Asymptotic direction of a ballistic random walk in a two-dimensional random environment with nonuniform mixing

Julien Allasia*

Abstract

In this paper, we study random walks evolving with a directional drift in a two-dimensional random environment with correlations that vanish polynomially. Using renormalization methods first employed for one-dimensional dynamic environments along with additional ideas specific to this new framework, we show that there exists an asymptotic direction for such a random walk. We also provide examples of classical models for which our results apply.

1 Introduction

Research on random walks in random environments (RWREs) has been active since the 1970s and has found its motivation in various applied fields. Typically, in a static framework, we allocate to each point in $\mathbb{Z}^d$ $(d \geqslant 1)$ a transition probability measure on the set of its neighbors, used to determine the law of the jump of a particle located at this site (we often refer to the random walker as a particle). Contrary to classical random walks, results as simple as laws of large numbers (LLNs) are often hard to obtain, and strong assumptions describing the dependencies in the environment and the ballisticity of the random walk usually have to be made.

The one-dimensional i.i.d. case is well understood, and a LLN was shown in [Sol75] using ergodicity arguments. In larger dimensions, it is possible to derive a LLN under ballisticity conditions. For instance, [Zer98] used a drift assumption that implied large deviation results. In [SZ99] and [Zer02], the authors introduced a seminal regeneration argument that gives a LLN under Kalikow's condition; see [Kal81]. In [Szn02], a weaker ballisticity condition known as Sznitman's condition (T) was introduced, which gives a LLN in the uniformly elliptic setting. In any cases, not even the i.i.d. framework is well understood without ballisticity assumptions, and our paper is no exception: we will make a drift assumption that is a strong version of ballisticity.

One can wonder if conditions on the dependencies of the environment weaker than the i.i.d. assumption would be sufficient to derive a LLN. In [CZ04], the authors managed to adapt the regeneration argument from [SZ99] when the environment is assumed to satisfy some uniform mixing conditions. Recent progress has also been made for one-dimensional dynamic random environments, in which ballisticity is automatic in the time direction. Similar mixing conditions to that of [CZ04] were used to derive a LLN; see for instance [AdHR11]. Asymptotic results were also shown for some particular environments using their specific properties, like the contact process in [dHdS14] and [MV15], or the environment given by independent simple random walks in [HdHS$^+$15].

In [BHT20] however, a LLN was shown for general environments satisfying a non-uniform polynomial mixing condition, using multi-scale renormalization methods inspired by percolation theory. The latter article fundamentally relies on a monotonicity property of the model (see (2.9) in [BHT20]), which is ensured by the dynamic framework and a nearest-neighbor assumption. Generalizing the methods of [BHT20] when this essential property is missing was already explored in [All23] by lifting the nearest-neighbor assumption. In the present paper, we keep this assumption but we move from the dynamic one-dimensional framework to a static two-dimensional one.

More precisely, we assume that we are given a polynomially correlated random environment μ on $\mathbb{Z}^2$, where for each site x in $\mathbb{Z}^2$, $\mu(x)$ gives the transition probabilities for the jump of a particle located at

*`allasia@math.univ-lyon1.fr`. Univ Lyon, CNRS, Université Claude Bernard Lyon 1, UMR5208, Institut Camille Jordan, F-69622 Villeurbanne, France.

x to one of the four nearest-neighbors of x. We consider X_n, the random walk starting at the origin in environment μ. In order to use the ideas of [BHT20], we give X_n a drift upwards by asking that $\mu_{x,x+e_2} \geqslant 1/2 + \varepsilon$ a.s. for every site x. This allows us to think of the vertical coordinate as roughly equivalent to time. Thus we will be able to show the existence of an asymptotic direction for our random walk, which is an almost sure limit of $X_n/|X_n|$ when n goes to infinity, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^2$; see Theorem 2.4. This could well be the first step towards showing a LLN in this framework, i.e. the almost sure convergence of X_n/n.

The question of the existence of an asymptotic direction for RWREs has already been discussed in the i.i.d. setup in [Sim07] and [DR10]. One important result is that if the random walk is transient in the neighborhood of a given direction, then an asymptotic direction can be found using renewal structures. But again these methods fail when we have weaker decorrelation assumptions for the environments.

Extending the ideas of [BHT20] does not merely consist in rewriting its arguments in a different framework. On top of the additional technical considerations about ballisticity (which is a given in the dynamic framework), it requires finding a way to generalize the lost monotonicity property. More precisely, we need to guarantee that if a particle starts on the left of another particle, it will remain on its left forever. This is made possible by choosing the right coupling for our random walks and proving a weaker "barrier" property: see Proposition 5.1 and Figure 4 for an illustration of what can happen in this new framework. Furthermore, the fact that our random walks can revisit their pasts calls for an argument to somehow split sample paths into different sections that do not meet. Since the classical argument of renewal times does not work with our weak decorrelation assumptions, we use a weaker notion of cut lines, presented in Section 3.5.

Outline of the paper. In Section 2, we define precisely the framework of this paper by defining static environments and random walks on them, before stating our main result, Theorem 2.4. Its proof is divided into two parts, which correspond respectively to Sections 4 and 5. In the first part, we show the existence of two limiting directions that bound the spatial behavior of our random walks in some sense. In the second part, we show that these two directions coincide, which will give the asymptotic direction that we are after. In Section 3, we introduce essential tools that will be instrumental in both parts of the proof. In Section 6, we give some ideas and problems that we are facing to show a complete LLN. In Section 7, we present some models for which our results apply.

Conventions.

- $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ respectively denote the set of natural integers (starting from 0), relative integers and real numbers. $\mathbb{N}^*$ denotes $\mathbb{N} \setminus \{0\}$, $\mathbb{R}_+$ denotes $\{x \in \mathbb{R},\, x \geqslant 0\}$ and $\mathbb{R}_+^*$ is $\mathbb{R}_+ \setminus \{0\}$. If $a < b$, $[a, b)$ is the interval $\{x \in \mathbb{R},\, a \leqslant x < b\}$. If $n \leqslant m$ are two integers, $[\![n, m]\!]$ is the set of integers $[n, m] \cap \mathbb{Z}$. For a couple $x = (a, b) \in \mathbb{R}^2$, we write $a = \pi_1(x)$ and $b = \pi_2(x)$, and we refer to them as the horizontal and vertical coordinates of x. The letter o denotes the origin $(0, 0) \in \mathbb{Z}^2$ and $\mathbf{0}$ the everywhere zero function of $\mathbb{N}^{\mathbb{Z}^2}$. If S is a finite set, $|S|$ and $\#S$ denote the cardinality of S. When we say that $a \in \mathbb{R}$ is "less than" or "at most" (resp. "greater than" or "at least") $b \in \mathbb{R}$, we mean $a \leqslant b$ (resp. $a \geqslant b$).

- c denotes a positive constant that can change throughout the paper and even from line to line. Constants that are used again later in the paper will be denoted with an index when they appear for the first time (for instance c_0, c_1...).

- The following letters will usually be used to denote the same kind of object: $n \in \mathbb{N}$ for an integer time quantity, $H \in \mathbb{N}$ for an integer space distance, $x, y, z \in \mathbb{Z}^2$ for a space location. Capital letters are usually used for events (A, F, E, $\mathcal{F}$...) or random variables (X, Y, Z, N, U...). Γ will denote a fixed history (see Section 2.4) while Λ will be a random history.

- Drawings across the paper are not to scale and they do not necessarily represent the random walks in an accurate way: they are only meant to make the reading easier. For instance, sample paths are depicted as smooth curves, although our random walks evolve on $\mathbb{Z}^2$.

2 Framework

2.1 Environment

Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Let $S = \{(p_i)_{i=1}^4 \in \mathbb{R}_+^4,\ \sum_{i=1}^4 p_i = 1\}$ and $\Omega_1 = S^{\mathbb{Z}^2}$. An element $\mu \in \Omega_1$ is called an environment. For $x \in \mathbb{Z}^2$, we will use the following notation:

$$\mu(x) = (\mu_{x,x+e_1}, \mu_{x,x-e_1}, \mu_{x,x-e_2}, \mu_{x,x+e_2}),$$

where, for example, $\mu_{x,x+e_2}$ will denote the probability for a particle located at x to jump to $x+e_2$.

We consider the topology on S induced by the canonical topology of $\mathbb{R}^4$, and the product topology on Ω_1. We denote by $\mathcal{T}_1$ the associated Borel σ-algebra. If $\mu \in \Omega_1$ and $y \in \mathbb{Z}^2$, we define the translated environment $\theta^y\mu : x \in \mathbb{Z}^2 \mapsto \mu(x+y)$. We also define, for $F \in \mathcal{T}_1$, the translated event $\theta^y F = \{\theta^y\mu,\ \mu \in F\}$.

Take a probability measure $\tilde{\mathbb{P}}$ on $(\Omega_1, \mathcal{T}_1)$. On $(\Omega_1, \mathcal{T}_1, \tilde{\mathbb{P}})$, the random variable id_{Ω_1} is called a static two-dimensional random environment with law $\tilde{\mathbb{P}}$. We denote it using the same letter μ by abuse of notation.

For the rest of the paper, we make the following assumptions on the random environment.

Assumption 2.1 (Translation invariance)**.** *For every $y \in \mathbb{Z}^2$ and $F \in \mathcal{T}_1$, we assume that*

$$\tilde{\mathbb{P}}(\theta^y F) = \tilde{\mathbb{P}}(F).$$

Assumption 2.2 (Drift)**.** *There exists $\varepsilon > 0$ and $\mathcal{A} \subseteq \Omega_1$ satisfying $\tilde{\mathbb{P}}(\mathcal{A}) = 1$ such that for every $\mu \in \mathcal{A}$,*

$$\forall x \in \mathbb{Z}^2,\ \mu_{x,x+e_2} \geqslant \frac{1}{2} + \varepsilon. \tag{2.1}$$

From now on, ε is fixed. In anticipation for Definition 3.15, we also fix an integer constant β satisfying

$$\beta > \frac{1/2 - \varepsilon}{2\varepsilon}. \tag{2.2}$$

All constants introduced from now on are allowed to depend on ε and β.

Assumption 2.3 (Vertical decoupling of the environment)**.** *Let $h > 0$. If B_1 and B_2 are 2-dimensional boxes (i.e. sets of $\mathbb{R}^2$ of the form $[a,b) \times [c,d)$ where $a < b$ and $c < d$), we say that they are h-separated if the vertical distance between B_1 and B_2 is at least h. We assume that there exist $c_0 > 0$ and $\alpha > 12$ such that for every $h > 0$, for every pair of h-separated boxes B_1 and B_2 with maximal side lengths $2(2\beta+1)h$, and for every pair of $\{0,1\}$-valued functions f_1 and f_2 on Ω_1 such that $f_1(\mu)$ is $\sigma(\mu|_{B_1})$-measurable and $f_2(\mu)$ is $\sigma(\mu|_{B_2})$-measurable,*

$$\mathrm{Cov}_{\tilde{\mathbb{P}}}\left(f_1(\mu), f_2(\mu)\right) \leqslant c_0\, h^{-\alpha}.$$

See Figure 1 for an illustration of this assumption: the environment inside box B_1 can be decoupled from that inside box B_2. We will come back to this property and this figure later, see Fact 2.9.

2.2 Random walker

We will work with random walks jumping at discrete times, but our results also hold in continuous time (in the Poissonian framework); see Remark 2.6. Mind that in [BHT20], using continuous time was crucial in the proof, because we needed that particles located at neighboring sites almost surely cannot jump simultaneously. However with this new model, time will not play such an important role in the coupling of particles. See Section 2.3 for more details.

We now define the random walk we are interested in and state our main results. For the sake of clarity, we define it in a simplified intuitive way before introducing a complete construction and a coupling in Section 2.3.

In a certain probability space with measure $\mathbb{P}$, we define the random walk $(X_n)_{n\in\mathbb{N}^*}$ as follows. The random walk starts at the origin of $\mathbb{Z}^2$: $X_0 = o$. Then, at each integer time n, the random walk jumps to one of the sites in $\{X_n + e_1, X_n - e_1, X_n - e_2, X_n + e_2\}$ with a probability given by $\mu(X_n)$, and this jump is independent of $\{X_k,\ k \leqslant n\}$ knowing $\mu(X_n)$.

The goal of this paper is to show the existence of an asymptotic direction for $X = (X_n)_{n\in\mathbb{N}}$. This is stated in the following theorem, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$ and $\mathbb{S}^1$ the unit Euclidean sphere centered at o.

Theorem 2.4 (Asymptotic direction)**.** *There exists $\chi \in \mathbb{S}^1$ with $\pi_2(\chi) > 0$ such that*

$$\mathbb{P}\text{-almost surely,} \quad \frac{X_n}{|X_n|} \xrightarrow[n\to\infty]{} \chi,$$

where $\frac{X_n}{|X_n|}$ is almost surely well-defined for n large. Moreover we have a polynomial rate of convergence:

$$\forall \xi > 0,\ \exists\, c_1 = c_1(\xi) > 0,\ \forall n \in \mathbb{N}^*, \quad \mathbb{P}\left(\big|X_n - |X_n|\chi\big| \geqslant \xi |X_n|\right) \leqslant c_1\, n^{-\alpha/4}.$$

It is straightforward to check that this result is a consequence of the following result. The latter is less appealing but its formulation is closer to the methods used in [BHT20], which is why we will focus on it from now on.

Theorem 2.5. *There exists $v \in \mathbb{R}$ such that*

$$\mathbb{P}\text{-almost surely,} \quad \frac{\pi_1(X_n)}{\pi_2(X_n)} \xrightarrow[n\to\infty]{} v,$$

where $\frac{\pi_1(X_n)}{\pi_2(X_n)}$ is almost surely well-defined for n large. Moreover we have a polynomial rate of convergence:

$$\forall \xi > 0,\ \exists\, c_2 = c_2(\xi) > 0,\ \forall n \in \mathbb{N}^*, \quad \mathbb{P}\left(|\pi_1(X_n) - v\,\pi_2(X_n)| \geqslant \xi\, |\pi_2(X_n)|\right) \leqslant c_2\, n^{-\alpha/4}. \tag{2.3}$$

Mind that in the rest of the paper, what we (abusively) call a direction is simply the relation between the two coordinates of a point in $\mathbb{Z}^2$. For instance, $\pi_1(X_n)/\pi_2(X_n)$ is the direction of X at time n. We will refer to v as the limiting direction of X. The link between v and χ from Theorems 2.4 and 2.5 is given by

$$\chi = \frac{(v,1)}{\sqrt{v^2+1}} \quad \text{and} \quad v = \frac{\pi_1(\chi)}{\pi_2(\chi)}.$$

Remark 2.6. Theorem 2.4 also holds for the random walk $(Y_t)_{t\geqslant 0}$ in $\mathbb{Z}^2$, started at o, in the following continuous time framework. Instead of jumping at integer times, we set a Poisson process $(T_n)_{n\in\mathbb{N}^*}$ of parameter 1 in $\mathbb{R}_+^*$ (independent of μ) and we allow Y_t to jump at each time given by this Poisson process; everything else is the same as in the discrete time framework. Then, $(X_n = Y_{T_n})_{n\in\mathbb{N}}$ (where $T_0 = 0$) satisfies Theorem 2.4. From there we can check that $Y_t/|Y_t|$ converges to the same asymptotic direction as $X_n/|X_n|$ when t goes to infinity.

2.3 Complete construction and coupling

Inspired by [BHT20], we want to define random walks starting from all possible starting points in $\mathbb{Z}^2$ and couple them in the following way: no matter its starting point, a random walk visiting a fixed site for the first time should jump to the same neighboring site. To define this properly, we first define a jump function $g : S \times [0,1] \to \{e_1, -e_1, -e_2, e_2\}$ by setting, for $p = (p_1,p_2,p_3,p_4) \in S$ and $u \in [0,1]$,

$$g(p,u) = \begin{cases} +e_1 & \text{if } u \in [0,p_1); \\ -e_1 & \text{if } u \in p_1 + [0,p_2); \\ -e_2 & \text{if } u \in p_1+p_2+[0,p_3); \\ +e_2 & \text{if } u \in [1-p_4, 1]. \end{cases} \tag{2.4}$$

Then, let $(U(x,i))_{x\in\mathbb{Z}^2,\, i\in\mathbb{N}^*}$ be a family of independent uniform random variables in $[0,1]$, defined on a probability space $(\Omega_2, \mathcal{T}_2, \hat{\mathbb{P}})$. The idea is that $U(x,i)$ will be the source of randomness used for the jump of a random walk visiting x for the i^{th} time. Let

$$\Omega = \Omega_1 \times \Omega_2, \quad \mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2, \quad \mathbb{P} = \tilde{\mathbb{P}} \otimes \hat{\mathbb{P}}. \tag{2.5}$$

We usually call $\mathbb{P}$ the annealed law. When $\mu \in \Omega_1$ is a fixed environment, $\mathbb{P}^\mu = \delta_{\{\mu\}} \otimes \hat{\mathbb{P}}$ is usually called the quenched law. We have $\mathbb{P}(\cdot) = \int \mathbb{P}^\mu(\cdot)\, \mathrm{d}\tilde{\mathbb{P}}(\mu)$.

In order to couple random walks, we have to count the number of times that each particle has visited each site. Therefore, for every starting point $y \in \mathbb{Z}^2$, we define simultaneously a random walk X^y and a counting process N^y, both as random variables on $(\Omega, \mathcal{T})$, by the following:

$$\begin{cases} X_0^y = y; \\ \forall x \in \mathbb{Z}^2,\ N_0^y(x) = 0; \\ \forall n \in \mathbb{N}, \forall x \in \mathbb{Z}^2,\ N_{n+1}^y(x) = N_n^y(x) + \delta_{x, X_n^y}; \\ \forall n \in \mathbb{N},\ X_{n+1}^y = X_n^y + g\left(\mu(X_n^y), U(X_n^y, N_{n+1}^y(X_n^y))\right), \end{cases} \tag{2.6}$$

where δ is the Kronecker symbol. Let us rephrase what these formulas mean. If a particle started at y reaches x for the first time at time n, then its jump at time n (namely $X_{n+1}^y - X_n^y$) is determined by $U(x,1)$. If it comes back to x later in time, it will use $U(x,2)$ to choose where to jump, and so on.

Note that when $y = o$, we do recover the law of random walk X introduced in Section 2.2, because our coupling ensures that the sequence of uniform variables used for the jumps is i.i.d. (for a detailed proof, see Proposition 3.3). Therefore, from now on, when working with $y = o$, the superscript y will be omitted, and X will denote the random walk $(X_n^o)_{n\in\mathbb{N}}$ defined in (2.6).

We will use a more practical notation for the uniform variables that are read by the random walker.

Notation 2.7. For $n \in \mathbb{N}^*$, we set $U_n^y = U(X_{n-1}^y, N_n^y(X_{n-1}^y))$.

With this notation, the induction formula that defines our random walks in 2.6 can be written in a more straightforward manner:

$$\forall n \in \mathbb{N},\ \ X_{n+1}^y = X_n^y + g\left(\mu(X_n^y), U_{n+1}^y\right). \tag{2.7}$$

Notation 2.8. Let $y \in \mathbb{Z}^2$ and P be a subset of $\mathbb{R}_+$. We define $X_P^y = \{X_s^y,\ s \in P \cap \mathbb{N}\}$ to be the sample path of X^y restricted to the times in $P \cap \mathbb{N}$.

In practice, we will use decoupling for events that involve our random walks, which are elements of the sigma-algebra that we denoted by $\mathcal{T}$ (recall (2.5)). This is actually not stronger than Assumption 2.3, because the uniform variables used for the jumps of our random walks are i.i.d., so two sets of uniform variables supported by disjoint boxes are independent. In practice, we will always use decoupling to upper bound the probability of the intersection of two events of $\mathcal{T}$. We will say that an event $A \in \mathcal{T}$ is measurable with respect to a set B if it is a measurable function of $\mu|_B$ and $\{U(x,i),\ i \in \mathbb{N}^*,\ x \in B\}$.

Fact 2.9 (Decoupling)**.** *Assume Assumption 2.3 is satisfied. Let $h > 0$. Let B_1 and B_2 be h-separated boxes with maximal side lengths $2(2\beta+1)h$. Let A_1 resp. A_2 be events of $\mathcal{T}$ that are measurable with respect to B_1 resp. B_2. We have*

$$\mathbb{P}(A_1 \cap A_2) \leqslant \mathbb{P}(A_1)\mathbb{P}(A_2) + c_0 h^{-\alpha}.$$

See Figure 1 for an illustration of this fact: events describing respectively the two sample paths drawn here can be decoupled using the decoupling property.

2.4 History

Let $n_0 \in \mathbb{N}^*$. Because of our coupling, the random walks given by $X_{n_0+\cdot}^y$ and $X^{X_{n_0}^y}$ do not necessarily have the same sample paths. Indeed, the first one has a non-empty history, in the sense that between time 0 and n_0, it has visited a certain number of sites and it has looked at n_0 random variables among the $\{U(x,i),\ x \in \mathbb{Z}^2,\ i \in \mathbb{N}^*\}$, which it will not look at again in the future. In order to address this issue, it will be convenient to define our random walks by adding an initial condition alongside the starting point, which we will call the initial history of the random walk.

Definition 2.10.

- *For $\Gamma : \mathbb{Z}^2 \to \mathbb{N}$, we define its support as* $\operatorname{Supp}\Gamma = \{x \in \mathbb{Z}^2,\ \Gamma(x) > 0\}$. *We let*

$$\mathcal{H} = \{\Gamma : \mathbb{Z}^2 \to \mathbb{N} \text{ such that } \operatorname{Supp}\Gamma \text{ is finite}\}.$$

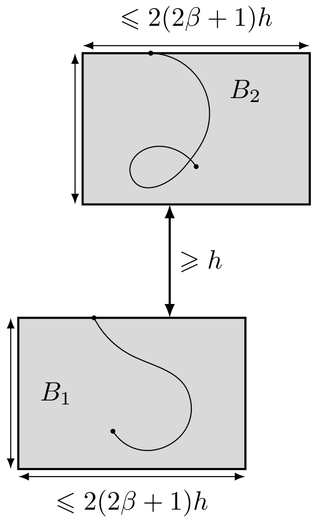


Figure 1: Illustration of the decoupling property.

- *Let $y \in \mathbb{Z}^2$ and $n \in \mathbb{N}$. The random variable N_n^y defined in (2.6), taking values in $\mathcal{H}$, is called the history of random walk X^y at time n.*
- *Let $y \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$. The random walk $X^{y,\Gamma}$ starting at y with history Γ is defined in the same way as before, except that in (2.6), we replace $U(x,i)$ by $U(x,i+\Gamma(x))$. We also define a process $N^{y,\Gamma}$ in the same way as before, and we use $U_n^{y,\Gamma}$ as in Notation 2.7. We extend the definitions of $\mathbb{P}^\mu$ and $\mathbb{P}$ to all the random walks $\{X^{y,\Gamma}, y \in \mathbb{Z}^2, \Gamma \in \mathcal{H}\}$.*

Note that we could have restricted ourselves to an even smaller subset of $\mathbb{N}^{\mathbb{Z}^2}$ for our set of histories. For instance, the support of a random walk's history has to be connected. Here we simply chose to define $\mathcal{H}$ as a simple countable subset of $\mathbb{N}^{\mathbb{Z}^2}$, in order to sum over possible outcomes $\Gamma \in \mathcal{H}$ without worrying about uncountability.

Definition 2.10 addresses the issue mentioned just before, for it ensures that for every $n_0 \in \mathbb{N}$, we have

$$\forall n \in \mathbb{N}, \quad X_{n_0+n}^y = X_n^{X_{n_0}^y, N_{n_0}^y}.$$

Using Definition 2.10, we recover (2.6) by noticing that $X_n^y = X_n^{y,\mathbf{0}}$. From now on, an omission of Γ in any notation that is defined using a history superscript Γ will always mean that we are considering $\Gamma = \mathbf{0}$. Also, as mentioned before, the omission of the starting point superscript y will mean that $y = o$.

The rest of the paper is dedicated to showing Theorem 2.5. Its final proof using lemmas that will be shown later can be found at the end of Section 3.4.2.

3 Key properties and tools

3.1 Lower-bound random walk

It will often be very handy to lower-bound the vertical position of our random walkers using Assumption 2.2. Recall Notation 2.7.

Definition 3.1. *Let $y \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$. We define the lower-bound random walk $\hat{X}^{y,\Gamma}$ as the random walk on $\mathbb{Z}$ defined by*

$$\begin{cases} \hat{X}_0^{y,\Gamma} = \pi_2(y); \\ \forall n \in \mathbb{N},\ \hat{X}_{n+1}^{y,\Gamma} = \hat{X}_n^{y,\Gamma} + \hat{g}(U_{n+1}^{y,\Gamma}), \end{cases} \tag{3.1}$$

where $\hat{g}(u) = \mathbf{1}_{u \geqslant 1/2-\varepsilon} - \mathbf{1}_{u < 1/2-\varepsilon}$.

The definition of $\hat{X}^{y,\Gamma}$ is made for the following properties to hold. First, $\hat{X}^{y,\Gamma}$ is simply a biased standard random walk, with transition coefficients given for every $n \in \mathbb{N}$ by

$$\mathbb{P}\left(\hat{X}^{y,\Gamma}_{n+1} = x+1 \,|\, \hat{X}^{y,\Gamma}_n = x\right) = \tfrac{1}{2} + \varepsilon;$$
$$\mathbb{P}\left(\hat{X}^{y,\Gamma}_{n+1} = x-1 \,|\, \hat{X}^{y,\Gamma}_n = x\right) = \tfrac{1}{2} - \varepsilon.$$

Second, it is coupled with $X^{y,\Gamma}$ in such a way that, for $\mu \in \mathcal{A}$, we have the following inclusion of events:

$$\left\{\hat{X}^{y,\Gamma}_{n+1} = \hat{X}^{y,\Gamma}_n + 1\right\} \subseteq \left\{X^{y,\Gamma}_{n+1} = X^{y,\Gamma}_n + e_2\right\}. \tag{3.2}$$

Indeed, assume $\hat{X}^{y,\Gamma}_{n+1} = \hat{X}^{y,\Gamma}_n + 1$. By definition of $\hat{g}$, this means that $U^{y,\Gamma}_{n+1} \geqslant 1/2 - \varepsilon$. Now, Assumption 2.2 ensures that for $\mu \in \mathcal{A}$, $\mu_{X^{y,\Gamma}_n, X^{y,\Gamma}_n + e_2} \geqslant 1/2 + \varepsilon$, so $U^{y,\Gamma}_{n+1} \geqslant 1 - \mu_{X^{y,\Gamma}_n, X^{y,\Gamma}_n + e_2}$, hence the result using (2.4) and (2.6).

This implies the following essential inequality between increments of $X^{y,\Gamma}$ and increments of $\hat{X}^{y,\Gamma}$.

Fact 3.2 (Increment inequality)**.** *For every $y \in \mathbb{Z}^2$, $\Gamma \in \mathcal{H}$, $n_0 \in \mathbb{N}$, $n \in \mathbb{N}^*$ and $\mu \in \mathcal{A}$,*

$$\hat{X}^{y,\Gamma}_{n_0+n} - \hat{X}^{y,\Gamma}_{n_0} \leqslant \pi_2\left(X^{y,\Gamma}_{n_0+n} - X^{y,\Gamma}_{n_0}\right). \tag{3.3}$$

This inequality is a mere consequence of (3.2). It justifies the name "lower-bound random walk" that was given to $\hat{X}$: its role is to lower-bound the vertical behavior of X.

3.2 Markov-type properties

Our coupling makes the definition of our random walks more complex than they usually are. Yet, as we already said, a single particle will behave just as in the usual framework, meaning that our random walks are Markov chains under a fixed environment. We make this more precise in the following proposition, whose proof can be found in the appendix.

Proposition 3.3. *Let $y \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$. Under either $\mathbb{P}$ or $\mathbb{P}^\mu$, the $(U^{y,\Gamma}_n)_{n\in\mathbb{N}^*}$ are independent uniform random variables in $[0,1]$.*

Corollary 3.4. *The law under $\mathbb{P}$ of $X^{y,\Gamma} - y$ and the law under $\mathbb{P}$ or $\mathbb{P}^\mu$ of $\hat{X}^{y,\Gamma} - \pi_2(y)$ do not depend on y and Γ.*

Proof. This is a consequence of Assumption 2.1, induction formulas in (2.7) and (3.1), and Proposition 3.3. □

Oftentimes, we will have to bound the probability of an event describing a random walk whose initial conditions (starting point and history) are random variables. To do this, we will need Markov-type properties.

In addition to the invariance property given by Corollary 3.4, Proposition 3.3 ensures that for any $y \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$, $X^{y,\Gamma}$ is a Markov chain under the quenched law. Nonetheless in general this Markov chain is obviously not time-homogeneous, since its transition matrices depend on the location of the random walker at each step. However, $\hat{X}^{y,\Gamma}$ is indeed a time-homogeneous Markov chain (under either $\mathbb{P}$ or $\mathbb{P}^\mu$), and so we have the strong Markov property given by Corollary 3.6.

Definition 3.5. *Let $y \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$. We say that a random variable τ is a stopping time for $X^{y,\Gamma}$ if for every $t \in \mathbb{N}$, $\{\tau = t\}$ is measurable with respect to μ and $\{U^{y,\Gamma}_n,\, n \leqslant t\}$.*

Corollary 3.6. *Let $y \in \mathbb{Z}^2$, $\Gamma \in \mathcal{H}$ and let τ be a stopping time for $X^{y,\Gamma}$. Then, conditioned on $\tau < \infty$, $\{\hat{X}_n,\, n \leqslant \tau\}$ is independent from $\{\hat{X}_n - \hat{X}_\tau,\, n > \tau\}$ (under either $\mathbb{P}$ or $\mathbb{P}^\mu$).*

However, mind that even if we are working with a deterministic time $\tau = t \in \mathbb{N}$ and under the quenched law $\mathbb{P}^\mu$, one cannot generalize Corollary 3.6 by substituting $\hat{X}$ with X, because of inhomogeneity. Indeed, the jumps of the process given by $\{X_n - X_\tau,\, n > \tau\}$ do not involve uniform variables only, but also the past of the random walk. For instance, the jump of the random walk between time t and $t+1$ is given by considering U_{t+1} and $\mu(X_t)$: even if μ is fixed, we still need to know X_t, which is clearly not independent of $\{X_n,\, n \leqslant t\}$. This is an obstacle to studying the probability of an event describing a random walk whose initial conditions are given by its past. Nonetheless, we do have the following proposition, which will be very useful in the future. Recall the definition of $\mathcal{A}$ from Assumption 2.2, as well as Definition 3.5.

Proposition 3.7. *Let $y_0 \in \mathbb{Z}^2$, $\Gamma_0 \in \mathcal{H}$ and let τ be a stopping time for X^{y_0,Γ_0}. For $y \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$, let $A^{y,\Gamma}$ be an event that is measurable with respect to μ and $(U_n^{y,\Gamma})_{n\in\mathbb{N}^*}$. Then*

$$\mathbb{P}\left(A^{X_\tau^{y_0,\Gamma_0},N_\tau^{y_0,\Gamma_0}}\right) \leqslant \sup_{\mu\in\mathcal{A}} \sup_{y,\Gamma} \mathbb{P}^\mu(A^{y,\Gamma}).$$

Therefore, if we have an upper bound of $\mathbb{P}^\mu(A^{y,\Gamma})$ that is uniform in $\mu \in \mathcal{A}$, $y \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$, it is also an upper bound of $\mathbb{P}(A^{X_\tau^{y_0,\Gamma_0},N_\tau^{y_0,\Gamma_0}})$.

Mind that *a priori* we may not replace $\sup_{y,\Gamma}\mathbb{P}^\mu(A^{y,\Gamma})$ by $\mathbb{P}^\mu(A^{o,\mathbf{0}})$. Indeed, although in the quenched setting $A^{y,\Gamma}$ is a measurable function of $(U_n^{y,\Gamma})_n$, whose law does not depend on (y,Γ), the function itself may depend on y and Γ. However we will usually not use Proposition 3.7 in that case and so we will usually simply use a supremum over μ: see for instance the proof of (3.14).

Proof. For the sake of simplicity, we write the proof for $y_0 = o$ and $\Gamma_0 = \mathbf{0}$. Let us fix an environment μ, $y \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$. First, note that the $(U_n^{y,\Gamma})_{n\in\mathbb{N}^*}$ are measurable with respect to $\{U(x,i),\, x\in\mathbb{Z}^2,\, i > \Gamma(x)\}$. Furthermore, we claim that for every $t \in \mathbb{N}$, $\{X_t = y,\, N_t = \Gamma,\, \tau = t\}$ is measurable with respect to $\{U(x,i),\, x \in \mathbb{Z}^2,\, i \leqslant \Gamma(x)\}$. Let us prove this claim. Since τ is a stopping time for X, for every $t \in \mathbb{N}$, there exists some $\{0,1\}$-valued measurable function f_t on $[0,1]^t$ such that $\{\tau = t\} = \{f_t(U_1,\ldots,U_t) = 1\}$. Therefore, we can write

$$\begin{aligned}
&\{X_t = y,\, N_t = \Gamma,\, \tau = t\} \\
&= \bigcup_{\substack{o=y_0,\ldots,y_t=y \\ 1=n_0,\ldots,n_{t-1}=\Gamma(y_{t-1})}} \begin{array}{c} \{X_1 = y_1,\ldots,X_t = y_t\} \cap \{N_1(y_0) = n_0,\ldots N_t(y_{t-1}) = n_{t-1}\} \\ \cap \{N_t = \Gamma\} \cap \{f_t(U(y_0,n_0),\ldots,U(y_{t-1},n_{t-1})) = 1\}. \end{array}
\end{aligned}$$

In the union above, all the choices of y_j and n_j (where $0 \leqslant j < t$) such that $n_j > \Gamma(y_j)$ give an empty contribution. Indeed, each choice of y_j and n_j corresponds to an event that is included in $\{N_{j+1}(y_j) = n_j,\, N_t = \Gamma\}$; therefore, if $n_j > \Gamma(y_j)$, then $N_{j+1}(y_j) > N_t(y_j)$, which is impossible since $s \mapsto N_s(x)$ is non-decreasing for any $x \in \mathbb{Z}^2$. Considering that each event in the union above is measurable with respect to $\{U(y_0,n_0),\ldots,U(y_{t-1},n_{t-1})\}$, the claim is proven.

Consequently, by Proposition 3.3, $\{X_t = y,\, N_t = \Gamma,\, \tau = t\}$ is $\mathbb{P}^\mu$-independent from the σ-algebra generated by $(U_n^{y,\Gamma})_{n\in\mathbb{N}^*}$, so it is $\mathbb{P}^\mu$-independent of $A^{y,\Gamma}$. As a result, we have

$$\begin{aligned}
\mathbb{P}\left(A^{X_\tau,N_\tau}\right) &= \sum_{y\in\mathbb{Z}^2,\,\Gamma\in\mathcal{H}} \int_{\Omega_1} \sum_{t\in\mathbb{N}} \mathbb{P}^\mu(A^{y,\Gamma},\, X_t = y,\, N_t = \Gamma,\, \tau = t)\,\mathrm{d}\tilde{\mathbb{P}}(\mu) \\
&= \sum_{y\in\mathbb{Z}^2,\,\Gamma\in\mathcal{H}} \int_{\Omega_1} \mathbb{P}^\mu(A^{y,\Gamma})\,\mathbb{P}^\mu(X_\tau = y,\, N_\tau = \Gamma)\,\mathrm{d}\tilde{\mathbb{P}}(\mu) && \text{by independence} \\
&= \sum_{y\in\mathbb{Z}^2,\,\Gamma\in\mathcal{H}} \int_{\mathcal{A}} \mathbb{P}^\mu(A^{y,\Gamma})\,\mathbb{P}^\mu(X_\tau = y,\, N_\tau = \Gamma)\,\mathrm{d}\tilde{\mathbb{P}}(\mu) && \text{since } \tilde{\mathbb{P}}(\mathcal{A}) = 1 \\
&\leqslant \sup_{\mu\in\mathcal{A}} \sup_{y,\Gamma} \mathbb{P}^\mu(A^{y,\Gamma}),
\end{aligned}$$

concluding the proof of the proposition. □

3.3 2D simplification

As explained in the introduction, the idea of our proof is to adapt arguments from the framework of one-dimensional dynamic environments from [BHT20]. The idea is therefore to treat the vertical coordinate as a time coordinate somehow. We will forget about the actual time variable and "hide" the time information by only considering hitting times of horizontal lines. In other words, we work in two dimensions instead of three (2 space + 1 time dimension), as was the case in [BHT20] (1 space + 1 time dimension).

Definition 3.8. *Let $H \in \mathbb{N}$, $y \in \mathbb{Z}^2$, $w \in \mathbb{R} \times \mathbb{Z}$ and $\Gamma \in \mathcal{H}$. The reaching time of height $\pi_2(w) + H$ by $X^{y,\Gamma}$ is defined by*

$$\tau_{H,w}^{y,\Gamma} = \begin{cases} \inf\{n \in \mathbb{N},\ \pi_2(X_n^{y,\Gamma}) = \pi_2(w) + H\} & \textit{if } \pi_2(y) \leqslant \pi_2(w) + H; \\ 0 & \textit{otherwise,} \end{cases}$$

where the infimum is in $\mathbb{N} \cup \{+\infty\}$.

In $\tau_{H,w}^{y,\Gamma}$, w is a reference point (whose horizontal coordinate does not play any role). It will be very useful in the future, because we will want to stop our random walks on a lattice centered at w, and we will have $\pi_2(y)$ slightly larger that $\pi_2(w)$ (see Definition 3.15 and the proof of Lemma 4.4). Note that when $y = w$, $\tau_{H,y}^{y,\Gamma}$ is simply the time that $X^{y,\Gamma}$ needs to go up H times.

Notations can get very heavy and so we introduce several conventions:

- Consistently with previous conventions, $\tau_H^{y,\Gamma}$ will mean $\tau_{H,o}^{y,\Gamma}$, not $\tau_{H,y}^{y,\Gamma}$. τ_H will simply be $\tau_{H,o}^{o,\mathbf{0}}$.
- We will write $X_{\tau_{H,w}^{y,\Gamma}}^{y,\Gamma}$ without specifying what $X_\infty^{y,\Gamma}$ means - any arbitrary value would work, since $\tau_{H,w}^{y,\Gamma} < \infty$ almost surely (see Section 3.4.1).
- We will write $X_{\tau_{H,w}}^{y,\Gamma}$ instead of $X_{\tau_{H,w}^{y,\Gamma}}^{y,\Gamma}$, and we will write $X_{[0,\tau_{H,w}]}^{y,\Gamma}$ instead of $X_{[0,\tau_{H,w}^{y,\Gamma}]}^{y,\Gamma}$. Mind that in these special cases, the omission of y and Γ does not mean $y = o$ and $\Gamma = \mathbf{0}$, contrary to the general rule we gave. Anyway things should be clear with the context.

We also define a stopping time for $\hat{X}^{y,\Gamma}$ as follows.

Definition 3.9. *Let $y \in \mathbb{Z}^2$, $\Gamma \in \mathcal{H}$ and $H \in \mathbb{N}$. We let*

$$\hat{\tau}_H^{y,\Gamma} = \inf\{n \in \mathbb{N},\ \hat{X}_n^{y,\Gamma} = \pi_2(y) + H\} \in \mathbb{N} \cup \{+\infty\}. \tag{3.4}$$

Mind that $\hat{\tau}_H^{y,\Gamma}$ is the equivalent for $\hat{X}^{y,\Gamma}$ of $\tau_{H,y}^{y,\Gamma}$, not $\tau_H^{y,\Gamma}$.

In order to show Theorem 2.5, it will actually be sufficient to show an almost sure asymptotic estimate for X along the subsequence given by $(\tau_H)_{H\in\mathbb{N}}$. This is what the following lemma is about.

Lemma 3.10. *There exists $v \in \mathbb{R}$ such that*

$$\mathbb{P}\textit{-almost surely,}\quad \frac{\pi_1(X_{\tau_H})}{H} \xrightarrow[H\to\infty]{} v.$$

The proof of Lemma 3.10 is the purpose of Sections 4 and 5. The fact that Lemma 3.10 implies Theorem 2.5 is shown at the end of Section 3.4.2.

3.4 Localization properties

3.4.1 Ballisticity

Recall Definition 3.1. Classically, we have, for any $y \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$, the almost sure divergence $\hat{X}_n^{y,\Gamma} \xrightarrow[n\to\infty]{} +\infty$. Therefore, because of (3.3), we also have

$$\pi_2(X_n^{y,\Gamma}) \xrightarrow[n\to\infty]{} +\infty \quad \mathbb{P}\text{-almost surely.} \tag{3.5}$$

In other words, we have directional transience for $X^{y,\Gamma}$ along the e_2 direction. Actually we have a much stronger ballisticity property that gives a minimum speed along the vertical coordinate, which is one of the key properties usually required to get a LLN for a RWRE.

Proposition 3.11. *For any $\xi > 0$, there exists a constant $c_3 = c_3(\xi) > 0$ such that for every $n \in \mathbb{N}$, $y \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$, we have*

$$\mathbb{P}\left(\left|\hat{X}_n^{y,\Gamma} - \pi_2(y) - 2\varepsilon n\right| \geqslant \xi n\right) \leqslant c_3^{-1} e^{-c_3 n}, \tag{3.6}$$

and the inequality is also true when replacing $\mathbb{P}$ by $\mathbb{P}^\mu$ for $\mu \in \mathcal{A}$.

Naturally, the limiting speed 2ε in Proposition 3.11 is simply the minimal possible expected value for the vertical jump of X, that is $(\frac{1}{2}+\varepsilon)-(\frac{1}{2}-\varepsilon)=2\varepsilon$, according to Assumption 2.2. The proof of Proposition 3.11 is based on a very classical Chernoff bound, so we choose to leave it out. From Proposition 3.11 and inequality (3.3), we can easily deduce the following ballisticity property for $X^{y,\Gamma}$.

Corollary 3.12 (Ballisticity)**.** *For any $\zeta \in (0, 2\varepsilon)$, $y \in \mathbb{Z}^2$, $\Gamma \in \mathcal{H}$ and $n \in \mathbb{N}$,*

$$\mathbb{P}\left(\pi_2(X_n^{y,\Gamma}) - \pi_2(y) \leqslant \zeta n\right) \leqslant c_3^{-1} e^{-c_3 n},$$

where $c_3 = c_3(2\varepsilon - \zeta)$ is the constant defined in Proposition 3.11. Moreover, the inequality is also true when replacing $\mathbb{P}$ by $\mathbb{P}^\mu$, for $\mu \in \mathcal{A}$.

3.4.2 Vertical lower bound

Another key property of biased random walks is the gambler's ruin estimate (see for instance [GS01], Section 3.9), that gives a formula for the probability of exiting a section of $\mathbb{Z}$ by either of the two sides. This will allow us to have a global lower bound for the second coordinate of $X^{y,\Gamma}$.

We define an event guaranteeing that $X^{y,\Gamma}$ stays above a certain horizontal line by setting, for $H \in \mathbb{N}$,

$$E_H^{y,\Gamma} = \left\{\forall n \in \mathbb{N},\ \pi_2(X_n^{y,\Gamma}) \geqslant \pi_2(y) - H\right\}. \tag{3.7}$$

Proposition 3.13. *There exists $c_4 > 0$ such that for every $H \in \mathbb{N}$,*

$$\mathbb{P}\left((E_H^{y,\Gamma})^c\right) \leqslant e^{-c_4 H}. \tag{3.8}$$

Moreover when $H = 0$, we even have

$$\mathbb{P}\left((E_0^{y,\Gamma})^c\right) \leqslant 1 - 2\varepsilon. \tag{3.9}$$

Both inequalities are also true when replacing $\mathbb{P}$ by $\mathbb{P}^\mu$ for $\mu \in \mathcal{A}$.

The proof of Proposition 3.13 can be found in the appendix.

The localization properties given by Propositions 3.12 and 3.13 allow us to prove that Lemma 3.10 is sufficient to prove Theorem 2.5, using an argument of interpolation.

Proof of Theorem 2.5. Let v be as in Lemma 3.10. Let $n \in \mathbb{N}^*$ be such that $\pi_2(X_n) > 0$ (which, by (3.5), happens for n large enough $\mathbb{P}$-almost surely). Let $H_n \in \mathbb{N}$ be such that $\tau_{H_n} \leqslant n < \tau_{H_n+1}$. Note that, using (3.5) again,

$$H_n \xrightarrow[n\to\infty]{a.s} +\infty. \tag{3.10}$$

Also, note that since $\pi_2(X_n) < H_n + 1$, we have, for n large enough,

$$\mathbb{P}(H_n < \varepsilon n) \leqslant \mathbb{P}(\pi_2(X_n) < \varepsilon n + 1) \leqslant \mathbb{P}\left(\pi_2(X_n) \leqslant \frac{3\varepsilon}{2} n\right) \leqslant c_3(\varepsilon/2)^{-1}\, e^{-c_3(\varepsilon/2)n}. \tag{3.11}$$

Now, note that

$$\frac{\pi_1(X_n)}{\pi_2(X_n)} = \frac{\pi_1(X_{\tau_{H_n}})}{H_n} + \frac{\pi_1(X_n) - \pi_1(X_{\tau_{H_n}})}{H_n} + \pi_1(X_n)\left(\frac{1}{\pi_2(X_n)} - \frac{1}{H_n}\right). \tag{3.12}$$

First, using (3.10) and Lemma 3.10, we have

$$\frac{\pi_1(X_{\tau_{H_n}})}{H_n} \xrightarrow[n\to\infty]{a.s.} v. \tag{3.13}$$

As for the second term on the right-hand side of (3.12), let us fix $a > 0$ and note that

$$\mathbb{P}\left(|\pi_1(X_n) - \pi_1(X_{\tau_{H_n}})| \geqslant aH_n\right) \leqslant \mathbb{P}(\tau_{H_n+1} - \tau_{H_n} \geqslant aH_n)$$

$$\leqslant \mathbb{P}\left(\tau_{1,X_{\tau_{H_n}}}^{X_{\tau_{H_n}},N_{\tau_{H_n}}} \geqslant a\varepsilon n\right) + \mathbb{P}(H_n < \varepsilon n).$$

Now, if we fix $\mu \in \mathcal{A}$, we have

$$\begin{aligned}
\mathbb{P}^\mu(\tau_1 \geqslant a\varepsilon n) &\leqslant \mathbb{P}^\mu(X_{\lfloor a\varepsilon n\rfloor} \leqslant \varepsilon\lfloor a\varepsilon n\rfloor) && \text{for } n \text{ large enough}\\
&\leqslant c_3(\varepsilon)^{-1} e^{-c_3(\varepsilon)\lfloor a\varepsilon n\rfloor} && \text{using Corollary 3.12}\\
&\leqslant c^{-1}e^{-cn}.
\end{aligned}$$

Therefore, using Proposition 3.7 and (3.11),

$$\mathbb{P}\left(|\pi_1(X_n) - \pi_1(X_{\tau_{H_n}})| \geqslant aH_n\right) \leqslant c^{-1}e^{-cn}, \tag{3.14}$$

which is summable in n. Using Borel-Cantelli, we obtain that

$$\frac{\pi_1(X_n) - \pi_1(X_{\tau_{H_n}})}{H_n} \xrightarrow[n\to\infty]{a.s.} 0. \tag{3.15}$$

In order to estimate the third term on the right-hand side of (3.12), first note that if $\pi_2(X_n) \geqslant H_n - H_n^{1/2}$ and $H_n \geqslant \varepsilon n$, then we have, for n large enough,

$$\left|\pi_1(X_n)\left(\frac{1}{\pi_2(X_n)} - \frac{1}{H_n}\right)\right| = \left|\frac{\pi_1(X_n)}{\pi_2(X_n)}\right| \frac{H_n - \pi_2(X_n)}{H_n} \leqslant \frac{n}{H_n/2}\frac{H_n^{1/2}}{H_n} = 2nH_n^{-3/2}.$$

In the first equality, we used that $H_n - \pi_2(X_n) \geqslant 0$, since $n < \tau_{H_n+1}$. In the inequality, we used that $|\pi_1(X_n)| \leqslant n$, and that for n large enough, $H_n \geqslant \varepsilon n \geqslant 4$, so $H_n - H_n^{1/2} \geqslant H_n/2$. Therefore, if we fix $a > 0$, we have, for n large enough,

$$\begin{aligned}
\mathbb{P}\left(\left|\pi_1(X_n)\left(\frac{1}{\pi_2(X_n)} - \frac{1}{H_n}\right)\right| \geqslant a\right) &\leqslant \mathbb{P}(\pi_2(X_n) < H_n - H_n^{1/2}) + \mathbb{P}(H_n \leqslant (2n/a)^{2/3}) + \mathbb{P}(H_n < \varepsilon n)\\
&\leqslant \mathbb{P}\left(\left(E_{\lfloor(\varepsilon n)^{1/2}\rfloor}^{X_{\tau_{H_n}},N_{\tau_{H_n}}}\right)^c\right) + 3\mathbb{P}(H_n < \varepsilon n)\\
&\leqslant e^{-cn^{1/2}} + 3c^{-1}e^{-cn},
\end{aligned}$$

using Propositions 3.13 and 3.7, as well as (3.11). Applying the Borel-Cantelli lemma once more, we obtain that

$$\pi_1(X_n)\left(\frac{1}{\pi_2(X_n)} - \frac{1}{H_n}\right) \xrightarrow[n\to\infty]{a.s.} 0. \tag{3.16}$$

Putting together (3.12), (3.13), (3.15) and (3.16), we obtain that

$$\frac{\pi_1(X_n)}{\pi_2(X_n)} \xrightarrow[n\to\infty]{a.s.} v,$$

concluding the proof of Theorem 2.5. □

3.4.3 Horizontal bounds

It will also be essential to control the horizontal behavior of the random walk. The lack of intrinsic information on the horizontal jumps of the random walks does not allow us to get a global horizontal bound as in Section 3.4.2. However, what we can do using Assumption 2.2 is bound the horizontal displacement of the random walk by the time it reaches a certain height. To that end, we define the following event, for $y \in \mathbb{Z}^2$, $\Gamma \in \mathcal{H}$ and $H \in \mathbb{N}^*$:

$$D_H^{y,\Gamma} = \left\{\forall n \in [\![0, \tau_{H,y}^{y,\Gamma}]\!],\ \left|\pi_1(X_n^{y,\Gamma}) - \pi_1(y)\right| \leqslant \beta H\right\}. \tag{3.17}$$

Proposition 3.14. *There exists $c_5 > 0$ such that for every $y \in \mathbb{Z}^2$, $\Gamma \in \mathcal{H}$ and $H \in \mathbb{N}^*$,*

$$\mathbb{P}\left((D_H^{y,\Gamma})^c\right) \leqslant c_5^{-1}e^{-c_5 H}, \tag{3.18}$$

and the same estimate holds with $\mathbb{P}^\mu$ for any $\mu \in \mathcal{A}$.

We refer to the appendix for a proof of Proposition 3.14.

3.4.4 Localization in boxes

In order to apply Fact 2.9, we will have to localize events in boxes. In practice, this will be done by working on large probability events that ensure that our random walks stay in certain boxes before reaching a certain height, or, in other words, that they exit those boxes through the top side. This simply requires to put together the results of Sections 3.4.2 and 3.4.3. However, we actually want something stronger: we want to control the behavior of a lot of particles simultaneously. This will be instrumental for Section 5.

Recall the definition of β in (2.2). We will also often use the following notation, for $H \in \mathbb{R}_+^*$,

$$H' = \left\lceil H^{1/2} \right\rceil . \tag{3.19}$$

We will also use this notation with specific values of H: for instance in the future we will write H'_k for $\left\lceil H_k^{1/2} \right\rceil$ or $(hL_k)'$ for $\lceil (hL_k)^{1/2} \rceil$.

Definition 3.15. *Let $H \in \mathbb{R}_+^*$ and $w \in \mathbb{R} \times \mathbb{Z}$. We define*

$$\begin{cases} I_H(w) = (w + [0,H) \times [0,H')) \cap \mathbb{Z}^2; \\ \mathcal{I}_H(w) = (w + [0,H) \times \{0\}) \cap \mathbb{Z}^2; \\ B_H(w) = w + [-\beta H, (\beta+1)H) \times [-H', H] \subseteq \mathbb{R}^2. \end{cases} \tag{3.20}$$

We also define the following events, for $H \in \mathbb{N}^*$ and $w \in \mathbb{R} \times \mathbb{Z}$:

$$F_H(w) = \bigcap_{y \in I_H(w)} \left\{ X^y_{[0,\tau_{H,w}]} \subseteq B_H(w) \right\} . \tag{3.21}$$

As usual, $I_H = I_H(o)$, $B_H = B_H(o)$ and $F_H = F_H(o)$.

See Figure 2 for an illustration of those definitions.

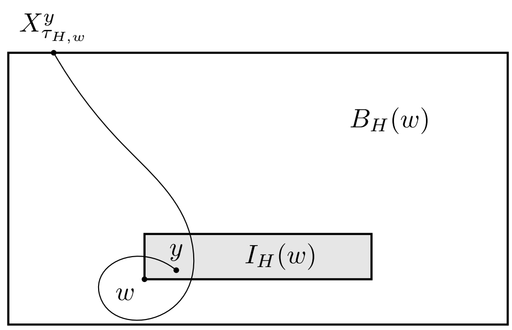


Figure 2: Illustration of Definition 3.15. On $F_H(w)$, random walk X^y has to exit $B_H(w)$ through the top side.

Note that in Definition 3.15, we used a real parameter $H > 0$, while H is usually an integer. This is because we will use the objects defined above with non-integer parameters as of Section 5.2.

Naturally, the choice of $H^{1/2}$ and β are made in order for our random walks to exit the boxes that we defined through the top side with large probability. In fact, we have the following estimates.

Proposition 3.16. *There exists $c_6 > 0$ such that for every $w \in \mathbb{R} \times \mathbb{Z}$ and $H \in \mathbb{N}^*$, we have*

$$\mathbb{P}\left(F_H(w)^c\right) \leqslant c_6^{-1} e^{-c_6 H^{1/2}} . \tag{3.22}$$

This is also true when replacing $\mathbb{P}$ by $\mathbb{P}^\mu$ for $\mu \in \mathcal{A}$.

Proof. This is a direct consequence of Propositions 3.13 and 3.14, along with a union bound. □

Remark 3.17. For the rest of the paper, we fix $\mathbf{H}_0$ an integer constant satisfying

$$\forall H \geqslant \mathbf{H}_0,\ H' \leqslant \min(H/2, 2\beta H).$$

Why is that? We will often use Fact 2.9 with $B_H(w)$ and $h = H/2$. The horizontal size of boxes $B_H(w)$ is precisely $(2\beta+1)H = 2(2\beta+1)h$. As for the vertical size, it is equal to $H + H'$, and we want it to be at most $(2\beta+1)H$, so we want $H' \leqslant 2\beta H$. As for the $H' \leqslant H/2$ condition, it is because we will encounter boxes that are $(H-H')$-separated; in order to apply the decoupling property with vertical separation $h = H/2$, we therefore need $H' \leqslant H/2$. For the rest of the paper, we will work with $H \geqslant \mathbf{H}_0$.

3.5 Cut lines

When trying to adapt the ideas of [BHT20], the history that our random walk accumulates will raise issues (see for instance Section 4.2.3). Therefore, it will be very useful to find a time after which our random walk does not revisit the sites visited in the past. In this sense, everything will be as if, considering the random walk after this time, its initial history is everywhere zero.

Definition 3.18. *Let $z \in \mathbb{N}$.*

- *Let $Z = (Z_n)_{n\in\mathbb{N}}$ be a random walk in $\mathbb{Z}$ and let T_z denote the first hitting time of $\{z\}$ by Z. We say that z is a cut point for Z if $T_z < \infty$ and for every $n \geqslant T_z$, $Z_n \geqslant z$. In other words, the sample path of the random walk Z can be split into two parts with each part contained in a half-line delimited by z. We set*

$$\Theta(Z) = \inf\{a \in \mathbb{N},\ Z_0 + a \text{ is a cut point for } Z\};$$
$$T_c(Z) = T_{Z_0+\Theta(Z)}.$$

- *Let now $Z = (Z_n)_{n\in\mathbb{N}}$ be a random walk in $\mathbb{Z}^2$. We say that $\mathbb{R}\times\{z\}$ is a cut line for Z if z is a cut point for $\pi_2(Z)$. We extend the previous definitions by setting $\Theta(Z) = \Theta(\pi_2(Z))$ and $T_c(Z) = T_c(\pi_2(Z))$.*

As before, we start by showing estimates on the lower-bound random walk (recall Section 3.1). We refer to the appendix for a proof of the next Lemma.

Lemma 3.19. *There exists $c_7 > 0$ such that for every $y \in \mathbb{Z}^2$, $\Gamma \in \mathcal{H}$ and $a \in \mathbb{N}$,*

$$\mathbb{P}\left(\Theta(\hat{X}^{y,\Gamma}) > a\right) \leqslant c_7^{-1} e^{-c_7 a^{1/2}}.$$

The inequality is also true when replacing $\mathbb{P}$ by $\mathbb{P}^\mu$ for $\mu \in \mathcal{A}$.

Proposition 3.20. *There exists $c_8 > 0$ such that for every $y \in \mathbb{Z}^2$, $\Gamma \in \mathcal{H}$ and $a \in \mathbb{N}$,*

$$\mathbb{P}\left(\Theta(X^{y,\Gamma}) > a\right) \leqslant c_8^{-1} e^{-c_8 a^{1/2}}.$$

This estimate is also true when replacing $\mathbb{P}$ by $\mathbb{P}^\mu$ for $\mu \in \mathcal{A}$.

Proof. We write the proof for $y = o$ and $\Gamma = \mathbf{0}$ for simplicity. Let $\mu \in \mathcal{A}$ and $a \in \mathbb{N}$. The crucial idea here is that $X_{T_c(\hat{X})} + \mathbb{R}\times\{0\}$ is a cut line for X. Indeed, using increment inequality (3.3), we have:

- For $n \in \mathbb{N}$, $\pi_2(X_{T_c(\hat{X})+n}) \geqslant \pi_2(X_{T_c(\hat{X})}) + \hat{X}_{T_c(\hat{X})+n} - \hat{X}_{T_c(\hat{X})} \geqslant \pi_2(X_{T_c(\hat{X})})$;
- For $0 < n \leqslant T_c(\hat{X})$, $\pi_2(X_{T_c(\hat{X})-n}) \leqslant \pi_2(X_{T_c(\hat{X})}) + \hat{X}_{T_c(\hat{X})-n} - \hat{X}_{T_c(\hat{X})} < \pi_2(X_{T_c(\hat{X})})$.

Using this observation, if we let $b = \lfloor \varepsilon a \rfloor$, we get

$$\begin{aligned}\mathbb{P}^\mu(\Theta(X) > a) &\leqslant \mathbb{P}^\mu(\pi_2(X_{T_c(\hat{X})}) > a)\\ &\leqslant \mathbb{P}^\mu(\pi_2(X_{T_c(\hat{X})}) > a,\, \Theta(\hat{X}) \leqslant \lfloor \varepsilon a\rfloor) + \mathbb{P}^\mu(\Theta(\hat{X}) > \lfloor \varepsilon a \rfloor).\end{aligned}$$

Now, using Lemma 3.19, $\mathbb{P}^\mu(\Theta(\hat{X}) > \lfloor \varepsilon a\rfloor) \leqslant c_7^{-1} e^{-c_7 \lfloor \varepsilon a \rfloor^{1/2}}$, and

$$\mathbb{P}^\mu(\pi_2(X_{T_c(\hat{X})}) > a,\, \Theta(\hat{X}) \leqslant \lfloor \varepsilon a \rfloor)$$

$$
\begin{aligned}
&\leqslant \mathbb{P}^\mu(\pi_2(X_{\hat\tau_{\lfloor \varepsilon a\rfloor}}) > a) && \text{using increment inequality (3.3)}\\
&\leqslant \mathbb{P}^\mu(\hat\tau_{\lfloor \varepsilon a\rfloor} > a) \\
&\leqslant c_3(\varepsilon)^{-1} e^{-c_3(\varepsilon)\lceil a\rceil} && \text{using Proposition 3.11}
\end{aligned}
$$

hence the result by adjusting c_8. □

Corollary 3.21. *There exists $c_9 > 0$ such that for every $y \in \mathbb{Z}^2$, $\Gamma \in \mathcal{H}$ and $n \in \mathbb{N}$, we have*

$$\mathbb{P}\left(T_c(X^{y,\Gamma}) > n\right) \leqslant c_9^{-1} e^{-c_9 n^{1/2}}.$$

This estimate is also true when replacing $\mathbb{P}$ by $\mathbb{P}^\mu$, for $\mu \in \mathcal{A}$.

Proof. Let $y = o$, $\Gamma = \mathbf{0}$, $n \in \mathbb{N}$ and $\mu \in \mathcal{A}$. Using Propositions 3.12 and 3.20,

$$
\begin{aligned}
\mathbb{P}^\mu(T_c(X) > n) &\leqslant \mathbb{P}^\mu(\Theta(X) > \lfloor \varepsilon n\rfloor) + \mathbb{P}^\mu(T_c(X) > n,\ \Theta(X) \leqslant \lfloor \varepsilon n\rfloor)\\
&\leqslant c_8^{-1} e^{-c_8 \lfloor \varepsilon n\rfloor^{1/2}} + \mathbb{P}^\mu(\tau_{\lfloor \varepsilon n\rfloor} > n)\\
&\leqslant c^{-1} e^{-c n^{1/2}} + c_3(\varepsilon)^{-1} e^{-c_3(\varepsilon) n},
\end{aligned}
$$

hence the result by choosing c_9 properly. □

3.6 The multi-scale renormalization method

The proofs of several major propositions in the rest of the paper are based on the fundamental idea of multi-scale renormalization, which gives a practical method for using decoupling property (2.9). We now give a general idea of how such a proof works, and we will often refer to it in the future.

Suppose we want to show an estimate for the probability of a certain family of "bad" events $(A_H)_{H\in\mathbb{N}}$.

- We start by focusing on a certain subsequence $(A_{H_k})_k$, $(H_k)_{k\in\mathbb{N}}$ being a sequence of scales. We set $p_k = \mathbb{P}(A_{H_k})$. We show that $A_{H_{k+1}}$ is included in two events of probability p_k supported by R_k-separated boxes of maximal side lengths $2(2\beta+1)R_k$.
- We deduce the desired estimate for $(p_k)_{k\in\mathbb{N}}$.
  - Using Fact 2.9 and a union bound, we get an inequality
    $$p_{k+1} \leqslant C_k\,(p_k^2 + c_0 R_k^{-\alpha}),$$
    where C_k is a certain integer.
  - From this inequality we deduce the desired estimate of p_k by induction on k. For this to work, the scales and the bound to show have to be chosen properly. The base case of the induction (often referred to as "triggering") requires arguments that are specific to each case.
- We conclude by interpolating the estimate from the $(H_k)_{k\in\mathbb{N}}$ to any parameter H.

In order to accommodate to the polynomial decoupling, it will be useful to use the following scales. Recall the definition of $\mathbf{H}_0$ from Remark 3.17.

Definition 3.22. *We set $L_0 = \max(10^{10}, \mathbf{H}_0)$ and, for $k \geqslant 0$,*

$$L_{k+1} = l_k\, L_k, \quad \textit{where} \quad l_k = \lfloor L_k^{1/4}\rfloor.$$

The choice 10^{10} will become clearer in the proof of Proposition 5.12.

The rest of this paper will be dedicated to showing Lemma 3.10. To do this, we strongly rely on methods developed in [BHT20]. First, in Section 4, we will show that there exist limiting directions v_- and v_+ that bound the asymptotic behavior of our random walk with high probability. This requires to adapt the methods in [BHT20] by addressing two technical issues: the deterministic drift in the time direction is lost in the static framework, and the random walks can revisit their paths. Then, in Section 5, we will show that these two directions actually coincide, which will give us the limiting direction v in Lemma 3.10. It is in this part of the proof that introducing a weaker "barrier" property as a replacement of the monotonicity property of [BHT20] will be instrumental.

4 Limiting directions

4.1 Definitions and main results

Recall that for us a direction is simply the inverse slope of a line in $\mathbb{R}^2$; for instance, all points $x \in \mathbb{R}^2$ satisfying $\pi_1(x) = a\pi_2(x)$ with $\pi_2(x) > 0$ have direction a. The goal of this section is to show that there exist two directions v_- and v_+ that somehow bound the spatial behavior of our random walker in the long run. This property is made clearer in Lemma 4.4. It will consist of the first part of the proof of Lemma 3.10, and we will show that in fact $v_- = v_+$ in Section 5, thus concluding the proof.

As a matter of fact, we aim at showing a stronger version of Lemma 3.10 by considering not only one fixed particle but all the particles starting simultaneously in $I_H(w)$ from Definition 3.15. This will be instrumental in Section 5, where we will need to control the directions of lots of particles at once. Recall also notation $\tau_{H,w}^{y,\Gamma}$ from Definition 3.8.

Definition 4.1. *Let $w \in \mathbb{R} \times \mathbb{Z}$ and $H \in \mathbb{N}^*$. Let $y \in I_H(w)$ and $\Gamma \in \mathcal{H}$. We define the empirical direction of $X^{y,\Gamma}$ at height H with reference point w to be*

$$V_{H,w}^{y,\Gamma} = \frac{1}{H}\left(\pi_1(X_{\tau_{H,w}}^{y,\Gamma}) - \pi_1(y)\right).$$

As usual, when w or y are not mentioned, it means that we are considering the origin, and an omission of Γ means $\Gamma = \mathbf{0}$. Now let $v \in \mathbb{R}$. We consider the following events:

$$A_{H,w}(v) = \left\{\exists\, y \in I_H(w),\, V_{H,w}^y \geqslant v\right\};$$
$$\tilde{A}_{H,w}(v) = \left\{\exists\, y \in I_H(w),\, V_{H,w}^y \leqslant v\right\}.$$

As usual, $A_H(v) = A_{H,o}(v)$ and $\tilde{A}_H(v) = \tilde{A}_{H,o}(v)$. We set

$$p_H(v) = \mathbb{P}(A_H(v));$$
$$\tilde{p}_H(v) = \mathbb{P}(\tilde{A}_H(v)).$$

We define the limiting directions by setting

$$v_+ = \inf\left\{v \in \mathbb{R},\, \liminf_{H\to\infty} p_H(v) = 0\right\};$$
$$v_- = \sup\left\{v \in \mathbb{R},\, \liminf_{H\to\infty} \tilde{p}_H(v) = 0\right\}.$$

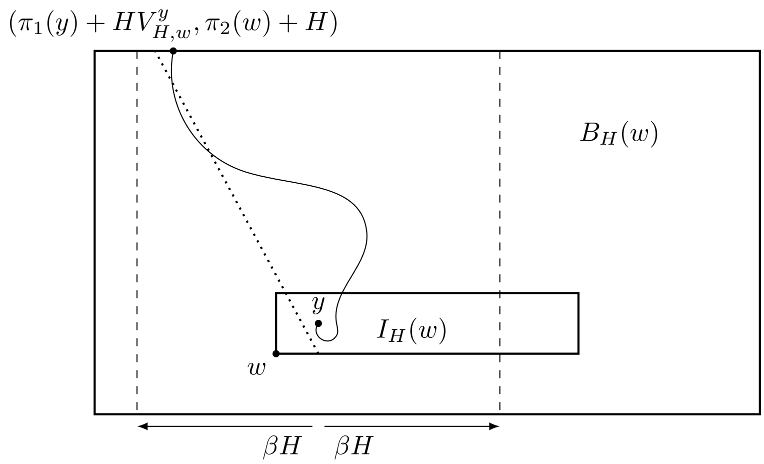


Figure 3: Illustration of $A_{H,w}(v)$. The sample path started at y reaches height $\pi_2(w) + H$ with a direction $V_{H,w}^y$ larger than the direction v given by the dotted line.

Note that when $\pi_2(y) = \pi_2(w)$, $V_{H,w}^{y,\Gamma} = V_{H,y}^{y,\Gamma}$ is simply the inverse slope of the line connecting y and $X_{\tau_{H,w}}^{y,\Gamma}$: it is the direction of $X^{y,\Gamma}$ at height H. Mind that when $\pi_2(y) > \pi_2(w)$ however, this is not exactly true anymore.

Note that because of translation invariance, $\mathbb{P}(A_{H,w}(v))$ and $\mathbb{P}(\tilde{A}_{H,w}(v))$ actually do not depend on w, which is why we only considered the origin for the definitions of v_- and v_+. Indeed, we can first restrict ourselves to $w \in (-1, 0] \times \{0\}$, using Corollary 3.4. Then, H being an integer here, $I_H(w) = I_H(o)$ for every $w \in (-1, 0] \times \{0\}$, so that $A_{H,w}(v) = A_H(v)$. This would be wrong if H was any positive real number, and that is why we will have to be more careful later, in Lemma 5.4.

It may sound unclear why we use liminfs in the definitions of v_- and v_+, instead of limsups. In fact, this will be required in order to get a much needed uniform lower bound on the probability for the random walk to attain average direction greater but close to v_- over long time intervals (see Lemma 5.4).

Note that we never stated that $v_- \leqslant v_+$, although it would be very tempting to say that it is obvious. In fact, it is not an obvious consequence of their definitions, but it will be a consequence of Lemma 4.4.

Fact 4.2. *We have the following bounds on v_- and v_+:*

$$\begin{cases} -\beta \leqslant v_+ \leqslant \beta; \\ -\beta \leqslant v_- \leqslant \beta. \end{cases} \tag{4.1}$$

Proof. The proof being symmetric, let us just focus on the bounds for v_+.

- If $v < -\beta$, then using Proposition 3.14,

$$p_H(v) = \mathbb{P}(\exists\, y \in I_H,\, V_H^y \geqslant v) \geqslant \mathbb{P}(V_H \geqslant -\beta) \geqslant \mathbb{P}(D_H) \geqslant 1 - c_5^{-1} e^{-c_5 H} \xrightarrow[H\to\infty]{} 1.$$

- If $v > \beta$, using Proposition 3.14 again,

$$\begin{aligned} p_H(v) &= \mathbb{P}(\exists\, y \in I_H,\, V_H^y \geqslant v) \leqslant \mathbb{P}(\exists\, y \in I_H,\, V_H^y > \beta) \\ &\leqslant HH' \sup_{y \in I_H(w)} \mathbb{P}\left((D_H^y)^c\right) \leqslant c_5^{-1} HH' e^{-c_5 H^{1/2}} \xrightarrow[H\to\infty]{} 0. \end{aligned}$$

□

Remark 4.3. Note that $v \in \mathbb{R} \mapsto p_H(v)$ is a non-increasing function. Therefore, for $v > v_+$, we must have $\liminf_{H\to\infty} p_H(v) = 0$. Similarly, for $v < v_-$, $\liminf_{H\to\infty} \tilde{p}_H(v) = 0$.

In spite of Remark 4.3, the definitions that we gave for v_- and v_+ are quite weak at first glance, because we only have information on the liminfs. Our goal now is to show that for $v > v_+$ and $v < v_-$, the liminfs given in Remark 4.3 are actual limits, and we will even prove a precise estimate for $p_H(v)$ and $\tilde{p}_H(v)$ when H goes to infinity.

Lemma 4.4 (Deviation bounds)**.** *For every $\xi > 0$, there exists $c_{10} = c_{10}(\xi) > 0$ such that for every $H \in \mathbb{N}^*$,*

$$\begin{cases} p_H(v_+ + \xi) \leqslant c_{10} H^{-\alpha/4}; \\ \tilde{p}_H(v_- - \xi) \leqslant c_{10} H^{-\alpha/4}. \end{cases}$$

The proof of Lemma 4.4 is the goal of Section 4.2.

Corollary 4.5. *The two limiting directions satisfy $v_- \leqslant v_+$.*

We will then show the following result, using Corollary 4.5.

Lemma 4.6. *We have $v_- = v_+$. We call this quantity v.*

The proof of Lemma 4.6 is the goal of Section 5 and can be found more precisely in Section 5.5. For now, let us now prove Lemma 3.10 as a consequence of Lemmas 4.4 and 4.6.

Proof of Lemma 3.10. Let $\xi > 0$. Combining Lemma 4.4 with Lemma 4.6, we have, for every $H \in \mathbb{N}^*$,

$$\mathbb{P}\left(\left|\frac{X_{\tau_H}}{H} - v\right| \geqslant \xi\right) \leqslant 2\, c_{10}(\xi)\, H^{-\alpha/4}.$$

Therefore, since $\alpha > 4$, $\sum_{n\in\mathbb{N}} \mathbb{P}\left(\left|\frac{X_n}{n} - v\right| \geqslant \xi\right) < \infty$. As a consequence, Borel-Cantelli's lemma ensures that

$$\mathbb{P}\text{-almost surely,} \quad \frac{X_{\tau_H}}{H} \xrightarrow[H\to\infty]{} v,$$

concluding the proof of Lemma 3.10. □

4.2 Deviation bounds: proof of Lemma 4.4

4.2.1 Ideas of the proof

Let us first give some heuristic insight on how the proof is going to unfold.

- We will only show the estimate for $p_H(v)$, where $v > v_+$. The estimate for $\tilde{p}_H(v)$ with $v < v_-$ is shown in the same way, the proof being symmetric.
- The road map for the proof is given by the renormalization method explained in Section 3.6, with a sequence of scales given by $(h_0 L_k)_{k\geqslant k_0}$, where h_0 and k_0 will have to be chosen properly. In the induction that will give an estimate on this sequence of scales, the choice of k_0 and the definition of $(L_k)_{k\in\mathbb{N}}$ will be instrumental in the induction step, while h_0 is chosen for the base case to work.
- We are going to work with the sequence of events $(A_{h_0L_k}(v_k))_{k\geqslant k_0}$ with an appropriate choice of $(v_k)_{k\geqslant k_0}$. The goal is to show that with good probability, on $A_{h_0L_{k+1}}(v_{k+1})$, we can find events $A_{h_0L_k,w_1}(v_k)$ and $A_{h_0L_k,w_2}(v_k)$ with certain base points w_1, w_2 located on a grid whose cardinality does not depend on h_0. The challenge is that we asked those two events to have everywhere-zero histories. One way to find them is to look for the two starting points y_1 and y_2 (from the definitions of $A_{h_0L_k,w_1}(v_k)$ and $A_{h_0L_k,w_2}(v_k)$) on cut lines that we ask to be at vertical distance less than $(h_0L_k)^{1/2}$ of two points w_1 and w_2 on our grid. This is the whole reason why in our paper, $I_H(w)$ is a flattened rectangle, instead of being a true horizontal interval as in [BHT20].

4.2.2 Choice of h_0 and k_0

Let us fix $v > v_+$. Recall Definition 3.22. We let $k_0 = k_0(v) \in \mathbb{N}^*$ be such that

$$\sum_{k\geqslant k_0}\left(\frac{2\beta}{H'_k} + \frac{6\beta}{l_k}\right) < \frac{v - v_+}{2}. \tag{4.2}$$

We also set $v_{k_0} = \frac{v+v_+}{2}$. Using Remark 4.3, note that since $v_{k_0} > v_+$,

$$\liminf_{H\to\infty} p_H(v_{k_0}) = 0.$$

Therefore there exists $H \geqslant L_{k_0}$ such that $p_H(v_{k_0}) \leqslant L_{k_0}^{-\alpha/2}$. Let $h_0 = H/L_{k_0} \in [1,\infty)$. By definition, we have $h_0L_k \in \mathbb{N}$ for all $k \geqslant k_0$, and

$$p_{h_0L_{k_0}}(v_{k_0}) \leqslant L_{k_0}^{-\alpha/2}.$$

This will be the base case for our estimate on $p_H(v)$. Now that h_0 is fixed, we let

$$H_k = h_0 L_k \text{ for all } k \geqslant k_0. \tag{4.3}$$

Recall notation H'_k from (3.19). We now define a sequence $(v_k)_{k\geqslant k_0}$ by setting, for $k \geqslant k_0$,

$$\begin{cases} v'_k = v_k + \dfrac{2\beta}{H'_k}; \\ v_{k+1} = v'_k + \dfrac{6\beta}{l_k}. \end{cases}$$

This definition combined with (4.2) and the fact that $h_0 \geqslant 1$ ensures that $v_k \xrightarrow[k\to\infty]{\nearrow} v_\infty < v$. Therefore, if we show that

$$\forall k \geqslant k_0,\ \ p_{H_k}(v_k) \leqslant L_k^{-\alpha/2}, \tag{4.4}$$

then we get, using Remark 4.3,

$$\forall k \geqslant k_0,\ \ p_{H_k}(v) \leqslant L_k^{-\alpha/2}. \tag{4.5}$$

So the second step of the proof will be devoted to showing estimate (4.4) by induction on k.

4.2.3 Proof of (4.4)

Definition of the grid. Let us fix $k \geqslant k_0$. Recall (3.20). In order to link scales $h_0 L_k$ and $h_0 L_{k+1}$, we define the grid $\mathcal{C}_k \subseteq \mathbb{R} \times H_k\mathbb{Z}$ to be such that

$$\bigcup_{w\in\mathcal{C}_k} \mathcal{I}_{H_k}(w) = B_{H_{k+1}} \cap (\mathbb{Z}\times H_k\mathbb{Z})\,, \tag{4.6}$$

where the union above is disjoint (note that boxes $B_{H_k}(w)$ with $w \in \mathcal{C}_k$ are not disjoint though). The cardinality of $\mathcal{C}_k$ can be bounded from above by $c_{11} l_k^2$, where $c_{11} > 0$.

Localization at scale k. We first need to define an event $\mathcal{F}_k$ that guarantees that the random walks starting in $I_{H_{k+1}}$ will stay in $B_{H_{k+1}}$ and that their horizontal behaviours at scale H_k are properly bounded. To define this precisely, we set, for $y \in I_{H_{k+1}}$ and $j \in [\![0, l_k]\!]$,

$$\begin{cases} \mathcal{X}_j^y = X^y_{\tau_{jH_k}}; \\ \mathcal{N}_j^y = N^y_{\tau_{jH_k}}. \end{cases} \tag{4.7}$$

Note that $\mathcal{X}_0^y = y$ and $\mathcal{N}_0^y = \mathbf{0}$. Note also that for $j \geqslant 1$, $\tau^y_{jH_k} > 0$, since $\pi_2(I_{H_{k+1}})$ is included in $[0, H_k)$ (indeed, $H'_{k+1} = \lceil (h_0 L_{k+1})^{1/2}\rceil \leqslant h_0 L_k^{5/8} \leqslant H_k$). Therefore, indices satisfying $j \geqslant 1$ will not be a problem even when $\pi_2(y) > 0$, while $j = 0$ will be set aside in the next steps of the proof.

Recall Definitions (3.17) and (3.21). Let

$$\mathcal{F}_k = F_{H_{k+1}} \cap \bigcap_{y\in I_{H_{k+1}}} \bigcap_{j=0}^{l_k-1} D_{H_k}^{\mathcal{X}_j^y, \mathcal{N}_j^y}. \tag{4.8}$$

Note that in order to bound the horizontal displacement of X^y between times 0 and $\tau^y_{H_k}$, it would have been sufficient to consider $D^y_{H_k-\pi_2(y)}$ instead of $D^y_{H_k}$, but the stronger event given by (4.8) is more pleasant to write and work with.

For each y and j, in order to bound $\mathbb{P}\left((D_{H_k}^{\mathcal{X}_j^y, \mathcal{N}_j^y})^c\right)$, we use Proposition 3.7 with stopping time $\tau^y_{jH_k}$. Using Proposition 3.14, for every $\mu \in \mathcal{A}$, $z \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$, we have $\mathbb{P}^\mu\left((D_{H_k}^{z,\Gamma})^c\right) \leqslant c_5^{-1} e^{-c_5 H_k}$, which is uniform in μ, z and Γ. So, by Proposition 3.7, for every y and j, we have

$$\mathbb{P}\left((D_{H_k}^{\mathcal{X}_j^y, \mathcal{N}_j^y})^c\right) \leqslant c_5^{-1} e^{-c_5 H_k^{1/2}}.$$

In the end, using union bounds and Proposition 3.16,

$$\mathbb{P}(\mathcal{F}_k^c) \leqslant c_6^{-1} e^{-c_6 H_{k+1}^{1/2}} + H_{k+1} H'_{k+1} l_k c_5^{-1} e^{-c_5 H_k^{1/2}} \leqslant c^{-1} e^{-c H_k^{1/2}}, \tag{4.9}$$

where $c > 0$ does not depend on h_0 and k_0.

Link between scales k and $k+1$. We are now moving on to the crucial idea of the proof: on event $A_{H_{k+1}}(v_{k+1})$, which is observable at scale $k+1$, several similar events occur at scale k.

Let us fix $y \in I_{H_{k+1}}$. We claim to have the following inclusion of events:

$$\left\{V^y_{H_{k+1}} \geqslant v_{k+1}\right\} \cap \mathcal{F}_k \subseteq \left\{ \begin{array}{c} \text{there exist three } j \in [\![1, l_k-1]\!] \\ \text{such that } \pi_1(\mathcal{X}^y_{j+1}) \geqslant \pi_1(\mathcal{X}^y_j) + v'_k H_k \end{array} \right\}. \tag{4.10}$$

Indeed, let us argue by contraposition and assume that $\pi_1(\mathcal{X}^y_{j+1}) \geqslant \pi_1(\mathcal{X}^y_j) + v'_k H_k$ for at most two $j \in [\![1, l_k-1]\!]$. The horizontal displacement of X^y between times 0 and $\tau^y_{H_{k+1}}$ is the sum of l_k horizontal displacements, $l_k - 3$ of which we can now bound by $v'_k H_k$, and the three remaining ones can be bounded using $D^{\mathcal{X}^y_j, \mathcal{N}^y_j}_{H_k}$. More precisely, on $\mathcal{F}_k$,

$$\begin{aligned}
\pi_1\left(X^y_{\tau_{H_{k+1}}}\right) = \pi_1(\mathcal{X}^y_{l_k}) &= \pi_1(\mathcal{X}^y_1) + \sum_{j=1}^{l_k-1} \left(\pi_1(\mathcal{X}^y_{j+1}) - \pi_1(\mathcal{X}^y_j)\right) \\
&< \pi_1(y) + (l_k - 3) v'_k H_k + 3\beta H_k \\
&= \pi_1(y) + \left(v'_k + \frac{3(\beta - v'_k)}{l_k}\right) H_{k+1} \\
&< \pi_1(y) + \left(v'_k + \frac{6\beta}{l_k}\right) H_{k+1} \\
&= \pi_1(y) + v_{k+1} H_{k+1},
\end{aligned}$$

where in the last inequality, we used bounds (4.1) to get that $v'_k > v_k > v_+ \geqslant -\beta$. This concludes the proof of (4.10).

Removal of histories. The issue now is that in (4.10), events $\left\{\pi_1(\mathcal{X}^y_{j+1}) \geqslant \pi_1(\mathcal{X}^y_j) + v'_k H_k\right\}$ implicitly feature a non-zero history $\mathcal{N}^y_j$, while our goal is to get zero-history events $A_{H_k,w}(v_k)$ for two $w \in \mathcal{C}_k$. In order to get those, we use cut lines as defined in Section 3.5, which requires defining a new event of large probability that will fulfill the technical requirements for the rest of the argument, namely that we find cut lines quickly enough, that before then the random walks do not go too far horizontally, and that they all stay in boxes allowing us to use decoupling. Recall notation H'_k defined in (3.19). We let

$$\mathcal{G}_k = \bigcap_{y \in I_{H_{k+1}}} \bigcap_{j=1}^{l_k-1} \left(D^{\mathcal{X}^y_j, \mathcal{N}^y_j}_{H'_k} \cap \left\{\Theta(X^{\mathcal{X}^y_j, \mathcal{N}^y_j}) < H'_k\right\}\right) \cap \bigcap_{w \in \mathcal{C}_k} F_{H_k}(w). \tag{4.11}$$

In order to control the probability of $\mathcal{G}_k$, we use Proposition 3.7 again, as well as Propositions 3.16 and 3.20. Using a union bound, we have

$$\begin{aligned}
\mathbb{P}\left(\mathcal{G}^c_k\right) &\leqslant H_{k+1} H'_{k+1} l_k \sup_{y,j} \left[\mathbb{P}\left((D^{\mathcal{X}^y_j, \mathcal{N}^y_j}_{H'_k})^c\right) + \mathbb{P}\left(\Theta(X^{\mathcal{X}^y_j, \mathcal{N}^y_j}) > H'_k/2\right)\right] + c_{11} l_k^2 c_6^{-1} e^{-c_6 H_k^{1/2}} \\
&\leqslant H_{k+1} H'_{k+1} l_k \left(c_5^{-1} e^{-c_5 H_k^{1/2}} + c_8^{-1} e^{-c_8 2^{-1/2} H_k^{1/4}}\right) + c^{-1} e^{-c H_k^{1/2}} \\
&\leqslant c^{-1} e^{-c H_k^{1/4}},
\end{aligned} \tag{4.12}$$

where $c > 0$ does not depend on h_0 and k_0.

Let $j \in [\![1, l_k-1]\!]$, and let θ^y_j be the location of X^y on the first cut line reached after height jH_k, that is

$$\theta^y_j = X^{\mathcal{X}^y_j, \mathcal{N}^y_j}_{T_c(X^{\mathcal{X}^y_j, \mathcal{N}^y_j})} \tag{4.13}$$

(recall notations from Definition 3.18). On $\mathcal{G}_k \cap \left\{\pi_1(\mathcal{X}^y_{j+1}) \geqslant \pi_1(\mathcal{X}^y_j) + v'_k H_k\right\}$, we have

$$\pi_1\left(X^{\theta^y_j}_{\tau_{H_k, \mathcal{X}^y_j}}\right) = \pi_1(\mathcal{X}^y_{j+1}) \qquad \text{by definition of a cut line}$$

$$\begin{aligned}
&\geqslant \pi_1(\mathcal{X}^y_j) + v'_k H_k && \\
&\geqslant \pi_1(\theta^y_j) + v'_k H_k - \beta H'_k && \text{using } D^{\mathcal{X}^y_j,\mathcal{N}^y_j}_{H'_k} \\
&\geqslant \pi_1(\theta^y_j) + v_k H_k && \text{by definition of } v'_k,
\end{aligned}$$

so, in other words, $V^{\theta^y_j}_{H_k,\mathcal{X}^y_j} \geqslant v_k$. Using (4.10), this means that we have found three points (given by θ^y_j for three values of $j \in [\![1, l_k - 1]\!]$) with the right lower bound on their directions and with everywhere-zero initial histories. Furthermore, on $\mathcal{F}_k \cap \mathcal{G}_k$, we have $\pi_2(\theta^y_j) < \pi_2(\mathcal{X}^y_j) + H'_k = jH_k + H'_k$ (since for $j \geqslant 1$, $\tau^y_{jH_k} > 0$). Therefore the θ^y_j are located in three rectangles $I_{H_k}(w_i)$ for $w_i \in \mathcal{C}_k$ satisfying $|\pi_2(w_i) - \pi_2(w_j)| \geqslant H_k$ for $i < j$. As a result,

$$\mathcal{F}_k \cap \mathcal{G}_k \cap A_{H_{k+1}}(v_{k+1}) \subseteq \bigcup_{\substack{w_1,w_2 \in \mathcal{C}_k \\ |\pi_2(w_1)-\pi_2(w_2)|\geqslant 2H_k}} \begin{array}{l} (A_{H_k,w_1}(v_k) \cap F_{H_k}(w_1)) \\ \cap (A_{H_k,w_2}(v_k) \cap F_{H_k}(w_2)). \end{array} \tag{4.14}$$

Now, events $A_{H_k,w_1}(v_k) \cap F_{H_k}(w_1)$ and $A_{H_k,w_2}(v_k) \cap F_{H_k}(w_2)$ above are respectively measurable with respect to boxes $B_{H_k}(w_1)$ and $B_{H_k}(w_2)$, which have maximum side lengths $(2\beta+1)H_k$ and are $H_k/2$-separated under condition $|\pi_2(w'_1) - \pi_2(w'_2)| \geqslant 2H_k$ (recall Remark 3.17). Therefore, we can use Fact 2.9 to get

$$\begin{aligned}
\mathbb{P}\left(\mathcal{F}_k \cap \mathcal{G}_k \cap A_{H_{k+1}}(v_{k+1})\right) &\leqslant |\mathcal{C}_k|^2 \left(p_{H_k}(v_k)^2 + c_0 (H_k/2)^{-\alpha}\right) \\
&\leqslant c_{11}^2\, l_k^4 \, \left(p_{H_k}(v_k)^2 + c_0 (H_k/2)^{-\alpha}\right).
\end{aligned}$$

In the end, using bounds (4.9) and (4.12), we get

$$\begin{aligned}
\mathbb{P}(A_{H_{k+1}}(v_{k+1})) &\leqslant c_{11}^2\, l_k^4 \, \left(p_{H_k}(v_k)^2 + c_0 (H_k/2)^{-\alpha}\right) + \mathbb{P}\left(\mathcal{F}^c_k\right) + \mathbb{P}\left(\mathcal{G}^c_k\right) \\
&\leqslant c_{12}\, l_k^4 \left(p_{H_k}(v_k)^2 + c_{12} L_k^{-\alpha}\right),
\end{aligned} \tag{4.15}$$

for a certain constant $c_{12} > 0$ that does not depend on h_0 and k_0 (to which we gave a name because we will need it again for the proof of Proposition 5.12 at the end of our paper). By induction, we can conclude that if $p_{H_k}(v_k) \leqslant L_k^{-\alpha/2}$, then

$$\frac{p_{H_{k+1}}(v_{k+1})}{L_{k+1}^{-\alpha/2}} \leqslant c\, L_{k+1}^{\alpha/2}\, l_k^4\, L_k^{-\alpha} \leqslant c\, L_k^{\frac{-3\alpha+8}{8}},$$

for a well-chosen constant $c > 0$ that does not depend on h_0 and k_0. Since $\alpha > 3$, up to taking an even larger k_0 (independently on h_0), we can assume that this is less than 1, which concludes the induction and the proof of estimate (4.4).

4.2.4 Interpolation

In Section 4.2.3, we proved (4.4), which, as we explained in Section 4.2.2, implies estimate (4.5). To sum up, so far we have shown that

$$\forall v > v_+,\ \exists\, k_0(v) \in \mathbb{N}^*,\ \exists\, h_0 = h_0(v) \geqslant 1,\ \forall k \geqslant k_0,\ \ p_{h_0 L_k}(v) \leqslant L_k^{-\alpha/2}. \tag{4.16}$$

We want to interpolate this estimate to show that

$$\forall v > v_+,\ \exists\, c_{10} = c_{10}(v) > 0,\ \forall H \in \mathbb{N}^*,\ \ p_H(v) \leqslant c_{10} H^{-\alpha/4}. \tag{4.17}$$

Let $v > v_+$. Set $v' = \frac{v_+ + v}{2}$, $v'' = \frac{v' + v}{2}$, $h_0 = h_0(v')$ and $k_1 \geqslant k_0(v')$ be such that

$$L_{k_1}^{1/10} > \frac{2\beta}{v - v'}; \tag{4.18}$$

$$\frac{2\beta}{L_{k_1}^{1/2}} + \frac{2|v'|}{L_{k_1}^{1/10}} \leqslant v'' - v'. \tag{4.19}$$

Let $H \geqslant (h_0 L_{k_1})^{11/10}$, and let $k_2 \geqslant k_1$ be such that

$$(h_0 L_{k_2})^{11/10} \leqslant H < (h_0 L_{k_2+1})^{11/10}. \tag{4.20}$$

Since $k_2 \geqslant k_0(v')$ and $v' > v_+$, using (4.16),

$$p_{h_0 L_{k_2}}(v') \leqslant L_{k_2}^{-\alpha/2}. \tag{4.21}$$

Recall notation H_{k_2} from (4.3). Note that (4.20) implies that $H' < H_{k_2}$, therefore every $y \in I_H$ satisfies $\tau^y_{H_{k_2}} > 0$. Now let $\bar{H} = \lfloor H/H_{k_2} \rfloor \, H_{k_2}$ be the last multiple of H_{k_2} before H. For $j \in [\![0, \lfloor H/H_{k_2} \rfloor - 1]\!]$ and $y \in I_H$, we let $\mathcal{X}^y_j = X^y_{\tau_{jH_{k_2}}}$ and $\mathcal{N}^y_j = X^y_{\tau_{jH_{k_2}}}$. Let us define $\hat{\mathcal{C}}$ to be a minimal set satisfying

$$\bigcup_{w \in \hat{\mathcal{C}}} \mathcal{I}_{H_{k_2}}(w) = B_H \cap (\mathbb{Z} \times H_{k_2}\mathbb{Z}) \, .$$

We will work with the following events:

$$\begin{aligned}
\mathcal{A}_1 &= \bigcap_{w \in \hat{\mathcal{C}}} A_{H_{k_2}, w}(v')^c; \\
\mathcal{A}_2 &= \bigcap_{y \in I_H} \left\{ \pi_1(X^y_{\tau_H}) - \pi_1(X^y_{\tau_{\bar{H}}}) < (v - v'')H - \beta H_{k_2} \right\}; \\
\mathcal{F} &= F_H \cap \bigcap_{y \in I_H} D^y_{H_{k_2}} \\
\mathcal{G} &= \bigcap_{y \in I_H} \bigcap_{j=1}^{\lfloor H/H_{k_2} \rfloor - 1} \left(D^{\mathcal{X}^y_j, \mathcal{N}^y_j}_{H'_{k_2}} \cap \left\{ \Theta\left(X^{\mathcal{X}^y_j, \mathcal{N}^y_j} \right) < H'_{k_2} \right\} \right).
\end{aligned}$$

For any $y \in I_H$, events $\mathcal{A}_1$ and $\mathcal{G}$ (as well as F_H) allow us to bound the displacement of X^y between times $\tau^y_{jH_{k_2}}$ and $\tau^y_{(j+1)H_{k_2}}$ for $j \in [\![1, \lfloor H/H_{k_2} - 1 \rfloor]\!]$. Indeed, the conditions on cut lines given by $\mathcal{G}$ allow us to find a point inside $I_{H_{k_2}}(w)$, for a certain $w \in \hat{\mathcal{C}}$, for which we can use $A_{H_{k_2},w}(v')^c$ given by $\mathcal{A}_1$. More precisely, on $\mathcal{A}_1 \cap \mathcal{F} \cap \mathcal{G}$ and for every $y \in I_H$, we have

$$\begin{aligned}
\pi_1(X^y_{\tau_{\bar{H}}}) &= \pi_1(X^y_{\tau_{H_{k_2}}}) + \sum_{j=1}^{\lfloor H/H_{k_2} \rfloor - 1} \left(\pi_1(X^y_{\tau_{(j+1)H_{k_2}}}) - \pi_1(X^y_{\tau_{jH_{k_2}}}) \right) \\
&\leqslant \pi_1(X^y_{\tau_{H_{k_2}}}) + (\lfloor H/H_{k_2} \rfloor - 1) \left(\beta H'_{k_2} + v' H_{k_2} \right) \\
&\leqslant \pi_1(X^y_{\tau_{H_{k_2}}}) + \frac{H}{H_{k_2}} (\beta H'_{k_2} + v' H_{k_2}) + 2|v'| H_{k_2} \\
&\leqslant \pi_1(X^y_{\tau_{H_{k_2}}}) + v'' H,
\end{aligned}$$

where in the last line we used (4.19) and (4.20). Therefore, on $\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{F} \cap \mathcal{G}$, we have, for every $y \in I_H$,

$$\begin{aligned}
\pi_1(X^y_{\tau_H}) - \pi_1(y) &= \left(\pi_1(X^y_{\tau_{H_{k_2}}}) - \pi_1(y) \right) + \left(\pi_1(X^y_{\tau_{\bar{H}}}) - \pi_1(X^y_{\tau_{H_{k_2}}}) \right) + \left(\pi_1(X^y_{\tau_H}) - \pi_1(X^y_{\tau_{\bar{H}}}) \right) \\
&< \beta H_{k_2} + v'' H + (v - v'')H - \beta H_{k_2} = vH.
\end{aligned}$$

As a result,

$$A_H(v) \subseteq \mathcal{A}_1^c \cup \mathcal{A}_2^c \cup \mathcal{F}^c \cup \mathcal{G}^c. \tag{4.22}$$

Now, note that

$$\mathbb{P}\left(\mathcal{A}_1^c\right) \leqslant c \left(H/H_{k_2} \right)^2 L_{k_2}^{-\alpha/2}, \tag{4.23}$$

using (4.16). Also, by (4.18) and the fact that $k_2 \geqslant k_1$, as well as (4.20), we have $(v - v')H > 2\beta H_{k_2}$, therefore

$$\begin{aligned}
\mathbb{P}\left(\mathcal{A}_2^c\right) &\leqslant HH'\,\mathbb{P}\left(\pi_1\left(X^{X^y_{\tau_{\bar{H}}},N^y_{\tau_{\bar{H}}}}_{\tau_{H-\bar{H},X^y_{\tau_{\bar{H}}}}}\right) - \pi_1(X^y_{\tau_{\bar{H}}}) > \beta H_{k_2}\right) \\
&\leqslant HH' \sup_{\mu\in\mathcal{A}} \sup_{z,\Gamma} \mathbb{P}^{\mu}\left(\left(D^{z,\Gamma}_{H_{k_2}}\right)^c\right) \\
&\leqslant HH' c_5^{-1} e^{-c_5 H_{k_2}}, \qquad (4.24)
\end{aligned}$$

using Propositions 3.14 and 3.7. We also have

$$\mathbb{P}(\mathcal{F}^c) \leqslant c_6^{-1} e^{-c_6 H^{1/2}} + HH' c_5^{-1} e^{-c_5 H_{k_2}}; \qquad (4.25)$$

$$\mathbb{P}(\mathcal{G}^c) \leqslant HH' \frac{H}{H_{k_2}} \left(c_5^{-1} e^{-c_5 H_{k_2}^{1/2}} + c_8^{-1} e^{-c_8 2^{-1/2} H_{k_2}^{1/4}}\right). \qquad (4.26)$$

Using (4.20), we can see that the upper bounds given by (4.24), (4.25) and (4.26) are all negligible with respect to that given by (4.23), so we get

$$\begin{aligned}
\mathbb{P}(A_H(v)) &\leqslant c\,(H/H_{k_2})^2\, L_{k_2}^{-\alpha/2} \\
&\leqslant cH^2 L_{k_2}^{-\alpha/2} \\
&\leqslant cH^2 H^{-5\alpha/7} && \text{using (4.20)} \\
&\leqslant cH^{-\alpha/4} && \text{using that } \alpha \geqslant 5.
\end{aligned}$$

By adjusting c to accommodate small values of H, this concludes the proof of (4.17) and therefore the proof of Lemma 4.4.

5 Equality of the limiting directions: proof of Lemma 4.6

The goal of this section is to show Lemma 4.6. The heuristic idea behind the proof is the following. The definitions of v_+ and v_- ensure that the random walk often has directions close to v_- and v_+. On the other hand, the probability that the random walk has a direction larger than $v_+ + \xi$ (where $\xi > 0$ is fixed) decreases quickly, as was shown in Lemma 4.4. Therefore, assuming by contradiction that $v_+ > v_-$, the moments when its direction stays close to v_- may prevent it from reaching a direction close to v_+ in the future, which would be a contradiction. However, the random walk might be able to compensate by going faster than $v_+ + \xi(H)$ for some well-chosen $\xi(H)$. This is why we need precise estimates, and these will be given by a notion of trap that we will introduce further on.

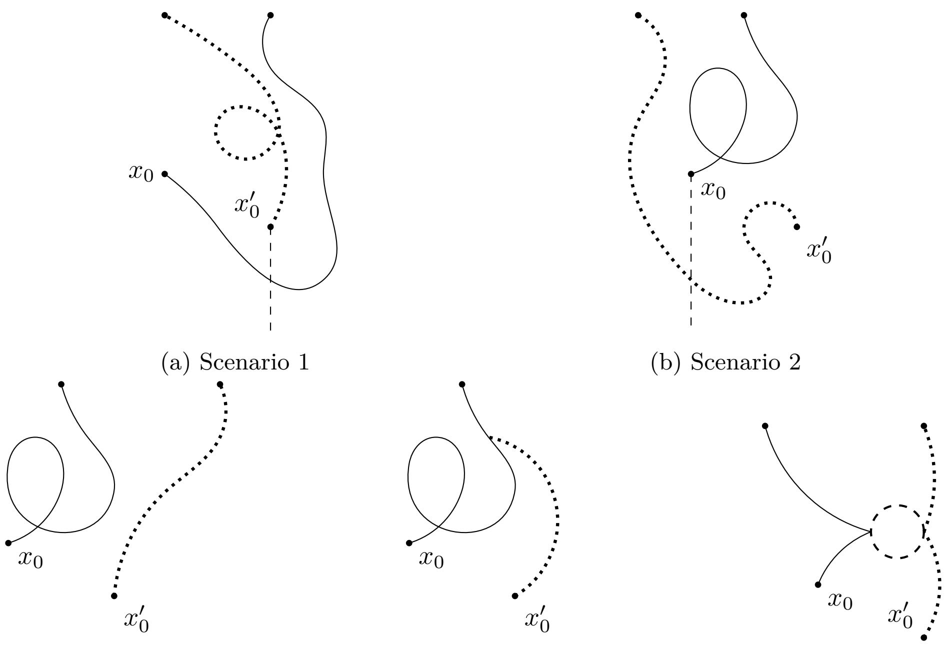
We start by presenting a major property of our model, which comes from the coupling of random walks that we chose. The choice of the coupling is actually made in order to get this property, which is inspired by the arguments from [BHT20]. Roughly, it says that particles block each other in some weak sense: a random walk can always bypass another random walk, but this happens with low probability.

5.1 Barrier property

Proposition 5.1. *Let $x_0, x_0' \in \mathbb{Z}^2$ with $\pi_2(x_0) \geqslant \pi_2(x_0')$ and $\pi_1(x_0) < \pi_1(x_0')$. Let $H \in \mathbb{N}^*$. Let $\Gamma \in \mathcal{H}$ such that $\operatorname{Supp}\Gamma \cap X^{x_0'}_{\mathbb{N}} = \emptyset$. Assume that $\tau^{x_0,\Gamma}_{H,x_0} < \infty$ and $\tau^{x_0'}_{H,x_0} < \infty$ (which happens almost surely). Then at least one of the following scenarios occurs:*

1. *$X^{x_0,\Gamma}$ visits the half-line $x_0' + \{0\} \times (-\infty, 0)$;*
2. *$X^{x_0'}$ visits the half-line $x_0 + \{0\} \times (-\infty, 0)$;*
3. *We have $\pi_1\big(X^{x_0,\Gamma}_{\tau_{H,x_0}}\big) \leqslant \pi_1\big(X^{x_0'}_{\tau_{H,x_0}}\big)$.*

The statement is also true when we replace x_0, x_0' and Γ by values of random variables satisfying the same assumptions.



(a) Scenario 1 (b) Scenario 2

(c) Scenario 3.a (d) Scenario 3.b (e) Scenario 3.c

Figure 4: Illustrations of what can happen in each of the three scenarios of Proposition 5.1. In scenarios 1 and 2, one particle goes around the other one. In scenario 3, they end up with the same horizontal order because either they never meet or go around each other (scenario 3.a), or they meet but coalesce (scenario 3.b), or they meet on a loop that they both visit entirely before carrying on separately (scenario 3.c).

Proof. Replacing x_0, x'_0 and Γ by values of random variables does not change the proof, since the statement is deterministic. Even so, the proof is subtle. A lot of different things can happen, as is illustrated by Figure 4. We start by discussing the issues and ideas of the proof, in order to give some motivations for the next steps.

Heuristics of the proof. The main problem of the proof is that even if two particles meet, they may not coalesce (*i.e.* they may not stay together forever from then on), since the two random walks do not necessarily look at the same uniform variables every step of the way afterwards. However, we will show when two particles meet, either they coalesce (as in scenario 3.b)) or they end up splitting up without having swapped their initial horizontal order (as in scenario 3.c)). In the latter case, what actually happens is that both particles visit the same loop, and removing this loop is tantamount to adding the same history to both random walks. This prompts us to show a stronger version of the proposition by adding a common history Γ_0 to both random walks, which will allow us to apply our line of reasoning inductively by removing loops one by one.

Stronger claim. We now add more history. We fix x_0, x'_0, H and Γ as in the statement of the proposition, and we let $\Gamma_0 \in \mathcal{H}$. From now on, we use simpler notations: τ for $\tau^{x_0,\Gamma+\Gamma_0}_{H,x_0}$, τ' for $\tau^{x'_0,\Gamma_0}_{H,x_0}$, X for $X^{x_0,\Gamma+\Gamma_0}$ and X' for $X^{x'_0,\Gamma_0}$. Assume that both τ and τ' are finite. We argue by contradiction and assume that

$$(5.1) \qquad \begin{cases} X \text{ does not visit } x'_0 + \{0\} \times (-\infty, 0) & (1) \\ X' \text{ does not visit } x_0 + \{0\} \times (-\infty, 0) & (2) \\ \pi_1(X_\tau) > \pi_1(X'_{\tau'}) & (3) \end{cases}$$

We want to get to a contradiction from this, and then choosing $\Gamma_0 = \mathbf{0}$ will give the desired result.

Loop removal algorithm. We now define an algorithm allowing us to remove all loops from the paths of our random walks, so that we can focus on the zero-loop case later on. We consider a path defined by a parametrization $f : P \to \mathbb{Z}^2$, where P is a bounded subset of $\mathbb{N}$, satisfying for every $s, t \in P$, $|t-s| = 1 \Rightarrow \|f(t) - f(s)\| = 1$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. If f is not injective, we define

$$\begin{aligned}&T_1^{out}(f) = \min\{t \in P,\ f(t) \in \{f(s), s \in P, s < t\}\};\\&T_1^{in}(f) = \min\{t \in P,\ f(t) = f(T_1^{out}(f))\};\\&P_1(f) = [\![T_1^{in}(f), T_1^{out}(f) - 1]\!] \cap P;\\&L_1(f) = f(P_1(f)).\end{aligned}$$

We call $L_1(f)$ the first loop of f. The times $T_1^{in}(f)$ and $T_1^{out}(f)$ are called the first entry and exit times of $L_1(f)$. We define by induction the other loops of f, if they exist, by defining, for $i \geqslant 2$,

$$\begin{aligned}&T_i^{out}(f) = T_1^{out}\left(f|_{P\setminus\cup_{j<i}P_j(f)}\right);\\&T_i^{in}(f) = T_1^{in}\left(f|_{P\setminus\cup_{j<i}P_j(f)}\right);\\&P_i(f) = [\![T_i^{in}(f), T_i^{out}(f) - 1]\!] \cap (P \setminus \cup_{j<i}P_j(f))\\&L_i(f) = f(P_i(f)).\end{aligned}$$

Mind that here we consider functions defined on subsets of P that are not necessarily connected in P, which is why we did not assume P to be connected in $\mathbb{N}$ in the first place.

If there are no more loops, we just set $P_i(f) = \emptyset$, $L_i(f) = \emptyset$ and $T_i^{in}(f) = T_i^{out}(f) = \infty$.

We also define, for such a function f, its interpolated sample path as the curve in $\mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R}$ obtained by joining each pair of points $\{f(t), f(t+1)\}$ (for t and $t+1 \in P$) by a segment. We denote it by $\mathrm{int}(f)$.

The two sample paths meet. Recall our assumption (5.1), from which we want to get to a contradiction. Let $C = \mathrm{int}\left(X|_{[\![0,\tau]\!]}\right)$ and $C' = \mathrm{int}\left(X'|_{[\![0,\tau']\!]}\right)$. For now, we want to show that C and C' meet at some point of $\mathbb{R}^2$, which implies, by construction, that the two sample paths $X_{[0,\tau]}$ and $X'_{[0,\tau']}$ meet at some point of $\mathbb{Z}^2$. To show that, we first form a closed simple curve C_0 of $\mathbb{R}^2$ as shown in Figure 5. First we consider

$$C'_* = \mathrm{int}\left(X'|_{[\![0,\tau']\!]\setminus\cup_{i\in\mathbb{N}^*}P_i(X')}\right),$$

which is C' from which we removed all the loops and which we interpolated. Then we join the two extreme points x'_0 and x'_1 of C' (note that they both have to be on C'_*) using horizontal and vertical segments that go low enough and left enough so that they do not meet $C \cup C'$ except at x'_0 and x'_1 (if C'_* intersects $x'_0 + \{0\} \times (-\infty, 0)$, we remove the initial part of C'_* so that x'_0 is replaced by the lowest point on $C'_* \cap (x'_0 + \{0\} \times (-\infty, 0))$). This is possible because $C \cup C'$ is a compact set and because of (1) and (3) in (5.1). By construction, C_0 is a closed simple curve, so we can apply Jordan's theorem to C_0. Point $x_1 = X^{x_0,\Gamma+\Gamma_0}_{\tau_{H,x_0}}$ is in the unbounded component, because the half-line $x_1 + (0,\infty) \times \{0\}$ cannot meet C_0. On the contrary, point x_0 has to be in the bounded component, because the vertical segment joining x_0 to a lower point x_2 in the unbounded component meets C_0 only once, because of (2) in (5.1) (here we use the sometimes called even-odd rule that can be found in [Shi62]). Therefore, curve C has to meet C_0, and by construction of C_0, it has to meet it on C'_*. Therefore C and C' intersect.

Zero-loop case. First consider the simpler case where C' has no loops intersecting C. By the previous point, C' meets C, so we can consider

$$\hat{t} = \max\{t \in [\![0,\tau']\!],\ X'_t \in C\},$$

and $\hat{x} = X'_{\hat{t}}$. Because of (3) in (5.1), $\hat{t} < \tau'$. The uniform variable that X' uses to jump at time $\hat{t}$ is $U(\hat{x}, \Gamma_0(\hat{x}) + 1)$ (for there is no loop on C' intersecting C, so $\hat{x}$ cannot be on a loop of C'). The same uniform variable is used by X when it gets to $\hat{x}$ for the first time, since by assumption $\mathrm{Supp}\,\Gamma \cap C' = \emptyset$. Therefore, $X'_{\hat{t}+1} \in C$, which contradicts the definition of $\hat{t}$. At the end of the day, we have a contradiction, so our assumption (5.1) was false.

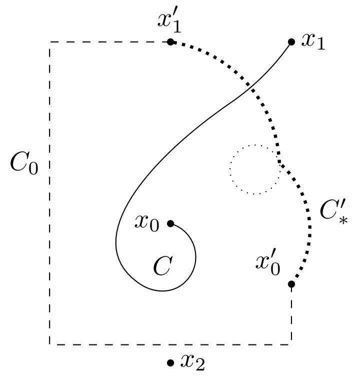


Figure 5

General case. Consider now the case where C' has a loop intersecting C. For $i \geqslant 1$, let $P_i = P_i(X')$, $L_i = L_i(X')$, $T_i^{in} = T_i^{in}(X')$ and $T_i^{out} = T_i^{out}(X')$. We let

$$n_1 = \min\{i \in \mathbb{N}^*, L_i \cap C \neq \emptyset\}.$$

Let T be the first hitting time of L_{n_1} by X and $T' \in [\![T_{n_1}^{in}, T_{n_1}^{out} - 1]\!]$ be the first time such that $X'_{T'} = X_T$. Let us show by induction on $t \leqslant T_{n_1}^{out} - T'$ that

$$(X_T, X_{T+1}, \ldots, X_{T+t}) = \left(X'_{T'}, X'_{T'+1}, \ldots, X'_{T'+t}\right). \tag{5.2}$$

The case $t = 0$ follows from the fact that $X'_{T'} = X_T$. Suppose (5.2) is true for $t < T_{n_1}^{out} - T'$. We need to show that $X_{T+t+1} = X'_{T'+t+1}$.

- First, note that X' cannot have visited $X'_{T'+t}$ before time $T'+t$. Indeed, suppose that it has; then there exists $i < n_1$ such that $X'_{T'+t} \in L_i$, now $X'_{T'+t} = X_{T+t}$ so $L_i \cap C \neq \emptyset$, which contradicts the definition of n_1. Therefore, the uniform variable X' uses to jump at time $T'+t$ is $U(X'_{T'+t}, \Gamma_0(X'_{T'+t}) + 1)$.
- This also means that $X'_{T'+t} = X_{T+t}$ is not among $\{X'_{T'}, \ldots, X'_{T'+t-1}\}$, which is the same set as $\{X_T, \ldots, X_{T+t-1}\}$ by the induction assumption. Therefore, using also the definition of T, X has not visited site X_{T+t} before time $T+t$, so the uniform variable it uses to jump at time $T+t$ is $U(X_{T+t}, \Gamma_0(X_{T+t}) + 1) = U(X'_{T'+t}, \Gamma_0(X'_{T'+t}) + 1)$ (we also use the fact that $\operatorname{Supp}\Gamma \cap C' = \emptyset$).

Both random walks use the same uniform variable, therefore $X_{T+t+1} = X'_{T'+t+1}$, which shows (5.2) for $t+1$ and ends the induction. Applying equality (5.2) with $t = T_{n_1}^{out} - T'$ yields

$$\left(X_T, X_{T+1}, \ldots, X_{T+T_{n_1}^{out}-T'}\right) = \left(X'_{T'}, X'_{T'+1}, \ldots, X'_{T_{n_1}^{out}}\right). \tag{5.3}$$

With the same arguments, we can show that we also have

$$\left(X_{T+T_{n_1}^{out}-T'}, \ldots, X_{T+T_{n_1}^{out}-T_{n_1}^{in}}\right) = \left(X'_{T_{n_1}^{in}}, X'_{T_{n_1}^{in}+1}, \ldots, X'_{T'}\right). \tag{5.4}$$

Also, remark that in $(X'_{T_{n_1}^{in}}, \ldots, X'_{T_{n_1}^{out}-1})$, we have $T_{n_1}^{out} - T_{n_1}^{in} + 1$ distinct points of L_{n_1} (by definition of $T_{n_1}^{out}$), so we have all the points in L_{n_1} exactly once. Therefore, putting together (5.3) and (5.4), and considering that $X'_{T_{n_1}^{in}} = X'_{T_{n_1}^{out}}$, we see that between times T and $T + T_{n_1}^{out} - T_{n_1}^{in}$, X visits all the sites in L_{n_1} exactly once too.

Set $\Gamma_1 = \Gamma_0 + \sum_{x \in L_{n_1}} \delta_{\{x\}}$ and $\tilde{P}_1 = [\![T, T + T_{n_1}^{out} - T_{n_1}^{in} - 1]\!]$. Separating what happens before time T resp. $T_{n_1}^{in}$ and what happens after time $T + T_{n_1}^{out} - T_{n_1}^{in}$ resp. $T_{n_1}^{out}$, we get

$$X_{[0,\tau_{H,x_0}]}^{x_0,\Gamma+\Gamma_1} = X_{[0,\tau_{H,x_0}]\setminus\tilde{P}_1}^{x_0,\Gamma+\Gamma_0};$$
$$X_{[0,\tau_{H,x_0}]}^{x_0',\Gamma_1} = X_{[0,\tau_{H,x_0}]\setminus P_{n_1}}^{x_0',\Gamma_0}.$$

Let K be the number of loops of C' that intersect C (K is finite). By applying the same line of reasoning inductively on the next loops of C' that intersect C (which we denote by $L_{n_2}, \ldots, L_{n_K}$), we can construct a history Γ_K such that

$$X^{x_0,\Gamma+\Gamma_K}_{[0,\tau_{H,x_0}]} = X^{x_0,\Gamma+\Gamma_0}_{[0,\tau_{H,x_0}]\setminus(\tilde{P}_1\cup\ldots\cup\tilde{P}_K)};$$
$$X^{x_0',\Gamma_K}_{[0,\tau_{H,x_0}]} = X^{x_0',\Gamma_0}_{[0,\tau_{H,x_0}]\setminus(P_{n_1}\cup\ldots\cup P_{n_K})}.$$

Therefore, by construction, the sample path of X^{x_0',Γ_K} has no loops intersecting that of $X^{x_0,\Gamma+\Gamma_K}$, so we can apply the previous zero-loop case by replacing Γ_0 by Γ_K. Assumptions from (5.1) are still satisfied, and $\operatorname{Supp}\Gamma_K \cap X^{x_0',\Gamma_K}_{\mathbb{N}} = \emptyset$ by construction, so we do recover a contradiction. $\square$

5.2 Trapped points

Let us move on to the proof of Lemma 4.6. Recall that $v_- \leqslant v_+$, on account of Corollary 4.5, so we now argue by contradiction and assume that $v_- < v_+$. In the rest of this section, we set

$$\delta = \frac{v_+ - v_-}{4(\beta+1)}. \tag{5.5}$$

Note that $\delta \in (0, 1/2]$, using the bounds in (4.1).

The crucial idea of our proof is given by Proposition 5.1, which implies that a particle can be "trapped" by another particle. We want to ensure that trapped particles will experience a delay with respect to v_+, which motivates the first definition below.

Let $H \in \mathbb{N}^*$ and $w \in \mathbb{R}\times\mathbb{Z}$. Recall notation H' from (3.19). We define

$$z_w = w + (\delta H + 4\beta H', -2H') \in \mathbb{R}\times\mathbb{Z}; \tag{5.6}$$
$$R_H(w) = w + ((-\infty, \delta H) \times (-\infty, H') \cup [\delta H, +\infty) \times (-\infty, -3H')) \subseteq \mathbb{R}^2. \tag{5.7}$$

See Figure 6 for an illustration of these notations.

Definition 5.2 (Trap)**.** *Let $H \in \mathbb{N}^*$ and $w \in \mathbb{R}\times\mathbb{Z}$. w is said to be H-trapped if there exists $y \in I_{\delta H/2}(z_w)$ such that:*

1. *$V^y_{H+2H',z_w} \leqslant v_- + \delta/2$;*
2. *X^y does not visit $R_H(w)$.*

Let us explain heuristically the idea behind this definition. Condition 2 ensures that the random walk started at y passes the point $w + (\delta H, H')$ on the right only. This will guarantee, using the barrier property (Proposition 5.1), that the sample path started at y is a barrier for any random walk starting in $w + (-\infty, \delta H) \times [0, H')$. Condition 1 gives quantitative information about this barrier at height $\pi_2(w) + H$. See Figure 6 for an illustration of Definition 5.2.

Remark 5.3. Note that event $\{w \text{ is } H\text{-trapped}\}$ is measurable with respect to the horizontal strips between heights $\pi_2(w) - 3H'$ and $\pi_2(w) + H$. Indeed, the definition of a trap implies that we can define an algorithm to decide if w is H-trapped or not, only looking at the environment and the uniform variables outside $R_H(w)$ and below height $\pi_2(w) + H$.

5.2.1 Probability of being trapped

Of course, we will not be able to show that a point is trapped with a high probability, for point 1 in Definition 5.2 is very demanding. However, the definition of v_- will allow us to show that we can reach any distance close to but greater than v_- with a positive probability, so we will be able to get a uniform lower bound on the probability of being trapped. This is what the following lemma expresses. Recall the definition of $\mathbf{H}_0$ from Remark 3.17.

Lemma 5.4. *There exists an integer constant $\mathbf{H}_1 \geqslant \mathbf{H}_0$, depending on δ, such that*

$$\inf_{H \geqslant \mathbf{H}_1} \inf_{w \in \mathbb{R}\times\mathbb{Z}} \mathbb{P}(w \textit{ is } H\textit{-trapped}) > 0.$$

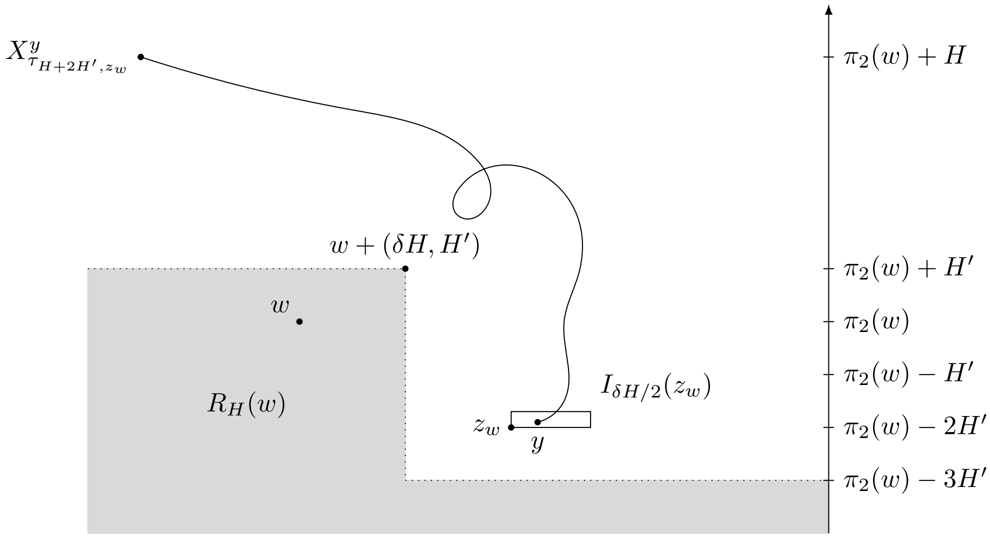


Figure 6: Illustration of w being H-trapped. The line connecting $(\pi_1(y),\pi_2(z_w))$ and $X^y_{\tau_{H+2H',z_w}}$ has direction less than $v_-+\delta/2$.

Proof. Let us first study condition 1 in Definition 5.2. Recall notation $\tilde{p}$ from Definition 4.1. We claim that there exist two positive constants c_{14} and c_{15} such that for H large enough,

$$\inf_{w\in\mathbb{R}\times\mathbb{Z}} \mathbb{P}\left(\exists\, y\in I_{\delta H/2}(z_w),\ V^y_{H+2H',z_w}\leqslant v_-+\delta/2\right) \geqslant c_{14}^{-1}\,\tilde{p}_H(v_-+\delta/4)-c_{13}^{-1}e^{-c_{13}H^{1/2}}. \tag{5.8}$$

Let us prove this claim. Let us fix $H\geqslant 4/\delta$. We have

$$\begin{aligned}
&\inf_{w\in\mathbb{R}\times\mathbb{Z}} \mathbb{P}\left(\exists\, y\in I_{\delta H/2}(z_w),\ V^y_{H+2H',z_w}\leqslant v_-+\delta/2\right)\\
&=\inf_{w\in\mathbb{R}\times\mathbb{Z}} \mathbb{P}\left(\exists\, y\in I_{\delta H/2}(w),\ V^y_{H+2H',w}\leqslant v_-+\delta/2\right)\\
&=\inf_{w\in[-1,0)\times\{0\}} \mathbb{P}\left(\exists\, y\in I_{\delta H/2}(w),\ V^y_{H+2H',w}\leqslant v_-+\delta/2\right)\\
&\geqslant\sup_{w\in[0,1)\times\{0\}} \mathbb{P}\left(\exists\, y\in I_{\delta H/4}(w),\ V^y_{H+2H',w}\leqslant v_-+\delta/2\right)\\
&=\sup_{w\in\mathbb{R}\times\mathbb{Z}} \mathbb{P}\left(\exists\, y\in I_{\delta H/4}(w),\ V^y_{H+2H',w}\leqslant v_-+\delta/2\right).
\end{aligned}$$

In the first equality, we used that $w\mapsto z_w$ is a bijection of $\mathbb{R}\times\mathbb{Z}$. In the second and last equalities, we used Corollary 3.4. In the inequality, we used that since $H\geqslant 4/\delta$, for any $w\in[-1,0)\times\{0\}$ and $w'\in[0,1)\times\{0\}$, $I_{\delta H/4}(w')$ is included in $I_{\delta H/2}(w)$.

Now, we want to replace $H+2H'$ by H in the parameter of the direction. Indeed, the information we have on v_- is a liminf when H goes to infinity, and it could be that we are unlucky and this liminf is reached on a subsequence that is not eventually in the image of $H\mapsto H+2H'$. In order to do this, we work on

$$\bigcap_{y\in I_{\delta H/4}(w)} D_{2H'}^{X^y_{\tau_{H,w}},N^y_{\tau_{H,w}}},$$

which, using Propositions 3.7 and 3.14, has probability at least $1-c_{15}^{-1}e^{-c_{15}H^{1/2}}$, where c_{15} is a positive constant that does not depend on H. On this event, provided that $2\beta H'\leqslant\delta(H+2H')/6$, we have that if $V^y_{H,w}\leqslant v_-+\delta/3$, then $V^y_{H+2H',w}\leqslant v_-+\delta/2$. Therefore, for H large enough,

$$\sup_{w\in\mathbb{R}\times\mathbb{Z}} \mathbb{P}\left(\exists\, y\in I_{\delta H/4}(w),\ V^y_{H+2H',w}\leqslant v_-+\delta/2\right)$$

$$\geqslant \sup_{w\in\mathbb{R}\times\mathbb{Z}} \mathbb{P}\left(\exists\, y \in I_{\delta H/4}(w),\ V^y_{H,w} \leqslant v_- + \delta/3\right) - c_{15}^{-1} e^{-c_{15} H^{1/2}}.$$

Now, in order to recover parameter H in the size of the rectangle too, we consider I_H and split it into rectangles $I_{\delta H/4}(w)$ for a certain number c_{14} (which does not depend on H) of values of $w \in \mathbb{R}\times\mathbb{Z}$ satisfying $0 \leqslant \pi_2(w) < H'$. Let us fix such a w and $y \in I_H$. In order to link V^y_H with $V^y_{H,w}$, we work on

$$\bigcap_{y\in I_H} D_{H'}^{X^y_{\tau_H}, N^y_{\tau_H}},$$

which, using Propositions 3.7 and 3.14, has probability at least $1 - c_{16}^{-1} e^{-c_{16} H^{1/2}}$ for a certain constant $c_{16} > 0$ that does not depend on H. On this event, the displacement of X^y between times τ^y_H and $\tau^y_{H,w}$ is less than $\beta H'$, which is less than $\delta H/12$ for H large enough. In the end, using a union bound, we have

$$\begin{aligned}
&\sup_{w\in\mathbb{R}\times\mathbb{Z}} \mathbb{P}\left(\exists\, y \in I_{\delta H/4}(w),\ V^y_{H,w} \leqslant v_- + \delta/3\right)\\
&\qquad \geqslant c_{14}^{-1} \left(\mathbb{P}\left(\exists\, y \in I_H,\ V^y_H \leqslant v_- + \delta/4\right) - c_{16}^{-1} e^{-c_{16} H^{1/2}}\right)\\
&\qquad = c_{14}^{-1} \left(\tilde{p}_H(v_- + \delta/4) - c_{16}^{-1} e^{-c_{16} H^{1/2}}\right).
\end{aligned}$$

Putting all inequalities together, we derive our claim (5.8) with c_{13} depending on c_{14}, c_{16} and c_{15}. Now, $\liminf_{H\to\infty} \tilde{p}_H(v_- + \delta/4) > 0$, since $v_- + \delta/4 > v_-$ (recall Definition 4.1). Therefore, using (5.8),

$$\liminf_{H\to\infty} \inf_{w\in\mathbb{R}\times\mathbb{Z}} \mathbb{P}\left(\exists\, y \in I_{\delta H/2}(z_w),\ V^y_{H+2H',z_w} \leqslant v_- + \delta/2\right) > 0.$$

This implies that there exists $\mathbf{H}_1 > \max(4, \mathbf{H}_0)$ such that

$$c_{17} := \inf_{H\geqslant \mathbf{H}_1} \inf_{w\in\mathbb{R}\times\mathbb{Z}} \mathbb{P}\left(\exists\, y \in I_{\delta H/2}(z_w),\ V^y_{H+2H',z_w} \leqslant v_- + \delta/2\right) > 0.$$

As for condition 2 in Definition 5.2, we can notice that a scenario on which it is satisfied is when events $E^y_{H'}$, $D^y_{4H'}$ and $E_{H'}^{X^y_{\tau_{4H',z_w}}, N^y_{\tau_{4H',z_w}}}$ occur for every $y \in I_{\delta H/2}(z_w)$ (recall (3.7) and (3.17)). Indeed:

- Being on $E^y_{H'}$ ensures that X^y stays outside $w + [\delta H, +\infty) \times (-\infty, -3H')$;
- X^y also stays outside $w + (-\infty, \delta H) \times (-\infty, H')$. Indeed, the horizontal distance between y and $w + (\delta H, H')$ is at least $4\beta H'$ (by definition of z_w), so on $D^y_{4H'}$, X^y passes $w + (\delta H, H')$ on the right, and $E^{\mathcal{X},\mathcal{N}}_{H'}$ ensures that it never comes back to height $\pi_2(w) + H'$ afterwards.

That being said, for every $H \geqslant \mathbf{H}_1$ and $w \in \mathbb{R}\times\mathbb{Z}$, we get

$$\begin{aligned}
&\mathbb{P}(w \text{ is } H\text{-trapped})\\
&\geqslant c_{17} - c\, H^{3/2} \sup_{y\in I_{\delta H/2}(z_w)} \left(\mathbb{P}((E^y_{H'})^c) + \mathbb{P}((D^y_{3H'})^c) + \mathbb{P}\left(\left(E_{H'}^{X^y_{\tau_{4H',z_w}}, N^y_{\tau_{4H',z_w}}}\right)^c\right)\right)\\
&\geqslant c_{17} - c\, H^{3/2} \left(2e^{-c_4 H^{1/4}} + c_5^{-1} e^{-c_5\sqrt{3} H^{1/2}}\right)\\
&\geqslant \frac{c_{17}}{2} \qquad \text{if } \mathbf{H}_1 \text{ is large enough.}
\end{aligned}$$

where in the second-to-last inequality, we used Propositions 3.16 and 3.13 as well as Proposition 3.7. Since $\frac{c_{17}}{2} > 0$, this yields the result. $\square$

We stated the above result as a lemma because it will later appear as a mere first step towards a stronger result, Proposition 5.7. The same holds for the next lemma, which is the first step towards Proposition 5.9.

5.2.2 Delay near a trapped point

The following lemma explains why the name "trap" was chosen: heuristically speaking, when we start a random walk X^{x_0} near an H-trapped point, with high probability it is delayed by the time it reaches height H.

Recall the definition of $\mathbf{H}_1$ in Lemma 5.4. For the next lemma, we need another technical requirement on H. Note that, since $\beta\delta > 0$, there exists $\mathbf{H}_2 > \mathbf{H}_1$, which depends on v_- and v_+, such that

$$\forall H \geqslant \mathbf{H}_2,\ 4\beta\delta H - (4\beta - (8\beta+7)\delta + 2v_+)H' \geqslant 0. \tag{5.9}$$

Lemma 5.5. *Let $H \geqslant \mathbf{H}_2$, $w \in \mathbb{R}\times\mathbb{Z}$, $x_0 \in w + (-\infty, \delta H) \times [0, H')$ and $\Gamma \in \mathcal{H}$ whose support satisfies* $\operatorname{Supp}\Gamma \subseteq R_H(w)$. *Suppose that w is H-trapped and that $E_{H'}^{x_0,\Gamma}$ occurs. Then, we have*

$$\pi_1\big(X^{x_0,\Gamma}_{\tau_{H,w}}\big) \leqslant \pi_1(w) + (v_+ - 2\delta)H.$$

Again, the statement is also true when we replace x_0, w and Γ by values of random variables satisfying the same assumptions.

Proof. Let H, w, x_0 and Γ be as in the statement of the lemma. Suppose w is H-trapped and $E_{H'}^{x_0,\Gamma}$ occurs. By definition, there exists $y \in I_{\delta H/2}(z_w)$ such that $V^y_{H+2H',z_w} \leqslant v_- + \delta/2$ and X^y does not visit $R_H(w)$. Let us apply the barrier property (Proposition 5.1) with x_0, Γ and y (replacing x_0' by y and H by $H+2H'$).

- Since X^y does not visit $R_H(w)$ and $\operatorname{Supp}\Gamma \subseteq R_H(w)$, we have $\operatorname{Supp}\Gamma \cap X^y_{\mathbb{N}} = \emptyset$.
- Since X^y does not visit $R_H(w)$ and the half-line $x_0 + \{0\}\times(-\infty,0)$ is included in $R_H(w)$, X^y cannot visit that half-line.
- Since $E_{H'}^{x_0,\Gamma}$ occurs and $\pi_2(y) < \pi_2(x_0) - H'$, X^{x_0} cannot visit the half-line $y + \{0\}\times(-\infty,0)$ either.

Therefore we must have

$$\begin{aligned}
\pi_1\big(X^{x_0,\Gamma}_{\tau_{H,w}}\big) &\leqslant \pi_1\big(X^y_{\tau_{H+2H',z_w}}\big) \\
&\leqslant \pi_1(y) + \left(v_- + \frac{\delta}{2}\right)(H+2H') && \text{since } V^y_{H+2H',z_w} \leqslant v_- + \delta/2 \\
&\leqslant \pi_1(z_w) + \frac{\delta H}{2} + \left(v_- + \frac{\delta}{2}\right)(H+2H') && \text{since } y \in I_{\delta H/2}(z_w) \\
&\leqslant \pi_1(w) + \delta H + 4\beta H' + \frac{\delta H}{2} + \left(v_- + \frac{\delta}{2}\right)(H+2H') && \text{using (5.6)} \\
&= \pi_1(w) + (v_+ - 2\delta)H - 4\beta\delta H + (4\beta - (8\beta+7)\delta + 2v_+)H' && \text{using (5.5)} \\
&\leqslant \pi_1(w) + (v_+ - 2\delta)H && \text{using (5.9).}
\end{aligned}$$

□

The interest of traps becomes clear with Lemma 5.5. The issue however is that the probability of being trapped cannot be made arbitrarily close to 1 when H goes to infinity; we only know, thanks to Lemma 5.4, that it is uniformly positive. Therefore, we need to introduce another notion in which we will allow some entropy on where to find a trap.

5.3 Threatened points

The problem with traps is that the probability of being trapped may be very small; however we will see that it is sufficient to have a trapped point along a line segment of slope v_+ in order to experience the delay, which motivates the new definition below.

Definition 5.6 (Threat)**.** *Let $H \in \mathbb{N}^*$, $r \in \mathbb{N}^*$ and $w \in \mathbb{R}\times\mathbb{Z}$. w is said to be (H,r)-threatened if one of the points $w_j = w + jH(v_+,1)$, where $j \in [\![0, r-1]\!]$, is H-trapped.*

5.3.1 Probability of being threatened

When r increases (keep in mind that r is the vertical length of the line segment along which we look for trapped points), it is clear that the probability that w is threatened increases. We now show that it goes to 1 when $r \to \infty$, and quantify the convergence using α. This is the major interest of the notion of threats. Recall constant $\mathbf{H}_1$ from Lemma 5.4.

Proposition 5.7. *There exists $c_{18} = c_{18}(\delta) > 0$ such that for every $H \geqslant \mathbf{H}_1$ and $r \in \mathbb{N}^*$,*

$$\sup_{w \in \mathbb{R}\times\mathbb{Z}} \mathbb{P}(w \textit{ is not } (H,r)\textit{-threatened}) \leqslant c_{18}\, r^{-\alpha}. \tag{5.10}$$

Proof. We follow again the structure of proof given in Section 3.6 (only here the scale parameter is r and not H). Mind that here we will need to apply the renormalization method twice to get the desired estimate.

First estimate. We start by considering only $r = 3^k$ for $k \in \mathbb{N}$. We set

$$q_k = q_k(H) = \sup_{w\in\mathbb{R}\times\mathbb{Z}} \mathbb{P}(w \text{ is not } (H,3^k)\text{-threatened}).$$

Let us start by showing that q_k converges to 0 when $k \to \infty$, uniformly in H large enough. More precisely, we show that there exists $c_{19} \in [1/3, 1)$ and $k_3 \in \mathbb{N}$ such that

$$\forall k \geqslant 2,\ \forall H \geqslant \mathbf{H}_1,\ q_{k_3+k} \leqslant c_{19}^k. \tag{5.11}$$

Note that the problem with this bound is that it does not involve α, which is why we will need to show a second estimate after this one. To prove (5.11), we use induction on $k \geqslant 2$. Let us fix $k_3 \in \mathbb{N}$ (we will choose it later in the proof).

Base case. If a point is not (H,r)-threatened, in particular it is not H-trapped, so, by Lemma 5.4,

$$\sup_{H\geqslant \mathbf{H}_1} q_{k_3+2} \leqslant \sup_{H\geqslant \mathbf{H}_1} \sup_{w\in\mathbb{R}\times\mathbb{Z}} \mathbb{P}(w \text{ is not } H\text{-trapped}) < 1.$$

Therefore there exists $c_{19} \in [1/3, 1)$ such that the case $k = 2$ in (5.11) is satisfied, namely $q_{k_3+2} \leqslant c_{19}^2$ for all $H \geqslant \mathbf{H}_1$, and the choice of c_{19} can be made independently of k_3.

Induction step. Fix $k \geqslant 2$ and suppose that

$$\sup_{H\geqslant \mathbf{H}_1} q_{k_3+k} \leqslant c_{19}^k. \tag{5.12}$$

Fix an integer $H \geqslant \mathbf{H}_1$ and $w \in \mathbb{R}\times\mathbb{Z}$. Note that event $\{w$ is not $(H, 3^{k_3+k+1})$-threatened$\}$ is included in the events given by

$$\mathcal{A}_k = \bigcap_{j=0}^{3^{k_3+k}-1} \{w_j \text{ is not } H\text{-trapped}\} \quad\text{and}\quad \mathcal{A}'_k = \bigcap_{j=2\cdot 3^{k_3+k}}^{3^{k_3+k+1}-1} \{w_j \text{ is not } H\text{-trapped}\}.$$

Using Remark 5.3, those events are measurable with respect to horizontal strips separated in time by $3^{k_3+k}H - 3H'$, which is larger than $3^{k_3+k}H/2$. In order to replace those strips by boxes of side lengths at most $(2\beta+1)\cdot 3^{k_3+k}H$ (anticipating the use of Fact 2.9), first note that by definition, $\{w_j$ is H-trapped$\}$ is measurable with respect to the sigma-algebra generated by

$$\left\{ X^y_{\left[0,\tau_{H+2H',z_{w_j}}\right]},\ y \in I_{\delta H/2}(z_{w_j}) \right\},$$

which motivates the introduction of the following events:

$$\mathcal{O}_k = \bigcap_{j=0}^{3^{k_3+k}-1} G_j(w) \ \text{ and }\ \mathcal{O}'_k = \bigcap_{j=2\cdot 3^{k_3+k}}^{3^{k_3+k+1}-1} G_j(w_{2\cdot 3^{k_3+k}})$$

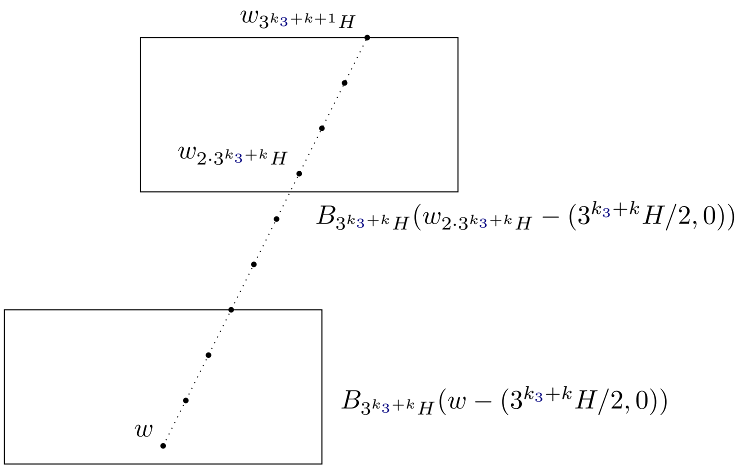


Figure 7

where, for $\bar{w} \in \mathbb{R} \times \mathbb{Z}$,

$$G_j(\bar{w}) = \bigcap_{y \in I_{\delta H/2}(z_{w_j})} \left\{ X^y_{\left[0,\tau_{H+2H',z_{w_j}}\right]} \subseteq B_{3^{k_3+k}H}(\bar{w} - (3^{k_3+k}H/2, 0)) \right\}.$$

Now, $\mathcal{A}_k \cap \mathcal{O}_k$ is measurable with respect to box $B_{3^{k_3+k}H}(w-(3^{k_3+k}H/2,0))$, and $\mathcal{A}'_k \cap \mathcal{O}'_k$ is measurable with respect to box $B_{3^{k_3+k}H}(w_{2\cdot 3^{k_3+k}} - (3^{k_3+k}H/2,0))$. Those two boxes are $(3^{k_3+k}H/2)$-separated and have maximum side lengths $(2\beta+1)\cdot 3^{k_3+k}H$.

Let us now bound the probability of $\big(G_j(w)\big)^c$. For $j \in [\![0, 3^{k_3+k}-1]\!]$, we have

$$\bigcap_{y \in I_{\delta H/2}(z_{w_j})} D^y_{3^{k_3+k-1}H/\beta} \subseteq G_j(w). \tag{5.13}$$

Indeed, let us assume that k_3 is large enough so that for every H we have $3^{k_3+k-1}H/\beta \geqslant H + 2H'$. Let $j \in [\![0, 3^{k_3+k}-1]\!]$, $y \in I_{\delta H/2}(z_{w_j})$ and $n \in [\![0, \tau^y_{H+2H',z_{w_j}}]\!]$. On the event on the left-hand side of (5.13), we have

$$\begin{aligned} |\pi_1(X^y_n) - \pi_1(w)| &\leqslant |\pi_1(X^y_n) - \pi_1(y)| + |\pi_1(y) - \pi_1(w)| \\ &\leqslant 3^{k_3+k-1}H + jH|v_+| + \delta H + 4\beta H' + \frac{\delta H}{2} \\ &\leqslant 3^{k_3+k-1}H + 3^{k_3+k}\beta H + \delta H + 4\beta H' + \frac{\delta H}{2} && \text{using (4.1)} \\ &< 3^{k_3+k}\beta H + \frac{3^{k_3+k}H}{2}, \end{aligned}$$

provided that k_3 is large enough (independently of H), which gives a first condition to choose k_3. From these horizontal bounds, noting that the vertical bounds are always satisfied by construction, we obtain

$$X^y_{\left[0,\tau_{H+2H',z_{w_j}}\right]} \subseteq B_{3^{k_3+k}H}(w - (3^{k_3+k}H/2, 0)),$$

which ends the proof of (5.13). Similarly, with the exact same arguments, for $j \in [\![2\cdot 3^{k_3+k}, 3^{k_3+k+1}-1]\!]$, we have

$$\bigcap_{y \in I_{\delta H/2}(z_{w_j})} D^y_{3^{k_3+k-1}H/\beta} \subseteq G_j(w_{2\cdot 3^{k_3+k}}). \tag{5.14}$$

Using (5.13) and (5.14) along with union bounds and Proposition 3.14, we have

$$\mathbb{P}\left(\mathcal{O}_k^c\right) \leqslant c_5^{-1}\, 3^{k_3+k} \left(\frac{\delta H}{2}\right)^2 e^{-c_5 3^{k_3+k-1} H/\beta} \leqslant c^{-1} e^{-c\, 3^{k_3+k} H},$$

where $c > 0$ does not depend on H, and the same holds for $\mathcal{O}'_k$. So, using Fact 2.9,

$$\begin{aligned} q_{k_3+k+1} &\leqslant \mathbb{P}\left((\mathcal{A}_k \cap \mathcal{O}_k) \cap (\mathcal{A}'_k \cap \mathcal{O}'_k)\right) + \mathbb{P}(\mathcal{O}_k^c) + \mathbb{P}((\mathcal{O}'_k)^c) \\ &\leqslant q_{k_3+k}^2 + c_1 \left(\frac{3^{k_3+k} H}{2}\right)^{-\alpha} + 2c^{-1} e^{-c\, 3^{k_3+k} H} \\ &\leqslant q_{k_3+k}^2 + c\, 3^{-(k_3+k)\alpha} \qquad \text{using that } H \geqslant 1. \end{aligned}$$

In the end, using induction assumption (5.12) as well as the fact that $1/3 \leqslant c_{19} < 1$, $\alpha \geqslant 1$ and $k \geqslant 2$,

$$\frac{q_{k_3+k+1}}{c_{19}^{k+1}} \leqslant c_{19}^{k-1} + c\, c_{19}^{k_3-1} \leqslant c_{19} + c\, c_{19}^{k_3-1} \leqslant 1,$$

provided that k_3 is chosen large enough (recall that the choice of c_{19} was independent of k_3).

Estimate on the subsequence. We now prove the desired estimate on the subsequence; more precisely, we prove that there exists $k_4 \in \mathbb{N}^*$ such that

$$\forall k \in \mathbb{N}^*,\ \forall H \geqslant \mathbf{H}_1,\ q_{k_4+k} \leqslant \frac{1}{2}\, 3^{-\alpha k}. \tag{5.15}$$

We use exactly the same method as in the proof of the first estimate (5.11). Since q_k goes to 0 uniformly in $H \geqslant \mathbf{H}_1$ (on account of (5.11)), we have, for any $k_4 \in \mathbb{N}$ large enough and $H \geqslant \mathbf{H}_1$,

$$q_{k_4+1} \leqslant \frac{1}{2}\, 3^{-\alpha}.$$

We now show by induction on $k \geqslant 1$ that $q_{k_4+k} \leqslant \frac{1}{2}\, 3^{-\alpha k}$. For the induction step, we obtain with the same arguments as before,

$$\frac{q_{k_4+k+1}}{\frac{1}{2} 3^{-\alpha(k+1)}} \leqslant 2 \cdot 3^{\alpha(k+1)} \left(\frac{1}{4} 3^{-2\alpha k} + c\, 3^{-\alpha(k_4+k)}\right),$$

which is less than 1 provided that k_4 is large enough. This gives a second condition to choose k_4. This constant being properly chosen, we get (5.15).

Interpolation. Let $H \geqslant \mathbf{H}_1$, $r \geqslant 3^{k_4+1}$ and $k \in \mathbb{N}^*$ such that $3^{k_4+k} \leqslant r < 3^{k_4+k+1}$. Then

$$\begin{aligned} \mathbb{P}(w \text{ is not } (H,r)\text{-threatened}) &\leqslant \mathbb{P}(w \text{ is not } (H, 3^{k_4+k})\text{-threatened}) && \text{by definition} \\ &\leqslant \frac{1}{2} 3^{-\alpha k} && \text{by (5.15)} \\ &\leqslant \frac{3^{\alpha(k_4+1)}}{2}\, r^{-\alpha}. \end{aligned}$$

It remains to tailor constant c_{18} in order for (5.10) to hold for every $r \in \mathbb{N}^*$. □

5.3.2 Delay near a threatened point

Now that we have shown that every point is threatened with a high probability, we need to quantify the delay caused by threats for the random walk, just as we did for traps with Lemma 5.5.

First a technical definition is required, because we do not want to look for threats among too many points for entropy reasons (see the proof of Lemma 5.10).

Definition 5.8. *Let $y \in \mathbb{Z}^2$ and $H \in \mathbb{N}$ such that $H \geqslant 4/\delta$ and $H' \geqslant 1$ (recall (3.19)). We denote by $\lfloor y \rfloor_H$ the point of $\lfloor \delta H/4 \rfloor \mathbb{Z} \times H'\mathbb{Z}$ given by*

$$\lfloor y \rfloor_H = \left(\left\lfloor \frac{\pi_1(y)}{\tilde{H}} \right\rfloor \tilde{H}, \left\lfloor \frac{\pi_2(y)}{H'} \right\rfloor H' \right), \quad \text{where } \tilde{H} = \lfloor \delta H/4 \rfloor.$$

Let $\mathbf{H}_3 \geqslant \mathbf{H}_2$ be an integer (depending on v_- and v_+) satisfying

$$\text{for every } H \geqslant \mathbf{H}_3, \begin{cases} 4H' < H; \\ H \geqslant 4/\delta; \\ H' \geqslant 1; \\ 4\beta H' \leqslant \delta H/5; \\ 4v_+ H' > -\delta H/20. \end{cases} \tag{5.16}$$

The second and third conditions ensure that Definition 5.8 can be used, and the others are technical requirements that will appear later on.

Proposition 5.9. *Let $H \geqslant \mathbf{H}_3$, $r \in \mathbb{N}^*$ and $y \in \mathbb{Z}^2$. We set $w = \lfloor y \rfloor_H$. Let $\Gamma \in \mathcal{H}$ be such that $\operatorname{Supp} \Gamma \subseteq R_H(w)$. For every $j \in [\![0, r]\!]$, set*

$$\begin{aligned} \mathcal{X}_j &= \mathcal{X}_j^{y,\Gamma} = X_{\tau_{jH,y}}^{y,\Gamma} && \text{and} \quad \mathcal{N}_j = \mathcal{N}_j^{y,\Gamma} = N_{\tau_{jH,y}}^{y,\Gamma}; \\ \tilde{\mathcal{X}}_j &= \tilde{\mathcal{X}}_j^{y,\Gamma} = X_{\tau_{jH,w}}^{y,\Gamma} && \text{and} \quad \tilde{\mathcal{N}}_j = \tilde{\mathcal{N}}_j^{y,\Gamma} = N_{\tau_{jH,w}}^{y,\Gamma}; \\ Z_j &= Z_j^{y,\Gamma} = X_{\tau_{(j+1)H-4H',y}}^{y,\Gamma} && \text{and} \quad \Lambda_j = \Lambda_j^{y,\Gamma} = N_{\tau_{(j+1)H-4H',y}}^{y,\Gamma}. \end{aligned}$$

Assume the following conditions are met:

1. *w is (H, r)-threatened;*
2. *For every $j \in [\![0, r-1]\!]$, $V_{H,\mathcal{X}_j}^{\mathcal{X}_j,\mathcal{N}_j} \leqslant v_+ + \frac{\delta}{2r}$;*
3. *For every $j \in [\![0, r-1]\!]$, $V_{H-4H',\mathcal{X}_j}^{\mathcal{X}_j,\mathcal{N}_j} \leqslant v_+ + \frac{\delta}{2r}$ and $D_{4H'}^{Z_j,\Lambda_j}$ occurs;*
4. *For every $j \in [\![0, r-1]\!]$, $E_{H'}^{\mathcal{X}_j,\mathcal{N}_j}$ occurs;*
5. *For every $j \in [\![1, r]\!]$, $D_{H'}^{\tilde{\mathcal{X}}_j,\tilde{\mathcal{N}}_j}$ occurs.*

Then we have $\pi_1(\mathcal{X}_r) \leqslant \pi_1(y) + \left(v_+ - \frac{\delta}{2r}\right) rH$.

Again, y and Γ can be replaced by random variables satisfying the same assumptions.

Let us explain the proposition heuristically. Suppose that a point w close to y is threatened (condition 1). Divide the strip $\pi_2^{-1}([\pi_2(y), \pi_2(y) + rH])$ into r strips of height H, and assume that the random walk started at y does not go too fast on each of these r strips (condition 2). The fact that a point w near y is threatened and that X^y does not go too fast will imply that X^y meets a potential barrier on its right (which is given by a certain trapped point $w_{j_0} = w + j_0 H(v_+, 1)$), and it cannot get around this barrier because of condition 4. Condition 3 ensures that X^y stays inside $R_H(w_{j_0})$, which is required to apply Lemma 5.5 (see Figures 8 and 9 for an illustration). So by combining the upper bound we have on each of the r strips, and the new upper bound that the barrier gives us on this particular strip, we get a global upper bound. Mind that in this proposition two grids coexist, one with lines at heights $\pi_2(y) + jH$ ($j \in [\![0, r]\!]$), where the $\mathcal{X}_j$ are, and one with lines at heights $\pi_2(w) + jH$ ($j \in [\![1, r]\!]$), where the $\tilde{\mathcal{X}}_j$ are. Condition 5 allows us to control the error of displacement between $\tilde{\mathcal{X}}_j$ and $\mathcal{X}_j$.

Proof. Let $H \geqslant \mathbf{H}_3$, $r \in \mathbb{N}^*$, $y \in \mathbb{Z}^2$, $w = \lfloor y \rfloor_H$, $\Gamma \in \mathcal{H}$ such that $\operatorname{Supp} \Gamma \subseteq R_H(w)$. Assume all five assumptions from the statement are satisfied. Because of condition 1, there exists $j_0 \in [\![0, r-1]\!]$ such that w_{j_0} is H-trapped. We want to apply Lemma 5.5 replacing w in the statement by w_{j_0}, x_0 by $\mathcal{X}_{j_0}$ and Γ by $\mathcal{N}_{j_0}$ (recall that the lemma was also true for a random choice of x_0, w and Γ), which requires justifying that

$$\mathcal{X}_{j_0} \in w_{j_0} + (-\infty, \delta H) \times [0, H'); \tag{5.17}$$

$$\operatorname{Supp} \mathcal{N}_{j_0} \subseteq R_H(w_{j_0}). \tag{5.18}$$

Note that the fact that $E_{H'}^{\mathcal{X}_{j_0},\mathcal{N}_{j_0}}$ occurs is a direct consequence of condition 4.

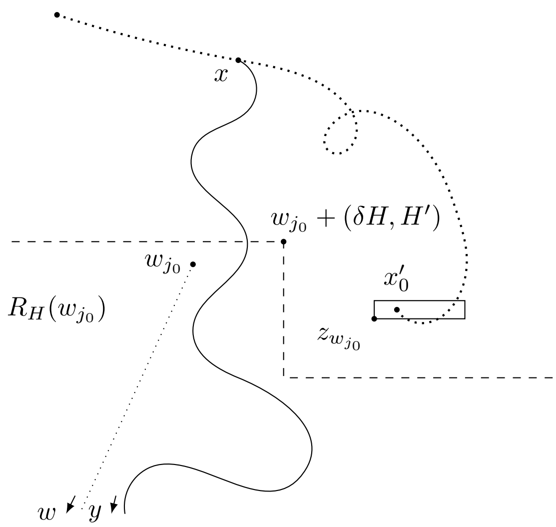


Figure 8: Illustration of $w = \lfloor y \rfloor_H$ being (H, r)-threatened and the random walk starting from y experiencing a delay on account of the trap. In this example, at the point labeled x, the sample path started at y coalesces with the dotted curve, which causes a delay.

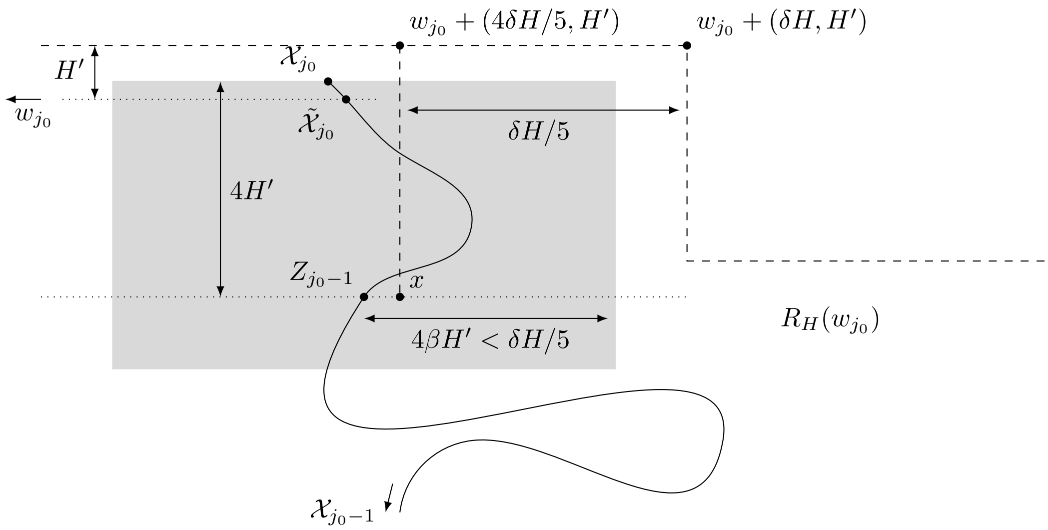


Figure 9: Illustration of condition 3 and the proof of (5.18). The fact that $V^{\mathcal{X}_{j_0-1}, \mathcal{N}_{j_0-1}}_{H-4H', \mathcal{X}_{j_0-1}} \leqslant v_+ + \frac{\delta}{2r}$ (along with condition 2) ensures that Z_{j_0-1} lies on the left of the point labeled x (see (5.20)). Then, staying in the gray box horizontally until time $\tau^{y,\Gamma}_{j_0 H, y}$ ensures that $X^{y,\Gamma}$ does not exit $R_H(w_{j_0})$ before that time.

Proof of (5.17). We compute

$$\begin{aligned}
\pi_1(\mathcal{X}_{j_0}) &= \pi_1(y) + \sum_{j=0}^{j_0-1} (\pi_1(\mathcal{X}_{j+1}) - \pi_1(\mathcal{X}_j)) \\
&\leqslant \pi_1(y) + \left(v_+ + \frac{\delta}{2r}\right) j_0 H \qquad \text{using condition 2}
\end{aligned}$$

$$\leqslant \pi_1(w_{j_0}) + \frac{\delta H}{4} + \frac{\delta H}{2} \qquad \text{by definition of } w = \lfloor y \rfloor_H \text{ and } w_{j_0} = w + j_0 H(v_+, 1) \tag{5.19}$$
$$< \pi_1(w_{j_0}) + \delta H.$$

As for the second coordinate, by definition of w we have

$$\pi_2(\mathcal{X}_{j_0}) = \pi_2(y) + j_0 H \in \pi_2(w) + j_0 H + [0, H') = \pi_2(w_{j_0}) + [0, H')$$

This proves (5.17).

Proof of (5.18). When $j_0 = 0$, $\mathcal{N}_{j_0} = \Gamma \subseteq R_H(w)$ by assumption, so (5.18) is satisfied. Suppose now that $j_0 \geqslant 1$. See Figure 9 for an illustration of the following arguments. Mind that without condition 3, in spite of (5.19), there is a possibility that between time $\tau^{y,\Gamma}_{(j_0-1)H}$ and time $\tau^{y,\Gamma}_{j_0 H}$, $X^{y,\Gamma}$ exits $R_H(w_{j_0})$. Note that since $\pi_2(Z_{j_0-1}) = \pi_2(\mathcal{X}_{j_0}) - 4H' \leqslant \pi_2(w_{j_0}) - 3H'$, we only need to check that for every $n \in [\![\tau^{y,\Gamma}_{j_0 H - 4H', y}, \tau^{y,\Gamma}_{j_0 H, y}]\!]$, we have $\pi_1(X^{y,\Gamma}_n) < \pi_1(w_{j_0}) + \delta H$. Now, condition 3 ensures that $V^{\mathcal{X}_{j_0-1}, \mathcal{N}_{j_0-1}}_{H-4H', \mathcal{X}_{j_0-1}} \leqslant v_+ + \frac{\delta}{2r}$, therefore,

$$\begin{aligned}
\pi_1(Z_{j_0-1}) &\leqslant \pi_1(\mathcal{X}_{j_0-1}) + \left(v_+ + \frac{\delta}{2r}\right)(H - 4H') \\
&\leqslant \pi_1(y) + \left(v_+ + \frac{\delta}{2r}\right)(j_0 - 1)H + \left(v_+ + \frac{\delta}{2r}\right)(H - 4H') && \text{using condition 2} \\
&= \pi_1(y) + \left(v_+ + \frac{\delta}{2r}\right) j_0 H - 4\left(v_+ + \frac{\delta}{2r}\right) H' \\
&\leqslant \pi_1(w_{j_0}) + \frac{3\delta H}{4} - 4\left(v_+ + \frac{\delta}{2r}\right) H' && \text{using the same ideas as in (5.19)} \\
&< \pi_1(w_{j_0}) + \frac{4\delta H}{5} && \text{using (5.16).} \qquad (5.20)
\end{aligned}$$

Now, condition 3 also ensures that $D^{Z_{j_0-1}, \Lambda_{j_0-1}}_{4H'}$ occurs. Now, by (5.16), we have $4\beta H' \leqslant \delta H/5$. Combining that with (5.20), we get that for every $n \in [\![\tau^{y,\Gamma}_{j_0 H - 4H', y}, \tau^{y,\Gamma}_{j_0 H, y}]\!]$,

$$\begin{aligned}
\pi_1(X^{y,\Gamma}_n) &\leqslant \pi_1(Z_{j_0-1}) + \frac{\delta H}{5} \\
&< \pi_1(w_{j_0}) + \frac{4\delta H}{5} + \frac{\delta H}{5} && \text{using (5.20)} \\
&= \pi_1(w_{j_0}) + \delta H.
\end{aligned}$$

Therefore, $X^{y,\Gamma}_{[0, \tau_{j_0 H, y})} \subseteq R_H(w_{j_0})$, and therefore $\mathrm{Supp}\, \mathcal{N}_{j_0} \subseteq R_H(w_{j_0})$.

Conclusion. By applying Lemma 5.5, we get that

$$\pi_1(\tilde{\mathcal{X}}_{j_0+1}) = \pi_1\left(X^{\mathcal{X}_{j_0}, \mathcal{N}_{j_0}}_{\tau_{H, w_{j_0}}}\right) \leqslant \pi_1(w_{j_0}) + (v_+ - 2\delta)H. \tag{5.21}$$

Therefore, using that we are on $D^{\tilde{\mathcal{X}}_{j_0+1}, \tilde{\mathcal{N}}_{j_0+1}}_{H'}$ (condition 5), we get

$$\begin{aligned}
\pi_1(\mathcal{X}_{j_0+1}) &\leqslant \pi_1(\tilde{\mathcal{X}}_{j_0+1}) + \beta H' \\
&\leqslant \pi_1(w_{j_0}) + (v_+ - 2\delta)H + \beta H' \\
&\leqslant \pi_1(w_{j_0}) + (v_+ - \delta)H,
\end{aligned}$$

using (5.16). Consequently, we have

$$\pi_1(\mathcal{X}_r) = \pi_1(\mathcal{X}_{j_0+1}) + \sum_{j=j_0+1}^{r-1} \left(\pi_1(\mathcal{X}_{j+1}) - \pi_1(\mathcal{X}_j)\right)$$

$$\begin{aligned}
&\leqslant \pi_1(w_{j_0}) + (v_+ - \delta)H + (r - j_0 - 1)\left(v_+ + \frac{\delta}{2r}\right)H && \text{using (5.21) and condition 2}\\
&\leqslant \pi_1(y) + j_0 v_+ H + (v_+ - \delta)H + (r - j_0 - 1)\left(v_+ + \frac{\delta}{2r}\right)H\\
&\leqslant \pi_1(y) + r v_+ H - \frac{\delta}{2}H\\
&= \pi_1(y) + \left(v_+ - \frac{\delta}{2r}\right) rH.
\end{aligned}$$

□

5.4 Threatened paths

We now know that when a particle passes near a threatened point, it will be delayed to the left with a high probability (Proposition 5.9), and that each point has a high probability of being threatened (Proposition 5.7). The goal of this section is to improve the latter result, by showing that with a high probability, every particle meets a lot of threatened points along its way. Mind that this is not a direct consequence of Proposition 5.7, because the random walk could unfortunately go precisely to areas where there are few threats. From now on, we will focus on specific values of the parameters introduced before : $H = hL_k$ with $k > k_5$ for a wise choice of $h \in \mathbb{N}^*$ and $k_5 \in \mathbb{N}$, and $r = l_{k_5}$.

Lemma 5.10. *There exists $k_5 \in \mathbb{N}$ and $c_{20} = c_{20}(\delta) > 0$ such that the following conditions are met:*

- $L_{k_5} \geqslant \mathbf{H}_3$;
- *For every $h \in \mathbb{N}^*$,*

$$\mathbb{P}\left(\exists\, y \in I_{hL_{k_5+1}},\ \lfloor y \rceil_{hL_{k_5}} \text{ is not } (hL_{k_5}, l_{k_5})\text{-threatened}\right) \leqslant c_{20} L_{k_5+1}^{-(2\alpha-3)/10}; \tag{5.22}$$

- *The following two technical requirements are satisfied*

$$49\beta l_{k_5} \leqslant \delta l_{k_5+1}; \tag{5.23}$$

$$c_{12}(c_{12} + c_{20}^2)\, L_{k_5}^{-(6\alpha-49)/40} \leqslant c_{20}, \tag{5.24}$$

where c_{12} was defined in (4.15).

Proof. Let $h \in \mathbb{N}^*$ and $k_5 \in \mathbb{N}$ satisfying $L_{k_5} \geqslant \mathbf{H}_3$ and (5.23). Then, using Proposition 5.7 and the fact that $\mathbf{H}_3 \geqslant \mathbf{H}_1$, we have

$$\begin{aligned}
&\mathbb{P}\left(\exists\, y \in I_{hL_{k_5+1}},\ \lfloor y \rceil_{hL_{k_5}} \text{ is not } (hL_{k_5}, l_{k_5})\text{-threatened}\right)\\
&\leqslant \left\lceil \frac{hL_{k_5+1}}{\left\lfloor \frac{\delta h L_{k_5}}{4} \right\rfloor} \right\rceil \left\lceil \frac{(hL_{k_5+1})'}{(hL_{k_5})'} \right\rceil c_{18}\, l_{k_5}^{-\alpha}\\
&\leqslant c\, L_{k_5+1}^{-(2\alpha-3)/10}.
\end{aligned}$$

Therefore, we do get inequality (5.22) with a certain constant $c_{20} = c_{20}(\delta) > 0$. Now that c_{20} is fixed, it suffices to take a larger k_5 so that inequality (5.24) holds as well, which is possible because $\alpha \geqslant 9$. □

Conditions (5.23) and (5.24) will appear naturally later in the proof. Also, note that considering only rounded points $\lfloor y \rceil_{hL_{k_5}}$ was crucial here to obtain a bound that is uniform in h.

Definition 5.11. *Let k_5 be defined as in Lemma 5.10. Let $k > k_5$, $w \in \mathbb{R} \times \mathbb{Z}$, $h \in \mathbb{N}^*$ and $y \in I_{hL_k}(w)$. We set the threatened density of random walk X^y to be*

$$D^y_{h,k}(w) = \frac{L_{k_5+1}}{L_k} \#\left\{0 \leqslant j < \frac{L_k}{L_{k_5+1}},\ \left\lfloor X^y_{\tau_{jhL_{k_5+1},w}} \right\rceil_{hL_{k_5}} \text{ is } (hL_{k_5}, l_{k_5})\text{-threatened}\right\}. \tag{5.25}$$

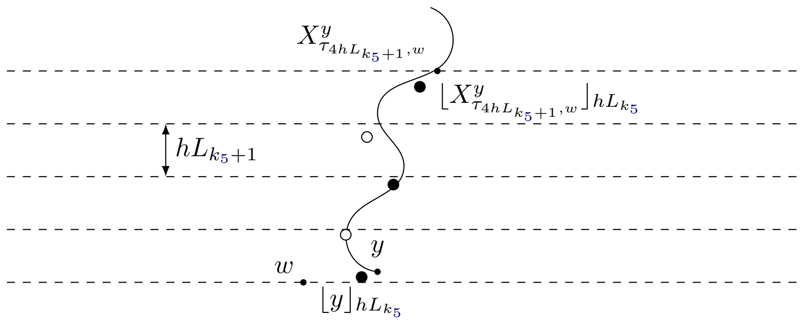


Figure 10: Illustration of a point $y \in I_{hL_k}(w)$ whose density is more than 1/2: there are more threatened points (the large filled circles) than non-threatened points (the empty large circles) along the way.

As usual, we also set $D^y_{h,k} = D^y_{h,k}(o)$.

Mind that contrary to what is depicted in Figure 10, we could have $\pi_1(y) - \pi_1(w) > hL_{k_5+1}$. In that case, whenever $jhL_{k_5+1} \leqslant \pi_1(y) - \pi_1(w)$, $\tau^y_{jhL_{k_5+1},w} = 0$, so $X^y_{\tau_{jhL_{k_5+1},w}} = y$. That is why, in (5.27), we are not interested in the $j = 0$ term.

Let us now state our final proposition before ending the proof of Lemma 4.6: with a high probability, our random walks encounter threats more than half of the time along the way.

Proposition 5.12. *For every $k > k_5$ and $h \in \mathbb{N}^*$,*

$$\mathbb{P}(\exists\, y \in I_{hL_k},\ D^y_{h,k} < 1/2) \leqslant c_{20}\, L_k^{-(2\alpha-3)/10}.$$

Proof. The proof uses again the renormalization method presented in Section 3.6 and is very similar to that of Lemma 4.4. Let us fix $k > k_5$ and $h \in \mathbb{N}^*$. We define a sequence of densities $(\rho_k)_{k \geqslant k_5}$ by setting

$$\begin{cases} \rho_{k_5} = 1; \\ \forall k \geqslant k_5,\ \rho_{k+1} = \rho_k - \frac{5}{l_k}. \end{cases}$$

One can check, using a computational knowledge engine, that since $L_0 \geqslant 10^{10}$, we have $\sum_{k \geqslant 1} \frac{5}{l_k} \leqslant \frac{1}{2}$, therefore we have $\rho_k \geqslant 1/2$ for every $k \geqslant k_5$. We define, for $w \in \mathbb{R} \times \mathbb{Z}$,

$$S_{h,k}(w) = \{\exists\, y \in I_{hL_k}(w),\ D^y_{h,k}(w) \leqslant \rho_k\}.$$

Since $\rho_k \geqslant 1/2$, it suffices to show that $s_{h,k} = \mathbb{P}(S_{h,k}(o))$ satisfies

$$s_{h,k} \leqslant c_{20}\, L_k^{-(2\alpha-3)/10}. \tag{5.26}$$

To do this, we use induction on $k > k_5$.

Base case. When $k = k_5 + 1$, the result follows directly from Definition 5.11 and Lemma 5.10.

Induction step. Assume that (5.26) has been shown for a fixed $k > k_5$, and fix $y \in I_{hL_{k+1}}$. Recall the definitions of $\mathcal{C}_k$, $\mathcal{F}_k$ and $\mathcal{G}_k$, from (4.6), (4.8) and (4.11), where H_k is replaced by hL_k. Recall also notations $\mathcal{X}^y_j$ and $\mathcal{N}^y_j$ used in (4.7), and θ^y_j used in (4.13). Note that in the definitions of $\mathcal{F}_k$ and $\mathcal{G}_k$, we will not use the part with events D (because contrary to Lemma 4.4, here we are not looking at horizontal displacements). We claim that

$$\mathcal{G}_k \cap \left\{D^y_{h,k+1} \leqslant \rho_{k+1}\right\} \subseteq \left\{\text{there exist three } j \in [\![1, l_k - 1]\!] \text{ such that } D^{\theta^y_j}_{h,k}(\mathcal{X}^y_j) \leqslant \rho_k\right\}. \tag{5.27}$$

Indeed, suppose that $\mathcal{G}_k$ occurs but it is not the case that there exist three $j \in [\![1, l_k - 1]\!]$ such that $D^{\theta^y_j}_{h,k}(\mathcal{X}^y_j) \leqslant \rho_k$. This means that for $l_k - 3$ values of $j \in [\![1, l_k - 1]\!]$, we have $D^{\theta^y_j}_{h,k}(\mathcal{X}^y_j) > \rho_k$, which means

$$\#\left\{0 \leqslant i < \frac{L_k}{L_{k_5+1}},\ \left\lfloor X^{\theta^y_j}_{\tau_{ihL_{k_5+1},\mathcal{X}^y_j}} \right\rfloor_{hL_{k_5}} \text{ is } (hL_{k_5}, l_{k_5})\text{-threatened}\right\} > \rho_k \frac{L_k}{L_{k_5+1}}.$$

Now, on $\mathcal{G}_k$, there are at most $\frac{(hL_k)'}{hL_{k_5+1}} \leqslant 2\frac{L_k^{1/2}}{L_{k_5+1}}$ indices $i \in [\![0, L_k/L_{k_5+1}]\!]$ such that $X^{\theta_j^y}_{\tau_{ihL_{k_5+1}},\mathcal{X}_j^y} = \theta_j^y$ (indeed, this occurs only when $ihL_{k_5+1} \leqslant \pi_2(\theta_j^y) - \pi_2(\mathcal{X}_j^y) \leqslant (hL_k)'$). Therefore,

$$\#\left\{0 \leqslant i < \frac{L_k}{L_{k_5+1}},\ \left\lfloor X^{\mathcal{X}_j^y,\mathcal{N}_j^y}_{\tau_{ihL_{k_5+1}},\mathcal{X}_j^y} \right\rfloor_{hL_{k_5}} \text{ is } (hL_{k_5}, l_{k_5})\text{-threatened}\right\} > \rho_k \frac{L_k}{L_{k_5+1}} - 2\frac{L_k^{1/2}}{L_{k_5+1}}.$$

In the end,

$$\begin{aligned} D^y_{h,k+1} &= \frac{L_{k_5+1}}{L_{k+1}} \#\left\{0 \leqslant j < \frac{L_{k+1}}{L_{k_5+1}},\ \left\lfloor X^y_{\tau_{jhL_{k_5+1}},\mathcal{X}_j^y} \right\rfloor_{hL_{k_5}} \text{ is } (hL_{k_5}, l_{k_5})\text{-threatened}\right\} \\ &\geqslant \frac{L_{k_5+1}}{L_{k+1}} (l_k - 3) \left(\rho_k \frac{L_k}{L_{k_5+1}} - 2\frac{L_k^{1/2}}{L_{k_5+1}}\right) \\ &= \left(1 - \frac{3}{l_k}\right)\left(\rho_k - \frac{2}{L_k^{1/2}}\right) \\ &> \rho_k - \frac{5}{l_k} = \rho_{k+1}, \end{aligned}$$

which proves (5.27).

Now, note that on $\mathcal{F}_k \cap \mathcal{G}_k$, for every $j \in [\![0, l_k - 1]\!]$, θ_j^y is in a $I_{hL_k}(w)$ with $w \in \mathcal{C}_k$. Therefore, in a similar way as in (4.14), we get

$$\mathcal{F}_k \cap \mathcal{G}_k \cap S_{h,k+1} \subseteq \bigcup_{\substack{w_1,w_2 \in \mathcal{C}_k \\ |\pi_2(w_1)-\pi_2(w_2)| \geqslant 2hL_k}} \begin{array}{c} (S_{h,k}(w_1) \cap F_{hL_k}(w_1)) \\ \cap (S_{h,k}(w_2) \cap F_{hL_k}(w_2)). \end{array}$$

Recall constant c_{12} from (4.15). Here again we get

$$s_{h,k+1} \leqslant c_{12}\, l_k^4\, (s_{h,k}^2 + c_{12} L_k^{-\alpha}),$$

and so, using the induction assumption and (5.24), we get

$$\begin{aligned} \frac{s_{h,k+1}}{L_{k+1}^{-(2\alpha-3)/10}} &\leqslant c_{12}(c_{12} + c_{20}^2)\, L_k^{(2\alpha-3)/8}\, l_k^4\, L_k^{-(2\alpha-3)/5} \\ &\leqslant c_{12}(c_{12} + c_{20}^2)\, L_k^{-(6\alpha-49)/40} \leqslant c_{20}. \end{aligned}$$

This concludes the induction and thus the proof of (5.26). □

5.5 Final proof of Lemma 4.6.

Recall that we argued by contradiction and assumed that $v_- < v_+$, therefore $\delta = \frac{v_+ - v_-}{4(\beta+1)} > 0$. Let $\eta = \frac{\delta}{4l_{k_5}} > 0$ where k_5 is defined as in Lemma 5.10. We are going to show that

$$p_{L_k^2}\left(v_+ - \frac{\eta}{6}\right) \xrightarrow[k\to\infty]{} 0, \tag{5.28}$$

which contradicts the definition of v_+. From now on, we fix $k > k_5 + 1$, and we work with $h = h_k = L_k$, which is why it was important for our previous estimates to hold uniformly on $h \geqslant 1$. We let $H_k = h_k L_{k_5} = L_k L_{k_5}$ and $r = l_{k_5}$. In order to prove (5.28), we consider the large box $B_{L_k^2}$, which we pave using small sub-boxes $B_{H_k}(y)$ for $y \in \hat{\mathcal{C}}_k$, where $\hat{\mathcal{C}}_k$ is the minimal set satisfying

$$\bigcup_{w \in \hat{\mathcal{C}}_k} \mathcal{I}_{H_k}(w) = B_{L_k^2} \cap (\mathbb{Z} \times H_k\mathbb{Z}).$$

Recall notations $\mathcal{X}_j^{z,\Gamma}$, $\mathcal{N}_j^{z,\Gamma}$, $\tilde{\mathcal{X}}_j^{z,\Gamma}$, $\tilde{\mathcal{N}}_j^{z,\Gamma}$, $Z_j^{z,\Gamma}$ and $\Lambda_j^{z,\Gamma}$ from the statement of Proposition 5.9, where $z \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$. For $y \in I_{L_k^2}$ and $1 \leqslant i \leqslant L_k/L_{k_5+1}$, we set $\mathscr{X}_i^y = X^y_{\tau_{irH_k}}$ and $\mathscr{N}_i^y = N^y_{\tau_{irH_k}}$, and for

$0 \leqslant j \leqslant r$, we set $\mathcal{X}_{i,j}^y = \mathcal{X}_j^{\mathscr{X}_i^y, \mathscr{N}_i^y}$. In the same way, we define $\mathcal{N}_{i,j}^y$, $\tilde{\mathcal{X}}_{i,j}^y$, $\tilde{\mathcal{N}}_{i,j}^y$, $Z_{i,j}^y$ and $\Lambda_{i,j}^y$. We define the following events:

$$\begin{aligned}
\hat{\mathcal{A}}_k &= \bigcap_{w \in \hat{\mathcal{C}}_k} \left(A_{H_k,w}(v_+ + \eta)^c \cap A_{H_k - 4H_k', w}(v_+ + \eta)^c \right); \\
\hat{\mathcal{F}}_k &= F_{L_k^2} \cap \bigcap_{y \in I_{L_k^2}} \left(D_{rH_k}^y \cap \bigcap_{i=1}^{L_k/L_{k_5}-1} \bigcap_{j=0}^{r} D_{4H_k'}^{Z_{i,j}^y, \Lambda_{i,j}^y} \cap E_{H_k'}^{\mathcal{X}_{i,j}^y, \mathcal{N}_{i,j}^y} \cap D_{H_k'}^{\tilde{\mathcal{X}}_{i,j}^y, \tilde{\mathcal{N}}_{i,j}^y} \right); \\
\hat{\mathcal{G}}_k &= \bigcap_{y \in I_{L_k^2}} \bigcap_{i=1}^{L_k/L_{k_5}-1} \bigcap_{j=0}^{r-1} \left(D_{H_k'}^{\mathcal{X}_{i,j}^y, \mathcal{N}_{i,j}^y} \cap \left\{ \Theta\left(X^{\mathcal{X}_{i,j}^y, \mathcal{N}_{i,j}^y} \right) < H_k' \right\} \right); \\
\hat{\mathcal{H}}_k &= \bigcap_{y \in I_{L_k^2}} \left\{ D_{L_k,k}^y \geqslant 1/2 \right\}.
\end{aligned}$$

Using Lemma 4.4 and the fact that $\alpha > 8$, we have

$$\mathbb{P}(\hat{\mathcal{A}}_k^c) \leqslant c \left(\frac{L_k}{L_{k_5}} \right)^2 \left(c_{10}(\eta) H_k^{-\alpha/4} + c_{10}(\delta/2r)(H_k - 4H_k')^{-\alpha/4} \right) \leqslant c L_k^{-(\alpha-8)/4} \xrightarrow[k \to \infty]{} 0. \tag{5.29}$$

Using Propositions 3.7, 3.14 and 3.13, we have

$$\mathbb{P}(\hat{\mathcal{F}}_k^c) \leqslant c_6^{-1} e^{-c_6 L_k} + c L_k^4 \left(2 c_5^{-1} e^{-c_5 H_k^{1/2}} + e^{-c_4 H_k^{1/2}} \right) \xrightarrow[k \to \infty]{} 0. \tag{5.30}$$

Using Propositions 3.7, 3.14 and 3.20, we have

$$\mathbb{P}(\hat{\mathcal{G}}_k^c) \leqslant c L_k^4 \left(c_5^{-1} e^{-2c_5 H_k^{1/2}} + c_8^{-1} e^{-c_8 H_k^{1/4}} \right) \xrightarrow[k \to \infty]{} 0. \tag{5.31}$$

By Proposition 5.12, we have, using that $\alpha \geqslant 2$,

$$\mathbb{P}(\hat{\mathcal{H}}_k^c) \leqslant c_{20} \, L_k^{-(2\alpha-3)/10} \xrightarrow[k \to \infty]{} 0. \tag{5.32}$$

The goal now is to show that on the four events defined above, we have, for every $y \in I_{L_k^2}$,

$$V_{L_k^2}^y = \frac{1}{L_k^2} \left(\pi_1 \left(\mathscr{X}_{L_k/L_{k_5+1}}^y \right) - \pi_1(y) \right) < v_+ - \eta/3. \tag{5.33}$$

First note that since $\lceil (L_k^2)^{1/2} \rceil = L_k < H_k < rH_k$, we have $\tau_{irH_k} > 0$ for every $i \geqslant 1$. Therefore, we will only isolate $i = 0$ and simply use $D_{rH_k}^y$ to bound the displacement $\pi_1(\mathscr{X}_1^y) - \pi_1(y)$. Let us now focus on bounding $\pi_1(\mathscr{X}_{i+1}^y) - \pi_1(\mathscr{X}_i^y)$ where $1 \leqslant i < L_k/L_{k_5+1}$. First note that $\hat{\mathcal{A}}_k$ along with $F_{L_k^2}$ and $\hat{\mathcal{G}}_k$ allow us to bound the displacements of the random walk, similarly to what we did in Section 4.2.4. We have, for $1 \leqslant i < L_k/L_{k_5+1}$ and $0 \leqslant j < r$,

$$\begin{aligned}
\pi_1(\mathcal{X}_{i,j+1}^y) - \pi_1(\mathcal{X}_{i,j}^y) &\leqslant \beta H_k' + (v_+ + \eta) H_k \\
&\leqslant \left(v_+ + \frac{3\eta}{2} \right) H_k && \text{using (5.23)} \qquad (5.34) \\
&\leqslant \left(v_+ + \frac{\delta}{2r} \right) H_k && \text{by definition of } \eta. \qquad (5.35)
\end{aligned}$$

We can use (5.34) for indices i such that $\lfloor \mathscr{X}_i^y \rfloor$ is not (H_k, r)-threatened. As for the remaining indices i, we will use Proposition 5.9, replacing in the statement of the proposition H by H_k, y by $\mathscr{X}_i^y$ and Γ by $\mathscr{N}_i^y$. Assumption 2 in Proposition 5.9 is satisfied using (5.35), and we can show in a similar way, using events $A_{H_k - 4H_k', w}(v_+ + \eta)^c$ in $\hat{\mathcal{A}}_k$, that we have

$$\pi_1(Z_{i,j}^y) - \pi_1(\mathcal{X}_{i,j}^y) \leqslant \left(v_+ + \frac{\delta}{2r} \right) (H - 4H'), \tag{5.36}$$

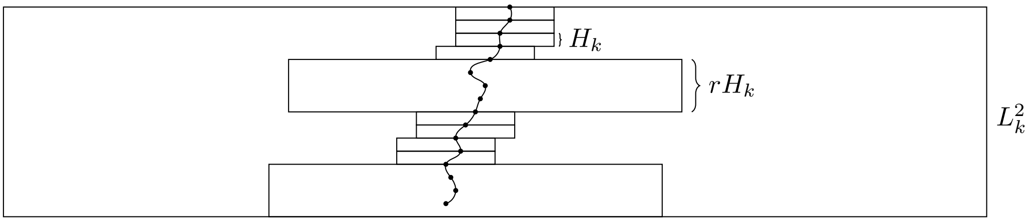


Figure 11: The final bound in the proof of Lemma 4.6. The large boxes correspond to the displacements for $i \in J_k^y$ and the specific case $i = 0$, and the small boxes for the rest.

so assumption 3 is satisfied too. Assumption 4 and 5 are satisfied using event $\hat{\mathcal{F}}_k$. Finally, the fact that $\operatorname{Supp} \mathcal{N}_i^y \subseteq R_H(\lfloor \mathcal{X}_i^y \rfloor)$ can be shown in the same way as (5.18) in the proof of Proposition 5.9, using (5.35), (5.36) and $\hat{\mathcal{F}}_k$. At the end of the day, Proposition 5.9 ensures that for indices i such that $\lfloor \mathcal{X}_i^y \rfloor$ is (H_k, r)-threatened, we have

$$\pi_1(\mathcal{X}_{i+1}^y) - \pi_1(\mathcal{X}_i^y) \leqslant \left(v_+ - \frac{\delta}{2r}\right) rH_k. \tag{5.37}$$

Denote by J_k^y the set of indices $i \in [\![1, L_k/L_{k_5+1} - 1]\!]$ such that $\lfloor \mathcal{X}_i^y \rfloor_{H_k}$ is (H_k, r)-threatened. By Definition 5.11, we have the inclusion of events

$$\hat{\mathcal{H}}_k \subseteq \bigcap_{y \in I_{L_k^2}} \left\{ |J_k^y| \geqslant \frac{L_k}{2L_{k_5+1}} - 1 \right\}. \tag{5.38}$$

Therefore,

$$\begin{aligned}
&\pi_1\left(\mathcal{X}_{L_k/L_{k_5+1}}^y\right) - \pi_1(y) \\
&= \pi_1(\mathcal{X}_1^y) - \pi_1(y) + \sum_{i=1}^{L_k/L_{k_5+1}-1} \left(\pi_1(\mathcal{X}_{i+1}^y) - \pi_1(\mathcal{X}_i^y)\right) \\
&\leqslant \beta r H_k + \sum_{i \in J_k^y} \left(\pi_1(\mathcal{X}_{i+1}^y) - \pi_1(\mathcal{X}_i^y)\right) + \sum_{i \notin J_k^y} \left(\pi_1(\mathcal{X}_{i+1}^y) - \pi_1(\mathcal{X}_i^y)\right) && \text{using } D_{rH_k}^y \\
&\leqslant \beta r H_k + |J_k^y| \left(v_+ - \frac{\delta}{2r}\right) rH_k + \left(\frac{L_k}{L_{k_5+1}} - 1 - |J_k^y|\right)\left(v_+ + \frac{3\eta}{2}\right) rH_k && \text{using (5.37) and (5.34)} \\
&\leqslant \beta r H_k + v_+ L_k^2 + \frac{3\eta}{2} L_k^2 - |J_k^y| \left(\frac{\delta}{2r} + \frac{3\eta}{2}\right) rH_k \\
&= \left(v_+ - \frac{\eta}{4} + \left(\frac{7\eta}{2} + \beta\right) \frac{L_{k_5+1}}{L_k}\right) L_k^2 && \text{by (5.38), } r = l_{k_5},\ \eta = \tfrac{\delta}{4 l_{k_5}} \\
&< \left(v_+ - \frac{\eta}{6}\right) L_k^2 && \text{using } k > k_5 + 1 \text{ and (5.23).}
\end{aligned}$$

See Figure 11 for an illustration of the above bounds. In the end, we do have (5.33), which is true for any $y \in I_{L_k^2}$, so

$$p_{L_k^2}\left(v_+ - \frac{\eta}{6}\right) = \mathbb{P}\left(\exists\, y \in I_{L_k^2},\, V_{L_k^2}^y \geqslant v_+ - \frac{\eta}{6}\right) \leqslant \mathbb{P}(\hat{\mathcal{A}}_k^c) + \mathbb{P}(\hat{\mathcal{F}}_k^c) + \mathbb{P}(\hat{\mathcal{G}}_k^c) + \mathbb{P}(\hat{\mathcal{H}}_k^c) \xrightarrow[k \to \infty]{} 0,$$

using (5.29), (5.30), (5.31) and (5.32). Therefore $\liminf_{H \to \infty} p_H(v_+ - \eta/6) = 0$, where $v_+ - \eta/6 < v_+$. This contradicts the definition of v_+, therefore, $v_- = v_+$.

6 Towards a complete LLN

The next step of our work would be to prove a LLN for the random walk defined in Section 2.2, as is expressed in the following conjecture.

Conjecture 1 (LLN)**.** *There exists* $\xi \in \mathbb{R}^2$ *such that*

$$\mathbb{P}\text{-almost surely,} \quad \frac{X_n}{n} \xrightarrow[n\to\infty]{} \xi.$$

With Theorem 2.4, we have a result that is weaker - albeit very interesting both in itself and in the methods used to prove it. Indeed, when assuming conjecture 1, we immediately get Theorem 2.4 with $\chi = \frac{\xi}{\|\xi\|}$. However, going from Theorem 2.4 to an actual LLN is not trivial at all. It is actually sufficient to show that $\frac{\tau_n}{n}$ converges almost surely to derive the LLN from Theorem 2.5. In other words, what we are missing at this point is the understanding of the temporal behavior of X.

This is *a priori* a hard question, because the environment from the point of view of the particle may not behave very nicely under our assumptions. We used renormalization methods to get around this issue, but it is unclear to which events describing the temporal behaviour of X we could apply a renormalization method.

7 Applications

In this section, we give examples of environments that satisfy the assumptions introduced in Section 2.1. These are taken from classical 1D dynamic or 2D static models for which we can control the vertical dependencies (provided that for a 1D dynamic model, the vertical coordinate is time).

Oftentimes, a static environment $\mu \in \Omega_1$ is constructed using a *background environment*, namely a random partition $\mathcal{P}$ of $\mathbb{Z}^2$ into sets $(O_i)_i$, and allocating to all the points $x \in O_i$ a common fixed value $\mu(x) = (p_1^{(i)}, \dots, p_4^{(i)}) \in S$. Typically the number of sets in partition $\mathcal{P}$ is finite, often with simply two sets. This construction ensures that μ is a deterministic function of $\mathcal{P}$, so that translation invariance and decoupling for the background environment implies the same for μ. As for the drift assumption, it suffices to demand that there exist $\varepsilon > 0$ such that $p_4^{(i)} \geqslant 1/2 + \varepsilon$ for every i. All the examples in this section fall under this framework. In the rest of this section, the subscript b will be used to indicate that we are working with the background environment.

7.1 One-dimensional dynamic environments

In [BHT20] are presented several models of 1D dynamic environments that have at most polynomial time correlations. More precisely, let $I \subseteq \mathbb{N}$. A one-dimensional dynamic environment is a random variable on a certain probability space $(\Omega_b, \mathcal{T}_b, \mathbb{P}_b)$ given by $\eta : (y,t) \in \mathbb{Z} \times \mathbb{R}_+ \mapsto \eta_t(y) \in I$ and taking values in $\mathcal{D}(\mathbb{R}_+, I^{\mathbb{Z}})$, the space of càdlàg functions from $\mathbb{R}_+$ to $I^{\mathbb{Z}}$. The state of environment η at time t and site y is described by $\eta_t(y)$. We assume that η is translation-invariant, that is

$$\begin{array}{c}\text{for every } (z,s) \in \mathbb{Z} \times \mathbb{R}_+,\ (\eta_t(y))_{(y,t)\in\mathbb{Z}\times\mathbb{R}_+} \text{ and} \\ (\eta_{s+t}(z+y))_{(y,t)\in\mathbb{Z}\times\mathbb{R}_+} \text{ have the same law under } \mathbb{P}_b.\end{array} \tag{7.1}$$

We also assume the following time-decoupling condition. There exists $\alpha > 0$ such that for every $A > 0$, there exists $c_{21} = c_{21}(A) > 0$ such that for every $h > 0$, for every pair of boxes B_1 and B_2 with maximal side lengths Ah that are h-separated, for all pairs of $[0,1]$-valued functions f_1 and f_2 on $\mathcal{D}(\mathbb{R}_+, I^{\mathbb{Z}})$ such that $f_1(\eta)$ is $\sigma(\eta|_{B_1})$-measurable and $f_2(\eta)$ is $\sigma(\eta|_{B_2})$-measurable,

$$\mathrm{Cov}_b\left(f_1(\eta), f_2(\eta)\right) \leqslant c_{21} h^{-\alpha}, \tag{7.2}$$

where Cov_b denotes the covariance with respect to $\mathbb{P}_b$.

This model consists of our background environment in the sense that it partitions $\mathbb{Z}^2$ into sets given by $\mathcal{O}_i = \{x = (y,t) \in \mathbb{Z}^2, \eta_t(y) = i\}$ for every $i \in I$. Assumption (7.2) implies the decoupling property we are after using the right choice of A and provided that α is large enough.

Examples of environments satisfying (7.1) and (7.2) are given in [BHT20]: the contact process, Markov processes with a positive spectral gap, the East model and independent renewal chains.

Mind that our contribution for random walks in those environments is quite different from what is done in [BHT20], even if we consider the random walks from the 1D dynamic setup as evolving in $\mathbb{R}^2$ by seeing time as a second spatial coordinate. For instance, these never go downwards. Another difference is that in [BHT20], in order to know where to jump, random walks are allowed to look at the environment not only where they are but in a horizontal interval of $\mathbb{R}^2$.

7.2 Boolean percolation

In [ATT18], the authors show a decoupling property for the Boolean percolation process in $\mathbb{R}^2$, a model first introduced in [Gil61]. Here is a brief account of what this model consists of and how it can be used in the framework of this paper.

Heuristically, Boolean percolation can be defined using a Poisson point process of intensity $\lambda > 0$ in $\mathbb{R}^2$, and allocating independently to each point in this point process a ball of random radius, sampled from a common distribution ν in $\mathbb{R}_+$. One way to make this more rigorous is that chosen in [ATT18].

For a subset $\eta \in \mathbb{R}^2 \times \mathbb{R}_+$, let

$$\mathcal{O}(\eta) = \bigcup_{(x,z)\in\eta} B(x,z),$$

where $B(x,z)$ is the Euclidean open ball of center x and radius z.

Let $\lambda > 0$ and ν be a probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. We assume that ν satisfies the following moment condition: there exists $\alpha > 0$ such that

$$\int_0^\infty z^{2+\alpha}\,\mathrm{d}\nu(z) = c_{22} < \infty. \tag{7.3}$$

This common assumption implies, using Markov's inequality, that the radii of our Boolean percolation have tails that decrease with a polynomial rate of exponent $\alpha + 2$.

Let η be a Poisson point process in $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity $\lambda\,\mathrm{d}x \otimes \mathrm{d}\nu(z)$, where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}^2$. Let $\mathbb{P}_b$ denote the law of this random variable. $\mathbb{E}_b$ and Cov_b denote the associated expectation and covariance.

Random variable $\mathcal{O} = \mathcal{O}(\eta)$ is called the Poisson-Boolean percolation of intensity λ and radius law ν. For every site $x \in \mathbb{Z}^2$, we say that x is occupied if $x \in \mathcal{O}$. Otherwise, we say that x is vacant. This model consists of our background environment in the sense that it partitions $\mathbb{Z}^2$ into two sets: the occupied sites and the vacant sites.

Let us now state a decoupling property for this environment. Proposition 7.1 gives a stronger property than the decoupling property we want to get, using translation invariance, the right choice of κ and provided that α is large enough. For $r > 0$, let $B^\infty(r) = [-r,r]^2$.

Proposition 7.1 ([ATT18], Proposition 2.2)**.** *Recall (7.3). For every $\kappa > 0$, there exists $c_{23} = c_{23}(\lambda,\nu,\kappa) > 0$ such that for all $r \geqslant 1$ and for all pairs of functions $f_1, f_2 : \mathcal{P}(\mathbb{R}^2) \to [-1,1]$ such that $f_1(\mathcal{O})$ is $\sigma(\mathcal{O} \cap B^\infty(r))$-measurable and $f_2(\mathcal{O})$ is $\sigma(\mathcal{O} \cap B^\infty(r(1+\kappa))^c)$-measurable, we have*

$$\mathrm{Cov}_b(f_1(\mathcal{O}), f_2(\mathcal{O})) \leqslant c_{23}\, r^{-\alpha}.$$

7.3 Gaussian fields

In [BHKT23] (Section 6.1), the authors introduce a background environment on $\mathbb{Z}^2$ using Gaussian fields. This environment satisfies a decoupling assumption that is stronger than ours. Here is a brief account of what we need from [BHKT23] in our framework.

Let $q : \mathbb{Z}^2 \to \mathbb{R}_+$ a non-zero function such that

$$\forall (x_1,x_2) \in \mathbb{Z}^2,\ q(x_1,x_2) = q(-x_1,x_2). \tag{7.4}$$

We also assume that there exists $\lambda > 2$ and $c_{24} > 0$ such that

$$\forall x \in \mathbb{Z}^2 \setminus \{0\},\ q(x) \leqslant c_{24}|x|^{-\lambda}. \tag{7.5}$$

We also consider a family $(W_x)_{x\in\mathbb{Z}^2}$ of i.i.d. standard normal random variables and we define the Gaussian field $(g_x)_{x\in\mathbb{Z}^2}$ by setting

$$g_x = \sum_{y\in\mathbb{Z}^2} q(x-y)\,W_y.$$

The background environment we are interested in is given by $(\eta_x)_{x\in\mathbb{Z}^2}$ where η_x is the sign of g_x (that is, $\eta_x \in \{\pm 1\}$). By construction $(\eta_x)_{x\in\mathbb{Z}^2}$ is translation-invariant. It remains to check that it satisfies our decoupling assumption. The authors of [BHKT23] show the stronger property that follows.

Proposition 7.2 ([BHKT23], Lemma 6.2)**.** *Recall (7.4) and (7.5). There exists $c_{25} > 0$ such that for every integer $r \geqslant 2$ and every box $C = [a, a+w] \times [b, b+h] \subseteq \mathbb{Z}^2$ with $w, h \geqslant 1$, there exists a coupling between η and a field $\eta^{C,r}$ such that*

$$\mathbb{P}_b(\eta \neq \eta^{C,r}) \leqslant c_{25}(wh + (w+h)r + r^2)\, r^{-\lambda+3/2}$$

and, if $A \subseteq C$ and $B \subseteq \mathbb{Z}^2$ satisfy $d(A, B) > r$, then $\eta^{C,r}|_A$ and $\eta^{C,r}|_B$ are independent.

This decoupling property implies that if B_1 and B_2 are two boxes with maximum side lengths $2(2\beta+1)h$ that are h-separated, and if f_1 and f_2 are two measurable functions on $\{\pm 1\}^{\mathbb{Z}^2}$ such that $f_1(\eta)$ is $\sigma(\eta|_{B_1})$-measurable and $f_2(\eta)$ is $\sigma(\eta|_{B_2})$-measurable,

$$\mathrm{Cov}_b(f_1(\eta), f_2(\eta)) \leqslant c\, h^{-\alpha}$$

for a certain constant $c > 0$, where $\alpha = -(2 - \lambda + 3/2)$. Therefore we have $\alpha > 12$ provided that $\lambda > 31/2$.

7.4 Factors of i.i.d. with light-tail finite radii

Let $Y = (Y_x)_{x \in \mathbb{Z}^2}$ be a family of i.i.d. random variables in $[0,1]$ with law $\mathbb{P}_b$ (as usual, $\mathbb{E}_b$ and Cov_b will denote the associated expectation and covariance). Let $\eta : \mathbb{Z}^2 \to \{0,1\}$ be a random variable. We say that η is a factor of Y with finite radius if there exist two measurable functions $\phi : [0,1]^{\mathbb{Z}^2} \to \{0,1\}$ and $\rho : [0,1]^{\mathbb{Z}^2} \to \mathbb{R}_+$ such that:

- For all $x \in \mathbb{Z}^2$, $\eta(x) = \phi(\theta^x Y)$, where $\theta^x \mathbf{y} = (y_{x+v})_{v \in \mathbb{Z}^2}$ for every $\mathbf{y} = (y_v) \in [0,1]^{\mathbb{Z}^2}$;
- For $\mathbb{P}_b$-almost all $\mathbf{y}, \mathbf{y}' \in [0,1]^{\mathbb{Z}^2}$ that coincide outside of $B(o, \rho(\mathbf{y}))$, $\phi(\mathbf{y})$ and $\phi(\mathbf{y}')$ are equal at o.

This implies that we only need to look at Y in a ball of radius $\rho(Y)$ around a site $x \in \mathbb{Z}^2$ to determine $\eta(x)$. Random variable $R = \rho(Y)$ is called the radius of η.

Such a process η can be seen as a background environment. It is translation-invariant by construction. In order to show a decoupling property for η, we need to make an additional assumption on the radius: we assume that there exist $\alpha > 0$ and $c_{26} > 0$ such that for all $r > 0$,

$$\mathbb{P}_b(R > r) \leqslant c_{26}\, r^{-\alpha}. \tag{7.6}$$

Proposition 7.3. *There exists $c_{27} > 0$ such that for every $h > 0$, for every pair of h-separated boxes B_1 and B_2 of $\mathbb{Z}^2$, for all pairs of $[0,1]$-valued functions f_1 and f_2 on $\{0,1\}^{\mathbb{Z}^2}$ such that $f_1(\eta)$ is $\sigma(\eta|_{B_1})$-measurable and $f_2(\eta)$ is $\sigma(\eta|_{B_2})$-measurable,*

$$\mathrm{Cov}_b\left(f_1(\eta), f_2(\eta)\right) \leqslant c_{27}\, h^{-\alpha}. \tag{7.7}$$

Proof. Let us define, for $i \in \{1,2\}$, a box $\tilde{B}_i = (B_i + [-h/3, h/3]^2) \cap \mathbb{Z}^2$ and a function $\psi_i : [0,1]^{\tilde{B}_i} \to [0,1]^{\mathbb{Z}^2}$ by setting, for $\mathbf{y} = (y_v)_{v \in \tilde{B}_i} \in [0,1]^{\tilde{B}_i}$ and $x \in \mathbb{Z}^2$, $\psi_{\tilde{B}_i}(y)_x = y_x \mathbf{1}_{\tilde{B}_i}(x)$. Also, we define $g_i = f_i \circ \phi \circ \psi_{\tilde{B}_i}$, which takes arguments in $[0,1]^{\tilde{B}_i}$. The crucial idea is that if $R \leqslant h/3$, we have $f_i(\eta) = g_i(Y_{\tilde{B}_i})$, where $Y_{\tilde{B}_i} = (Y_v)_{v \in \tilde{B}_i}$. Now, remark that $Y_{\tilde{B}_1}$ and $Y_{\tilde{B}_2}$ are independent, since $\tilde{B}_1$ and $\tilde{B}_2$ are disjoint. Therefore,

$$\begin{aligned}
\mathbb{E}_b[f_1(\eta)\, f_2(\eta)] &\leqslant \mathbb{E}_b[g_1(Y_{\tilde{B}_1})\, g_2(Y_{\tilde{B}_2})\, \mathbf{1}_{R \leqslant h/3}] + \mathbb{P}_b(R > h/3) \\
&\leqslant \mathbb{E}_b[g_1(Y_{\tilde{B}_1})]\, \mathbb{E}_b[g_2(Y_{\tilde{B}_2})] + c_{26}\, h^{-\alpha} \\
&\leqslant (\mathbb{E}_b[f_1(\eta)] + c_{26}\, h^{-\alpha})\, (\mathbb{E}_b[f_2(\eta)] + c_{26}\, h^{-\alpha}) + c_{26}\, h^{-\alpha} \\
&\leqslant \mathbb{E}_b[f_1(\eta)]\, \mathbb{E}_b[f_2(\eta)] + c\, h^{-\alpha}.
\end{aligned}$$

□

Taking a closer look at this proof, it is clear that in fact the decoupling property holds not only for boxes that are vertically separated, but for any two sets of $\mathbb{Z}^2$ between which the distance is at least h.

In the end we do recover the decoupling property that we want, provided that $\alpha > 12$.

Remark 7.4. To make things simple, we used a factor of i.i.d. that takes values in $\{0,1\}$, but this does not affect the proof of the decoupling property. We could work with a much bigger set I and use the background environment given by $\mathcal{O}_i = \{x \in \mathbb{Z}^2,\, \eta(x) = i\}$ for $i \in I$. Furthermore, one way to construct a random environment directly (that is, without using an intermediary background environment as we have been doing so far), would be to take for I the set S defined in Section 2.1. In that case, if we add the drift condition, $\mu = \phi(Y)$ is a random environment satisfying our assumptions.

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Appendix

Proof of Proposition 3.3. Let $y \in \mathbb{Z}^2$ and $\Gamma \in \mathcal{H}$. We want to show that for every $k \in \mathbb{N}^*$ and $f_1, \dots, f_k$ measurable non-negative functions on $[0,1]$, we have

$$\mathbb{E}\left[f_1(U_1^{y,\Gamma}) \cdots f_k(U_k^{y,\Gamma})\right] = \int_0^1 f_1(u)\,\mathrm{d}u \,\cdots\, \int_0^1 f_k(u)\,\mathrm{d}u. \tag{7.8}$$

We show this by induction on k. The case $k=1$ simply follows from the fact that $U_1^{y,\Gamma} = U(y, \Gamma(y)+1)$. Assume (7.8) is true for a fixed $k \in \mathbb{N}^*$. Let $f_1, \dots, f_{k+1}$ be measurable non-negative functions on $[0,1]$. Set $n_0 = 1$ and $x_0 = y$. We have

$$\begin{aligned}
&\mathbb{E}\left[f_1(U_1^{y,\Gamma}) \cdots f_{k+1}(U_{k+1}^{y,\Gamma})\right] \\
&= \sum_{\substack{x_1,\dots,x_k \in \mathbb{Z}^2 \\ n_1,\dots,n_k \in \mathbb{N}^*}} \mathbb{E}\left[f_1(U(x_0,\Gamma(x_0)+n_0)) \cdots f_{k+1}(U(x_k,\Gamma(x_k)+n_k)) \prod_{j=1}^{k} \mathbf{1}_{X_j^{y,\Gamma}=x_j}\, \mathbf{1}_{N_{j+1}^{y,\Gamma}(x_j)=n_j}\right].
\end{aligned}$$

Now in each term of this sum, the variable $f_{k+1}(U(x_k,\Gamma(x_k)+n_k))$ is independent from all the other variables that appear, for those are all measurable with respect to μ and

$$\{U(x_0,\Gamma(x_0)+n_0), \dots, U(x_{k-1},\Gamma(x_{k-1})+n_{k-1})\},$$

where, for every $j \in [\![0,k-1]\!]$, either $x_k \neq x_j$ or $\Gamma(x_k)+n_k \neq \Gamma(x_j)+n_j$. Now for any $x_k \in \mathbb{Z}^2$ and $n_k \in \mathbb{N}^*$, $\mathbb{E}[f_{k+1}(U(x_k,\Gamma(x_k)+n_k))] = \int_0^1 f_{k+1}(u)\,\mathrm{d}u$. Therefore

$$\mathbb{E}\left[f_1(U_1^{y,\Gamma}) \cdots f_{k+1}(U_{k+1}^{y,\Gamma})\right] = \mathbb{E}\left[f_1(U_1^{y,\Gamma}) \cdots f_k(U_k^{y,\Gamma})\right] \int_0^1 f_{k+1}(u)\,\mathrm{d}u,$$

and then using the induction assumption allows us to conclude. The exact same arguments work when replacing $\mathbb{P}$ by $\mathbb{P}^\mu$. $\square$

Proof of Proposition 3.13. We write the proof for $y = o$ and $\Gamma = \mathbf{0}$ for the sake of simplicity (initial conditions play no part in our reasoning). Let $\mu \in \mathcal{A}$. Recall Definition 3.1 for the lower-bound random walk, as well as (3.4). Using increment inequality (3.3), we have

$$\begin{aligned}
\mathbb{P}^\mu(E_H^c) &= \mathbb{P}^\mu(\exists\, n \in \mathbb{N}^*,\ \pi_2(X_n) < -H) \\
&\leqslant \mathbb{P}^\mu(\exists\, n \in \mathbb{N}^*,\ \hat{X}_n < -H)
\end{aligned}$$

$$\leqslant \mathbb{P}^\mu(\hat{\tau}_{-H} < \infty).$$

Now, under $\mathbb{P}^\mu$, $\hat{X}$ is a standard biased 1D random walk with probability $1/2+\varepsilon$ of going up and $1/2-\varepsilon$ of going down. Applying the gambler's ruin estimate, we get

$$\mathbb{P}^\mu\left(\hat{\tau}_{-H} < \infty\right) = \left(\frac{1/2-\varepsilon}{1/2+\varepsilon}\right)^H,$$

hence the result for $\mathbb{P}^\mu$, and integrate to get the result for $\mathbb{P}$. For the case $H=0$, we can write

$$\begin{aligned}
\mathbb{P}^\mu(E_0) &\geqslant \mathbb{P}^\mu\left(\{X_1 = e_2\} \cap E_1^{e_2,\mathbf{1}_{\{o\}}}\right) \\
&= \mathbb{P}^\mu(X_1 = e_2)\,\mathbb{P}^\mu\left(E_1^{e_2,\mathbf{1}_{\{o\}}}\right) && \text{by Proposition 3.3} \\
&\geqslant \left(\frac{1}{2}+\varepsilon\right)\mathbb{P}^\mu\left(E_1^{e_2,\mathbf{1}_{\{o\}}}\right).
\end{aligned}$$

Now we can use the gambler's ruin estimate again: $\mathbb{P}^\mu\left(E_1^{e_2,\mathbf{1}_{\{o\}}}\right) \geqslant \mathbb{P}^\mu\left(\hat{\tau}_{-1}^{e_2,\mathbf{1}_{\{o\}}} = +\infty\right) = \frac{2\varepsilon}{1/2+\varepsilon}$. This yields the result for $\mathbb{P}^\mu$, and we integrate to get the result for $\mathbb{P}$. □

Proof of Proposition 3.14. Let $\mu \in \mathcal{A}$. Again, we only show the case $y = o$ and $\Gamma = \mathbf{0}$ for the sake of conciseness. Now, in order to study the horizontal behavior of X, let us define, in the same fashion as in Definition 3.1, a lazy biased 1D random walk $\tilde{X}$ coupled to X in the following way:

$$\begin{cases} \tilde{X}_0 = 0; \\ \forall n \in \mathbb{N},\ \tilde{X}_{n+1} = \tilde{X}_n + \mathbf{1}_{U_{n+1} \leqslant 1/2-\varepsilon}. \end{cases}$$

We can check that this random walk satisfies, for every $n \in \mathbb{N}$,

$$\begin{array}{l} \mathbb{P}^\mu\left(\tilde{X}_{n+1} = x+1 \,|\, \tilde{X}_n = x\right) = \frac{1}{2} - \varepsilon; \\ \mathbb{P}^\mu\left(\tilde{X}_{n+1} = x \,|\, \tilde{X}_n = x\right) = \frac{1}{2} + \varepsilon; \\ \left\{\tilde{X}_{n+1} = \tilde{X}_n + 1\right\} \supseteq \left\{X_{n+1} = X_n + e_1\right\}. \end{array} \tag{7.9}$$

We also associate a stopping time $\tilde{\tau}_H$ for every $H \in \mathbb{N}^*$, in the same way as $\hat{\tau}_H$ was associated to $\hat{X}$. The mean speed of this new random walk is $1/2-\varepsilon$, so we can obtain a ballisticity property similar to that of Proposition 3.11 where we replace 2ε by $1/2-\varepsilon$. More precisely, for any $\xi > 0$, there exists a constant $c_{28} = c_{28}(\xi) > 0$ such that for every $n \in \mathbb{N}$, we have

$$\mathbb{P}^\mu\left(\left|\tilde{X}_n - \tilde{X}_0 - \left(\frac{1}{2}-\varepsilon\right)n\right| \geqslant \xi n\right) \leqslant c_{28}^{-1} e^{-c_{28} n}. \tag{7.10}$$

We could define another random walk for when X goes left, but using only $\tilde{X}$ is sufficient by symmetry of the problem. Actually, if we fix a parameter $\zeta > 0$ to be chosen later, we have

$$\mathbb{P}^\mu\left(D_H^c\right) \leqslant 2\mathbb{P}^\mu(\tilde{\tau}_{\beta H} \leqslant \hat{\tau}_H) \leqslant 2\mathbb{P}^\mu\left(\hat{\tau}_H \geqslant \lceil H/\zeta \rceil\right) + 2\mathbb{P}^\mu\left(\tilde{\tau}_{\beta H} \leqslant \lceil H/\zeta \rceil\right).$$

For the first term above, we use Proposition 3.11 and write, for $\zeta < 2\varepsilon$,

$$\begin{aligned}
\mathbb{P}^\mu\left(\hat{\tau}_H \geqslant \lceil H/\zeta \rceil\right) &\leqslant \mathbb{P}^\mu(\hat{X}_{\lceil H/\zeta \rceil} \leqslant H) \\
&\leqslant c_3^{-1} e^{-c_3 \lceil H/\zeta \rceil} && \text{where } c_3 = c_3(2\varepsilon - \zeta) \\
&\leqslant c^{-1} e^{-cH},
\end{aligned}$$

for a certain constant $c = c(\zeta) > 0$. In the same way, using (7.10), $\mathbb{P}^\mu\left(\tilde{\tau}_{\beta H} \leqslant \lceil H/\zeta \rceil\right) \leqslant c^{-1} e^{-cH}$ when $\beta > \frac{1/2-\varepsilon}{\zeta}$, that is $\zeta > \frac{1/2-\varepsilon}{\beta}$. This means that our proof works whenever $\zeta \in \left(\frac{1/2-\varepsilon}{\beta}, 2\varepsilon\right)$, which is non-empty because of (2.2). Therefore, estimate (3.18) is shown. □

Proof of Lemma 3.19. Let $\mu \in \mathcal{A}$. By Corollary 3.4, we can assume that $y = o$ and $\Gamma = \mathbf{0}$. We fix $a \in \mathbb{N}$ and $k = k(a)$ an integer to be chosen later. Let us consider hitting times $\hat{\tau}_{jk}$ for $j \in \mathbb{N}$ (recall (3.4)). Let us consider the following events (with the convention that if $\hat{\tau}_{jk} = \infty$, we just take the empty set):

$$A_{j,k} = \left\{ \left(\hat{X}_{\hat{\tau}_{jk}+n} \right)_{n\in\mathbb{N}} \text{ does not return below } jk \right\} = \bigcap_{n\in\mathbb{N}} \left\{ \hat{X}_{\hat{\tau}_{jk}+n} \geqslant jk \right\};$$

$$\tilde{A}_{j,k} = \left\{ \left(\hat{X}_{\hat{\tau}_{jk}+n} \right)_{n\in\mathbb{N}} \text{ does not return below } jk \text{ within } k \text{ steps} \right\} = \bigcap_{n=0}^{k} \left\{ \hat{X}_{\hat{\tau}_{jk}+n} \geqslant jk \right\}.$$

Remark that, since $\hat{X}$ jumps at range 1, for every $j \in \mathbb{N}$ we have $\hat{\tau}_{(j+1)k} \geqslant \hat{\tau}_{jk} + k$. Therefore, using Corollary 3.6 along with an induction argument, the $(\tilde{A}_{j,k})_{j\in\mathbb{N}}$ are independent events. Moreover they all have the same probability

$$p_k = \mathbb{P}^\mu \left(\tilde{A}_{j,k} \right) \geqslant \mathbb{P}^\mu \left(A_{j,k} \right) = \mathbb{P}^\mu(A_{0,k}) \geqslant 2\varepsilon,$$

using Proposition 3.3 and the same line of reasoning as in the end of the proof of Proposition 3.13. Therefore the random variable given by $G_k = \inf\{j \in \mathbb{N},\ \tilde{A}_{j,k} \text{ occurs}\}$ is a geometric variable of parameter p_k. Now, we have

$$\begin{aligned}
\mathbb{P}^\mu(\Theta(\hat{X}) > a) &\leqslant \mathbb{P}^\mu(\forall j \leqslant a/k,\ jk \text{ is not a cut point of } \hat{X}) \\
&\leqslant \mathbb{P}^\mu \left(\cap_{j\leqslant a/k} A_{j,k}^c \right) \\
&\leqslant \mathbb{P}^\mu \left(\cap_{j\leqslant a/k} \tilde{A}_{j,k}^c \right) + \mathbb{P}^\mu \left(\cup_{j\leqslant a/k} A_{j,k}^c \cap \tilde{A}_{j,k} \right). \qquad (7.11)
\end{aligned}$$

The first term in the last line above can be bounded from above by

$$\mathbb{P}^\mu(G_k > \lfloor a/k \rfloor) = (1 - p_k)^{\lfloor a/k \rfloor + 1} \leqslant (1 - 2\varepsilon)^{a/k}. \qquad (7.12)$$

As for the second term, we use a union bound and remark that, for any $j \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{P}^\mu(A_{j,k}^c \cap \tilde{A}_{j,k}) &= \mathbb{P}^\mu \left((\hat{X}_{\hat{\tau}_{jk}+n})_{n\in\mathbb{N}} \text{ returns below } jk \text{ but not within } k \text{ steps} \right) \\
&= \sum_{t\in\mathbb{N}} \mathbb{P}^\mu \left((\hat{X}_{t+n})_{n\in\mathbb{N}} \text{ returns below } \hat{X}_t \text{ but not within } k \text{ steps},\ \hat{\tau}_{jk} = t \right).
\end{aligned}$$

In each term of the sum above, the two events between parentheses are independent, using Corollary 3.6. Now the probability of the first event actually does not depend on t, using Proposition 3.3, so

$$\begin{aligned}
\mathbb{P}^\mu(A_{j,k}^c \cap \tilde{A}_{j,k}) &= \mathbb{P}^\mu(\hat{X} \text{ returns below } 0 \text{ but not within } k \text{ steps}) \\
&\leqslant \mathbb{P}^\mu(\hat{X} \text{ returns below } 0 \text{ but not within } k \text{ steps},\ \hat{X}_k \geqslant \varepsilon k) + \mathbb{P}^\mu(\hat{X}_k < \varepsilon k).
\end{aligned}$$

Using Proposition 3.11, $\mathbb{P}^\mu(\hat{X}_k < \varepsilon k) \leqslant c_3(\varepsilon)^{-1} e^{-c_3(\varepsilon)k}$. To study the other term, we remark that it is less than $\mathbb{P}^\mu \left(\hat{\tau}_{-\lfloor \varepsilon k \rfloor}^{X_k, N_k} < \infty \right)$. Now, we can apply Proposition 3.7 to estimate this, considering that for any $\mu \in \mathcal{A}$, by the gambler's ruin estimate, we have

$$\mathbb{P}^\mu \left(\hat{\tau}_{-\lfloor \varepsilon k \rfloor} < \infty \right) = \left(\frac{1/2 - \varepsilon}{1/2 + \varepsilon} \right)^{\lfloor \varepsilon k \rfloor} \leqslant e^{-ck},$$

where $c > 0$ is uniform in μ. Therefore this bound also holds for $\mathbb{P}^\mu \left(\hat{\tau}_{-\lfloor \varepsilon k \rfloor}^{X_k, N_k} < \infty \right)$. At the end of the day, combining this with (7.11) and (7.12), we get the desired result by choosing $k = k(a) = \lfloor a^{1/2} \rfloor$ and adjusting c_7 properly. To get the same estimate with $\mathbb{P}$, we integrate over μ. $\square$

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