Quantum inverse scattering for time-decaying harmonic oscillators

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Abstract

Different from the usual harmonic oscillator, the time-decaying harmonic oscillator accelerates particles and generates scattering states. We study one of the multidimensional inverse scatterings in this two-body quantum system perturbed by short-range interaction potentials that have a bounded part and a locally singular part. Applying the Enss–Weder time-dependent method, we prove that the scattering operator determines the interaction potentials uniquely.

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1 Introduction

Let $n \geqslant 2$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathrm{i}p = (\partial_{x_1}, \ldots, \partial_{x_n})$ with $\mathrm{i} = \sqrt{-1}$. In this paper, we consider the quantum system governed by the following time-dependent free Hamiltonian

$$H_0(t) = p^2/2 + k(t)x^2/2 \tag{1.1}$$

acting on $L^2(\mathbb{R}^n)$, where the time-decay coefficient of the harmonic term is

$$k(t) = \begin{cases} \omega^2 & \text{if} \quad |t| < r_0, \\ \sigma/t^2 & \text{if} \quad |t| \geqslant r_0. \end{cases} \tag{1.2}$$

for $0 < \sigma < 1/4$, $\omega > 0$ and $r_0 > 0$. For simplicity, we write

$$0 < \lambda = (1 - \sqrt{1 - 4\sigma})/2 < 1/2. \tag{1.3}$$

We now state the assumptions imposed on the external potentials as multiplication operators that perturb $H_0(t)$.

Assumption 1.1. *The potential function V is decomposed into a bounded part and a singular part,*

$$V(x) = V^{\mathrm{bdd}}(x) + V^{\mathrm{sing}}(x). \tag{1.4}$$

$V^{\mathrm{bdd}} \in L^\infty(\mathbb{R}^n)$ satisfies

$$|V^{\mathrm{bdd}}(x)| \lesssim \langle x\rangle^{-\rho} \tag{1.5}$$

almost everywhere, with $x \in \mathbb{R}^n$, $\rho > 1/(1-\lambda)$, $\langle\cdot\rangle = \sqrt{1+|\cdot|^2}$ and $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leqslant CB$. $V^{\mathrm{sing}} \in L^q(\mathbb{R}^n)$ is compactly supported, where q satisfies

$$\infty > q \begin{cases} = 2 & \text{if} \quad n \leqslant 3, \\ > n/2 & \text{if} \quad n \geqslant 4. \end{cases} \tag{1.6}$$

The part V^{sing} is well known to be p^2-bounded infinitesimally (Reed–Simon [25, Theorems X.15 and X.20]). We define the full Hamiltonian such that

$$H(t) = H_0(t) + V(x). \tag{1.7}$$

The Newton equation of classical mechanics

$$(\mathrm{d}^2/\mathrm{d}t^2)x(t) = -k(t)x(t) \tag{1.8}$$

has general solution $x(t) = c_1 t^{1-\lambda} + c_2 t^\lambda$ for $t \geqslant r_0$ and the classical trajectory of the free particle behaves like $x(t) = O(t^{1-\lambda})$ as $t \to \infty$. From this classical motion of the particle, Ishida–Kawamoto [11, Theorems 1 and 2] proved that the threshold between short- and long-range is $-1/(1-\lambda)$.

By virtue of Yajima [32, Theorem 6 and Remark (a)], the existence of the propagators uniquely generated by $H_0(t)$ and $H(t)$ is guaranteed under Assumption 1.1. We denote these propagators by $U_0(t,s)$ and $U(t,s)$, respectively. The wave operators

$$W^\pm = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0) \tag{1.9}$$

then exist by [11, Theorem 1] and the scattering operator is defined such that

$$S(V) = (W^+)^* W^-. \tag{1.10}$$

Remark 1.2. *We can also treat the following combination of V^{bdd}- and V^{sing}-type potentials $V = V_{\mathrm{sing}}$ that satisfy $V_{\mathrm{sing}} \in L^q(\mathbb{R}^n)$ with* (1.6)*, and $\langle x\rangle^\rho V_{\mathrm{sing}}(x)\langle p\rangle^{-2}$ is the bounded operator on $L^2(\mathbb{R}^n)$ for $\rho > 1/(1-\lambda)$. To prove Theorem 1.5 for this $V = V_{\mathrm{sing}}$, it suffices to modify slightly the proof of Lemma 3.2 (specifically,* (3.11)*,* (3.20)*, and* (3.32)*).*

Remark 1.3. *[11, Theorem 1] proves the existence of* (1.9) *only for $V = V^{\mathrm{bdd}}$. We can immediately prove the existence of* (1.9) *for $V = V^{\mathrm{bdd}} + V^{\mathrm{sing}}$ and $V = V_{\mathrm{sing}}$ using propagation estimates [11, Proposition 2] and the standard Cook-Kuroda method [26, Theorem XI.4].*

Remark 1.4. *If the constants ω, r_0 and λ satisfy the following relation*

$$\omega r_0 \tan \omega r_0 = -\lambda, \tag{1.11}$$

the ordinary differential equation (1.8) *has two fundamental solutions for all $t \in \mathbb{R}$, and by [20, Theorem 1.2] (see also [21, Lemma 3.1]), the $L^{p_1}L^{p_2}$-type estimates*

$$\|U_0(t,0)\phi\|_{L^{p_1}} \lesssim |t|^{-n(1-\lambda)(1/2-1/p_1)} \|\phi\|_{L^{p_2}} \tag{1.12}$$

holds for $\phi \in \mathscr{S}(\mathbb{R}^n)$, which is the rapid decreasing function space, where $p_1 \geqslant 2$ and p_2 is the Hölder conjugate of p_1. Note that we do not assume any relations among ω, r_0 and λ in this paper.

Applying the Enss–Weder time-dependent method [6], we prove the following theorem that claims that the scattering operator determines the potential functions uniquely.

Theorem 1.5. *Let V_1 and V_2 satisfy Assumption 1.1. If $S(V_1) = S(V_2)$, then $V_1 = V_2$ holds.*

Since the Enss–Weder time-dependent method was devised, many authors have applied it to establish the uniqueness of the interaction potentials for various quantum models. [1], [2], [3], [10], [22], [23], [27], and [31] investigated the models with external electric fields, whereas [8] and [24] studied repulsive Hamiltonians, and [9] and [13] studied fractional and relativistic Laplacians. [28], [29], and [30] applied the method to the non-linear Schrödinger equations and the Hartree-Fock equations.

The time-decaying harmonic oscillator has been an interesting topic for research in both mathematical and physical aspects. For the usual harmonic oscillator, there are no scattering states and all of its spectrum is covered by the infinite discrete pure points. However, if the term x^2 has a time-decay coefficient of a specified order, the situation changes completely. This time-decay coefficient accelerates the particles and generates the scattering states. From this perspective, [11] and [12] discussed whether the wave operators exist. [11] proved that $V(x) = O(|x|^{-\rho_{\mathrm{L}}})$ as $|x| \to \infty$ with $0 < \rho_{\mathrm{L}} \leqslant 1/(1-\lambda)$ has to belong to the long-range class and proposed Dollard-type modified wave operators. If $\sigma = 1/4$ in (1.2), the circumstances of scattering change considerably. [12] found that the classical trajectory has order $x(t) = \sqrt{t}\log t$ as $t \to \infty$ and clarified the threshold between short- and long-range. In contrast, [20] and [16] constructed the Strichartz estimates, and recent studies of non-linear analysis [14], [15], [16], [17], [18] and [19] have shown progress.

By the definition of $k(t)$, $H_0(t) \equiv H_0 = p^2/2 + \omega^2 x^2/2$ is a time-independent harmonic oscillator for $|t| < r_0$. Let us here state the well-known Mehler formula for H_0 [4, Theorem 12.63]; specifically, the time evolution for $0 < |t| < r_0$ is represented as

$$e^{-\mathrm{i}tH_0} = \mathscr{M}(\tan\omega t)\mathscr{D}(\sin\omega t)\mathscr{F}\mathscr{M}(\tan\omega t) \tag{1.13}$$

where $\mathscr{M}$ denotes multiplication and $\mathscr{D}$ dilation,

$$\mathscr{M}(t)\phi(x) = e^{\mathrm{i}x^2/(2t)}\phi(x), \tag{1.14}$$

$$\mathscr{D}(t)\phi(x) = (\mathrm{i}t)^{-n/2}\phi(x/t), \tag{1.15}$$

and $\mathscr{F}$ denotes the Fourier transform over $L^2(\mathbb{R}^n)$. A straightforward calculation yields

$$\mathscr{D}(\sin\omega t) = \mathrm{i}^{n/2}\mathscr{D}(\cos\omega t)\mathscr{D}(\tan\omega t), \tag{1.16}$$

$$\mathscr{M}(\tan\omega t)\mathscr{D}(\cos\omega t)\mathscr{M}(-\tan\omega t) = \mathscr{M}(-\cot\omega t)\mathscr{D}(\cos\omega t) \tag{1.17}$$

and

$$e^{-\mathrm{i}tH_0} = \mathrm{i}^{n/2}\mathscr{M}(-\cot\omega t)\mathscr{D}(\cos\omega t)e^{-\mathrm{i}\tan\omega t p^2/2}, \tag{1.18}$$

because

$$e^{-\mathrm{i}tp^2/2} = \mathscr{M}(t)\mathscr{D}(t)\mathscr{F}\mathscr{M}(t). \tag{1.19}$$

The formula (1.18) was derived originally in Ishida [8] for the repulsive Hamiltonian. On the other hand, $U_0(t,s)$ and $U(t,s)$ also have the convenient factorizations for $t,s \geqslant r_0$ or $t,s \leqslant -r_0$, that were proved by [11, Proposition 1]. We define

$$\tilde{U}_0(t) = e^{\mathrm{i}\lambda x^2/(2t)}e^{-\mathrm{i}\lambda\log tA}e^{-\mathrm{i}t^{1-2\lambda}p^2/(2(1-2\lambda))} \tag{1.20}$$

if $t \geqslant r_0$ and

$$\tilde{U}_0(t) = e^{\mathrm{i}\lambda x^2/(2t)}e^{-\mathrm{i}\lambda\log(-t)A}e^{\mathrm{i}(-t)^{1-2\lambda}p^2/(2(1-2\lambda))} \tag{1.21}$$

if $t \leqslant -r_0$, where $A = (p\cdot x + x\cdot p)/2$. Then

$$U_0(t,s) = \tilde{U}_0(t)\tilde{U}_0(s)^* \tag{1.22}$$

and

$$U(t,s) = e^{\mathrm{i}\lambda x^2/(2t)}e^{-\mathrm{i}\lambda\log|t|A}\tilde{U}(t,s)e^{\mathrm{i}\lambda\log|s|A}e^{-\mathrm{i}\lambda x^2/(2s)} \tag{1.23}$$

hold for $t,s \geqslant r_0$ or $t,s \leqslant -r_0$, where $\tilde{U}(t,s)$ is the propagator generated by

$$\tilde{H}(t) = p^2/(2|t|^{2\lambda}) + V(|t|^\lambda x). \tag{1.24}$$

We additionaly define

$$\tilde{U}_0(t) = e^{-itH_0} \tag{1.25}$$

if $|t| < r_0$. The following strong limits

$$\tilde{W}^\pm = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^*\tilde{U}_0(t) \tag{1.26}$$

exist because (1.9) exist and we define

$$\tilde{S}(V) = (\tilde{W}^{+})^{*}\tilde{W}^{-}. \tag{1.27}$$

Noting that $W^{\pm} = \tilde{W}^{\pm}\tilde{U}_0(s_{\pm})^{*}U_0(s_{\pm},0)$ for $s_+ > r_0$ and $s_- < -r_0$, we easily find that S and $\tilde{S}$ the relation

$$S(V) = U_0(s_+,0)^{*}\tilde{U}_0(s_{\pm})\tilde{S}(V)\tilde{U}_0(s_-)^{*}U_0(s_-,0), \tag{1.28}$$

and that $S(V_1) = S(V_2)$ is equivalent to $\tilde{S}(V_1) = \tilde{S}(V_2)$. To analyze the time-evolution by $U_0(t,0)$ for all $t \in \mathbb{R}$ directly is difficult but can be overcome by pursuing the evolution of $e^{-\mathrm{i}\tan\omega t p^2/2}$ if $|t| < r_0$ and $e^{\mp \mathrm{i}|t|^{1-2\lambda}p^2/(2(1-2\lambda))}$ if $|t| \geqslant r_0$. While the particles escape with order $x(t) = O(|t|^{1-\lambda})$ through the potential effects when $|t| \geqslant r_0$, the particles cannot scatter far away from the time-independent harmonic oscillator when $|t| < r_0$. We consequently reconstruct the potential functions from the time-independent harmonic oscillator (see the proofs of Theorems 2.1 and 3.1).

Throughout this paper, we use the following notation; $\|\cdot\|$ denotes the L^2-norm and operator norm on $L^2(\mathbb{R}^n)$, $(\cdot,\cdot)$ the scalar product of $L^2(\mathbb{R}^n)$, and $F(\cdots)$ the characteristic function of the set $\{\cdots\}$.

2 Bounded case

We first consider the instances for which $V^{\mathrm{sing}} \equiv 0$, that is, $V = V^{\mathrm{bdd}}$ and prove the following reconstruction formula.

Theorem 2.1. *Let $\Phi_0 \in \mathscr{S}(\mathbb{R}^n)$ such that $\mathscr{F}\Phi_0 \in C_0^{\infty}(\mathbb{R}^n)$. For $v \in \mathbb{R}^n$, its normalization is $\hat{v} = v/|v|$. Let $\Phi_v = e^{\mathrm{i}v\cdot x}\Phi_0$ and Ψ_v have the same properties. Then*

$$\lim_{|v|\to\infty} |v|(\mathrm{i}(\tilde{S}(V^{\mathrm{bdd}}) - 1)\Phi_v, \Psi_v) = \int_{-\infty}^{\infty} (V^{\mathrm{bdd}}(x + \hat{v}\omega t)\Phi_0, \Psi_0)\mathrm{d}t \tag{2.1}$$

holds.

We now prepare to prove Theorem 2.1. The following propagation estimates for the free evolution $e^{-\mathrm{i}tp^2/2}$ [5, Proposition 2.10] is very useful in some of our estimates.

Proposition 2.2. *Let M and M' be measurable subsets of $\mathbb{R}^n$ and $f \in C_0^{\infty}(\mathbb{R}^n)$ have* $\operatorname{supp} f \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leqslant \eta\}$ *for some $\eta > 0$. Then*

$$\|F(x \in M')e^{-\mathrm{i}tp^2/2}f(p)F(x \in M)\| \lesssim_{N,f} (1 + |t| + r)^{-N} \tag{2.2}$$

for $t \in \mathbb{R}$ and $N \in \mathbb{N}$, where $r = \operatorname{dist}(M', M) - \eta|t|$ and $\lesssim_{N,f}$ means that the constant depends on N and f.

The following Lemma is the key propagation estimate in this section.

Lemma 2.3. *Let Φ_v be as in Theorem 2.1. Then*

$$\int_{-\infty}^{\infty} \|V^{\mathrm{bdd}}(x)\tilde{U}_0(t)\Phi_v\|\mathrm{d}t = O(|v|^{-1}) \tag{2.3}$$

holds as $|v| \to \infty$.

Proof. We can take $f \in C_0^\infty(\mathbb{R}^n)$ such that $\Phi_0 = f(p)\Phi_0$ and $\mathrm{supp}\, f \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leqslant \eta\}$ with some $\eta > 0$. We separete the integral such that

$$\int_{-\infty}^{\infty} = \int_{|t|<r_0} + \int_{|t|\geqslant r_0} \tag{2.4}$$

and consider $|t| < r_0$ first. By (1.18) and the relation

$$e^{-\mathrm{i}v\cdot x}e^{-\mathrm{i}\tan\omega t p^2/2}e^{\mathrm{i}v\cdot x} = e^{-\mathrm{i}\tan\omega t|v|^2/2}e^{-\mathrm{i}\tan\omega t p\cdot v}e^{-\mathrm{i}\tan\omega t p^2/2}, \tag{2.5}$$

we have

$$\begin{aligned} &\|V^{\mathrm{bdd}}(x)e^{-\mathrm{i}tH_0}\Phi_v\| = \|V^{\mathrm{bdd}}(\cos\omega t x)e^{-\mathrm{i}\tan\omega t p^2/2}\Phi_v\| \\ &= \|V^{\mathrm{bdd}}(\cos\omega t x + \sin\omega t v)e^{-\mathrm{i}\tan\omega t p^2/2}\Phi_0\| \leqslant I_1 + I_2 + I_3, \end{aligned} \tag{2.6}$$

where we put

$$\begin{aligned} I_1 &= \|V^{\mathrm{bdd}}(x)\|\|F(|x| \geqslant |\tan\omega t||v|/2)e^{-\mathrm{i}\tan\omega t p^2/2}f(p) \\ &\qquad \times F(|x| \leqslant |\tan\omega t||v|/4)\|\|\Phi_0\|, \\ I_2 &= \|V^{\mathrm{bdd}}(x)\|\|F(|x| \geqslant |\tan\omega t||v|/2)e^{-\mathrm{i}\tan\omega t p^2/2}f(p) \\ &\qquad \times F(|x| > |\tan\omega t||v|/4)\langle x\rangle^{-2}\|\|\langle x\rangle^2\Phi_0\|, \\ I_3 &= \|V^{\mathrm{bdd}}(\cos\omega t x + \sin\omega t v)F(|x| < |\tan\omega t||v|/2)\|\|\Phi_0\| \end{aligned} \tag{2.7}$$

as in the proof of [8, Proposition 2.2] (see also [1], [2], [3], [6], [9], [10], [27], and [31]). Because of the periodicity of $\tan\omega t$, we can assume that

$$\pi/(2\omega) \leqslant r_0 < \pi/\omega \tag{2.8}$$

without loss of generality. Moreover, if $r_0 < \pi/(2\omega)$, we can demonstrate our proofs much more simply. We state this details in Remark 3.3. Using Proposition 2.2 for I_1, we have

$$\int_{|t|<r_0} (I_1 + I_2)\mathrm{d}t \lesssim \int_0^{\pi/(2\omega)} + \int_{\pi/(2\omega)}^{r_0} \langle\tan\omega t v\rangle^{-2}\mathrm{d}t. \tag{2.9}$$

When $0 \leqslant t < \pi/(2\omega)$, $\tan \omega t \geqslant \omega t$ and

$$\int_0^{\pi/(2\omega)} \langle \tan \omega t v \rangle^{-2} \mathrm{d}t \leqslant \int_0^{\pi/(2\omega)} \langle \omega t v \rangle^{-2} \mathrm{d}t = |v|^{-1} \int_0^{\pi|v|/(2\omega)} \langle \omega \tau \rangle^{-2} \mathrm{d}\tau = O(|v|^{-1}) \tag{2.10}$$

hold by changing $\tau = t|v|$. When $\pi/(2\omega) \leqslant t < r_0$, $|\tan \omega t| > \pi - \omega t$ and

$$\begin{aligned} \int_{\pi/(2\omega)}^{r_0} \langle \tan \omega t v \rangle^{-2} \mathrm{d}t &\leqslant \int_{\pi/(2\omega)}^{r_0} \langle (\pi - \omega t) v \rangle^{-2} \mathrm{d}t \\ &= |v|^{-1} \int_{(\pi/\omega - r_0)|v|}^{\pi|v|/(2\omega)} \langle \omega \tau \rangle^{-2} \mathrm{d}\tau = O(|v|^{-2}) \end{aligned} \tag{2.11}$$

hold by changing $\tau = (\pi/\omega - t)|v|$. As for I_3, when $|x| < |\tan \omega t||v|/2$,

$$|\cos \omega t x + \sin \omega t v| > |\sin \omega t||v|/2 \tag{2.12}$$

and

$$I_3 \leqslant \|V^{\mathrm{bdd}}(x) F(|x| > |\sin \omega t||v|/2)\| \|\Phi_0\| \tag{2.13}$$

hold. Assuming (1.5), we have

$$\int_{|t|<r_0} I_3 \mathrm{d}t \lesssim \int_0^{\pi/(2\omega)} + \int_{\pi/(2\omega)}^{r_0} \langle \sin \omega t v \rangle^{-\rho} \mathrm{d}t = O(|v|^{-1}) + O(|v|^{-\rho}), \tag{2.14}$$

noting that $\rho > 1/(1-\lambda) > 1$ because $\sin \omega t \geqslant \omega t/2$ when $0 \leqslant t < \pi/(2\omega)$, and $\sin \omega t > (\pi - \omega t)/2$ when $\pi/(2\omega) \leqslant t < r_0$. We next consider the integral over $|t| \geqslant r_0$, in particular, we consider $t \geqslant r_0$. Integral over $t \leqslant -r_0$ can be estimated in the same way with $t \geqslant r_0$. By (1.20) and relation

$$\begin{aligned} &e^{-\mathrm{i}v\cdot x} e^{-\mathrm{i}t^{1-2\lambda} p^2/(2(1-2\lambda))} e^{\mathrm{i}v\cdot x} \\ &= e^{-\mathrm{i}t^{1-2\lambda}|v|^2/(2(1-2\lambda))} e^{-\mathrm{i}t^{1-2\lambda} p\cdot v/(1-2\lambda)} e^{-\mathrm{i}t^{1-2\lambda} p^2/(2(1-2\lambda))}, \end{aligned} \tag{2.15}$$

we have

$$\begin{aligned} &\|V^{\mathrm{bdd}}(x) \tilde{U}_0(t) \Phi_v\| = \|V^{\mathrm{bdd}}(t^\lambda x) e^{-\mathrm{i}t^{1-2\lambda} p^2/(2(1-2\lambda))} \Phi_v\| \\ &= \|V^{\mathrm{bdd}}(t^\lambda x + t^{1-\lambda} v/(1-2\lambda)) e^{-\mathrm{i}t^{1-2\lambda} p^2/(2(1-2\lambda))} \Phi_0\| \lesssim I_4 + I_5 + I_6, \end{aligned} \tag{2.16}$$

where we put, with $N \in \mathbb{N}$,

$$\begin{aligned} I_4 &= \|F(|x| \geqslant t^{1-2\lambda}|v|/(2(1-2\lambda))) e^{-\mathrm{i}t^{1-2\lambda} p^2/(2(1-2\lambda))} f(p) \\ &\qquad \times F(|x| \leqslant t^{1-2\lambda}|v|/(4(1-2\lambda)))\|, \\ I_5 &= \|F(|x| > t^{1-2\lambda}|v|/(4(1-2\lambda))) \langle x \rangle^{-N}\|, \\ I_6 &= \|V^{\mathrm{bdd}}(t^\lambda x + t^{1-\lambda} v/(1-2\lambda)) F(|x| < t^{1-2\lambda}|v|/(2(1-2\lambda)))\| \end{aligned} \tag{2.17}$$

as in (2.7). Using Proposition 2.2 for I_4, we have

$$\int_{t\geqslant r_0}(I_4+I_5)\mathrm{d}t \lesssim \int_{r_0}^{\infty}\langle t^{1-2\lambda}v\rangle^{-N}\mathrm{d}t$$
$$= (|v|^{-1/(1-2\lambda)}/(1-2\lambda))\int_{r_0^{1-2\lambda}|v|}^{\infty}\langle\tau\rangle^{-N}\tau^{2\lambda/(1-2\lambda)}\mathrm{d}\tau = O(|v|^{-N}), \tag{2.18}$$

where we changed $\tau = t^{1-2\lambda}|v|$ and chose $N \gg 1$ such that $-N+2\lambda/(1-2\lambda) < -1$. As for I_6, when $|x| < t^{1-2\lambda}|v|/(2(1-2\lambda))$,

$$|t^{\lambda}x + t^{1-\lambda}v/(1-2\lambda)| > t^{1-\lambda}|v|/(2(1-2\lambda)) \tag{2.19}$$

and

$$I_6 \leqslant \||V^{\mathrm{bdd}}(x)F(|x| > t^{1-\lambda}|v|/(2(1-2\lambda)))\| \tag{2.20}$$

hold. By the assumption of V^{bdd} (1.5), we have

$$\int_{r_0}^{\infty} I_6\mathrm{d}t \lesssim \int_{r_0}^{\infty}\langle t^{1-\lambda}v\rangle^{-\rho}\mathrm{d}t \tag{2.21}$$
$$= (|v|^{-1/(1-\lambda)}/(1-\lambda))\int_{r_0^{1-\lambda}|v|}^{\infty}\langle\tau\rangle^{-\rho}\tau^{\lambda/(1-\lambda)}\mathrm{d}\tau = O(|v|^{-\rho}), \tag{2.22}$$

where we changed $\tau = t^{1-\lambda}|v|$ and used $-\rho + \lambda/(1-\lambda) < -1$. Equations (2.10), (2.11), (2.14), (2.18), and (2.22) imply (2.3). □

Lemma 2.4. *Let Φ_v be as in Theorem 2.1. Then*

$$\sup_{t\in\mathbb{R}}\|(U(t,0)\tilde{W}^{-} - \tilde{U}_0(t))\Phi_v\| = O(|v|^{-1}) \tag{2.23}$$

holds as $|v| \to \infty$.

Proof. This proof is taken from [6, Corollary 2.3] (see also [1], [2], [3], [8], [9], [10], [22], [23], [24], [27], and [31]). We calculate

$$\tilde{W}^{-} - U(t,0)^*\tilde{U}_0(t) = -\int_{-\infty}^{t}(\mathrm{d}/\mathrm{d}\tau)U(\tau,0)^*\tilde{U}_0(\tau)\mathrm{d}\tau$$
$$= -\mathrm{i}\int_{-\infty}^{t}U(\tau,0)^*V^{\mathrm{bdd}}(x)\tilde{U}_0(\tau)\mathrm{d}\tau. \tag{2.24}$$

We thus have

$$\|(W^{-} - U(t,0)^*\tilde{U}_0(t))\Phi_v\| \leqslant \int_0^{\infty}\|V^{\mathrm{bdd}}(x)\tilde{U}_0(\tau)\Phi_v\|\mathrm{d}\tau = O(|v|^{-1}) \tag{2.25}$$

as $|v| \to \infty$ by Lemma 2.3. This completes the proof. □

Proof of Theorem 2.1. It follows from

$$\mathrm{i}(\tilde{S}-1)=\mathrm{i}(\tilde{W}^{+}-\tilde{W}^{-})^{*}\tilde{W}^{-}=\int_{-\infty}^{\infty}\tilde{U}_0(t)^{*}V^{\mathrm{bdd}}(x)U(t,0)W^{-}\mathrm{d}\tau \tag{2.26}$$

that

$$|v|(\mathrm{i}(\tilde{S}-1)\Phi_v,\Psi_v)=|v|\int_{-\infty}^{\infty}(V^{\mathrm{bdd}}(x)\tilde{U}_0(t)\Phi_v,\tilde{U}_0(t)\Psi_v)\mathrm{d}t+R(v) \tag{2.27}$$

where

$$R(v)=|v|\int_{-\infty}^{\infty}((U(t,0)\tilde{W}^{-}-\tilde{U}_0(t))\Phi_v,V^{\mathrm{bdd}}(x)\tilde{U}_0(t)\Psi_v)\mathrm{d}t=O(|v|^{-1}) \tag{2.28}$$

as $|v|\to\infty$ by virtue of Lemmas 2.3 and 2.4. We separate the integral on the right-hand side of (2.27) such that

$$\int_{-\infty}^{\infty}=\int_{|t|<\pi/(2\omega)}+\int_{\pi/(2\omega)\leqslant|t|<r_0}+\int_{|t|\geqslant r_0} \tag{2.29}$$

and consider the part $|t|<\pi/(2/\omega)$ first. By (1.18) and (2.5), we have

$$e^{-\mathrm{i}v\cdot x}e^{\mathrm{i}tH_0}V^{\mathrm{bdd}}(x)e^{-\mathrm{i}tH_0}e^{\mathrm{i}v\cdot x}=e^{\mathrm{i}tH_0}V^{\mathrm{bdd}}(x+\sin\omega t v)e^{-\mathrm{i}tH_0}. \tag{2.30}$$

We thus have

$$\begin{aligned}
&|v|\int_{|t|<\pi/(2\omega)}(V^{\mathrm{bdd}}(x)e^{-\mathrm{i}tH_0}\Phi_v,e^{-\mathrm{i}tH_0}\Psi_v)\mathrm{d}t\\
&=|v|\int_{|t|<\pi/(2\omega)}(V^{\mathrm{bdd}}(x+\sin\omega t v)e^{-\mathrm{i}tH_0}\Phi_0,e^{-\mathrm{i}tH_0}\Psi_0)\mathrm{d}t\\
&=\int_{|\tau|<|v|/\omega}(1/\sqrt{1-(\omega\tau/|v|)^2})(V^{\mathrm{bdd}}(x+\hat{v}\omega\tau)e^{-\mathrm{i}\arcsin(\omega\tau/|v|)H_0/\omega}\Phi_0,\\
&\qquad e^{-\mathrm{i}\arcsin(\omega\tau/|v|)H_0/\omega}\Psi_0)\mathrm{d}\tau
\end{aligned} \tag{2.31}$$

by changing $\tau=\sin\omega t|v|/\omega$. Because $e^{-\mathrm{i}tH_0}$ is strongly continuous at $t=0$, we have

$$\begin{aligned}
&(1/\sqrt{1-(\omega\tau/|v|)^2})(V^{\mathrm{bdd}}(x+\hat{v}\omega\tau)e^{-\mathrm{i}\arcsin(\omega\tau/|v|)H_0/\omega}\Phi_0,e^{-\mathrm{i}\arcsin(\omega\tau/|v|)H_0/\omega}\Psi_0)\\
&\qquad\to(V^{\mathrm{bdd}}(x+\hat{v}\omega\tau)\Phi_0,\Psi_0)
\end{aligned} \tag{2.32}$$

as $|v|\to\infty$ pointwisely in $\tau\in\mathbb{R}$. In addition, we have

$$\begin{aligned}
&|v|\int_{|t|<\pi/(2\omega)}|(V^{\mathrm{bdd}}(x)e^{-\mathrm{i}tH_0}\Phi_v,e^{-\mathrm{i}tH_0}\Psi_v)|\mathrm{d}t\\
&=\int_{|\tau|<\pi|v|/(2\omega)}|(V^{\mathrm{bdd}}(x)e^{-\mathrm{i}(\tau/|v|)H_0}\Phi_v,e^{-\mathrm{i}(\tau/|v|)H_0}\Psi_v)|\mathrm{d}\tau
\end{aligned} \tag{2.33}$$

by changing $\tau = |v|t$. It follows from the calculations in the proof of Lemma 2.3 that

$$\begin{aligned}|(V^{\mathrm{bdd}}(x)e^{-\mathrm{i}(\tau/|v|)H_0}\Phi_v, e^{-\mathrm{i}(\tau/|v|)H_0}\Psi_v)| &\leqslant \|V^{\mathrm{bdd}}(x)e^{-\mathrm{i}(\tau/|v|)H_0}\Phi_v\|\|\Psi_0\| \\ &\lesssim \langle\tan\omega(\tau/|v|)v\rangle^{-2} + \langle\sin\omega(\tau/|v|)v\rangle^{-\rho} \lesssim \langle\tau\rangle^{-2} + \langle\tau\rangle^{-\rho}.\end{aligned} \tag{2.34}$$

We therefore obtain

$$|v|\int_{|t|<\pi/(2\omega)}(V^{\mathrm{bdd}}(x)e^{-\mathrm{i}tH_0}\Phi_v, e^{-\mathrm{i}tH_0}\Psi_v)\mathrm{d}t \to \int_{-\infty}^{\infty}(V^{\mathrm{bdd}}(x+\hat{v}\omega\tau)\Phi_0, \Psi_0)\mathrm{d}t \tag{2.35}$$

as $|v| \to \infty$ by the Lebesgue dominated convergence theorem. To complete our proof, we prove that the second and third integrals of (2.29) converge to zero as $|v| \to \infty$. This was almost proved already in Lemma 2.3. Indeed, for the second integral over $\pi/(2\omega) \leqslant |t| < r_0$, we find that $|(V^{\mathrm{bdd}}(x)e^{-\mathrm{i}tH_0}\Phi_v, e^{-\mathrm{i}tH_0}\Psi_v)| \leqslant \|V^{\mathrm{bdd}}(x)e^{-\mathrm{i}tH_0}\Phi_v\|\|\Psi_0\|$ and

$$\begin{aligned}&|v|\int_{\pi/(2\omega)\leqslant|t|<r_0}|(V^{\mathrm{bdd}}(x)e^{-\mathrm{i}tH_0}\Phi_v, e^{-\mathrm{i}tH_0}\Psi_v)|\mathrm{d}t \\ \lesssim\ &|v|\int_{\pi/(2\omega)}^{r_0}(\langle\tan\omega tv\rangle^{-2} + \langle\sin\omega tv\rangle^{-\rho})\mathrm{d}t = O(|v|^{-1}) + O(|v|^{-\rho+1})\end{aligned} \tag{2.36}$$

by (2.11) and (2.14). For the third integral on $|t| \geqslant r_0$, we find that

$$|v|\int_{|t|\geqslant r_0}|(V^{\mathrm{bdd}}(x)\tilde{U}_0(t)\Phi_v, \tilde{U}_0(t)\Psi_v)|\mathrm{d}t = O(|v|^{-N+1}) + O(|v|^{-\rho+1}) \tag{2.37}$$

by (2.18) and (2.22). With $N \geqslant 2$ and $\rho > 1$, equations (2.35), (2.36) and (2.37) complete the proof. □

3 Singular case

We now consider the instances $V^{\mathrm{sing}} \not\equiv 0$ and prove the following reconstruction formula. At the end of this section, we finally complete the proof of Theorem 1.5.

Theorem 3.1. *Let Φ_v and Ψ_v be as in Theorem 2.1. Then*

$$\lim_{|v|\to\infty}|v|(\mathrm{i}(\tilde{S}(V)-1)\Phi_v, \Psi_v) = \int_{-\infty}^{\infty}(V(x+\hat{v}\omega t)\Phi_0, \Psi_0)\mathrm{d}t \tag{3.1}$$

holds.

To prove Theorem 3.1, we prepare the following Lemma 3.2, which is the singular version of Lemma 2.3.

Lemma 3.2. *Let Φ_v be as in Theorem 2.1. Then*

$$\int_{-\infty}^{\infty} \|V^{\mathrm{sing}}(x)\tilde{U}_0(t)\Phi_v\|\mathrm{d}t = O(|v|^{-1}) \tag{3.2}$$

holds as $|v| \to \infty$.

Proof. As in the proof of Lemma 2.3, we take $f \in C_0^\infty(\mathbb{R}^n)$ such that $\Phi_0 = f(p)\Phi_0$ and $\operatorname{supp} f \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leqslant \eta\}$ with some $\eta > 0$ and assume that (2.8). Separating the integral such that

$$\int_{-\infty}^{\infty} = \int_{|t|\leqslant \pi/(4\omega)} + \int_{\pi/(4\omega)<|t|<r_0} + \int_{|t|\geqslant r_0} \tag{3.3}$$

and first consider the part $|t| \leqslant \pi/(4\omega)$. Similar to (2.6) and (2.7), we have

$$\begin{aligned} \|V^{\mathrm{sing}}(x)e^{-\mathrm{i}tH_0}\Phi_v\| &= \|V^{\mathrm{sing}}(\cos\omega t x + \sin\omega t v)\langle p/\cos\omega t\rangle^{-2} \\ &\quad \times e^{-\mathrm{i}\tan\omega t p^2/2}\langle p/\cos\omega t\rangle^2\Phi_0\| \leqslant I_1 + I_2 + I_3, \end{aligned} \tag{3.4}$$

where we put

$$\begin{aligned} I_1 &= \|V^{\mathrm{sing}}(\cos\omega t x + \sin\omega t v)\langle p/\cos\omega t\rangle^{-2}\|\|\langle p/\cos\omega t\rangle^2\Phi_0\| \\ &\quad \times \|F(|x| \geqslant |\tan\omega t||v|/2)e^{-\mathrm{i}\tan\omega t p^2/2}f(p)F(|x| \leqslant |\tan\omega t||v|/4)\|, \\ I_2 &= \|V^{\mathrm{sing}}(\cos\omega t x + \sin\omega t v)\langle p/\cos\omega t\rangle^{-2}\|\|\langle x\rangle^2\langle p/\cos\omega t\rangle^2\Phi_0\| \\ &\quad \times \|F(|x| \geqslant |\tan\omega t||v|/2)e^{-\mathrm{i}\tan\omega t p^2/2}f(p)F(|x| > |\tan\omega t||v|/4)\langle x\rangle^{-2}\|, \\ I_3 &= \|V^{\mathrm{sing}}(\cos\omega t x + \sin\omega t v)\langle p/\cos\omega t\rangle^{-2} \\ &\quad \times F(|x| < |\tan\omega t||v|/2)\|\|\langle p/\cos\omega t\rangle^2\Phi_0\| \end{aligned} \tag{3.5}$$

as in the proof of [8, Proposition 2.3]. Noting that

$$\begin{aligned} &\|V^{\mathrm{sing}}(\cos\omega t x + \sin\omega t v)\langle p/\cos\omega t\rangle^{-2}\| \\ &= \|V^{\mathrm{sing}}(\cos\omega t x)\langle p/\cos\omega t\rangle^{-2}\| = \|V(x)\langle p\rangle^{-2}\| \end{aligned} \tag{3.6}$$

and that

$$\|\langle p/\cos\omega t\rangle^2\Phi_0\| \leqslant \|\langle\sqrt{2}p\rangle^2\Phi_0\| \tag{3.7}$$

because $|t| \leqslant \pi/(4\omega)$, we have

$$\int_{|t|\leqslant\pi/(4\omega)} (I_1 + I_2)\mathrm{d}t \lesssim \int_0^{\pi/(4\omega)} \langle\omega t v\rangle^{-2}\mathrm{d}t = O(|v|^{-1}) \tag{3.8}$$

as in the proof of Lemma 2.3. Because

$$|\cos\omega t x + \sin\omega t v| > |\sin\omega t||v|/2 \geqslant |t||v|/4 \tag{3.9}$$

holds when $|x| < |\tan \omega t||v|/2$, we have

$$\int_{|t|\leqslant\pi/(4\omega)} I_3 \mathrm{d}t \lesssim \int_0^{\pi/(4\omega)} \|V^{\mathrm{sing}}(x)\langle p\rangle^{-2}F(|x| \geqslant |v|t/4)\|\mathrm{d}t$$
$$= |v|^{-1}\int_0^1 + |v|^{-1}\int_1^{\pi|v|/(4\omega)} \|V^{\mathrm{sing}}(x)\langle p\rangle^{-2}F(|x| \geqslant \tau/4)\|\mathrm{d}\tau \tag{3.10}$$

by changing $\tau = |v|t$. The first integral over interval $0 \leqslant \tau < 1$ clearly has order $O(|v|^{-1})$. For the second integral over $1 \leqslant \tau \leqslant \pi|v|/(4\omega)$, we take $\chi \in C^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ if $|x| \geqslant 1$ and $\chi(x) = 0$ if $|x| \leqslant 1/2$. We then have

$$\|V^{\mathrm{sing}}(x)\langle p\rangle^{-2}F(|x| \geqslant \tau/4)\| \leqslant \|V^{\mathrm{sing}}(x)\langle p\rangle^{-2}\chi(4x/\tau)\|$$
$$\lesssim \|V^{\mathrm{sing}}(x)\chi(4x/\tau)\langle p\rangle^{-2}\| + \tau^{-1}\|V^{\mathrm{sing}}(x)(\nabla\chi)(4x/\tau)\langle p\rangle^{-2}\| + \tau^{-2}\|V^{\mathrm{sing}}(x)\langle p\rangle^{-2}\| \tag{3.11}$$

by calculating the commutator $[\langle p\rangle^{-2}, \chi(4x/\tau)]$. Noting that V^{sing} is compactly supported and that the integral intervals of the first and second terms of (3.11) are finite for $|v| \gg 1$, we have

$$\int_{|t|\leqslant\pi/(4\omega)} I_3 \mathrm{d}t = O(|v|^{-1}). \tag{3.12}$$

We next consider the integral over $\pi/(4\omega) < |t| < r_0$. The strategy for the estimates of the integral terms from the proof of [8, Proposition 2.3] (see also [24, Lemma 4]). Using Mehler formula (1.13)

$$e^{-\mathrm{i}tH_0}\Phi_v = \mathscr{M}(\tan\omega t)e^{-\mathrm{i}\sin\omega t v\cdot p}\mathscr{D}(\sin\omega t)\mathscr{F}\mathscr{M}(\tan\omega t)\Phi_0$$
$$= e^{-\mathrm{i}\sin\omega t v\cdot p}e^{\mathrm{i}\cos\omega t\sin\omega t v^2/2}e^{\mathrm{i}\cos\omega t v\cdot x}e^{-\mathrm{i}tH_0}\Phi_0 \tag{3.13}$$

holds. Therefore we have

$$\|V^{\mathrm{sing}}(x)\tilde{U}_0(t)\Phi_v\| = \|V^{\mathrm{sing}}(x + \sin\omega t v)e^{-\mathrm{i}tH_0}\Phi_0\|$$
$$= \|V^{\mathrm{sing}}(\sin\omega t(x+v))\mathscr{F}\mathscr{M}(\tan\omega t)\Phi_0\| \leqslant I_4 + I_5, \tag{3.14}$$

where we put

$$I_4 = \|V^{\mathrm{sing}}(\sin\omega t(x+v))\langle p/\sin\omega t\rangle^{-2}F(|x| \leqslant |v|/2)\|\|\langle x/\sin\omega t\rangle^2\Phi_0\|,$$
$$I_5 = \|V^{\mathrm{sing}}(\sin\omega t(x+v))\langle p/\sin\omega t\rangle^{-2}\|$$
$$\times \|F(|x| > |v|/2)\mathscr{F}\mathscr{M}(\tan\omega t)\langle x/\sin\omega t\rangle^2\Phi_0\|. \tag{3.15}$$

Clearly

$$\|\langle x/\sin\omega t\rangle^2\Phi_0\| \lesssim \|\langle x\rangle^2\Phi_0\| \tag{3.16}$$

holds because $\pi/(4\omega) < |t| < r_0$ and

$$0 < \min\{1/\sqrt{2}, \sin\omega r_0\} < |\sin\omega t| \tag{3.17}$$

noting $r_0 < \pi/\omega$. When $|x| \leqslant |v|/2$, there exists a small constant $c > 0$ such that

$$|\sin\omega t(x+v))| \geqslant |\sin\omega t||v|/2 \geqslant c|t||v| \tag{3.18}$$

again noting $r_0 < \pi/\omega$. We thus have

$$\begin{aligned}\int_{\pi/(4\omega)<|t|<r_0} I_4 \mathrm{d}t &\lesssim \int_{\pi/(4\omega)}^{r_0} \|V^{\mathrm{sing}}(x)\langle p\rangle^{-2} F(|x| \geqslant ct|v|)\| \mathrm{d}t \\ &= |v|^{-1} \int_{\pi|v|/(4\omega)}^{r_0|v|} \|V^{\mathrm{sing}}(x)\langle p\rangle^{-2} F(|x| \geqslant c\tau)\| \mathrm{d}\tau\end{aligned} \tag{3.19}$$

by changing $\tau = |v|t$. For $\tau > \pi|v|/(4\omega) \gg 1$, we have

$$\|V^{\mathrm{sing}}(x)\langle p\rangle^{-2} F(|x| \geqslant c\tau)\| \lesssim \tau^{-2} \|V^{\mathrm{sing}}(x)\langle p\rangle^{-2}\| \tag{3.20}$$

as in (3.11) noting that V^{sing} is compactly supported. Therefore, we can obtain

$$\int_{\pi/(4\omega)<|t|<r_0} I_4 \mathrm{d}t = O(|v|^{-2}). \tag{3.21}$$

For the integral I_5, we write

$$\begin{aligned}&\mathscr{F}\mathscr{M}(\tan\omega t)\langle x/\sin\omega t\rangle^2 \Phi_0 \\ &= \int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot y} e^{\mathrm{i}y^2/(2\tan\omega t)} \langle y/\sin\omega t\rangle^2 \Phi_0(y) \mathrm{d}y/(2\pi)^{n/2}.\end{aligned} \tag{3.22}$$

Using the relation $e^{-\mathrm{i}x\cdot y} = \langle x\rangle^{-2}(1 + \mathrm{i}x\cdot\nabla_y)e^{-\mathrm{i}x\cdot y}$ and integrating by parts, we have

$$\begin{aligned}\mathscr{F}\mathscr{M}(\tan\omega t)\langle x/\sin\omega t\rangle^2 \Phi_0 &= \langle x\rangle^{-2} \mathscr{F}\mathscr{M}(\tan\omega t)\langle x/\sin\omega t\rangle^2 \Phi_0 \\ &+(1/\tan\omega t)\langle x\rangle^{-2} x\cdot \mathscr{F} x\mathscr{M}(\tan\omega t)\langle x/\sin\omega t\rangle^2 \Phi_0 \\ &-\mathrm{i}\langle x\rangle^{-2} x\cdot \mathscr{F}\mathscr{M}(\tan\omega t)\nabla_x \langle x/\sin\omega t\rangle^2 \Phi_0.\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}&\|F(|x| > |v|/2)\mathscr{F}\mathscr{M}(\tan\omega t)\langle x/\sin\omega t\rangle^2 \Phi_0\| \\ &\lesssim |v|^{-2}\|\langle x\rangle^2 \Phi_0\| + |v|^{-1}(\|\langle x\rangle^3 \Phi_0\| + \|\langle x\rangle^2 \nabla\Phi_0\|).\end{aligned} \tag{3.24}$$

It follows from (3.24) and

$$\|V^{\mathrm{sing}}(\sin\omega t(x+v))\langle p/\sin\omega t\rangle^{-2}\| = \|V^{\mathrm{sing}}(x)\langle p\rangle^{-2}\| \tag{3.25}$$

that

$$\int_{\pi/(4\omega)<|t|<r_0} I_5 \mathrm{d}t = O(|v|^{-1}). \tag{3.26}$$

We consider the final integral over $|t| \geqslant r_0$, in particular $t \geqslant r_0$. In the same way with (2.16), (2.17), and (3.5), we have

$$\begin{aligned} \|V^{\mathrm{sing}}(x)e^{-\mathrm{i}H_0}\Phi_v\| = \|V^{\mathrm{sing}}(t^\lambda x + t^{1-\lambda}v/(1-2\lambda))\langle p/t^\lambda\rangle^{-2} \\ \times e^{-\mathrm{i}t^{1-2\lambda}p^2/(2(1-2\lambda))}\langle p/t^\lambda\rangle^2\Phi_0\| \lesssim I_6 + I_7 + I_8, \end{aligned} \tag{3.27}$$

where we put, with $N \in \mathbb{N}$,

$$\begin{aligned} I_6 &= \|F(|x| \geqslant t^{1-2\lambda}|v|/(2(1-2\lambda)))e^{-\mathrm{i}t^{1-2\lambda}p^2/(2(1-2\lambda))}f(p) \\ &\quad \times F(|x| \leqslant t^{1-2\lambda}|v|/(2(1-2\lambda)))\|, \\ I_7 &= \|F(|x| > t^{1-2\lambda}|v|/(4(1-2\lambda)))\langle x\rangle^{-N}\|, \\ I_8 &= V^{\mathrm{sing}}(t^\lambda x + t^{1-\lambda}v/(1-2\lambda))\langle p/t^\lambda\rangle^{-2}F(|x| < t^{1-2\lambda}|v|/(2(1-2\lambda)))\|. \end{aligned} \tag{3.28}$$

We here used $\|V^{\mathrm{sing}}(t^\lambda x)\langle p/t^\lambda\rangle^{-2}\| = \|V^{\mathrm{sing}}(x)\langle p\rangle^{-2}\|$, $\|\langle p/t^\lambda\rangle^2\Phi_0\| \leqslant \|\langle p/r_0^\lambda\rangle^2\Phi_0\|$ and

$$\|\langle x\rangle^N\langle p/t^\lambda\rangle^2\Phi_0\| \leqslant \|\langle p/t^\lambda\rangle^2\langle x\rangle^N\Phi_0\| + \|[\langle x\rangle^N, \langle p/t^\lambda\rangle^2]\Phi_0\| \lesssim 1 \tag{3.29}$$

in (3.28). We immediately have

$$\int_{r_0}^\infty (I_6 + I_7)\mathrm{d}t = O(|v|^{-N}) \tag{3.30}$$

as in (2.18) for $N \gg 1$ such that $-N + 2\lambda/(1-2\lambda) < -1$. Because (2.19) holds when $|x| < t^{1-2\lambda}|v|/(2(1-2\lambda))$, we have

$$\begin{aligned} \int_{r_0}^\infty I_8 \mathrm{d}t &\leqslant \int_{r_0}^\infty \|V^{\mathrm{sing}}(x)\langle p\rangle^{-2}F(|x| > t^{1-\lambda}|v|/(2(1-2\lambda)))\|\mathrm{d}t \\ \lesssim |v|^{-1/(1-\lambda)} &\int_{r_0^{1-\lambda}|v|}^\infty \tau^{\lambda/(1-\lambda)}\|V^{\mathrm{sing}}(x)\langle p\rangle^{-2}F(|x| > \tau/(2(1-2\lambda)))\|\mathrm{d}\tau \end{aligned} \tag{3.31}$$

by changing $\tau = t^{1-\lambda}|v|$. As in (3.20), we thus have

$$\int_{r_0}^\infty I_8 \mathrm{d}t \lesssim |v|^{-1/(1-\lambda)} \int_{r_0^{1-\lambda}|v|}^\infty \tau^{\lambda/(1-\lambda)-2}\mathrm{d}\tau = O(|v|^{-2}) \tag{3.32}$$

noting that $\lambda/(1-\lambda) - 2 < -1$. Equations (3.8), (3.12), (3.21), (3.26), (3.30), and (3.32) imply (3.2). $\square$

Proof of Theorem 3.1. Note that Lemma 2.4 also holds for $V = V^{\mathrm{bdd}} + V^{\mathrm{sing}}$ by virtue of Lemma 3.2. We therefore have

$$|v|(\mathrm{i}(\tilde{S}-1)\Phi_v, \Psi_v) = |v| \int_{-\infty}^{\infty} (V(x)\tilde{U}_0(t)\Phi_v, \tilde{U}_0(t)\Psi_v)\mathrm{d}t + O(|v|^{-1}). \tag{3.33}$$

Because we have already proved

$$|v| \int_{-\infty}^{\infty} (V^{\mathrm{bdd}}(x)\tilde{U}_0(t)\Phi_v, \tilde{U}_0(t)\Psi_v)\mathrm{d}t \to \int_{-\infty}^{\infty} (V^{\mathrm{bdd}}(x+\hat{v}\omega\tau)\Phi_0, \Psi_0)\mathrm{d}t \tag{3.34}$$

as $|v| \to \infty$ in the proof of Theorem 2.1, it suffices to prove

$$|v| \int_{-\infty}^{\infty} (V^{\mathrm{sing}}(x)\tilde{U}_0(t)\Phi_v, \tilde{U}_0(t)\Psi_v)\mathrm{d}t \to \int_{-\infty}^{\infty} (V^{\mathrm{sing}}(x+\hat{v}\omega\tau)\Phi_0, \Psi_0)\mathrm{d}t \tag{3.35}$$

as $|v| \to \infty$. We separate the integral such that

$$\int_{-\infty}^{\infty} = \int_{|t|\leqslant \pi/(4\omega)} + \int_{\pi/(4\omega)<|t|<r_0} + \int_{|t|\geqslant r_0} \tag{3.36}$$

and first consider the integral over $|t| \leqslant \pi/(4\omega)$. As in (2.31), we have

$$\begin{aligned}|v| \int_{|t|\leqslant \pi/(4\omega)} (V^{\mathrm{sing}}(x)e^{-\mathrm{i}tH_0}\Phi_v, e^{-\mathrm{i}tH_0}\Psi_v)\mathrm{d}t = \int_{|\tau|\leqslant |v|/(\sqrt{2}\omega)} (1/\sqrt{1-(\omega\tau/|v|)^2}) \\ \times (V^{\mathrm{sing}}(x+\hat{v}\omega\tau)e^{-\mathrm{i}\arcsin(\omega\tau/|v|)H_0/\omega}\Phi_0, e^{-\mathrm{i}\arcsin(\omega\tau/|v|)H_0/\omega}\Psi_0)\mathrm{d}\tau.\end{aligned} \tag{3.37}$$

Because

$$p_j e^{-\mathrm{i}tH_0}\Phi_0 = \sin\omega t e^{-\mathrm{i}tH_0} x_j\Phi_0 + \sin\omega t \tan\omega t e^{-\mathrm{i}tH_0} p_j \Phi_0 + e^{-\mathrm{i}tH_0} p_j\Phi_0/\cos\omega t \tag{3.38}$$

for $1 \leqslant j \leqslant n$ by (1.18), we have

$$\|\langle p\rangle^2 e^{-\mathrm{i}\arcsin(\omega\tau/|v|)H_0/\omega}\Phi_0\| \lesssim 1 \tag{3.39}$$

for $|\tau| \leqslant |v|/(\sqrt{2}\omega)$ and

$$\begin{aligned}(1/\sqrt{1-(\omega\tau/|v|)^2})(V^{\mathrm{sing}}(x+\hat{v}\omega\tau)e^{-\mathrm{i}\arcsin(\omega\tau/|v|)H_0/\omega}\Phi_0, e^{-\mathrm{i}\arcsin(\omega\tau/|v|)H_0/\omega}\Psi_0) \\ \to (V^{\mathrm{sing}}(x+\hat{v}\omega\tau)\Phi_0, \Psi_0)\end{aligned} \tag{3.40}$$

as $|v| \to \infty$ pointwisely in $\tau \in \mathbb{R}$. In addition, we have

$$\begin{aligned}&|v| \int_{|t|\leqslant \pi/(4\omega)} |(V^{\mathrm{sing}}(x)e^{-\mathrm{i}tH_0}\Phi_v, e^{-\mathrm{i}tH_0}\Psi_v)|\mathrm{d}t \\ &= \int_{|\tau|\leqslant \pi|v|/(4\omega)} |(V^{\mathrm{sing}}(x)e^{-\mathrm{i}(\tau/|v|)H_0}\Phi_v, e^{-\mathrm{i}(\tau/|v|)H_0}\Psi_v)|\mathrm{d}\tau\end{aligned} \tag{3.41}$$

by changing $\tau = |v|t$. From the calculations developed in the proof of Lemma 3.2, we find

$$\begin{aligned} |(V^{\mathrm{sing}}(x)e^{-\mathrm{i}(\tau/|v|)H_0}\Phi_v, e^{-\mathrm{i}(\tau/|v|)H_0}\Psi_v)| &\leqslant \|V^{\mathrm{sing}}(x)e^{-\mathrm{i}(\tau/|v|)H_0}\Phi_v\|\|\Psi_0\| \\ &\lesssim \langle\tau\rangle^{-2} + \|V^{\mathrm{sing}}(x)\langle p\rangle^{-2}F(|x| \geqslant |\tau|/4)\|. \end{aligned} \tag{3.42}$$

The right-hand side of (3.42) is integrable for τ independently of v (see (3.11)). We therefore obtain

$$|v|\int_{|t|\leqslant\pi/(4\omega)} (V^{\mathrm{sing}}(x)e^{-\mathrm{i}tH_0}\Phi_v, e^{-\mathrm{i}tH_0}\Psi_v)\mathrm{d}t \to \int_{-\infty}^{\infty}(V^{\mathrm{sing}}(x+\hat{v}\tau)\Phi_0, \Psi_0)\mathrm{d}t \tag{3.43}$$

as $|v| \to \infty$ by the Lebesgue dominated convergence theorem. For the integral over $\pi/(4\omega) < |t| < r_0$, integrating by parts in (3.23) once more, we find that (3.26) has order $O(|v|^{-2})$. we thus have

$$|v|\int_{\pi/(4\omega)<|t|<r_0} |(V^{\mathrm{sing}}(x)e^{-\mathrm{i}tH_0}\Phi_v, e^{-\mathrm{i}tH_0}\Psi_v)|\mathrm{d}t = O(|v|^{-1}) \tag{3.44}$$

as $|v| \to \infty$ by using calculations obtained in the proof of Lemma 3.2 (see also (3.21)). Finally, for the integral over $|t| \geqslant r_0$, we also have

$$|v|\int_{|t|\geqslant r_0} |(V^{\mathrm{sing}}(x)\tilde{U}_0(t)\Phi_v, \tilde{U}_0(t)\Psi_v)|\mathrm{d}t = O(|v|^{-N+1}) + O(|v|^{-1}) \tag{3.45}$$

as $|v| \to \infty$ by (3.30) and (3.32). With $N \geqslant 2$, equations (3.43), (3.44) and (3.45) imply (3.35). □

Remark 3.3. *In our proofs of Theorems 2.1, 3.1, Lemmas 2.3 and 3.2, we partitioned the integrals at points $\pi/(4\omega)$, $\pi/(2\omega)$, and r_0. However, if we assume $0 < r_0 < \pi/(2\omega)$, it suffices to separate the integrals such that*

$$\int_{-\infty}^{\infty} = \int_{|t|<r_0} + \int_{|t|\geqslant r_0} \tag{3.46}$$

in these proofs. We especially do not have to consider the integrals over $\pi/(4\omega) < |t| < r_0$ in the proofs of Theorem 3.1 and Lemma 3.2 even if $r_0 > \pi/(4\omega)$.

Proof of Theorem 1.5. From Theorem 3.1 and the Plancherel formula associated with the Radon transform (see [7, Theorem 2.17 in Chap.1]), $V_1 = V_2$ can be proved similarly as in the proof of [6, Theorem 1.1]. □

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