\int_0^t\|u_s\cdot\nabla\xi_s\|_{H^{-1}}\,ds &\lesssim\|\xi_0\|_{L^2}\int_0^t\|\xi_s\|_{H^1}^{\gamma}\,\|\xi_s\|_{L^2}^{1-\gamma}\,ds\\ &\le\|\xi_0\|_{L^2}^{2-\gamma}\Big(\int_0^t 1\,ds\Big)^{1-\frac{\gamma}{2}}\Big(\int_0^t\|\xi_s\|_{H^1}^2\,ds\Big)^{\frac{\gamma}{2}}\\ &\lesssim\kappa^{-\frac{\gamma}{2}}\,t^{1-\frac{\gamma}{2}}\,\|\xi_0\|_{L^2}^2.\end{aligned}\tag{4.9}$$

Inserting (4.8) and (4.9) into (4.7), we take supremum and deduce

$$\sup_{t\in[0,T]}\|f_t\|_{H^{-1}}\lesssim 2\|\xi_0\|_{L^2}+(\kappa^{\frac12}+\nu\kappa^{-\frac12})\,T^{\frac12}\|\xi_0\|_{L^2}+\kappa^{-\frac{\gamma}{2}}T^{1-\frac{\gamma}{2}}\|\xi_0\|_{L^2}^2.$$

The proposition follows due to the choices of parameters. □

With the above preparations in mind, we can prove the first main theorem now.

Proof of Theorem 1.1. By the definition of f, we have

$$\xi_t=\xi_0+(\kappa+\nu)\int_0^t\Delta\xi_s\,ds-\int_0^t u_s\cdot\nabla\xi_s\,ds+f_t,$$

while by (1.4), it holds

$$\bar\xi_t=\xi_0+(\kappa+\nu)\int_0^t\Delta\bar\xi_s\,ds-\int_0^t\bar u_s\cdot\nabla\bar\xi_s\,ds.$$

Define

$$X_t:=f_t-\int_0^t(u_s\cdot\nabla\xi_s-\bar u_s\cdot\nabla\bar\xi_s)\,ds,$$

then the difference $\xi-\bar\xi$ satisfies

$$\xi_t-\bar\xi_t=(\kappa+\nu)\int_0^t\Delta(\xi_s-\bar\xi_s)ds+X_t.\tag{4.10}$$

We first prove the theorem in the Sobolev spaces $H^{-\tilde\vartheta}$ with $\tilde\vartheta>1$; without loss of generality we can assume $\tilde\vartheta\in(1,\frac32)$. Recall that we have already obtained in Proposition 4.2 an estimate on $\|f\|_{C^\rho([0,T],H^{-\beta})}$ for $\beta > 3$; we shall fix such a β in the sequel. Besides, Proposition 4.3 gives us a bound on $\sup_{t\in[0,T]} \|f_t\|_{H^{-1}}$, hence by Lemma 2.4, for any $\tilde\vartheta \in (1, \frac{3}{2})$, there exists $\zeta \in (0,1)$ satisfying $\beta\zeta + (1-\zeta) = \tilde\vartheta$, such that

$$\|f_t - f_s\|_{H^{-\tilde\vartheta}} \le \|f_t - f_s\|_{H^{-1}}^{1-\zeta} \|f_t - f_s\|_{H^{-\beta}}^{\zeta}.$$

Furthermore, for $\rho \in (0, \frac{1}{4})$, we can calculate $\|f\|_{C^{\rho\zeta}([0,T],H^{-\tilde\vartheta})}$ as

$$\begin{aligned}
\|f\|_{C^{\rho\zeta}([0,T],H^{-\tilde\vartheta})} &\sim \sup_{0\le s<t\le T} \frac{\|f_t - f_s\|_{H^{-\tilde\vartheta}}}{|t-s|^{\rho\zeta}} \\
&\le \sup_{0\le s<t\le T} \frac{\|f_t - f_s\|_{H^{-1}}^{1-\zeta} \|f_t - f_s\|_{H^{-\beta}}^{\zeta}}{|t-s|^{\rho\zeta}} \\
&\lesssim \Big(\sup_{t\in[0,T]} \|f_t\|_{H^{-1}}^{1-\zeta}\Big) \|f\|_{C^\rho([0,T],H^{-\beta})}^{\zeta}.
\end{aligned} \tag{4.11}$$

Proposition 4.2 implies that, $\mathbb{P}$-a.s., $f \in C^{\rho\zeta}([0,T], H^{-\tilde\vartheta})$. Next, we have

$$\Big\| \int_0^t u_r \cdot \nabla\xi_r \, dr - \int_0^s u_r \cdot \nabla\xi_r \, dr \Big\|_{H^{-\tilde\vartheta}} \le \int_s^t \|u_r \cdot \nabla\xi_r\|_{H^{-\tilde\vartheta}} \, dr \lesssim \int_s^t \|u_r \xi_r\|_{H^{1-\tilde\vartheta}} \, dr$$

and by Lemma 2.2, $\|u_r\xi_r\|_{H^{1-\tilde\vartheta}} \lesssim \|\xi_r\|_{L^2} \|u_r\|_{H^{2-\tilde\vartheta}} \lesssim \|\xi_r\|_{L^2}^2 \le \|\xi_0\|_{L^2}^2$; thus, the function $t \mapsto \int_0^t u_r \cdot \nabla\xi_r \, dr \in H^{-\tilde\vartheta}$ is Lipschitz continuous. With slightly more effort, one can show that it actually belongs to $C^1([0,T], H^{-\tilde\vartheta})$ by using the fact $\xi \in C([0,T], L^2)$; similar result holds for $t \mapsto \int_0^t \bar u_r \cdot \nabla\bar\xi_r \, dr \in H^{-\tilde\vartheta}$.

Summarizing the above arguments, for every $\tilde\vartheta \in (1, \frac{3}{2})$ and $\rho \in (0, \frac{1}{4})$, we deduce that, $\mathbb{P}$-a.s., $X \in C^{\rho\zeta}([0,T], H^{-\tilde\vartheta})$ for all $T \ge 1$, where $\zeta \in (0,1)$ is defined as above. Therefore, according to [27, Theorem 1], there exists a linear map $\mathcal{G}$, such that

$$\xi - \bar\xi = \mathcal{G}(X) = \mathcal{G}(f) - \mathcal{G}\Big(\int_0^\cdot (u_s \cdot \nabla\xi_s - \bar u_s \cdot \nabla\bar\xi_s) \, ds \Big); \tag{4.12}$$

furthermore, the following result holds:

$$\sup_{t\in[0,T]} \|\mathcal{G}(f_t)\|_{H^{-\tilde\vartheta}} \lesssim \|f\|_{C^{\rho\zeta}([0,T],H^{-\tilde\vartheta})}. \tag{4.13}$$

Now we will deal with the last term in (4.12). As $\int_0^\cdot (u_s \cdot \nabla\xi_s - \bar u_s \cdot \nabla\bar\xi_s) \, ds$ belongs to $C^1([0,T], H^{-\tilde\vartheta})$, then by [27, Theorem 1], it holds

$$\begin{aligned}
&\mathcal{G}\Big(\int_0^\cdot (u_s \cdot \nabla\xi_s - \bar u_s \cdot \nabla\bar\xi_s) \, ds \Big)(t) \\
&= \int_0^t e^{(\kappa+\nu)(t-s)\Delta} (u_s \cdot \nabla\xi_s - \bar u_s \cdot \nabla\bar\xi_s) \, ds \\
&= \int_0^t e^{(\kappa+\nu)(t-s)\Delta} \big[(u_s - \bar u_s) \cdot \nabla\xi_s\big] ds + \int_0^t e^{(\kappa+\nu)(t-s)\Delta} \big[\bar u_s \cdot \nabla(\xi_s - \bar\xi_s)\big] ds.
\end{aligned} \tag{4.14}$$

For the first term, we can use the standard heat kernel estimate (see e.g. [12, Section 2]) to get

$$\Big\| \int_0^t e^{(\kappa+\nu)(t-s)\Delta} \big[(u_s - \bar u_s) \cdot \nabla\xi_s\big] ds \Big\|_{H^{-\tilde\vartheta}}^2 \lesssim \frac{1}{\kappa+\nu} \int_0^t \|(u_s - \bar u_s) \cdot \nabla\xi_s\|_{H^{-\tilde\vartheta-1}}^2 \, ds. \tag{4.15}$$

Noting that $\|(u_s - \bar{u}_s) \cdot \nabla \xi_s\|_{H^{-\tilde{\vartheta}-1}} = \|(u_s - \bar{u}_s)\, \xi_s\|_{H^{-\tilde{\vartheta}}}$, then for $\tilde{\vartheta} \in (1, \frac{3}{2})$ and $\phi \in C^\infty(\mathbb{T}^2)$, Lemma 2.2 yields

$$\big|\langle (u_s - \bar{u}_s)\, \xi_s, \phi\rangle\big| \le \|u_s - \bar{u}_s\|_{H^{1-\tilde{\vartheta}}} \|\xi_s \phi\|_{H^{\tilde{\vartheta}-1}} \lesssim \|\xi_s - \bar{\xi}_s\|_{H^{-\tilde{\vartheta}}} \|\xi_s\|_{H^{\frac{1}{2}}} \|\phi\|_{H^{\tilde{\vartheta}-\frac{1}{2}}}.$$

Therefore, by duality of Sobolev norms and Lemma 2.4,

$$\|(u_s - \bar{u}_s)\, \xi_s\|_{H^{-\tilde{\vartheta}}} \lesssim \|\xi_s - \bar{\xi}_s\|_{H^{-\tilde{\vartheta}}} \|\xi_s\|_{H^{\frac{1}{2}}} \lesssim \|\xi_s - \bar{\xi}_s\|_{H^{-\tilde{\vartheta}}} \|\xi_s\|_{H^1}^{\frac{1}{2}} \|\xi_0\|_{L^2}^{\frac{1}{2}}.$$

Applying the above result to (4.15), we obtain

$$\Big\| \int_0^t e^{(\kappa+\nu)(t-s)\Delta} \big[(u_s - \bar{u}_s) \cdot \nabla \xi_s\big]\, ds \Big\|_{H^{-\tilde{\vartheta}}}^2 \lesssim \frac{1}{\kappa+\nu} \int_0^t \|\xi_s - \bar{\xi}_s\|_{H^{-\tilde{\vartheta}}}^2 \|\xi_0\|_{L^2} \|\xi_s\|_{H^1}\, ds. \tag{4.16}$$

The latter term of (4.14) can be treated similarly as follows:

$$\Big\| \int_0^t e^{(\kappa+\nu)(t-s)\Delta} \big[\bar{u}_s \cdot \nabla (\xi_s - \bar{\xi}_s)\big]\, ds \Big\|_{H^{-\tilde{\vartheta}}}^2 \lesssim \frac{1}{\kappa+\nu} \int_0^t \big\|\bar{u}_s\, (\xi_s - \bar{\xi}_s)\big\|_{H^{-\tilde{\vartheta}}}^2\, ds. \tag{4.17}$$

Let $\phi \in C^\infty(\mathbb{T}^2)$ be a test function; for any fixed $s \in [0, T]$, we denote $A(\bar{u}_s \phi) := \int_{\mathbb{T}^2} (\bar{u}_s \phi)(x)\, dx$. As $\int_{\mathbb{T}^2} (\xi_s - \bar{\xi}_s)(x)\, dx = 0$, we have $\langle \xi_s - \bar{\xi}_s, A(\bar{u}_s \phi)\rangle = 0$, and therefore

$$\big|\langle \bar{u}_s\, (\xi_s - \bar{\xi}_s), \phi\rangle\big| = \big|\langle \xi_s - \bar{\xi}_s, \bar{u}_s \phi - A(\bar{u}_s \phi)\rangle\big| \le \|\xi_s - \bar{\xi}_s\|_{H^{-\tilde{\vartheta}}} \|\bar{u}_s \phi - A(\bar{u}_s \phi)\|_{H^{\tilde{\vartheta}}}.$$

By Poincaré's inequality, Lemmas 2.2 and 2.3, for $\tilde{\vartheta} \in (1, \frac{3}{2})$ and $\varepsilon \in (0, \frac{1}{2})$, we get

$$\begin{aligned} \|\bar{u}_s \phi - A(\bar{u}_s \phi)\|_{H^{\tilde{\vartheta}}} \lesssim \|\nabla (\bar{u}_s \phi)\|_{H^{\tilde{\vartheta}-1}} &\le \|(\nabla \bar{u}_s) \phi\|_{H^{\tilde{\vartheta}-1}} + \|\bar{u}_s \cdot \nabla \phi\|_{H^{\tilde{\vartheta}-1}} \\ &\lesssim \|\nabla \bar{u}_s\|_{H^{\frac{1}{2}}} \|\phi\|_{H^{\tilde{\vartheta}-\frac{1}{2}}} + \|\bar{u}_s\|_{C^{\tilde{\vartheta}-1+\varepsilon}} \|\nabla \phi\|_{H^{\tilde{\vartheta}-1}} \\ &\lesssim \|\bar{u}_s\|_{H^2} \|\phi\|_{H^{\tilde{\vartheta}}}. \end{aligned}$$

Summarizing these arguments leads to

$$\|\bar{u}_s\, (\xi_s - \bar{\xi}_s)\|_{H^{-\tilde{\vartheta}}} \lesssim \|\xi_s - \bar{\xi}_s\|_{H^{-\tilde{\vartheta}}} \|\bar{\xi}_s\|_{H^1}.$$

Inserting the above estimate to (4.17), we obtain

$$\Big\| \int_0^t e^{(\kappa+\nu)(t-s)\Delta} \big[\bar{u}_s \cdot \nabla (\xi_s - \bar{\xi}_s)\big]\, ds \Big\|_{H^{-\tilde{\vartheta}}}^2 \lesssim \frac{1}{\kappa+\nu} \int_0^t \|\xi_s - \bar{\xi}_s\|_{H^{-\tilde{\vartheta}}}^2 \|\bar{\xi}_s\|_{H^1}^2\, ds. \tag{4.18}$$

Combining (4.16) and (4.18), by (4.14) we get

$$\begin{aligned} &\Big\| \mathcal{G}\Big(\int_0^{\cdot} (u_s \cdot \nabla \xi_s - \bar{u}_s \cdot \nabla \bar{\xi}_s)\, ds \Big)(t) \Big\|_{H^{-\tilde{\vartheta}}}^2 \\ &\lesssim \frac{1}{\kappa+\nu} \int_0^t \|\xi_s - \bar{\xi}_s\|_{H^{-\tilde{\vartheta}}}^2 \big(\|\xi_0\|_{L^2} \|\xi_s\|_{H^1} + \|\bar{\xi}_s\|_{H^1}^2 \big)\, ds. \end{aligned} \tag{4.19}$$

According to (4.12), for $\tilde{\vartheta} \in (1, \frac{3}{2})$, we have

$$\|\xi_t - \bar{\xi}_t\|_{H^{-\tilde{\vartheta}}}^2 \lesssim \|\mathcal{G}(f_t)\|_{H^{-\tilde{\vartheta}}}^2 + \frac{1}{\kappa+\nu} \int_0^t \|\xi_s - \bar{\xi}_s\|_{H^{-\tilde{\vartheta}}}^2 \big(\|\xi_0\|_{L^2} \|\xi_s\|_{H^1} + \|\bar{\xi}_s\|_{H^1}^2 \big)\, ds.$$

By Grönwall's inequality, it holds

$$\sup_{t\in[0,T]}\|\xi_t-\bar{\xi}_t\|^2_{H^{-\tilde{\vartheta}}}\lesssim\Big(\sup_{t\in[0,T]}\|\mathcal{G}(f_t)\|^2_{H^{-\tilde{\vartheta}}}\Big)\exp\bigg(\frac{1}{\kappa+\nu}\int_0^T\big(\|\xi_0\|_{L^2}\|\xi_s\|_{H^1}+\|\bar{\xi}_s\|^2_{H^1}\big)\,ds\bigg).\tag{4.20}$$

Notice that ξ and $\bar{\xi}$ satisfy (2.1) and (2.2) respectively, then Hölder's inequality yields

$$\begin{aligned}\int_0^T\big(\|\xi_0\|_{L^2}\|\xi_s\|_{H^1}+\|\bar{\xi}_s\|^2_{H^1}\big)\,ds&\lesssim T^{\frac12}\|\xi_0\|_{L^2}\Big(\int_0^T\|\xi_s\|^2_{H^1}\,ds\Big)^{\frac12}+\int_0^T\|\bar{\xi}_s\|^2_{H^1}\,ds\\&\lesssim\kappa^{-\frac12}T^{\frac12}\|\xi_0\|^2_{L^2}+(\kappa+\nu)^{-1}\|\xi_0\|^2_{L^2}.\end{aligned}$$

Substituting this estimate into (4.20) and setting

$$C=\frac{\kappa^{-\frac12}T^{\frac12}+(\kappa+\nu)^{-1}}{\kappa+\nu},$$

we arrive at

$$\sup_{t\in[0,T]}\|\xi_t-\bar{\xi}_t\|^2_{H^{-\tilde{\vartheta}}}\lesssim\Big(\sup_{t\in[0,T]}\|\mathcal{G}(f_t)\|^2_{H^{-\tilde{\vartheta}}}\Big)\exp\big(C\|\xi_0\|^2_{L^2}\big).$$

Furthermore, by (4.13), we have

$$\|\xi-\bar{\xi}\|_{C([0,T],H^{-\tilde{\vartheta}})}\lesssim\|f\|_{C^{\rho\zeta}([0,T],H^{-\tilde{\vartheta}})}\exp\Big(\frac{C}{2}\|\xi_0\|^2_{L^2}\Big).\tag{4.21}$$

Taking expectation and then applying (4.11), for $\tilde{\vartheta}\in(1,\frac32)$, Lemmas 4.2 and 4.3 yield

$$\begin{aligned}&\mathbb{E}\Big[\|\xi-\bar{\xi}\|_{C([0,T],H^{-\tilde{\vartheta}})}\Big]\\&\lesssim\mathbb{E}\Big[\|f\|_{C^{\rho}([0,T],H^{-\beta})}\Big]^{\zeta}\Big(\sup_{t\in[0,T]}\|f_t\|_{H^{-1}}\Big)^{1-\zeta}\exp\Big(\frac{C}{2}\|\xi_0\|^2_{L^2}\Big)\\&\lesssim T^{1+\frac{\gamma}{2}(\zeta-1)}\|\xi_0\|_{L^2}\big(\nu^{1+\frac{\gamma}{2}}\alpha^{-\epsilon}+\nu^{\frac12}\|\theta\|_{\ell^\infty}\big)^{\zeta}\big(\nu\kappa^{-\frac12}+\kappa^{-\frac{\gamma}{2}}\big)^{1-\zeta}\exp\Big(\big(1+\frac{C}{2}\big)\|\xi_0\|^2_{L^2}\Big),\end{aligned}\tag{4.22}$$

where the last step follows from

$$\big(1+\|\xi_0\|_{L^2}\big)^{1+\zeta}<\big(1+\|\xi_0\|_{L^2}\big)^2\le2\big(1+\|\xi_0\|^2_{L^2}\big)\le2\exp(\|\xi_0\|^2_{L^2}),\quad\zeta\in(0,1).$$

If we take $C_1\sim T^{1+\frac{\gamma}{2}(\zeta-1)}\big(\nu\kappa^{-\frac12}+\kappa^{-\frac{\gamma}{2}}\big)^{1-\zeta}$ and $C_2=1+\frac{C}{2}$, then (4.22) can be rewritten as

$$\mathbb{E}\Big[\|\xi-\bar{\xi}\|_{C([0,T],H^{-\tilde{\vartheta}})}\Big]\le C_1\|\xi_0\|_{L^2}\exp\big(C_2\|\xi_0\|^2_{L^2}\big)\big(\nu^{1+\frac{\gamma}{2}}\alpha^{-\epsilon}+\nu^{\frac12}\|\theta\|_{\ell^\infty}\big)^{\zeta}.\tag{4.23}$$

Once we have the estimate for $\tilde{\vartheta}>1$, then for $\hat{\vartheta}\in(0,1)$, we deduce from Lemma 2.4 that

$$\begin{aligned}\mathbb{E}\Big[\|\xi-\bar{\xi}\|_{C([0,T],H^{-\hat{\vartheta}})}\Big]&\le\mathbb{E}\Big[\|\xi-\bar{\xi}\|^{\hat{\vartheta}/\tilde{\vartheta}}_{C([0,T],H^{-\tilde{\vartheta}})}\Big]\,\|\xi_0\|^{1-\hat{\vartheta}/\tilde{\vartheta}}_{L^2}\\&\le C_1^{\hat{\vartheta}/\tilde{\vartheta}}\|\xi_0\|_{L^2}\exp\big(C_2(\hat{\vartheta}/\tilde{\vartheta})\|\xi_0\|^2_{L^2}\big)\,\big(\nu^{1+\frac{\gamma}{2}}\alpha^{-\epsilon}+\nu^{\frac12}\|\theta\|_{\ell^\infty}\big)^{\zeta\hat{\vartheta}/\tilde{\vartheta}}\\&\le C_1\|\xi_0\|_{L^2}\exp\big(C_2\|\xi_0\|^2_{L^2}\big)\big(\nu^{1+\frac{\gamma}{2}}\alpha^{-\epsilon}+\nu^{\frac12}\|\theta\|_{\ell^\infty}\big)^{\zeta},\end{aligned}\tag{4.24}$$

where in the last step we have used the fact that $C_1>1$, $C_2>0$ and $\hat{\vartheta}/\tilde{\vartheta}<1$. Hence it holds $C_1^{\hat{\vartheta}/\tilde{\vartheta}}<C_1$ and $C_2\,\hat{\vartheta}/\tilde{\vartheta}<C_2$. Combining (4.23) and (4.24), we obtain the final conclusion of Theorem 1.1 for every $\vartheta>0$. □

4.2 Proof of Theorem 1.2

In order to prove Theorem 1.2, we first present several simple results as follows.

Lemma 4.4. *For the solution of* (1.1)*, the energy equality holds with probability one:*

$$\frac{d}{dt}\|\xi\|_{L^2}^2 = -2\kappa\,\|\xi\|_{H^1}^2.$$

Proof. This result is well-known; we present the proof for completeness. Notice that

$$\frac{d}{dt}\|\xi\|_{L^2}^2 = \frac{d}{dt}\langle\xi,\xi\rangle = \int_{\mathbb{T}^2} 2\xi\,\frac{\partial\xi}{\partial t}\,dx.$$

By (1.1) , we can further get

$$\begin{aligned}\int_{\mathbb{T}^2} 2\xi\,\frac{\partial\xi}{\partial t}\,dx &= 2\kappa\int_{\mathbb{T}^2}\xi\,\Delta\xi\,dx - 2\int_{\mathbb{T}^2}\xi u\cdot\nabla\xi\,dx - 2\int_{\mathbb{T}^2}\xi\,\boldsymbol{b}\cdot\nabla\xi\,dx\\ &= -2\kappa\int_{\mathbb{T}^2}|\nabla\xi|^2\,dx + \int_{\mathbb{T}^2}\operatorname{div}(u)\,\xi^2\,dx + \int_{\mathbb{T}^2}\operatorname{div}(\boldsymbol{b})\,\xi^2\,dx\\ &= -2\kappa\,\|\xi\|_{H^1}^2.\end{aligned}$$

□

Recall that $\{\mathcal{F}_t\}_{t\ge 0}$ is the filtration on the probability space Ω. The following estimate is an easy consequence of Theorem 1.1.

Lemma 4.5. *For any $n\in\mathbb{N}$, let $\{\bar\xi_t^n\}_{t\ge n}$ be the solution to*

$$\begin{cases}\partial_t\bar\xi^n + \bar u^n\cdot\nabla\bar\xi^n = (\kappa+\nu)\Delta\bar\xi^n, & t\ge n,\\ \bar u^n = K*\bar\xi^n, \quad \bar\xi^n|_{t=n} = \xi_n.\end{cases} \tag{4.25}$$

Then it holds, $\mathbb{P}$-a.s.,

$$\mathbb{E}\Big[\sup_{t\in[n,n+1]}\|\xi_t-\bar\xi_t^n\|_{H^{-1}}\big|\mathcal{F}_n\Big] \le C_1\|\xi_n\|_{L^2}\exp\big(C_2\|\xi_0\|_{L^2}^2\big)\big(\nu^{1+\frac{\gamma}{2}}\alpha^{-\epsilon}+\nu^{\frac12}\|\theta\|_{\ell^\infty}\big)^{\zeta},$$

where C_1 and C_2 are defined as in Theorem 1.1 and are independent of n.

Proof. Notice that if we take $\vartheta = 1$ and $T = 1$ in Theorem 1.1, then we get a quantitative estimate on the distance between the solutions of (1.1) and (1.4), both with the same deterministic initial value ξ_0. Since the Ornstein-Uhlenbeck flow $\boldsymbol{b}$ in (1.1) is a stationary process, such estimate holds on any unit interval of the form $[n,n+1]$, as long as we restart (1.4) at the time $t=n$ with the same value ξ_n. However, as ξ_n is random, we need to take conditional expectation with respect to $\mathcal{F}_n$ and get the desired result. □

Lemma 4.6. *For all $n\in\mathbb{N}$, decay of L^2-norm of the solution to* (4.25) *satisfies*

$$\|\bar\xi_t^n\|_{L^2}^2 \le e^{-\lambda_1(t-n)}\,\|\xi_n\|_{L^2}^2,$$

where $\lambda_1 := 8\pi^2(\kappa+\nu)$ is the principal eigenvalue of $(\kappa+\nu)\Delta$ on $\mathbb{T}^2$.

Proof. By (4.25), we use similar method as Lemma 4.4 to get

$$\frac{d}{dt}\|\bar\xi^n\|_{L^2}^2 = -2(\kappa+\nu)\,\|\nabla\bar\xi^n\|_{L^2}^2.$$

Poincaré's inequality yields $\|\bar\xi^n\|_{L^2}^2 \le \frac{1}{4\pi^2}\,\|\nabla\bar\xi^n\|_{L^2}^2$, thus

$$\frac{d}{dt}\|\bar\xi^n\|_{L^2}^2 \le -8\pi^2(\kappa+\nu)\|\bar\xi^n\|_{L^2}^2.$$

Solving the differential inequality gives us the desired estimate. □

Remark 4.7. *In particular, if we consider* (4.25) *with initial time $n = 0$, then it reduces to equation* (1.4)*, and we get the decay rate of L^2-norm for the solution to* (1.4) *as*

$$\|\bar{\xi}_t\|_{L^2}^2 \le e^{-\lambda_1 t}\,\|\xi_0\|_{L^2}^2.$$

On the basis of the above results, now we can provide

Proof of Theorem 1.2. Since the proof is rather long, we divide it into the following four steps.

Step 1. Let $R > 0$ be given as in the statement of Theorem 1.2, and denote

$$c_1 := C_1^{1/2} \exp\Big(\frac{C_2 R^2}{2}\Big)\big(\nu^{1+\frac{\gamma}{2}}\alpha^{-\epsilon} + \nu^{\frac{1}{2}}\|\theta\|_{\ell^\infty}\big)^{\zeta/2},$$

which is sufficiently small by taking α big and $\|\theta\|_{\ell^\infty}$ small. Recall that we have assumed $\|\xi_0\|_{L^2} \le R$; then by Lemma 4.5, we have, $\mathbb{P}$-a.s.,

$$\mathbb{E}\Big[\sup_{t\in[n,n+1]} \|\xi_t - \bar{\xi}_t^n\|_{H^{-1}}\big|\mathcal{F}_n\Big] \le c_1^2\|\xi_n\|_{L^2}. \tag{4.26}$$

Define the event

$$A_n := \Big\{\omega \in \Omega : \sup_{t\in[n,n+1]} \|\xi_t(\omega) - \bar{\xi}_t^n(\omega)\|_{H^{-1}} > c_1\|\xi_n(\omega)\|_{L^2}\Big\},$$

and A_n^c is its complement; we want to prove

$$\mathbb{P}(A_n^c) > 1 - c_1, \tag{4.27}$$

which is an easy consequence of

$$\mathbb{P}\big(A_n|\mathcal{F}_n\big) = \mathbb{E}\big[\mathbf{1}_{A_n}|\mathcal{F}_n\big] \le c_1.$$

Indeed, for any $B \in \mathcal{F}_n$, it holds

$$\begin{aligned}
\int_B \mathbb{E}\big[\mathbf{1}_{A_n}|\mathcal{F}_n\big]\,d\mathbb{P} = \int_B \mathbf{1}_{A_n}\,d\mathbb{P} &\le \int_{A_n\cap B} \frac{\sup_{t\in[n,n+1]}\|\xi_t - \bar{\xi}_t^n\|_{H^{-1}}}{c_1\|\xi_n\|_{L^2}}\,d\mathbb{P}\\
&\le c_1^{-1}\int_B \|\xi_n\|_{L^2}^{-1} \sup_{t\in[n,n+1]}\|\xi_t - \bar{\xi}_t^n\|_{H^{-1}}\,d\mathbb{P}\\
&= c_1^{-1}\int_B \mathbb{E}\Big[\|\xi_n\|_{L^2}^{-1} \sup_{t\in[n,n+1]}\|\xi_t - \bar{\xi}_t^n\|_{H^{-1}}\big|\mathcal{F}_n\Big]\,d\mathbb{P}.
\end{aligned}$$

Then by the arbitrariness of B and (4.26), we obtain

$$\begin{aligned}
\mathbb{E}\big[\mathbf{1}_{A_n}|\mathcal{F}_n\big] &\le c_1^{-1}\mathbb{E}\Big[\|\xi_n\|_{L^2}^{-1} \sup_{t\in[n,n+1]}\|\xi_t - \bar{\xi}_t^n\|_{H^{-1}}\big|\mathcal{F}_n\Big]\\
&= c_1^{-1}\|\xi_n\|_{L^2}^{-1}\,\mathbb{E}\Big[\sup_{t\in[n,n+1]}\|\xi_t - \bar{\xi}_t^n\|_{H^{-1}}\big|\mathcal{F}_n\Big]\\
&\le c_1^{-1}\|\xi_n\|_{L^2}^{-1}\,c_1^2\|\xi_n\|_{L^2} = c_1.
\end{aligned} \tag{4.28}$$

Step 2. In order to estimate the decay rate of L^2-norm for ξ, we first try to find the relationship between $\|\xi_n\|_{L^2}$ and $\|\xi_{n+1}\|_{L^2}$ for any $n \in \mathbb{N}$, then we use iteration to extend the conclusion to $\|\xi_t\|_{L^2}$ with initial value ξ_0 for all $t \ge 0$.

Notice that for $t \in [n, n+1]$, Lemma 4.6 and inequality (4.27) yield, with probability no less than $1 - c_1$,

$$\|\xi_t\|_{H^{-1}}^2 \le 2\|\bar{\xi}_t^n\|_{H^{-1}}^2 + 2\|\xi_t - \bar{\xi}_t^n\|_{H^{-1}}^2 \le 2\|\xi_n\|_{L^2}^2 \big(e^{-\lambda_1(t-n)} + c_1^2\big).$$

Besides, according to Lemma 2.4, we have $\|\xi_t\|_{L^2}^2 \le \|\xi_t\|_{H^{-1}} \|\xi_t\|_{H^1}$. Hence, combining Lemma 4.4 with the above two inequalities leads to

$$\frac{d}{dt}\|\xi_t\|_{L^2}^2 \le -2\kappa \frac{\|\xi_t\|_{L^2}^4}{\|\xi_t\|_{H^{-1}}^2} \le -\frac{\kappa \|\xi_t\|_{L^2}^4}{\|\xi_n\|_{L^2}^2 \big(e^{-\lambda_1(t-n)} + c_1^2\big)}, \quad t \in [n, n+1].$$

Solving the differential inequality and then letting $t = n+1$, we get, on the event A_n^c,

$$\|\xi_{n+1}\|_{L^2}^2 \le \frac{\|\xi_n\|_{L^2}^2}{1 + \frac{\kappa}{\lambda_1 c_1^2} \log \frac{1 + c_1^2 e^{\lambda_1}}{1 + c_1^2}} =: c_2^2 \|\xi_n\|_{L^2}^2. \tag{4.29}$$

As the L^2-norm of ξ is decreasing, we additionally apply (4.29) to further get

$$\mathbb{E}\|\xi_{n+1}\|_{L^2} = \mathbb{E}\big[\|\xi_{n+1}\|_{L^2} \mathbf{1}_{A_n}\big] + \mathbb{E}\big[\|\xi_{n+1}\|_{L^2} \mathbf{1}_{A_n^c}\big] \le \mathbb{E}\big[\|\xi_n\|_{L^2} \mathbf{1}_{A_n}\big] + \mathbb{E}\big[c_2 \|\xi_n\|_{L^2}\big].$$

Using the property of conditional expectation, (4.28) yields

$$\begin{aligned} \mathbb{E}\|\xi_{n+1}\|_{L^2} &\le \mathbb{E}\Big[\mathbb{E}\big[\|\xi_n\|_{L^2} \mathbf{1}_{A_n} \big| \mathcal{F}_n\big]\Big] + c_2\, \mathbb{E}\|\xi_n\|_{L^2} \\ &= \mathbb{E}\Big[\|\xi_n\|_{L^2}\, \mathbb{E}\big[\mathbf{1}_{A_n} \big| \mathcal{F}_n\big]\Big] + c_2\, \mathbb{E}\|\xi_n\|_{L^2} \\ &\le (c_1 + c_2)\, \mathbb{E}\|\xi_n\|_{L^2}. \end{aligned}$$

Afterwards, we denote $c_0 := c_1 + c_2$ for simplicity. By induction, for any $n \in \mathbb{N}$, it holds

$$\mathbb{E}\|\xi_n\|_{L^2} \le c_0^n \|\xi_0\|_{L^2}. \tag{4.30}$$

Step 3. To show the enhanced dissipation property of Ornstein-Uhlenbeck flow, the constant $c_0 > 0$ has to be sufficiently small. We start with proving the following quantity in the definition of c_2 can be very large under suitable choice of parameters:

$$\frac{\kappa}{\lambda_1 c_1^2} \log \frac{1 + c_1^2 e^{\lambda_1}}{1 + c_1^2} = \frac{\kappa}{\lambda_1 c_1^2} \log \bigg(\frac{c_1^2}{1 + c_1^2} \big(e^{\lambda_1} - 1\big) + 1 \bigg).$$

First, fix a sufficiently large ν, then $\lambda_1 = 8\pi^2(\kappa + \nu)$ is also very large. Next, we let α be large and $\|\theta\|_{\ell^\infty}$ be small enough, and thus c_1 is sufficiently small by its definition. In particular, we can assume that $\frac{c_1^2}{1+c_1^2}(e^{\lambda_1} - 1) \in (0, 1]$. As $\log(1 + x) \ge x \log 2$ for $x \in (0, 1]$, we have

$$\frac{\kappa}{\lambda_1 c_1^2} \log \bigg(\frac{c_1^2}{1 + c_1^2} \big(e^{\lambda_1} - 1\big) + 1 \bigg) \ge \frac{\kappa \log 2}{\lambda_1 c_1^2} \cdot \frac{c_1^2}{1 + c_1^2} \big(e^{\lambda_1} - 1\big) \ge \frac{\kappa \log 2}{2} \cdot \frac{e^{\lambda_1} - 1}{\lambda_1},$$

where in the last step we have used the fact that $c_1^2 + 1 \le 2$. Since λ_1 is large, we deduce that the left-hand side is also very large; as a result, c_2 can be very small. Combined with the smallness of c_1, we conclude that $c_0 = c_1 + c_2$ is also a small constant.

Step 4. Based on the previous discussions, we can now prove the final conclusion of Theorem 1.2. Define $\lambda_0 := -\log c_0 > 0$, which can be assumed to be greater than $\lambda(1+p)$, where λ and p are given in the statement of Theorem 1.2. Then by (4.30), it holds

$$\mathbb{E}\Big[\sup_{t \in [n, n+1]} \|\xi_t\|_{L^2}\Big] = \mathbb{E}\|\xi_n\|_{L^2} \le e^{-\lambda_0 n} \|\xi_0\|_{L^2}.$$

We define the events

$$E_n := \Big\{\omega \in \Omega : \sup_{t\in[n,n+1]} \|\xi_t(\omega)\|_{L^2} > e^{-\lambda n}\|\xi_0\|_{L^2}\Big\}, \quad n \in \mathbb{N}.$$

By Markov's inequality, it holds

$$\sum_{n\in\mathbb{N}} \mathbb{P}(E_n) \le \sum_{n\in\mathbb{N}} \frac{e^{\lambda n}}{\|\xi_0\|_{L^2}} \mathbb{E}\Big[\sup_{t\in[n,n+1]} \|\xi_t\|_{L^2}\Big] \le \sum_{n\in\mathbb{N}} e^{(\lambda-\lambda_0)n} < +\infty.$$

Furthermore, Borel-Cantelli's lemma implies that for $\mathbb{P}$-a.s. $\omega \in \Omega$, there exists $N(\omega) \in \mathbb{N}$, such that for any $n > N(\omega)$,

$$\sup_{t\in[n,n+1]} \|\xi_t\|_{L^2} \le e^{-\lambda n}\|\xi_0\|_{L^2}.$$

For the case $0 \le n \le N(\omega)$, we have

$$\sup_{t\in[n,n+1]} \|\xi_t\|_{L^2} = \|\xi_n\|_{L^2} = e^{\lambda n} e^{-\lambda n}\|\xi_n\|_{L^2} \le e^{\lambda N(\omega)} e^{-\lambda n}\|\xi_0\|_{L^2}.$$

If we let $C(\omega) = e^{\lambda(1+N(\omega))}$, it is not difficult to verify that for $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$\|\xi_t\|_{L^2} \le C(\omega)e^{-\lambda t}\|\xi_0\|_{L^2}, \quad \forall\, t \ge 0. \tag{4.31}$$

As for the finite p-th moment of $C(\omega)$, a similar proof can be found in [12, Section 5.2], so we omit it here. □

5 Proof of Proposition 3.3

We devote this section to the proof of Proposition 3.3. We first divide the desired quantity into a summation part and two integrals. For the summation term, we follow the idea of [36] and use equation (1.1) to further decompose it. Then we will estimate each of the decomposed terms and the two integrals separately. Finally, in order to obtain the desired estimate, we need to make some restrictions on the parameters.

5.1 Decomposition

We decompose the quantity we want to estimate as follows:

$$\begin{aligned}
&\xi_{n\delta} - \xi_{m\delta} - (\kappa+\nu)\int_{m\delta}^{n\delta} \Delta\xi_s\, ds + \int_{m\delta}^{n\delta} u_s \cdot \nabla\xi_s\, ds \\
&= \xi_{n\delta} - \xi_{m\delta} - \delta(\kappa+\nu)\sum_{h=m}^{n-1} \Delta\xi_{h\delta} + \delta\sum_{h=m}^{n-1} (u_{h\delta}\cdot\nabla\xi_{h\delta}) + I_a + I_b,
\end{aligned} \tag{5.1}$$

where

$$I_a := (\kappa+\nu)\int_{m\delta}^{n\delta} \Delta\big(\xi_{[s]} - \xi_s\big)ds, \quad I_b := \int_{m\delta}^{n\delta} \big(u_s\cdot\nabla\xi_s - u_{[s]}\cdot\nabla\xi_{[s]}\big)ds,$$

and $[s] := \sup_{j\in\mathbb{N}}\{j\delta : j\delta \le s\}$. We first consider

$$\xi_{n\delta} - \xi_{m\delta} - \delta(\kappa+\nu)\sum_{h=m}^{n-1} \Delta\xi_{h\delta} + \delta\sum_{h=m}^{n-1} (u_{h\delta}\cdot\nabla\xi_{h\delta}), \quad n > m,$$

which can also be written as

$$\xi_{(n+1)\delta} - \xi_{m\delta} - \delta(\kappa+\nu)\sum_{h=m}^{n} \Delta\xi_{h\delta} + \delta\sum_{h=m}^{n}(u_{h\delta}\cdot\nabla\xi_{h\delta}), \quad n \geq m.$$

According to Definition 2.1, for every $h = 0, 1, \ldots, T/\delta - 1$, we have

$$\begin{aligned} \xi_{(h+1)\delta} - \xi_{h\delta} &= -\int_{h\delta}^{(h+1)\delta} u_s\cdot\nabla\xi_s\,ds - \int_{h\delta}^{(h+1)\delta} \boldsymbol{b}(s)\cdot\nabla\xi_s\,ds + \kappa\int_{h\delta}^{(h+1)\delta}\Delta\xi_s\,ds \\ &=: I_1(h) + I_2(h) + I_3(h). \end{aligned} \tag{5.2}$$

Furthermore, we can make the following decomposition for $I_2(h)$:

$$\begin{aligned} I_2(h) &= -\int_{h\delta}^{(h+1)\delta} \boldsymbol{b}(s)\cdot\nabla(\xi_s - \xi_{h\delta})\,ds - \int_{h\delta}^{(h+1)\delta}\boldsymbol{b}(s)\cdot\nabla\xi_{h\delta}\,ds \\ &=: I_{21}(h) + I_{22}(h) + I_{23}(h) + I_{24}(h) + I_{25}(h), \end{aligned}$$

where $I_{2i}(h), i = 1, \ldots, 5$ are defined as

$$\begin{aligned} I_{21}(h) &= \int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s} \boldsymbol{b}(s)\cdot\nabla(u_r\cdot\nabla\xi_r)\,drds, \\ I_{22}(h) &= \int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s} \boldsymbol{b}(s)\cdot\nabla\big(\boldsymbol{b}(r)\cdot\nabla(\xi_r - \xi_{h\delta})\big)\,drds, \\ I_{23}(h) &= \int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s} \boldsymbol{b}(s)\cdot\nabla\big(\boldsymbol{b}(r)\cdot\nabla\xi_{h\delta}\big)\,drds, \\ I_{24}(h) &= -\kappa\int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s} \boldsymbol{b}(s)\cdot\nabla(\Delta\xi_r)\,drds, \\ I_{25}(h) &= -\int_{h\delta}^{(h+1)\delta} \boldsymbol{b}(s)\cdot\nabla\xi_{h\delta}\,ds. \end{aligned}$$

By the definition of $\boldsymbol{b}$, the term $I_{23}(h)$ can be rewritten as follows:

$$\begin{aligned} I_{23}(h) &= 4\nu\sum_{k,k'\in\mathbb{Z}_0^2}\int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s}\theta_k\sigma_k\eta^{\alpha,k}(s)\cdot\nabla(\theta_{k'}\sigma_{k'}\eta^{\alpha,k'}(r)\cdot\nabla\xi_{h\delta})\,drds \\ &= 4\nu\sum_{k,k'\in\mathbb{Z}_0^2}\theta_k\sigma_k\cdot\nabla(\theta_{k'}\sigma_{k'}\cdot\nabla\xi_{h\delta})\bigg(\int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s}\eta^{\alpha,k}(s)\eta^{\alpha,k'}(r)\,drds - \delta_{k,k'}\frac{\delta}{2}\bigg) \\ &\quad + 2\nu\delta\sum_{k\in\mathbb{Z}_0^2}\theta_k\sigma_k\cdot\nabla(\theta_k\sigma_k\cdot\nabla\xi_{h\delta}) \\ &=: I_{231}(h) + I_{232}(h). \end{aligned}$$

Using the radial symmetry of $\theta \in \ell^2(\mathbb{Z}_0^2)$ and the expression of σ_k, it is not difficult to prove the following identity (see e.g. [18, Lemma 2.1])

$$\sum_{k\in\mathbb{Z}_0^2}\theta_k^2(\sigma_k\otimes\sigma_k)(x) = \frac{1}{2}\|\theta\|_{\ell^2}^2\,Id,$$

where Id is the two dimensional identity matrix; hence by $\mathrm{div}\sigma_k = 0$ and $\|\theta\|_{\ell^2}^2 = 1$, we have

$$\sum_{k\in\mathbb{Z}_0^2}\theta_k\sigma_k\cdot\nabla(\theta_k\sigma_k\cdot\nabla\xi_{h\delta}) = \mathrm{div}\Big(\sum_{k\in\mathbb{Z}_0^2}\theta_k^2\,(\sigma_k\otimes\sigma_k)\nabla\xi_{h\delta}\Big) = \frac{1}{2}\Delta\xi_{h\delta}. \tag{5.3}$$

Therefore, $I_{232}(h)=\nu\delta\Delta\xi_{h\delta}$. As for the term $I_3(h)$, we can divide it into two terms:

$$I_3(h)=\kappa\int_{h\delta}^{(h+1)\delta}\Delta(\xi_s-\xi_{h\delta})\,ds+\kappa\delta\Delta\xi_{h\delta}=:I_{31}(h)+I_{32}(h).$$

Taking the sum of (5.2) over $h=m,\dots,n$, and noticing that

$$I_{232}(h)+I_{32}(h)=\delta(\kappa+\nu)\Delta\xi_{h\delta},$$

we can finally get

$$\begin{aligned}&\xi_{(n+1)\delta}-\xi_{m\delta}-\delta(\kappa+\nu)\sum_{h=m}^{n}\Delta\xi_{h\delta}+\delta\sum_{h=m}^{n}(u_{h\delta}\cdot\nabla\xi_{h\delta})\\ =&\sum_{h=m}^{n}\big(I_1(h)+I_{21}(h)+I_{22}(h)+I_{231}(h)+I_{24}(h)+I_{25}(h)+I_{31}(h)+\delta(u_{h\delta}\cdot\nabla\xi_{h\delta})\big).\end{aligned}\tag{5.4}$$

In the following several subsections, we will estimate each term of the above formula. For readers' convenience, we give a brief introduction here. The estimates on terms $I_{21}(h)$, $I_{22}(h)$, $I_{24}(h)$ and $I_{31}(h)$ will be given in Section 5.2. As the terms $I_{231}(h)$ and $I_{25}(h)$ are more technical to deal with, we consider them in Sections 5.3 and 5.4, respectively. In Section 5.5, we treat $I_1(h)$ together with $\delta(u_{h\delta}\cdot\nabla\xi_{h\delta})$. We mention that the two remaining intergrals I_a and I_b in (5.1) are similar to $I_{31}(h)$ and $I_1(h)+\delta(u_{h\delta}\cdot\nabla\xi_{h\delta})$, respectively, hence we give their estimates in Section 5.6 without proof. Finally, we combine all the estimates and provide in Section 5.7 the proof of Proposition 3.3.

5.2 The terms $I_{21}(h), I_{22}(h), I_{24}(h), I_{31}(h)$

The estimates on these four terms are collected in the next lemma.

Lemma 5.1. *Let $\gamma\in(0,\frac13)$, $\beta>3$ and $T\geq1$, then we have the following estimates:*

$$\begin{aligned}\mathbb{E}\bigg[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n}I_{21}(h)\Big\|_{H^{-\beta}}\bigg]&\lesssim\delta\nu^{\frac12}\alpha^{\frac12}\,TC_{\theta,2-\gamma,2}^{1/2}\,\|\xi_0\|_{L^2}^2,\\ \mathbb{E}\bigg[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n}I_{22}(h)\Big\|_{H^{-\beta}}\bigg]&\lesssim\delta^{1+\gamma}\nu^{1+\frac{\gamma}{2}}\alpha^{1+\frac{\gamma}{2}}\,TC_{\theta,2-\gamma,4}^{\frac12+\frac{\gamma}{4}}\,\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)^{\gamma},\\ \mathbb{E}\bigg[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n}I_{24}(h)\Big\|_{H^{-\beta}}\bigg]&\lesssim\kappa^{\frac{1+\gamma}{2}}\delta^{\frac{1+\gamma}{2}}\nu^{\frac12}\alpha^{\frac12}\,TC_{\theta,2-\gamma,2}^{1/2}\,\|\xi_0\|_{L^2},\\ \mathbb{E}\bigg[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n}I_{31}(h)\Big\|_{H^{-\beta}}\bigg]&\lesssim\kappa\delta^{\gamma}\nu^{\frac{\gamma}{2}}\alpha^{\frac{\gamma}{2}}\,TC_{\theta,1+\gamma,2}^{\gamma/2}\,\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)^{\gamma}.\end{aligned}$$

Proof. First, we consider the term

$$I_{21}(h)=\int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s}\boldsymbol{b}(s)\cdot\nabla(u_r\cdot\nabla\xi_r)\,drds.$$

Taking a test function $\phi\in H^\beta(\mathbb{T}^2)$ and integrating by parts, it holds

$$\begin{aligned}\big|\langle\boldsymbol{b}(s)\cdot\nabla(u_r\cdot\nabla\xi_r),\phi\rangle\big|&=\big|\langle u_r\cdot\nabla(\boldsymbol{b}(s)\cdot\nabla\phi),\xi_r\rangle\big|\\ &\le\|u_r\cdot\nabla(\boldsymbol{b}(s)\cdot\nabla\phi)\|_{L^2}\|\xi_r\|_{L^2}\\ &\le\|u_r\|_{H^{1-\gamma}}\|\nabla(\boldsymbol{b}(s)\cdot\nabla\phi)\|_{H^{\gamma}}\|\xi_0\|_{L^2},\end{aligned}$$

where the last step follows from Lemma 2.2; again by Lemma 2.2, for $\gamma \in (0, \frac{1}{2})$, we have

$$\begin{aligned}\|\nabla(\boldsymbol{b}(s)\cdot\nabla\phi)\|_{H^\gamma} &\le \|\nabla\boldsymbol{b}(s)\cdot\nabla\phi\|_{H^\gamma} + \|\boldsymbol{b}(s)\cdot\nabla^2\phi\|_{H^\gamma}\\ &\lesssim \|\nabla\boldsymbol{b}(s)\|_{H^{1-\gamma}}\|\nabla\phi\|_{H^{2\gamma}} + \|\boldsymbol{b}(s)\|_{H^{1-\gamma}}\|\nabla^2\phi\|_{H^{2\gamma}}\\ &\lesssim \|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\|\phi\|_{H^{1+2\gamma}} + \|\boldsymbol{b}(s)\|_{H^{1-\gamma}}\|\phi\|_{H^{2+2\gamma}}\\ &\lesssim \|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\|\phi\|_{H^{2+2\gamma}}.\end{aligned} \tag{5.5}$$

Combining the above two estimates, we can easily get for $\beta > 3$,

$$\big\|\boldsymbol{b}(s)\cdot\nabla(u_r\cdot\nabla\xi_r)\big\|_{H^{-\beta}} \lesssim \|u_r\|_{H^{1-\gamma}}\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\|\xi_0\|_{L^2} \lesssim \|\xi_0\|_{L^2}^2\,\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}.$$

Furthermore,

$$\big\|I_{21}(h)\big\|_{H^{-\beta}} \lesssim \|\xi_0\|_{L^2}^2 \int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s}\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\,drds \lesssim \delta\,\|\xi_0\|_{L^2}^2\int_{h\delta}^{(h+1)\delta}\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\,ds.$$

Taking supremum and then expectation, Lemma 2.5 yields

$$\begin{aligned}\mathbb{E}\Big[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n} I_{21}(h)\Big\|_{H^{-\beta}}\Big] &\le \sum_{h=1}^{T/\delta-1}\mathbb{E}\Big[\big\|I_{21}(h)\big\|_{H^{-\beta}}\Big]\\ &\lesssim \delta\,\|\xi_0\|_{L^2}^2\int_0^T\Big(\mathbb{E}\big[\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}^2\big]\Big)^{\frac{1}{2}}ds\\ &\lesssim \delta\nu^{\frac{1}{2}}\alpha^{\frac{1}{2}}\,TC_{\theta,2-\gamma,2}^{1/2}\,\|\xi_0\|_{L^2}^2.\end{aligned}$$

Let us turn to the term

$$I_{22}(h) = \int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s}\boldsymbol{b}(s)\cdot\nabla\big(\boldsymbol{b}(r)\cdot\nabla(\xi_r-\xi_{h\delta})\big)\,drds.$$

For any test function $\phi \in H^\beta(\mathbb{T}^2)$ with $\beta > 3$, it holds

$$\big|\big\langle\boldsymbol{b}(s)\cdot\nabla\big(\boldsymbol{b}(r)\cdot\nabla(\xi_r-\xi_{h\delta})\big),\phi\big\rangle\big| = \big|\big\langle\boldsymbol{b}(r)\cdot\nabla(\boldsymbol{b}(s)\cdot\nabla\phi),\xi_r-\xi_{h\delta}\big\rangle\big|.$$

By Lemma 2.3 and Lemma 2.4, for $\gamma \in (0, \frac{1}{2})$, we have

$$\begin{aligned}&\big|\big\langle\boldsymbol{b}(r)\cdot\nabla(\boldsymbol{b}(s)\cdot\nabla\phi),\xi_r-\xi_{h\delta}\big\rangle\big|\\ &\le \|\boldsymbol{b}(r)\cdot\nabla(\boldsymbol{b}(s)\cdot\nabla\phi)\|_{H^\gamma}\|\xi_r-\xi_{h\delta}\|_{H^{-\gamma}}\\ &\lesssim \|\boldsymbol{b}(r)\|_{C^{2\gamma}}\|\nabla(\boldsymbol{b}(s)\cdot\nabla\phi)\|_{H^\gamma}\|\xi_r-\xi_{h\delta}\|_{H^{-\gamma}}\\ &\lesssim \|\boldsymbol{b}(r)\|_{H^{1+2\gamma}}\|\nabla(\boldsymbol{b}(s)\cdot\nabla\phi)\|_{H^\gamma}\|\xi_r-\xi_{h\delta}\|_{H^{-1}}^{\gamma}\|\xi_r-\xi_{h\delta}\|_{L^2}^{1-\gamma}.\end{aligned}$$

Then for $\gamma \in (0, \frac{1}{3})$, we use (5.5) to get

$$\big\|\boldsymbol{b}(s)\cdot\nabla\big(\boldsymbol{b}(r)\cdot\nabla(\xi_r-\xi_{h\delta})\big)\big\|_{H^{-\beta}} \lesssim \|\boldsymbol{b}(r)\|_{H^{2-\gamma}}\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\|\xi_r-\xi_{h\delta}\|_{H^{-1}}^{\gamma}\|\xi_r-\xi_{h\delta}\|_{L^2}^{1-\gamma}.$$

Hence we have

$$\begin{aligned}\big\|I_{22}(h)\big\|_{H^{-\beta}} &\lesssim \int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s}\|\boldsymbol{b}(r)\|_{H^{2-\gamma}}\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\|\xi_r-\xi_{h\delta}\|_{H^{-1}}^{\gamma}\|\xi_r-\xi_{h\delta}\|_{L^2}^{1-\gamma}\,drds\\ &\lesssim \|\xi_0\|_{L^2}^{1-\gamma}\int_{h\delta}^{(h+1)\delta}\|\boldsymbol{b}(r)\|_{H^{2-\gamma}}\|\xi_r-\xi_{h\delta}\|_{H^{-1}}^{\gamma}\,dr\int_{h\delta}^{(h+1)\delta}\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\,ds.\end{aligned}$$

We take expectation on the above formula and obtain

$$\mathbb{E}\Big[\big\|I_{22}(h)\big\|_{H^{-\beta}}\Big] \lesssim \|\xi_0\|_{L^2}^{1-\gamma}\bigg[\mathbb{E}\Big(\int_{h\delta}^{(h+1)\delta}\|\boldsymbol{b}(r)\|_{H^{2-\gamma}}\|\xi_r-\xi_{h\delta}\|_{H^{-1}}^{\gamma}\,dr\Big)^2\bigg]^{\frac12} \times\bigg[\mathbb{E}\Big(\int_{h\delta}^{(h+1)\delta}\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\,ds\Big)^2\bigg]^{\frac12}.$$

Now we use Hölder's inequality to further deal with the two terms respectively. By Lemma 2.5 and Lemma 3.1,

$$\begin{aligned}
&\mathbb{E}\Big(\int_{h\delta}^{(h+1)\delta}\|\boldsymbol{b}(r)\|_{H^{2-\gamma}}\|\xi_r-\xi_{h\delta}\|_{H^{-1}}^{\gamma}\,dr\Big)^2\\
&\le \mathbb{E}\bigg[\delta\int_{h\delta}^{(h+1)\delta}\|\boldsymbol{b}(r)\|_{H^{2-\gamma}}^2\|\xi_r-\xi_{h\delta}\|_{H^{-1}}^{2\gamma}\,dr\bigg]\\
&\le \delta\int_{h\delta}^{(h+1)\delta}\Big(\mathbb{E}\big[\|\boldsymbol{b}(r)\|_{H^{2-\gamma}}^4\big]\Big)^{\frac12}\Big(\mathbb{E}\big[\|\xi_r-\xi_{h\delta}\|_{H^{-1}}^{4\gamma}\big]\Big)^{\frac12}dr\\
&\lesssim \delta^{2(1+\gamma)}\nu^{1+\gamma}\alpha^{1+\gamma}\,C_{\theta,2-\gamma,4}^{1/2}\,C_{\theta,1+\gamma,2}^{\gamma}\,\|\xi_0\|_{L^2}^{2\gamma}\big(1+\|\xi_0\|_{L^2}\big)^{2\gamma},
\end{aligned}\tag{5.6}$$

to get the last line, we have used $\mathbb{E}\big[\|\xi_r-\xi_{h\delta}\|_{H^{-1}}^{4\gamma}\big]\le\Big(\mathbb{E}\big[\|\xi_r-\xi_{h\delta}\|_{H^{-1}}^{2}\big]\Big)^{2\gamma}$. Besides, for the second term, Lemma 2.5 yields

$$\mathbb{E}\Big(\int_{h\delta}^{(h+1)\delta}\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\,ds\Big)^2 \le \delta\int_{h\delta}^{(h+1)\delta}\mathbb{E}\big[\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}^2\big]ds \lesssim \delta^2\nu\alpha\,C_{\theta,2-\gamma,2}. \tag{5.7}$$

Hence we can combine (5.6) and (5.7) to get

$$\mathbb{E}\Big[\big\|I_{22}(h)\big\|_{H^{-\beta}}\Big] \lesssim \delta^{2+\gamma}\nu^{1+\frac{\gamma}{2}}\alpha^{1+\frac{\gamma}{2}}\,C_{\theta,2-\gamma,4}^{1/4}\,C_{\theta,1+\gamma,2}^{\gamma/2}\,C_{\theta,2-\gamma,2}^{1/2}\,\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)^{\gamma}.$$

Taking supremum, Remark 2.6 yields

$$\begin{aligned}
\mathbb{E}\bigg[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n}I_{22}(h)\Big\|_{H^{-\beta}}\bigg] &\le \sum_{h=1}^{T/\delta-1}\mathbb{E}\Big[\big\|I_{22}(h)\big\|_{H^{-\beta}}\Big]\\
&\lesssim \delta^{1+\gamma}\nu^{1+\frac{\gamma}{2}}\alpha^{1+\frac{\gamma}{2}}\,TC_{\theta,2-\gamma,4}^{\frac12+\frac{\gamma}{4}}\,\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)^{\gamma}.
\end{aligned}$$

As for the term

$$I_{24}(h) = -\kappa\int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s}\boldsymbol{b}(s)\cdot\nabla(\Delta\xi_r)\,drds,$$

notice that for every test function $\phi\in H^{\beta}(\mathbb{T}^2)$, it holds $\big|\langle\boldsymbol{b}(s)\cdot\nabla(\Delta\xi_r),\phi\rangle\big| = \big|\langle\Delta(\boldsymbol{b}(s)\cdot\nabla\phi),\xi_r\rangle\big|$; and by Lemma 2.4, for $\gamma\in(0,1)$, we have

$$\big|\langle\Delta(\boldsymbol{b}(s)\cdot\nabla\phi),\xi_r\rangle\big| \le \|\Delta(\boldsymbol{b}(s)\cdot\nabla\phi)\|_{H^{\gamma-1}}\|\xi_r\|_{H^{1-\gamma}} \lesssim \|\nabla(\boldsymbol{b}(s)\cdot\nabla\phi)\|_{H^{\gamma}}\|\xi_r\|_{H^1}^{1-\gamma}\|\xi_0\|_{L^2}^{\gamma}.$$

Then we can use (5.5) to obtain for $\beta>3$ and $\gamma\in(0,\frac12)$,

$$\big\|\boldsymbol{b}(s)\cdot\nabla(\Delta\xi_r)\big\|_{H^{-\beta}} \lesssim \|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\|\xi_r\|_{H^1}^{1-\gamma}\|\xi_0\|_{L^2}^{\gamma}.$$

Furthermore, Hölder's inequality and (2.1) yield

$$\begin{aligned}\big\|I_{24}(h)\big\|_{H^{-\beta}} &\lesssim \kappa\,\|\xi_0\|_{L^2}^{\gamma}\int_{h\delta}^{(h+1)\delta}\|\xi_r\|_{H^1}^{1-\gamma}\,dr\int_{h\delta}^{(h+1)\delta}\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\,ds\\ &\le \kappa\,\|\xi_0\|_{L^2}^{\gamma}\,\delta^{\frac{1+\gamma}{2}}\Big(\int_{h\delta}^{(h+1)\delta}\|\xi_r\|_{H^1}^2\,dr\Big)^{\frac{1-\gamma}{2}}\int_{h\delta}^{(h+1)\delta}\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\,ds\\ &\lesssim \kappa^{\frac{1+\gamma}{2}}\delta^{\frac{1+\gamma}{2}}\,\|\xi_0\|_{L^2}\int_{h\delta}^{(h+1)\delta}\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}\,ds.\end{aligned}$$

By Lemma 2.5, we take expectation and deduce

$$\begin{aligned}\mathbb{E}\Big[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n}I_{24}(h)\Big\|_{H^{-\beta}}\Big] &\le \sum_{h=1}^{T/\delta-1}\mathbb{E}\Big[\big\|I_{24}(h)\big\|_{H^{-\beta}}\Big]\\ &\lesssim \kappa^{\frac{1+\gamma}{2}}\delta^{\frac{1+\gamma}{2}}\,\|\xi_0\|_{L^2}\int_0^T\Big(\mathbb{E}\big[\|\boldsymbol{b}(s)\|_{H^{2-\gamma}}^2\big]\Big)^{\frac12}ds\\ &\lesssim \kappa^{\frac{1+\gamma}{2}}\delta^{\frac{1+\gamma}{2}}\nu^{\frac12}\alpha^{\frac12}\,TC_{\theta,2-\gamma,2}^{1/2}\,\|\xi_0\|_{L^2}.\end{aligned}$$

Finally, let us estimate the term

$$\big\|I_{31}(h)\big\|_{H^{-\beta}}\le\kappa\int_{h\delta}^{(h+1)\delta}\big\|\Delta(\xi_s-\xi_{h\delta})\big\|_{H^{-\beta}}\,ds\le\kappa\int_{h\delta}^{(h+1)\delta}\|\xi_s-\xi_{h\delta}\|_{H^{-\gamma}}\,ds,$$

for any $\gamma\in(0,1)$. Then Lemma 2.4 yields

$$\begin{aligned}\big\|I_{31}(h)\big\|_{H^{-\beta}} &\le \kappa\int_{h\delta}^{(h+1)\delta}\|\xi_s-\xi_{h\delta}\|_{H^{-1}}^{\gamma}\|\xi_s-\xi_{h\delta}\|_{L^2}^{1-\gamma}\,ds\\ &\lesssim \kappa\,\|\xi_0\|_{L^2}^{1-\gamma}\int_{h\delta}^{(h+1)\delta}\|\xi_s-\xi_{h\delta}\|_{H^{-1}}^{\gamma}\,ds.\end{aligned}$$

By Lemma 3.1, we arrive at

$$\begin{aligned}\mathbb{E}\Big[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n}I_{31}(h)\Big\|_{H^{-\beta}}\Big] &\le \sum_{h=1}^{T/\delta-1}\mathbb{E}\Big[\big\|I_{31}(h)\big\|_{H^{-\beta}}\Big]\\ &\lesssim \kappa\,\|\xi_0\|_{L^2}^{1-\gamma}\sum_{h=1}^{T/\delta-1}\int_{h\delta}^{(h+1)\delta}\Big(\mathbb{E}\big[\|\xi_{(h+1)\delta}-\xi_{h\delta}\|_{H^{-1}}^2\big]\Big)^{\frac{\gamma}{2}}ds\\ &\lesssim \kappa\delta^{\gamma}\nu^{\frac{\gamma}{2}}\alpha^{\frac{\gamma}{2}}\,TC_{\theta,1+\gamma,2}^{\gamma/2}\,\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)^{\gamma}.\end{aligned}$$

□

5.3 The term $I_{231}(h)$

In this section, we follow [36] and use Nakao's method (cf. [29]) to estimate the term

$$I_{231}(h)=4\nu\sum_{k,k'\in\mathbb{Z}_0^2}\theta_k\sigma_k\cdot\nabla(\theta_{k'}\sigma_{k'}\cdot\nabla\xi_{h\delta})\Big(\int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s}\eta^{\alpha,k}(s)\,\eta^{\alpha,k'}(r)\,drds-\delta_{k,k'}\frac{\delta}{2}\Big).$$

Lemma 5.2. *Let $\delta \in (0,1)$ satisfy $\delta^4\alpha^3 \lesssim 1$, then the following inequality holds for any $T \geq 1$:*

$$\mathbb{E}\bigg[\sup_{1\leq m<n\leq T/\delta-1}\Big\|\sum_{h=m}^{n} I_{231}(h)\Big\|_{H^{-\beta}}\bigg] \lesssim \nu\delta^{-1}\alpha^{-1}TD_{\theta,\gamma}\,\|\xi_0\|_{L^2},$$

where $D_{\theta,\gamma} := \big(\sum_{k\in\mathbb{Z}_0^2}|\theta_k|\,|k|^{2-\gamma}\big)^2$ is a finite constant depending on $\theta \in \ell^2(\mathbb{Z}_0^2)$ and γ.

Proof. For convenience, we define

$$c_{k,k'}(h) := \int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s}\eta^{\alpha,k}(s)\,\eta^{\alpha,k'}(r)\,drds.$$

By [36, Lemma 5.2], the conditional expectation of $c_{k,k'}(h)$ with respect to $\mathcal{F}_{h\delta}$ is

$$\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big] = \eta^{\alpha,k}(h\delta)\,\eta^{\alpha,k'}(h\delta)\,\frac{(1-e^{-\alpha\delta})^2}{2\alpha^2}+\delta_{k,k'}\bigg[\frac{\delta}{2}+\frac{1}{\alpha}\Big(e^{-\alpha\delta}-1+\frac{1}{4}(1-e^{-2\alpha\delta})\Big)\bigg]. \tag{5.8}$$

Now we define two processes as follows:

$$\begin{aligned} M_n &= \sum_{h=1}^{n-1}\sum_{k,k'\in\mathbb{Z}_0^2}\theta_k\sigma_k\cdot\nabla(\theta_{k'}\sigma_{k'}\cdot\nabla\xi_{h\delta})\Big(c_{k,k'}(h)-\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]\Big),\\ R_n &= \sum_{h=1}^{n-1}\sum_{k,k'\in\mathbb{Z}_0^2}\theta_k\sigma_k\cdot\nabla(\theta_{k'}\sigma_{k'}\cdot\nabla\xi_{h\delta})\Big(\mathbb{E}[c_{k,k'}(h)|\mathcal{F}_{h\delta}]-\delta_{k,k'}\frac{\delta}{2}\Big). \end{aligned}$$

Notice that $\{M_n\}_{n=1,\ldots,T/\delta}$ is a $H^{-\beta}$-valued discrete martingale with respect to $\{\mathcal{F}_{n\delta}\}_{n=1,\ldots,T/\delta}$, hence by Doob's maximal inequality,

$$\begin{aligned} &\mathbb{E}\bigg[\sup_{1<n\leq T/\delta}\big\|M_n\big\|_{H^{-\beta}}^2\bigg] \lesssim \mathbb{E}\Big[\big\|M_{T/\delta}\big\|_{H^{-\beta}}^2\Big]\\ &\leq \sum_{h=1}^{T/\delta-1}\mathbb{E}\bigg\|\sum_{k,k'\in\mathbb{Z}_0^2}\theta_k\sigma_k\cdot\nabla(\theta_{k'}\sigma_{k'}\cdot\nabla\xi_{h\delta})\Big(c_{k,k'}(h)-\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]\Big)\bigg\|_{H^{-\beta}}^2 \\ &\leq \sum_{h=1}^{T/\delta-1}\mathbb{E}\bigg(\sum_{k,k'\in\mathbb{Z}_0^2}\big\|\theta_k\sigma_k\cdot\nabla(\theta_{k'}\sigma_{k'}\cdot\nabla\xi_{h\delta})\big\|_{H^{-\beta}}\Big|c_{k,k'}(h)-\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]\Big|\bigg)^2. \end{aligned} \tag{5.9}$$

We first give the following estimate. For any test function $\phi \in H^\beta(\mathbb{T}^2)$, it holds

$$\big|\big\langle\theta_k\sigma_k\cdot\nabla(\theta_{k'}\sigma_{k'}\cdot\nabla\xi_{h\delta}),\phi\big\rangle\big| = \big|\big\langle\theta_{k'}\sigma_{k'}\cdot\nabla(\theta_k\sigma_k\cdot\nabla\phi),\xi_{h\delta}\big\rangle\big|;$$

by Lemma 2.2, for $\gamma \in (0,\frac{1}{2})$, we use (5.5) to obtain

$$\begin{aligned} \big|\big\langle\theta_{k'}\sigma_{k'}\cdot\nabla(\theta_k\sigma_k\cdot\nabla\phi),\xi_{h\delta}\big\rangle\big| &\lesssim \|\nabla(\theta_k\sigma_k\cdot\nabla\phi)\|_{L^2}\|\theta_{k'}\sigma_{k'}\xi_{h\delta}\|_{L^2}\\ &\lesssim \|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\phi\|_{H^{2+2\gamma}}\|\theta_{k'}\sigma_{k'}\|_{L^\infty}\|\xi_{h\delta}\|_{L^2}\\ &\lesssim \|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\phi\|_{H^{2+2\gamma}}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}\|\xi_0\|_{L^2}. \end{aligned}$$

Combining the above two results, we get for $\beta > 3$,

$$\big\|\theta_k\sigma_k\cdot\nabla(\theta_{k'}\sigma_{k'}\cdot\nabla\xi_{h\delta})\big\|_{H^{-\beta}} \lesssim \|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}\|\xi_0\|_{L^2}, \tag{5.10}$$

hence,

$$\begin{aligned}
&\mathbb{E}\bigg(\sum_{k,k'\in\mathbb{Z}_0^2}\big\|\theta_k\sigma_k\cdot\nabla(\theta_{k'}\sigma_{k'}\cdot\nabla\xi_{h\delta})\big\|_{H^{-\beta}}\Big|c_{k,k'}(h)-\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]\Big|\bigg)^2\\
\lesssim{}& \|\xi_0\|_{L^2}^2\,\mathbb{E}\bigg(\sum_{k,k'\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}\,\Big|c_{k,k'}(h)-\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]\Big|\bigg)^2.
\end{aligned}$$

We regard each term of $\|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}$ as the product of their square roots; then the Cauchy-Schwartz inequality and the projective property of conditional expectation yield

$$\begin{aligned}
&\mathbb{E}\bigg(\sum_{k,k'\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}\,\Big|c_{k,k'}(h)-\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]\Big|\bigg)^2\\
\le{}&\bigg(\sum_{k,k'\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}\bigg)\\
&\times\bigg(\sum_{k,k'\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}\mathbb{E}\Big|c_{k,k'}(h)-\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]\Big|^2\bigg)\\
\le{}&\bigg(\sum_{k\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\bigg)^2\bigg(\sum_{k,k'\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}\,\mathbb{E}\Big[c_{k,k'}(h)^2\Big]\bigg).
\end{aligned}$$

Notice that the following formula holds:

$$\begin{aligned}
\mathbb{E}\Big[c_{k,k'}(h)^2\Big]&=\mathbb{E}\bigg[\bigg(\int_{h\delta}^{(h+1)\delta}\int_{h\delta}^{s}\eta^{\alpha,k}(s)\,\eta^{\alpha,k'}(r)\,drds\bigg)^2\bigg]\\
&=\mathbb{E}\bigg[\bigg(\int_{h\delta}^{(h+1)\delta}\eta^{\alpha,k}(s)\Big(W_s^{k'}-W_{h\delta}^{k'}-\frac{1}{\alpha}\big(\eta^{\alpha,k'}(s)-\eta^{\alpha,k'}(h\delta)\big)\Big)ds\bigg)^2\bigg]\\
&\le\mathbb{E}\bigg[\delta\int_{h\delta}^{(h+1)\delta}\big|\eta^{\alpha,k}(s)\big|^2\Big(W_s^{k'}-W_{h\delta}^{k'}-\frac{1}{\alpha}\big(\eta^{\alpha,k'}(s)-\eta^{\alpha,k'}(h\delta)\big)\Big)^2ds\bigg].
\end{aligned}$$

Again by Cauchy's inequality, we have

$$\begin{aligned}
\mathbb{E}\Big[c_{k,k'}(h)^2\Big]&\le\delta\int_{h\delta}^{(h+1)\delta}\Big(\mathbb{E}\big[|\eta^{\alpha,k}(s)|^4\big]\Big)^{\frac12}\bigg(\mathbb{E}\Big[\Big(W_s^{k'}-W_{h\delta}^{k'}-\frac{1}{\alpha}\big(\eta^{\alpha,k'}(s)-\eta^{\alpha,k'}(h\delta)\big)\Big)^4\Big]\bigg)^{\frac12}ds\\
&\lesssim\delta\int_{h\delta}^{(h+1)\delta}\alpha\bigg(\mathbb{E}\Big[\big|W_{(h+1)\delta}^{k'}-W_{h\delta}^{k'}\big|^4\Big]+\frac{1}{\alpha^4}\mathbb{E}\Big[\big|\eta^{\alpha,k'}(s)\big|^4\Big]\bigg)^{\frac12}ds\\
&\lesssim\delta^3\alpha+\delta^2.
\end{aligned}$$

Summarizing the above estimates yields

$$\begin{aligned}
&\mathbb{E}\bigg(\sum_{k,k'\in\mathbb{Z}_0^2}\big\|\theta_k\sigma_k\cdot\nabla(\theta_{k'}\sigma_{k'}\cdot\nabla\xi_{h\delta})\big\|_{H^{-\beta}}\Big|c_{k,k'}(h)-\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]\Big|\bigg)^2\\
\lesssim{}&\|\xi_0\|_{L^2}^2\bigg(\sum_{k\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\bigg)^2\bigg(\sum_{k,k'\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}\,(\delta^3\alpha+\delta^2)\bigg)\\
={}&\|\xi_0\|_{L^2}^2\bigg(\sum_{k\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\bigg)^4(\delta^3\alpha+\delta^2).
\end{aligned}$$

Substituting this estimate into (5.9), we deduce

$$\begin{aligned}\mathbb{E}\Big[\sup_{1<n\le T/\delta}\big\|M_n\big\|_{H^{-\beta}}\Big] &\le \mathbb{E}\Big[\sup_{1<n\le T/\delta}\big\|M_n\big\|_{H^{-\beta}}^2\Big]^{\frac12}\\ &\lesssim T^{\frac12}\|\xi_0\|_{L^2}\Big(\sum_{k\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\Big)^2\big(\delta\alpha^{\frac12}+\delta^{\frac12}\big)\\ &\lesssim \delta\alpha^{\frac12}T^{\frac12}\|\xi_0\|_{L^2}\Big(\sum_{k\in\mathbb{Z}_0^2}|\theta_k|\,|k|^{2-\gamma}\Big)^2\\ &= \delta\alpha^{\frac12}T^{\frac12}D_{\theta,\gamma}\,\|\xi_0\|_{L^2}\,.\end{aligned}$$

Now let us turn to the term R_n, notice that (5.10) yields

$$\begin{aligned}\big\|R_n\big\|_{H^{-\beta}} &\le \sum_{h=1}^{n-1}\sum_{k,k'\in\mathbb{Z}_0^2}\big\|\theta_k\sigma_k\cdot\nabla(\theta_{k'}\sigma_{k'}\cdot\nabla\xi_{h\delta})\big\|_{H^{-\beta}}\Big|\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]-\delta_{k,k'}\frac{\delta}{2}\Big|\\ &\lesssim \|\xi_0\|_{L^2}\sum_{h=1}^{n-1}\sum_{k,k'\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}\Big|\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]-\delta_{k,k'}\frac{\delta}{2}\Big|.\end{aligned}$$

We use the same method as the term M_n to further deal with the above formula as follows:

$$\begin{aligned}&\sum_{k,k'\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}\Big|\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]-\delta_{k,k'}\frac{\delta}{2}\Big|\\ &=\sum_{k,k'\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}^{1/2}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}^{1/2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}^{1/2}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}^{1/2}\Big|\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]-\delta_{k,k'}\frac{\delta}{2}\Big|\\ &\le\Big(\sum_{k\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\Big)\Big(\sum_{k,k'\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\|\theta_{k'}\sigma_{k'}\|_{H^{2-\gamma}}\Big|\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]-\delta_{k,k'}\frac{\delta}{2}\Big|^2\Big)^{\frac12}.\end{aligned}$$

By (5.8), we can easily get

$$\begin{aligned}&\mathbb{E}\Big[\Big|\mathbb{E}\big[c_{k,k'}(h)|\mathcal{F}_{h\delta}\big]-\delta_{k,k'}\frac{\delta}{2}\Big|^2\Big]\\ &=\mathbb{E}\Big[\Big|\eta^{\alpha,k}(h\delta)\eta^{\alpha,k'}(h\delta)\frac{(1-e^{-\alpha\delta})^2}{2\alpha^2}+\delta_{k,k'}\frac{1}{\alpha}\big(e^{-\alpha\delta}-1+\frac14(1-e^{-2\alpha\delta})\big)\Big|^2\Big]\\ &\lesssim\alpha^{-2}.\end{aligned}$$

Combining the above results, we take supremum and then expectation on $\big\|R_n\big\|_{H^{-\beta}}$ to get

$$\begin{aligned}\mathbb{E}\Big[\sup_{1<n\le T/\delta}\big\|R_n\big\|_{H^{-\beta}}\Big] &\lesssim \alpha^{-1}\|\xi_0\|_{L^2}\sum_{h=1}^{T/\delta-1}\Big(\sum_{k\in\mathbb{Z}_0^2}\|\theta_k\sigma_k\|_{H^{2-\gamma}}\Big)^2\\ &\lesssim \delta^{-1}\alpha^{-1}\,TD_{\theta,\gamma}\,\|\xi_0\|_{L^2}.\end{aligned}$$

Taking our assumptions on the parameters into consideration, and noticing that

$$\sum_{h=m}^{n}I_{231}(h)=4\nu\big(M_{n+1}+R_{n+1}-M_m-R_m\big),$$

we complete the proof of Lemma 5.2. □

5.4 The term $I_{25}(h)$

In this section, we focus on the term

$$I_{25}(h) = -\int_{h\delta}^{(h+1)\delta} \boldsymbol{b}(s)\cdot\nabla\xi_{h\delta}\,ds = -\Big(\int_{h\delta}^{(h+1)\delta} \boldsymbol{b}(s)\,ds\Big)\cdot\nabla\xi_{h\delta}.$$

According to the definition of $\boldsymbol{b}(s)$, it holds

$$\int_{h\delta}^{(h+1)\delta} \boldsymbol{b}(s)\,ds = 2\sqrt{\nu}\sum_{k\in\mathbb{Z}_0^2}\theta_k\sigma_k\big(W^k_{(h+1)\delta}-W^k_{h\delta}\big)-2\sqrt{\nu}\alpha^{-1}\sum_{k\in\mathbb{Z}_0^2}\theta_k\sigma_k\big(\eta^{\alpha,k}((h+1)\delta)-\eta^{\alpha,k}(h\delta)\big),$$

then we can further decompose $I_{25}(h)$ as follows:

$$\begin{aligned}
I_{25}(h) &= -2\sqrt{\nu}\sum_{k\in\mathbb{Z}_0^2}\int_{h\delta}^{(h+1)\delta}\theta_k\sigma_k\cdot\nabla\xi_{h\delta}\,dW^k_s\\
&\quad + 2\sqrt{\nu}\alpha^{-1}\sum_{k\in\mathbb{Z}_0^2}\big(\theta_k\sigma_k\cdot\nabla\xi_{h\delta}\big)\big(\eta^{\alpha,k}((h+1)\delta)-\eta^{\alpha,k}(h\delta)\big)\\
&=: I_{251}(h)+I_{252}(h).
\end{aligned}$$

The following lemma gives the result for the term $I_{252}(h)$, as for the term $I_{251}(h)$, we will separately discuss it after the proof of Lemma 5.3.

Lemma 5.3. *Let $\gamma\in(0,\frac12)$, $\beta>3$ and $T\geq 1$, then we have*

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n} I_{252}(h)\Big\|_{H^{-\beta}}\Big]\\
&\lesssim \nu^{\frac{2+3\gamma}{2(1+\gamma)}}\big(\delta^{-\frac{\gamma}{2(1+\gamma)}}\alpha^{-\frac{\gamma}{2(1+\gamma)}}+\delta^{-\frac{\gamma}{1+\gamma}}\alpha^{-\frac{\gamma}{1+\gamma}}\big)\,TC_{\theta,2-\gamma,4}^{\frac{2+3\gamma}{4(1+\gamma)}}\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)^2\\
&\quad+\nu^{\frac12}\alpha^{-\frac12}\,C_{\theta,2-\gamma,2}^{1/2}\log^{\frac12}(1+\alpha T)\,\|\xi_0\|_{L^2}.
\end{aligned}$$

Proof. We first reformulate the sum as follows:

$$\begin{aligned}
\sum_{h=m}^{n} I_{252}(h) &= \alpha^{-1}\sum_{h=m}^{n}\big(\boldsymbol{b}((h+1)\delta)-\boldsymbol{b}(h\delta)\big)\cdot\nabla\xi_{h\delta}\\
&= -\alpha^{-1}\Big[\sum_{h=m+1}^{n}\boldsymbol{b}(h\delta)\cdot\nabla(\xi_{h\delta}-\xi_{(h-1)\delta})+\boldsymbol{b}(m\delta)\cdot\nabla\xi_{m\delta}-\boldsymbol{b}((n+1)\delta)\cdot\nabla\xi_{n\delta}\Big].
\end{aligned}$$

We will estimate each term of the above formula respectively. For the first term, notice that for any test function $\phi\in H^\beta(\mathbb{T}^2)$, it holds

$$\big|\big\langle\boldsymbol{b}(h\delta)\cdot\nabla(\xi_{h\delta}-\xi_{(h-1)\delta}),\phi\big\rangle\big| = \big|\big\langle\boldsymbol{b}(h\delta)\cdot\nabla\phi,\xi_{h\delta}-\xi_{(h-1)\delta}\big\rangle\big|;$$

meanwhile, (5.5) yields

$$\begin{aligned}
\big|\big\langle\boldsymbol{b}(h\delta)\cdot\nabla\phi,\xi_{h\delta}-\xi_{(h-1)\delta}\big\rangle\big| &\lesssim \|\nabla(\boldsymbol{b}(h\delta)\cdot\nabla\phi)\|_{H^\gamma}\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-1-\gamma}}\\
&\lesssim \|\boldsymbol{b}(h\delta)\|_{H^{2-\gamma}}\|\phi\|_{H^{2+2\gamma}}\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-1-\gamma}}.
\end{aligned}$$

Hence we can further get for $\beta>3$ and $\gamma\in(0,\frac12)$,

$$\big\|\boldsymbol{b}(h\delta)\cdot\nabla(\xi_{h\delta}-\xi_{(h-1)\delta})\big\|_{H^{-\beta}}\lesssim\|\boldsymbol{b}(h\delta)\|_{H^{2-\gamma}}\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-1-\gamma}}.$$

As for the second term, we can use Sobolev embedding theorem to get for $\phi \in H^\beta(\mathbb{T}^2)$,

$$\begin{aligned}\big|\langle \boldsymbol{b}(m\delta)\cdot\nabla\xi_{m\delta},\phi\rangle\big| = \big|\langle \boldsymbol{b}(m\delta)\cdot\nabla\phi,\xi_{m\delta}\rangle\big| &\lesssim \|\boldsymbol{b}(m\delta)\|_{L^2}\|\nabla\phi\|_{L^\infty}\|\xi_{m\delta}\|_{L^2}\\ &\lesssim \|\boldsymbol{b}(m\delta)\|_{H^{2-\gamma}}\|\phi\|_{H^{2+\gamma}}\|\xi_0\|_{L^2}.\end{aligned}$$

Hence $\big\|\boldsymbol{b}(m\delta)\cdot\nabla\xi_{m\delta}\big\|_{H^{-\beta}} \lesssim \|\boldsymbol{b}(m\delta)\|_{H^{2-\gamma}}\|\xi_0\|_{L^2}$ for any $\beta > 3$ and $\gamma \in (0,1)$.

Besides, the third term can be estimated in the same way as the second one and therefore we can get the similar result. Summarizing the above estimates, we obtain

$$\begin{aligned}\Big\|\sum_{h=m}^{n} I_{252}(h)\Big\|_{H^{-\beta}} \lesssim \alpha^{-1}\bigg(&\sum_{h=m+1}^{n}\|\boldsymbol{b}(h\delta)\|_{H^{2-\gamma}}\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-1-\gamma}}\\ &+\|\boldsymbol{b}(m\delta)\|_{H^{2-\gamma}}\|\xi_0\|_{L^2}+\|\boldsymbol{b}((n+1)\delta)\|_{H^{2-\gamma}}\|\xi_0\|_{L^2}\bigg).\end{aligned}$$

Then we take supremum and then expectation to further get

$$\begin{aligned}&\mathbb{E}\Big[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n} I_{252}(h)\Big\|_{H^{-\beta}}\Big]\\ &\lesssim \alpha^{-1}\sum_{h=1}^{T/\delta-1}\mathbb{E}\Big[\|\boldsymbol{b}(h\delta)\|_{H^{2-\gamma}}\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-1-\gamma}}\Big]+\alpha^{-1}\|\xi_0\|_{L^2}\,\mathbb{E}\Big[\sup_{1\le m<T/\delta-1}\|\boldsymbol{b}(m\delta)\|_{H^{2-\gamma}}\Big].\end{aligned}\tag{5.11}$$

By the Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}\Big[\|\boldsymbol{b}(h\delta)\|_{H^{2-\gamma}}\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-1-\gamma}}\Big] \le \Big(\mathbb{E}\big[\|\boldsymbol{b}(h\delta)\|_{H^{2-\gamma}}^2\big]\Big)^{\frac12}\Big(\mathbb{E}\big[\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-1-\gamma}}^2\big]\Big)^{\frac12}.$$

Considering the latter expectation, for $\gamma \in (0,1)$, Lemma 2.4 yields

$$\begin{aligned}\mathbb{E}\Big[\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-1-\gamma}}^2\Big] &\le \mathbb{E}\Big[\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-1}}^{\frac{2}{1+\gamma}}\,\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-2-\gamma}}^{\frac{2\gamma}{1+\gamma}}\Big]\\ &\le \bigg(\mathbb{E}\Big[\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-1}}^{\frac{4}{1+\gamma}}\Big]\bigg)^{\frac12}\bigg(\mathbb{E}\Big[\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-2-\gamma}}^{\frac{4\gamma}{1+\gamma}}\Big]\bigg)^{\frac12};\end{aligned}$$

moreover, to apply Lemma 3.2, we need to further estimate the second expectation of the last line as follows:

$$\mathbb{E}\Big[\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-2-\gamma}}^{\frac{4\gamma}{1+\gamma}}\Big] \le \mathbb{E}\Big[\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-2-\gamma}}^2\Big]^{\frac{2\gamma}{1+\gamma}}.$$

Combining the above results together, Lemma 3.1, Lemma 3.2 and Remark 2.6 yield

$$\begin{aligned}&\mathbb{E}\Big[\|\boldsymbol{b}(h\delta)\|_{H^{2-\gamma}}\|\xi_{h\delta}-\xi_{(h-1)\delta}\|_{H^{-1-\gamma}}\Big]\\ &\lesssim \nu^{\frac{2+3\gamma}{2(1+\gamma)}}\Big(\delta^{\frac{2+\gamma}{2(1+\gamma)}}\alpha^{\frac{2+\gamma}{2(1+\gamma)}}+\delta^{\frac{1}{1+\gamma}}\alpha^{\frac{1}{1+\gamma}}\Big)\,C_{\theta,2-\gamma,4}^{\frac{2+3\gamma}{4(1+\gamma)}}\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)^2.\end{aligned}\tag{5.12}$$

In addition, according to Lemma 2.9, we have

$$\mathbb{E}\Big[\sup_{1\le m<T/\delta-1}\|\boldsymbol{b}(m\delta)\|_{H^{2-\gamma}}\Big] \le \mathbb{E}\Big[\sup_{1\le m<T/\delta-1}\|\boldsymbol{b}(m\delta)\|_{H^{2-\gamma}}^2\Big]^{\frac12} \lesssim \nu^{\frac12}\alpha^{\frac12}\,C_{\theta,2-\gamma,2}^{1/2}\log^{\frac12}(1+\alpha T).\tag{5.13}$$

Inserting (5.12) and (5.13) into (5.11), we complete the proof. $\square$

Now let us deal with the term

$$\sum_{h=m}^{n-1} I_{251}(h) = -2\sqrt{\nu}\sum_{k\in\mathbb{Z}_0^2}\int_{m\delta}^{n\delta}\theta_k\sigma_k\cdot\nabla\xi_{[s]}\,dW_s^k,$$

where $[s]$ is defined as at the beginning of Section 5.1. By the Burkholder-Davis-Gundy's inequality, we have

$$\mathbb{E}\bigg[\Big\|2\sqrt{\nu}\sum_{k\in\mathbb{Z}_0^2}\int_{t_1}^{t_2}\theta_k\sigma_k\cdot\nabla\xi_{[s]}\,dW_s^k\Big\|_{H^{-\beta}}^4\bigg] \lesssim \nu^2\,\mathbb{E}\bigg[\Big(\sum_{k\in\mathbb{Z}_0^2}\int_{t_1}^{t_2}\big\|\theta_k\sigma_k\cdot\nabla\xi_{[s]}\big\|_{H^{-\beta}}^2\,ds\Big)^2\bigg]. \quad (5.14)$$

Let $e_k(x) = e^{2\pi ik\cdot x}$ and recall the definition of σ_k, then we have

$$\sum_{k\in\mathbb{Z}_0^2}\big\|\theta_k\sigma_k\cdot\nabla\xi_{[s]}\big\|_{H^{-\beta}}^2 \le \|\theta\|_{\ell^\infty}^2\sum_{k\in\mathbb{Z}_0^2}\big\|\sigma_k\,\xi_{[s]}\big\|_{H^{-\beta+1}}^2 \le \|\theta\|_{\ell^\infty}^2\sum_{k\in\mathbb{Z}_0^2}\big\|e_k\,\xi_{[s]}\big\|_{H^{-\beta+1}}^2;$$

furthermore, for $\beta > 3$, it holds

$$\sum_{k\in\mathbb{Z}_0^2}\big\|e_k\,\xi_{[s]}\big\|_{H^{-\beta+1}}^2 \lesssim \sum_{k\in\mathbb{Z}_0^2}\sum_{l\in\mathbb{Z}_0^2}\frac{1}{|l|^{2(\beta-1)}}\big|\langle\xi_{[s]},e_{l-k}\rangle\big|^2 = \big\|\xi_{[s]}\big\|_{L^2}^2\sum_{l\in\mathbb{Z}_0^2}\frac{1}{|l|^{2(\beta-1)}}.$$

Combing the above two estimates, we obtain

$$\sum_{k\in\mathbb{Z}_0^2}\big\|\theta_k\sigma_k\cdot\nabla\xi_{[s]}\big\|_{H^{-\beta}}^2 \lesssim \|\theta\|_{\ell^\infty}^2\|\xi_0\|_{L^2}^2\sum_{l\in\mathbb{Z}_0^2}\frac{1}{|l|^{2(\beta-1)}} \lesssim_\beta \|\theta\|_{\ell^\infty}^2\|\xi_0\|_{L^2}^2,$$

where we have used $\sum_{l\in\mathbb{Z}_0^2}|l|^{-2(\beta-1)} < \infty$. Hence (5.14) yields

$$\mathbb{E}\bigg[\Big\|2\sqrt{\nu}\sum_{k\in\mathbb{Z}_0^2}\int_{t_1}^{t_2}\theta_k\sigma_k\cdot\nabla\xi_{[s]}\,dW_s^k\Big\|_{H^{-\beta}}^4\bigg] \lesssim \nu^2\|\theta\|_{\ell^\infty}^4\|\xi_0\|_{L^2}^4\,|t_2-t_1|^2.$$

By the Kolmogorov Continuity Theorem, for every $\rho\in(0,\frac14)$, we arrive at

$$\mathbb{E}\Bigg[\sup_{0<t_1<t_2<T}\frac{\Big\|2\sqrt{\nu}\sum_{k\in\mathbb{Z}_0^2}\int_{t_1}^{t_2}\theta_k\sigma_k\cdot\nabla\xi_{[s]}\,dW_s^k\Big\|_{H^{-\beta}}}{|t_2-t_1|^\rho}\Bigg] \lesssim \nu^{\frac12}T^{\frac12-\rho}\|\theta\|_{\ell^\infty}\|\xi_0\|_{L^2}. \quad (5.15)$$

5.5 The term $I_1(h) + \delta(u_{h\delta}\cdot\nabla\xi_{h\delta})$

For the remaining two terms of (5.4), we will treat them together and prove

Lemma 5.4. *For $\gamma\in(0,1)$ and $\beta>3$, the following estimate holds for all $T\ge1$:*

$$\mathbb{E}\bigg[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n}\big(I_1(h)+\delta(u_{h\delta}\cdot\nabla\xi_{h\delta})\big)\Big\|_{H^{-\beta}}\bigg] \lesssim \delta\nu^{\frac12}\alpha^{\frac12}\,TC_{\theta,1+\gamma,2}^{1/2}\,\|\xi_0\|_{L^2}^2\big(1+\|\xi_0\|_{L^2}\big).$$

Proof. For the convenience of calculation, we make the following decomposition:

$$\begin{aligned} I_1(h)+\delta(u_{h\delta}\cdot\nabla\xi_{h\delta}) &= \int_{h\delta}^{(h+1)\delta}\big(u_{h\delta}\cdot\nabla\xi_{h\delta}-u_s\cdot\nabla\xi_s\big)\,ds \\ &= \int_{h\delta}^{(h+1)\delta}(u_{h\delta}-u_s)\cdot\nabla\xi_{h\delta}\,ds + \int_{h\delta}^{(h+1)\delta}u_s\cdot\nabla(\xi_{h\delta}-\xi_s)\,ds \\ &=: I_{11}(h)+I_{12}(h). \end{aligned}$$

We first consider the term $I_{11}(h)$. Notice that for every test function $\phi \in H^\beta(\mathbb{T}^2)$, it holds $\big|\big\langle (u_{h\delta}-u_s)\cdot\nabla\xi_{h\delta},\phi\big\rangle\big| = \big|\big\langle u_{h\delta}-u_s, \xi_{h\delta}\,\nabla\phi\big\rangle\big|$; besides, by Sobolev embedding theorem, for $\gamma\in(0,1)$, we have

$$\big|\big\langle u_{h\delta}-u_s, \xi_{h\delta}\,\nabla\phi\big\rangle\big| \le \|u_{h\delta}-u_s\|_{L^2}\|\xi_{h\delta}\|_{L^2}\|\nabla\phi\|_{L^\infty} \lesssim \|\xi_{h\delta}-\xi_s\|_{H^{-1}}\|\xi_0\|_{L^2}\|\phi\|_{H^{2+\gamma}}.$$

Then for $\beta>3$, we can further get

$$\big\|I_{11}(h)\big\|_{H^{-\beta}} \le \int_{h\delta}^{(h+1)\delta} \big\|(u_{h\delta}-u_s)\cdot\nabla\xi_{h\delta}\big\|_{H^{-\beta}}\,ds \lesssim \|\xi_0\|_{L^2}\int_{h\delta}^{(h+1)\delta}\|\xi_{h\delta}-\xi_s\|_{H^{-1}}\,ds.$$

Taking supremum and then expectation, Lemma 3.1 yields

$$\begin{aligned}\mathbb{E}\bigg[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n} I_{11}(h)\Big\|_{H^{-\beta}}\bigg] &\le \sum_{h=1}^{T/\delta-1}\mathbb{E}\Big[\big\|I_{11}(h)\big\|_{H^{-\beta}}\Big]\\ &\lesssim \|\xi_0\|_{L^2}\sum_{h=1}^{T/\delta-1}\int_{h\delta}^{(h+1)\delta}\Big(\mathbb{E}\big[\|\xi_{h\delta}-\xi_s\|_{H^{-1}}^2\big]\Big)^{\frac12}\,ds \\ &\lesssim \delta\nu^{\frac12}\alpha^{\frac12}\,TC_{\theta,1+\gamma,2}^{1/2}\,\|\xi_0\|_{L^2}^2\big(1+\|\xi_0\|_{L^2}\big).\end{aligned} \tag{5.16}$$

As for the term $I_{12}(h)$, we can use (3.4) to estimate it as follows: for any $\phi\in H^{2+\gamma}$,

$$\begin{aligned}\big|\big\langle u_s\cdot\nabla(\xi_{h\delta}-\xi_s),\phi\big\rangle\big| = \big|\big\langle u_s\cdot\nabla\phi, \xi_{h\delta}-\xi_s\big\rangle\big| &\lesssim \|\nabla(u_s\cdot\nabla\phi)\|_{L^2}\,\|\xi_{h\delta}-\xi_s\|_{H^{-1}}\\ &\lesssim \|\xi_0\|_{L^2}\|\phi\|_{H^{2+\gamma}}\|\xi_{h\delta}-\xi_s\|_{H^{-1}}.\end{aligned}$$

Hence for $\beta>3$, we have

$$\big\|I_{12}(h)\big\|_{H^{-\beta}} \le \int_{h\delta}^{(h+1)\delta}\big\|u_s\cdot\nabla(\xi_{h\delta}-\xi_s)\big\|_{H^{-\beta}}\,ds \lesssim \|\xi_0\|_{L^2}\int_{h\delta}^{(h+1)\delta}\|\xi_{h\delta}-\xi_s\|_{H^{-1}}\,ds.$$

Thus we can get the same estimate as the term $I_{11}(h)$, that is

$$\mathbb{E}\bigg[\sup_{1\le m<n\le T/\delta-1}\Big\|\sum_{h=m}^{n} I_{12}(h)\Big\|_{H^{-\beta}}\bigg] \lesssim \delta\nu^{\frac12}\alpha^{\frac12}\,TC_{\theta,1+\gamma,2}^{1/2}\,\|\xi_0\|_{L^2}^2\big(1+\|\xi_0\|_{L^2}\big). \tag{5.17}$$

Lemma 5.4 follows by combining (5.16) and (5.17). □

5.6 The terms I_a and I_b

Recall the definitions of I_a and I_b at the beginning of Section 5.1; since their treatments are similar to those involving $I_{31}(h)$ and $I_1(h)+\delta(u_{h\delta}\cdot\nabla\xi_{h\delta})$, respectively, we omit them here to save space.

Lemma 5.5. *Let $T\ge1$, $\beta>3$ and $\gamma\in(0,1)$, then the following estimates hold:*

$$\mathbb{E}\bigg[\sup_{1\le m<n\le T/\delta-1}\|I_a\|_{H^{-\beta}}\bigg] \lesssim \delta^\gamma\alpha^{\frac{\gamma}{2}}\big(\kappa\nu^{\frac{\gamma}{2}}+\nu^{1+\frac{\gamma}{2}}\big)\,TC_{\theta,1+\gamma,2}^{\gamma/2}\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)^\gamma,$$

$$\mathbb{E}\bigg[\sup_{1\le m<n\le T/\delta-1}\|I_b\|_{H^{-\beta}}\bigg] \lesssim \delta\nu^{\frac12}\alpha^{\frac12}\,TC_{\theta,1+\gamma,2}^{1/2}\,\|\xi_0\|_{L^2}^2\big(1+\|\xi_0\|_{L^2}\big).$$

5.7 Proof of Proposition 3.3

Now we will combine the results of Lemmas 5.1-5.5 and prove

$$\mathbb{E}\Big[\sup_{1\le m<n\le T/\delta-1}\frac{1}{(|n-m|\delta)^\rho}\Big\|\xi_{n\delta}-\xi_{m\delta}-(\kappa+\nu)\int_{m\delta}^{n\delta}\Delta\xi_s\,ds+\int_{m\delta}^{n\delta}u_s\cdot\nabla\xi_s\,ds\Big\|_{H^{-\beta}}\Big]$$
$$\lesssim T\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)^2\big(\nu^{1+\frac{\gamma}{2}}\alpha^{-\epsilon}+\nu^{\frac12}\|\theta\|_{\ell^\infty}\big).$$

Recalling the decompositions in (5.1) and (5.4), let us start with terms other than $I_{251}(h)$. Denote

$$I(h):=I_1(h)+I_{21}(h)+I_{22}(h)+I_{231}(h)+I_{24}(h)+I_{252}(h)+I_{31}(h)+\delta(u_{h\delta}\cdot\nabla\xi_{h\delta}),$$

then we want to prove

$$\mathbb{E}\Big[\sup_{1\le m<n\le T/\delta-1}\frac{1}{(|n-m|\delta)^\rho}\Big\|\sum_{h=m}^{n}I(h)+I_a+I_b\Big\|_{H^{-\beta}}\Big]\lesssim\nu^{1+\frac{\gamma}{2}}\alpha^{-\epsilon}\,T\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)^2. \tag{5.18}$$

As $\kappa<1$, then we can magnify it to 1 for convenience. Besides, observe that $\nu>1$ and the exponents of ν are all smaller than $1+\frac{\gamma}{2}$ in estimates in Sections 5.2–5.6, hence we keep $\nu^{1+\frac{\gamma}{2}}$ in the final result. Moreover, once $\theta\in\ell^2(\mathbb{Z}_0^d)$ is fixed, $C_{\theta,\tau,p}$ and $D_{\theta,\gamma}$ are finite constants which do not play a big role in our main results. As for the parts involving the L^2-norm of initial value ξ_0, they are all dominated by $\|\xi_0\|_{L^2}\big(1+\|\xi_0\|_{L^2}\big)^2$, so we mainly focus on parameters δ and α in the proof.

To make the term $I_{22}(h)$ satisfy (5.18), for fixed $\theta\in\ell^2(\mathbb{Z}_0^2)$, we need to make a restriction on δ and α as follows:

$$\delta^{1+\gamma-\rho}\alpha^{1+\frac{\gamma}{2}+\epsilon}\lesssim 1. \tag{5.19}$$

Moreover, to apply Lemmas 3.1 and 3.2 in the proofs of Lemmas 5.1–5.5, the following conditions are necessary:

$$\delta^4\alpha^3\lesssim 1,\quad \delta\alpha\gtrsim 1. \tag{5.20}$$

In order to verify that the above two conditions are consistent with each other, we present the specific choice of parameters. Indeed, for fixed $\gamma\in(0,\frac13)$, there exist sufficiently small ρ, $\epsilon>0$ such that

$$\epsilon+\rho<\frac14(1-\rho-2\epsilon)\gamma\quad\Leftrightarrow\quad(1+\rho)\Big(1+\frac{\gamma}{2}+\epsilon\Big)<(1-\epsilon)(1+\gamma-\rho);$$

therefore, for α big enough, we can choose δ satisfying

$$\alpha^{-\frac{1-\epsilon}{1+\rho}}\lesssim\delta\lesssim\alpha^{-\frac{1+\gamma/2+\epsilon}{1+\gamma-\rho}}. \tag{5.21}$$

With this choice, it is easy to see that (5.19) is verified. Next, we deduce from $\alpha^{-1}\le\alpha^{-\frac{1-\epsilon}{1+\rho}}\lesssim\delta$ that the second condition of (5.20) also holds. Finally, one has $\delta^4\alpha^3\lesssim\alpha^{\frac{-1+\gamma-4\epsilon-3\rho}{1+\gamma-\rho}}\le 1$ as the exponent is negative, which yields the first inequality of (5.20).

Based on the previous discussions, we choose several terms in the decomposition of $I(h)$ as examples to show that (5.18) holds under the condition (5.21), which implies (5.19) and (5.20).

(i) For the term $I_{24}(h)$, we only need to prove $\delta^{\frac{1+\gamma}{2}-\rho}\alpha^{\frac12+\epsilon}\lesssim 1$. By (5.19) and (5.20), we immediately deduce

$$\delta^{\frac{1+\gamma}{2}-\rho}\alpha^{\frac12+\epsilon}=\big(\delta^{1+\gamma-\rho}\alpha^{1+\frac{\gamma}{2}+\epsilon}\big)\big(\delta\alpha\big)^{-\frac{1+\gamma}{2}}\lesssim 1;$$

(ii) For the term $I_{231}(h)$, we want to verify that $\delta^{-1-\rho}\alpha^{-1+\epsilon} \lesssim 1$. Recalling our choice of δ in (5.21), we can further get

$$\delta^{-1-\rho}\alpha^{-1+\epsilon} \lesssim \alpha^{1-\epsilon}\alpha^{-1+\epsilon} = 1.$$

(iii) For the term $I_{252}(h)$, we only discuss the latter part here, that is:

$$\delta^{-\rho}\alpha^{-\frac{1}{2}+\epsilon}\log^{\frac{1}{2}}(1+\alpha T) \lesssim 1.$$

Note that $\log(1+\alpha T)$ is negligible with respect to $\alpha^{\frac{1}{2}}$ as α is sufficiently large, then by (5.21), the above inequality holds for small ϵ and ρ.

We can use similar method to prove that the remaining terms of Lemmas 5.1–5.5 satisfy (5.18). Finally we consider the term $I_{251}(h)$ separately, noticing that $T^{\frac{1}{2}-\rho} \leq T$ as $\rho \in (0, \frac{1}{4})$, then we can easily get the desired conclusion by combining (5.15) with (5.18).

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